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DEVELOPMENT OF THE DENSITY: A WIENER–CHAOS APPROACH

by

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1 Introduction

Let $(T, \mathcal{T}, \mu)$ be an atomless measure space with a σ-finite measure μ. Set $\mathcal{H} = L^2(T, \mathcal{T}, \mu)$ and let $W = \{W_h, h \in \mathcal{H}\}$ be a Gaussian zero-mean process with $E(W_h W_{h'}) = \langle h, h' \rangle_{\mathcal{H}}$ defined on some probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{F}$ be the σ-field generated by W. We consider a measurable mapping $F : \Omega \to \mathbb{R}^d$ belonging to $L^2(\Omega, \mathcal{F}, P)$ with Wiener-Chaos decomposition $F = E(F) + \sum_{n=1}^{\infty} I_n(f_n)$. Let $\{F^\varepsilon, \varepsilon \in (0, 1]\}$ be defined by $F^\varepsilon = E(F) + \sum_{n=1}^{\infty} \varepsilon^n I_n(f_n)$. We assume that the probability law of each F^ε is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. The purpose of this paper is to study the Taylor expansion of the density $p^\varepsilon(y)$ of F^ε at $\varepsilon = 0$, where $y = E(F) = E(F^\varepsilon)$. A similar problem has been widely studied for diffusion processes ([2], [4], [3], etc.) for the family F^ε, $\varepsilon \in (0, 1]$ obtained by changing the time t into εt. In this case, due to the scaling property of the Brownian motion, we obtain by probabilistic methods the behaviour of the density $p_t(y)$ of the diffusion X_t for small t. For general families of Wiener functionals the problem has been addressed in [14] and [13].

Our main goal is to give a precise description of the coefficients of the development using the Wiener-Chaos decomposition of F and the particular structure of the family $\{F^\varepsilon, \varepsilon \in (0, 1]\}$. They correspond to densities of completely explicit Radon measures. First we prove differentiability of the mapping $\varepsilon \mapsto F^\varepsilon$ on appropriate derivation spaces related with the Sobolev spaces $\mathbb{D}^{N,2}$ of Malliavin Calculus. The derivatives are expressed in terms of the multiple integrals $I_n(f_n)$. Then, using the approach of [7], [8] we obtain in Theorem 2.4 the Taylor expansion for the density via the development of $f(F^\varepsilon)$, for smooth f, and integration by parts. As for diffusions, the odd order coefficients of the expansion vanish and the non-null ones belong to a finite Wiener-Chaos.

The paper is divided into two sections. The first one is devoted to the proof of the main result described before; the second one contains two examples of hyperbolic stochastic partial differential equations where this results can be applied. As usually, all constants will be denoted by C independently of its value.

2 Development of the density

Let F be a $\mathbb{R}^d$-valued random vector defined on the abstract Wiener space $(\Omega, \mathcal{H}, P)$, belonging to $L^2(\Omega)$. Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ be its Wiener-chaos representation. For any $\varepsilon \in (0,1]$ we define $F^\varepsilon(\omega) = \sum_{n=0}^\infty \varepsilon^n I_n(f_n)$. Clearly, the series defining $F^\varepsilon(w)$ converges in $L^2(\Omega)$. The purpose of this section is to obtain an asymptotic development of the density of F^ε, $p^\varepsilon(y)$ at $y = E(F) = E(F^\varepsilon)$, whenever it exists. We will follow some ideas introduced in [7] (see also [8]). The first result establishes the smoothness of F^ε with respect to ε. To this end we first introduce some derivation spaces, which are related to the classical Sobolev spaces $\mathbb{D}^{k,p}$ of the Malliavin Calculus. For any $j \in \mathbb{Z}^+$, set

$$\Delta^{j,2} = \{F \in L^2(\Omega) : \sum_{k=j}^\infty \left(\frac{k\,!}{(k-j)\,!}\right)^2 k\,!\ \|f_k\|_2^2 < \infty\},$$

where $\|f_k\|_2$ denotes the norm of f_k in $L^2(T^k)$. Notice that $\Delta^{0,2} = L^2(\Omega)$ and $\Delta^{j,2}$ decreases as j increases.
In the next Proposition $d = 1$. For $d > 1$ the result applies componentwise.

Proposition 2.1 *Fix $j \geq 1$, and assume $F \in \Delta^{j+1,2}$. There exists a version of $\{F^\varepsilon,\ \varepsilon \in (0,1)\}$ which is of class C^j. Moreover,*

$$\frac{d^j F^\varepsilon}{d\,\varepsilon^j} = \sum_{k=j}^\infty \frac{k\,!}{(k-j)\,!}\ \varepsilon^{k-j}\ I_k(f_k)\,.$$

Proof. Consider first the case $j = 1$. For ε, ξ with $0 < \varepsilon + \xi < \varepsilon_0 < 1$ we have

$$\frac{F^{\varepsilon+\xi} - F^\varepsilon}{\xi} = \frac{\sum_{k=1}^\infty \sum_{i=0}^{k-1} \binom{k}{i}\ \varepsilon^i\ \xi^{k-i}\ I_k(f_k)}{\xi} = A_1^\varepsilon + \xi\, A_2^{\varepsilon,\xi}. \tag{2.1}$$

with

$$\begin{aligned} A_1^\varepsilon &= \sum_{k=1}^\infty k\,\varepsilon^{k-1}\ I_k(f_k)\,, \\ A_2^{\varepsilon,\xi} &= \sum_{k=2}^\infty \sum_{i=0}^{k-2} \binom{k}{i} \varepsilon^i\ \xi^{k-i-2}\ I_k(f_k)\,. \end{aligned}$$

Since $F \in \Delta^{1,2}$, the series defining A_1^ε converges in $L^2(\Omega)$. In addition,

$$\sup_\xi |A_2^{\varepsilon,\xi}| \leq C\,X_1\,,$$

with $X_1 := \left(\sum_{k=2}^{\infty} k^2(k-1)^2(I_k(f_k))^2 \right)^{1/2}$. Indeed, by Schwarz's inequality,

$$
\begin{aligned}
|A_2^{\varepsilon,\xi}| &\leq \sum_{k=2}^{\infty} \sum_{i=0}^{k-2} \binom{k-2}{i} \frac{k\,(k-1)}{(k-i)(k-i-1)}\ \varepsilon^i\, \xi^{k-i-2}\ |I_k(f_k)| \\
&\leq \frac{1}{2} \sum_{k=2}^{\infty} k\,(k-1)\ (\varepsilon+\xi)^{k-2}\ |I_k(f_k)| \\
&\leq \frac{1}{2} \Big(\sum_{k=0}^{\infty} (\varepsilon+\xi)^{2k} \Big)^{1/2} \Big(\sum_{k=2}^{\infty} k^2(k-1)^2 (I_k(f_k))^2 \Big)^{1/2} \\
&\leq \frac{1}{2}\ \frac{1}{(1-\varepsilon_0^2)^{1/2}}\ X_1 .
\end{aligned}
$$

Since $F \in \Delta^{2,2}$, X_1 is finite a.s. Consequently, from (2.1) we obtain

$$
\lim_{\xi \to 0} \frac{F^{\varepsilon+\xi} - F^{\varepsilon}}{\xi} = A_1^{\varepsilon}, \quad \text{a.s.}
$$

Let $j > 1$ and assume the statement holds for any $k \in \{1, \ldots, j-1\}$. Set

$$
d_{j-1}^{\varepsilon} = \frac{d^{j-1}\, F^{\varepsilon}}{d\,\varepsilon^{j}} = \sum_{k=j-1}^{\infty} \frac{k\,!}{(k-j+1)!}\ \varepsilon^{k-j+1}\ I_k(f_k) .
$$

Then,

$$
\frac{d_{j-1}^{\varepsilon+\xi} - d_{j-1}^{\varepsilon}}{\xi} = B_1^{\varepsilon} + \xi\, B_2^{\varepsilon,\xi} ,
$$

with

$$
\begin{aligned}
B_1^{\varepsilon} &= \sum_{k=j}^{\infty} \frac{k\,!}{(k-j)\,!}\ \varepsilon^{k-j}\ I_k(f_k) , \\
B_2^{\varepsilon,\xi} &= \sum_{k=j+1}^{\infty} \sum_{i=0}^{k-j-1} \frac{k\,!}{(k-j+1)\,!} \binom{k-j+1}{i}\ \varepsilon^i\ \xi^{k-j-i-1}\ I_k(f_k) .
\end{aligned}
$$

The series defining B_1^{ε} converges in $L^2(\Omega)$, because $F \in \Delta^{j,2}$. As for $A_2^{\varepsilon,\xi}$, we have

$$
|B_2^{\varepsilon,\xi}| \leq \frac{1}{2}\ \frac{1}{(1-\varepsilon_0^2)^{1/2}}\ X_j ,
$$

with $X_j := \left(\sum_{k=j+1}^{\infty}\left(\frac{k!}{(k-j-1)!}\right)^2 (I_k(f_k))^2\right)^{1/2}$. This random variable is finite a.s.. since $F \in \Delta^{j+1,2}$. Therefore

$$\lim_{\xi\to 0} \frac{d_{j-1}^{\varepsilon+\xi} - d_{j-1}^{\varepsilon}}{\xi} = B_1^{\varepsilon}, \quad \text{a.s.}$$

and the proof is complete. □

Remark. For $j \in \mathbb{Z}^+$, the Sobolev spaces $\mathbb{D}^{j,2}$ can be characterized as follows.

$$\mathbb{D}^{j,2} = \left\{F \in L^2(\Omega) : \sum_{k=j}^{\infty} \left(\frac{k!}{(k-j)!}\right)^2 (k-j)!\, \|f_k\|_2^2 < \infty\right\}.$$

Using the quotient criterium for comparison of series, one easily checks that $\mathbb{D}^{2j,2} = \Delta^{j,2}$, $\forall j \in \mathbb{Z}^+$. Hence, the preceding Proposition can be formulated in a more handly way, as follows.

Corollary 2.2 *Let $F \in \cap_{j=0}^{\infty} \mathbb{D}^{j,2}$. There exist a version of $\{F^{\varepsilon},\ \varepsilon \in (0,1)\}$ which is C^{∞} in ε and*

$$\frac{d^j F^{\varepsilon}}{d\varepsilon^j} = \sum_{k=j}^{\infty} \frac{k!}{(k-j)!}\, \varepsilon^{k-j}\, I_k(f_k)\,. \tag{2.2}$$

$j \in \mathbb{Z}^+$, where the series in (2.2) corverges in $L^2(\Omega)$.

In the proof of the main result of this section we deal with the random vector

$$\hat{F}^{\varepsilon} = \frac{F^{\varepsilon} - E(F)}{\varepsilon}, \quad 0 < \varepsilon < 1\,.$$

Corollary 2.2 yields the following

Corollary 2.3 *Let $F \in \cap_{j=0}^{\infty} \mathbb{D}^{j,2}$. There exists a version of $\{\hat{F}^{\varepsilon},\ \varepsilon \in (0,1)\}$ which is C^{∞} in ε and*

$$\frac{d^j \hat{F}^{\varepsilon}}{d\varepsilon^j} = \sum_{k=j+1}^{\infty} \frac{(k-1)!}{(k-(j+1))!}\, \varepsilon^{k-(j+1)}\, I_k(f_k)\,, \tag{2.3}$$

$$\left.\frac{d^j \hat{F}^{\varepsilon}}{d\varepsilon^j}\right|_{\varepsilon=0} := \lim_{\varepsilon\downarrow 0} \frac{d^j \hat{F}^{\varepsilon}}{d\varepsilon^j} = j!\, I_{j+1}(f_{j+1})\,, \tag{2.4}$$

$j \in \mathbb{Z}^+$. In particular, setting $\hat{F}^0 = \lim_{\varepsilon\downarrow 0} \hat{F}^{\varepsilon}$, we have $\hat{F}^0 = I_1(f_1)$.

Formula (2.3) can be checked by induction. using the same arguments as in the proof of Proposition 2.1.
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^∞ function with compact support. Leibniz formula yields. for $j \geq 1$

$$\frac{d^j}{d\varepsilon^j}\left(f(\hat{F}^\varepsilon)\right) = \sum^{(j)} (\nabla^k_\alpha f)\,(\hat{F}^\varepsilon)\,\nabla^{\beta_1}\hat{F}^{\varepsilon,\alpha_1}\ldots\nabla^{\beta_k}\hat{F}^{\varepsilon,\alpha_k}, \tag{2.5}$$

where the symbol $\sum^{(j)}$ is a shorthand for

$$\sum_{k=1}^{j}\ \sum_{\substack{\beta_1+\ldots+\beta_k=j\\ \beta_1,\ldots,\beta_k\geq 1}}\ \sum_{\substack{\alpha\in\{1,\ldots,d\}^k\\ \alpha=(\alpha_1,\ldots,\alpha_k)}} c_j(\beta_1,\ldots,\beta_k)$$

and the coefficients $c_j(\beta_1,\ldots,\beta_k)$ are obtained recurrently. as follows.

$$c_j(\beta_1,\ldots,\beta_k) = \sum_{i=1}^{k} c_{j-1}(\beta_1,\ldots,\beta_i-1,\ldots,\beta_k),$$

with $c_1(1)=1$; for $\beta_i = 1$ and $i<k$, $c_{j-1}(\beta_1,\ldots,\beta_i-1,\ldots,\beta_k)=0$: and for $\beta_k=1$, $c_{j-1}(\beta_1,\ldots,\beta_k-1)=c_{j-1}(\beta_1,\ldots,\beta_{k-1})$.
In the sequel we denote by Γ_G the Malliavin matrix of a Wiener functional $G:\Omega\to\mathbb{R}^d$. Let $\Phi\in\mathbb{D}^\infty(\mathbb{R}^d)$ with $\det\Gamma_\Phi^{-1}\in\cap_{p\geq 1}L^p$, $\Psi\in\mathbb{D}^\infty$; for a multiindex $\alpha\in\{1,\ldots,d\}^k$, $\alpha=(\alpha_1,\ldots,\alpha_k)$, $k\geq 1$, we define $H_\alpha(\Phi,\Psi)$ recurrently, as follows,

$$\begin{aligned} H_{(i)}(\Phi,\Psi) &= \sum_{j=1}^{d}\delta\left(\Psi(\Gamma_\Phi^{-1})^{ij}\,D\,\Phi^j\right),\\ H_\alpha(\Phi,\Psi) &= H_{(\alpha_k)}\left(\Phi, H_{(\alpha_1,\ldots,\alpha_{k-1})}(\Phi,\Psi)\right), \end{aligned} \tag{2.6}$$

where δ denotes the Skorohod integral.
This notation is appropriate to state the following version of the integration by parts formula

$$E\left[(\nabla^k_\alpha g)\,(\Phi)\Psi\right] = E[g(\Phi)\,H_\alpha(\Phi,\Psi)], \tag{2.7}$$

where g is any smooth function defined in $\mathbb{R}^d$.
A slight modification of Proposition 3.2.2 [9] yields the following estimate: For any $k\in\mathbb{N}$, $p\in[1,\infty)$ the exist, $k',b,b'\in(1,\infty)$, $d,d'\in\mathbb{N}$ such that

$$\|H_\alpha(\Phi,\Psi)\|_{k,p}\leq C\,(k,p,\alpha)\,(\|\Gamma_\Phi^{-1}\|_{k'}\;\|\Phi\|_{d,b}\;\|\Psi\|_{d'b'}). \tag{2.8}$$

The family of $\mathbb{R}^d$-valued random vectors $\{F^\varepsilon,\ \varepsilon\in(0,1]\}$ is said to be *uniformly non-degenerate* if the next two conditions are satisfied:

(i) $F^\varepsilon \in \mathbb{D}^\infty$, $\|\Gamma_{F^\varepsilon}^{-1}\|_p \le C\,\varepsilon^{-2}$, for any $\varepsilon \in (0,1]$, $p \in [1,\infty)$

(ii) $\sigma^2 := \det\Big(\mathrm{Cov}(I_1(f_1))\Big) > 0$.

We now give the main result of this section.

Theorem 2.4 *Let $\{F^\varepsilon,\ \varepsilon \in (0,1]\}$ be uniformly non-degenerate. The density $p^\varepsilon(y)$, for $y = E(F^\varepsilon) = E(F)$, has the Taylor expansion*

$$p^\varepsilon(y) = \frac{1}{\varepsilon^d}\Big\{\frac{1}{(2\pi)^{d/2}\,\sigma} + \sum_{j=1}^{N} \varepsilon^j\,\frac{1}{j!}\,p_j + \varepsilon^{N+1}\,\tilde{p}^{\,\varepsilon}_{N+1}\Big\}. \tag{2.9}$$

The coefficients p_j are null for odd j. For even $j \in \{1,2,\ldots,N\}$,

$$p_j = E\Big(1_{\{I_1(f_1)>0\}}\,P_j\Big), \tag{2.10}$$

with P_j belonging to $\oplus_{k=0}^{3j+d}\,\mathcal{H}_k$, $\mathcal{H}_k$ being the k-th Wiener chaos, and

$$P_j = \sum^{(j)} H_{(1,\ldots,d)}\Big(I_1(f_1),\ H_\alpha(I_1(f_1)),\ \prod_{\ell=1}^{k} \beta_\ell!\ I_{3_\ell+1}\,(f^{\alpha_\ell}_{3_\ell+1})\big)\Big). \tag{2.11}$$

In addition, if for any $j \in \mathbb{Z}^+$, $k \in \mathbb{N}$, $p \in [1,\infty)$

$$\sup_{\varepsilon\in(0,1]} \Big\|\frac{d^j}{d\,\varepsilon^j}\,\hat{F}^\varepsilon\Big\|_{k,p} \le C, \tag{2.12}$$

then, $\sup_{\varepsilon\in(0,1]} |\tilde{p}^{\,\varepsilon}_{N+1}|$ is finite.

Remarks.

(1) The identities (2.10), (2.11) express the fact that p_j, $j = 1,\ldots,N$, are the densities at $x = 0$ of the Radon measures defined by

$$g \longmapsto E\Big(g\,(I_1(f_1))\,\sum^{(j)} H_\alpha\Big(I_1(f_1),\prod_{\ell=1}^{k} \beta_\ell!\ I_{3_\ell+1}\,(f^{\alpha_p}_{3_\ell+1})\big)\Big)\Big).$$

for any smooth g (see, for instance, Corollary 3.2.1 [9]).

(2) As will become clear from the proof of Theorem 2.4, $\tilde{p}^{\,\varepsilon}_{N+1}$ is also the density of *a* Radon measure depending on ε. The last assertion of the Theorem gives a sufficient condition ensuring the uniform boundedness of this density. In this case, the last term in the development (2.9) is $O\,(\varepsilon^{N+1})$ as $\varepsilon \downarrow 0$.

Proof of Theorem 2.4. Let $\hat{p}^{\varepsilon}$ denote the density of $\hat{F}^{\varepsilon} = \frac{F^{\varepsilon}-E(F)}{\varepsilon}$. Clearly $p^{\varepsilon}(y) = \frac{1}{\varepsilon^d}\, \hat{p}^{\varepsilon}(0)$. Therefore we will find an expansion for $\hat{p}^{\varepsilon}(0)$. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $\mathcal{C}^{\infty}$ function with bounded support. The mapping $\varepsilon \longmapsto f(\hat{F}^{\varepsilon})$ is $\mathcal{C}^{\infty}$, a.s., therefore

$$
\begin{aligned}
f(\hat{F}^{\varepsilon}) &= f(\hat{F}^{0}) + \sum_{j=1}^{N} \frac{1}{j!}\, \varepsilon^j\, \frac{d^j}{d\varepsilon^j}\left(f(\hat{F}^{\varepsilon})\right)\Big|_{\varepsilon=0} \\
&\quad + \varepsilon^{N+1} \int_0^1 \frac{(1-t)^N}{N!}\, \frac{d^{N+1}}{d\eta^{N+1}}\left(f(\hat{F}^{\eta})\right)\Big|_{\eta=t\varepsilon}\, dt .
\end{aligned}
$$

Next, we take expectations in both sides of the preceding equality, we use (2.5), (2.4) and the integration by parts formula (2.7) to obtain

$$
\begin{aligned}
E\Big(f(\hat{F}^{\varepsilon})\Big) &= E\Big(f(I_1(f_1))\Big) \\
&\quad + \sum_{j=1}^{N} \frac{1}{j!}\, \varepsilon^j\, E\Big\{f(I_1(f_1)) \sum^{(j)} H_\alpha\Big(I_1(f_1), \prod_{\ell=1}^{k} \beta_\ell!\, I_{\beta_\ell+1}(f^{\alpha_\ell}_{\beta_\ell+1})\Big)\Big\} \\
&\quad + \varepsilon^{N+1} \int_0^1 \frac{(1-t)^N}{N!}\, E\Big\{f(\hat{F}^{\varepsilon t}) \sum^{(N+1)} H_\alpha\Big(\hat{F}^{\varepsilon t}, \prod_{\ell=1}^{k} \frac{d^{\beta_\ell} \hat{F}^{\eta,\alpha_\ell}}{d\eta^{\beta_\ell}}\Big|_{\eta=\varepsilon t}\Big)\Big\} dt.
\end{aligned}
\tag{2.13}
$$

The assumptions of the theorem ensure that the Radon measures defined by $E(f(\hat{F}^{\varepsilon}))$, $E(f(I_1(f_1)))$, $E\{f(I_1(f_1))\, Q^j\}$, $j = 1, \ldots, N$, $E\{f(\hat{F}^{\varepsilon t})\, Q^{N+1,\varepsilon}\}$, with

$$
\begin{aligned}
Q^j &= \sum^{(j)} H_\alpha\Big(I_1(f_1), \prod_{\ell=1}^{k} \beta_\ell!\, I_{\beta_\ell+1}(f^{\alpha_\ell}_{\beta_\ell+1})\Big) \\
Q^{N+1,\varepsilon} &= \sum^{(N+1)} H_\alpha\Big(\hat{F}^{\varepsilon t}, \prod_{\ell=1}^{k} \frac{d^{\beta_\ell}}{d\eta^{\beta_\ell}}\, \hat{F}^{\eta,\alpha_\ell}\Big|_{\eta=\varepsilon t}\Big)
\end{aligned}
\tag{2.14}
$$

possess $\mathcal{C}^{\infty}$ densities. Moreover, a new integration by parts in (2.13) yields (see Corollary 3.2.1 [9])

$$
\hat{p}^{\varepsilon}(0) = \frac{1}{(2\pi)^{d/2}\sigma} + \sum_{j=1}^{N} \frac{1}{j!}\, \varepsilon^j\, E\left\{1_{\{I_1(f_1)>0\}}\, P_j\right\} + \varepsilon^{N+1}\, \tilde{p}^{\varepsilon}_{N+1} .
$$

with

$$\tilde{p}^{\varepsilon}_{N+1} = \int_0^1 \frac{(1-t)^N}{N!} E\Big\{1_{\{\hat{F}^{\varepsilon t}>0\}} H_{(1,\dots,d)}\Big(\hat{F}^{\varepsilon t}, \sum^{(N+1)} H_\alpha\Big(\hat{F}^{\varepsilon t}, \prod_{\ell=1}^{k} \frac{d^{\beta_\ell}}{d\eta^{\beta_\ell}} \hat{F}^{\eta\alpha_\ell}\Big|_{\eta=\varepsilon t}\Big)\Big)\Big\}dt. \quad (2.15)$$

We prove in Lemma 2.5 that, for any odd number $j \in \{1, 2, \dots, N\}$

$$Q_j(\omega) = -Q_j(-\omega), \quad (2.16)$$

with Q_j defined in (2.14).
This shows $E\Big(f\,(I_1(f_1)\,)\,Q^j\Big) = 0$ for any smooth and symmetric function f and hence $p_j = 0$. Indeed, assume (2.16) holds. The Wiener measure P is invariant under the transformation $\mathcal{Z}(\omega) = -\omega$. Thus

$$E\,[\,f\,(I_1(f_1))\,Q_j] = E\,[\,(f\,(I_1(f_1))\,Q^j)\circ \mathcal{Z}\,] = -E\,[\,f\,(I_1(f_1))\,Q_j]\,.$$

The fact that P_j has a finite Wiener Chaos decomposition, more precisely, $P_j \in \mathcal{J}_{3j+d} = \oplus_{k=0}^{3j+d}\mathcal{H}_k$, follows from Lemma 2.6. Indeed, for any $k \in \{1, \dots, j\}$, $\Psi := \prod_{\ell=1}^{k} \beta_\ell!\; I_{\beta_\ell+1}(f^{\alpha_\ell}_{\beta_\ell+1}) \in \mathcal{J}_{2j}$, since $\beta_1 + \dots + \beta_k = j$. Consequently $Q_j \in \mathcal{J}_{3j}$, because the lenght of α is k. Finally, since $P_j = H_{(1,\dots,d)}\,(I_1(f_1), Q_j)$, Lemma 2.6 yields $P_j \in \mathcal{J}_{3j+d}$.
We want now to give a uniform bound for $\tilde{p}^{\varepsilon}_{N+1}$ (see (2.15)). Set $G^\varepsilon = \prod_{\ell=1}^{k} \frac{d^{\beta_\ell}}{d\eta^{\beta_\ell}}\,\hat{F}^{\varepsilon}$. Clearly, it suffices to show

$$\sup_{0<\varepsilon\leq 1} E\Big\{\Big|H_{(1,\dots,d)}\,(\hat{F}^{\varepsilon},\ H_\alpha(\hat{F}^{\varepsilon}, G^{\varepsilon}))\Big|\Big\} \leq C\,, \quad (2.17)$$

for any $\alpha \in \{1, \dots, d\}^k$, $\beta_1 + \dots + \beta_k = N+1$, $k = 1, \dots, N+1$, and some finite $C > 0$. The estimate (2.8) yields, for some $k, b, b' \in (1, \infty)$ and $d, d' \in \mathbb{N}$,

$$E\,\Big|H_{(1,\dots,d)}\Big(\hat{F}^{\varepsilon},\ H_\alpha(\hat{F}^{\varepsilon}, G^{\varepsilon})\Big)\Big| \leq C\,\Big(\,\|\Gamma^{-1}_{\hat{F}^{\varepsilon}}\|_k\ \|\hat{F}^{\varepsilon}\|_{d,b}\ \|G^{\varepsilon}\|_{d',b'}\Big).$$

Therefore, the non-degeneracy condition $\|\Gamma^{-1}_{F^\varepsilon}\|_p \leq C\,\varepsilon^{-2}$, $\forall p \in (1, \infty)$ together with condition (2.12) yields (2.17). This finishes the proof of the theorem. □

Lemma 2.5 *Let j be an odd natural number, Q_j the random vector defined by* (2.14) *and $\mathcal{Z}$ the transformation defined on the Wiener space by $\mathcal{Z}(\omega) = -\omega$. Then*

$$Q_j = -Q_j \circ \mathcal{Z}\,.$$

Proof. Fix $k \in \{1, \dots, j\}$, $\beta_1, \dots, \beta_k \geq 1$ with $\beta_1 + \dots + \beta_k = j$ and a multiindex $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, d\}^k$. We will prove

$$Q_{j,k} = -Q_{j,k} \circ \mathcal{Z}\,,$$

with

$$Q_{j,k} = H_\alpha\Big(I_1(f_1), \prod_{\ell=1}^{k} \beta_\ell !\; I_{\beta_\ell+1}\,(f^{\alpha_\ell}_{\beta_\ell+1})\Big).$$

Since

$$Q_j = \sum_{k=1}^{j} \sum_{\substack{\beta_1+\dots+\beta_k=j \\ \beta_1,\dots,\beta_k \geq 1}} \sum_{\substack{\alpha\in\{1,\dots,d\}^k \\ \alpha=(\alpha_1,\dots,\alpha_k)}} c_j(\beta_1, \dots, \beta_k)\; Q_{j,k},$$

this is enough for our purposes.

Let $(a^{ij})_{1\leq i,j\leq d}$ be the inverse of the covariance matrix of the Gaussian random vector $I_1(f_1)$. For any $\Psi \in \mathbb{D}^\infty$,

$$\begin{aligned} H_{(i)}\Big(I_1(f_1), \Psi\Big) &= \sum_{j=1}^{d} \delta\,(\Psi\; a^{ij}\; f_1^j(\cdot)\,) \\ &= \sum_{j=1}^{d} a^{ij}\, \{\Psi\, I_1(f_1^j) - \int_T D_r\, \Psi\, f_1^j(r)\, dr\}\,. \end{aligned} \qquad (2.18)$$

Consequently, for $\Psi = \prod_{\ell=1}^{k} \beta_\ell !\; I_{\beta_\ell+1}\,(f^{\alpha_\ell}_{\beta_\ell+1})$, it holds

$$\Psi \circ \mathcal{Z} = (-1)^{j+k}\; \Psi,\; (D_r\, \Psi) \circ \mathcal{Z} = (-1)^{j+k-1}\; D_r\, \Psi\,.$$

and therefore,

$$H_{(i)}\,(I_1\,(f_1), \Psi) \circ \mathcal{Z} = (-1)^{j+k+1}\; H_{(i)}\,(I_1(f_1), \Psi). \qquad (2.19)$$

The recurrent formula (2.6), (2.18) and (2.19) show

$$H_\alpha\Big(I_1(f_1), \prod_{\ell=1}^{k} \beta_\ell!\, I_{\beta_\ell+1}(f_{\beta_\ell+1}^{\alpha_\ell})\Big) \circ \mathcal{Z}$$

$$= (-1)^{j+2k} \times H_\alpha\Big(I_1(f_1), \prod_{\ell=1}^{k} \beta_\ell!\, I_{\beta_\ell+1}(f_{\beta_\ell+1}^{\alpha_\ell})\Big)$$

$$= -H_\alpha\Big(I_1(f_1), \prod_{\ell=1}^{k} \beta_\ell!\, I_{\beta_\ell+1}(f_{\beta_\ell+1}^{\alpha_\ell})\Big).$$

Therefore the lemma is proved. □

Lemma 2.6 *Let Φ be a non-degenerate d-dimensional Gaussian random vector, $\Psi \in \mathcal{J}_\ell$, $\ell \geq 0$. For any multiindex $\alpha = (\alpha_1, \ldots, \alpha_r) \in \{1, \ldots, d\}^r$, the random variable $H_\alpha(\Phi, \Psi)$ belongs to $\mathcal{J}_{\ell+r}$.*

Proof. We will proceed by induction on the lenght of α. Set $(b^{i,j})_{i,j=1,\ldots,d} = (\text{Cov}\ \Phi)^{-1}$ and $\Phi = I_1(f)$. Then, for any $i \in \{1, \ldots, d\}$

$$H_{(i)}(\Phi, \Psi) = \sum_{j=1}^{d} b^{ij}\delta(\Psi f) \in \mathcal{J}_{\ell+1}\,.$$

Assume the statement holds for any multiindex of lenght $r-1$. Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \{1, \ldots, d\}^r$. By (2.6)

$$H_\alpha(\Phi, \Psi) = H_{(\alpha_r)}\,(\Phi, \tilde{\Psi}) = \sum_{j=1}^{d} b^{\alpha_r, j}\ \delta(\tilde{\Psi} f)\,,$$

with $\tilde{\Psi} \in \mathcal{J}_{\ell+r-1}$. Thus $H_\alpha(\Phi, \Psi) \in \mathcal{J}_{\ell+r}$ and the proof is complete. □

Remark. Let $\Phi(h) = E(F) + \sum_{n=1}^{\infty} \int_{T^n} f_n(s_1, \ldots, s_n)\, dh_{s_1} \ldots dh_{s_n}$, $h \in \mathcal{H}$. Notice that, the series defining $\Phi(h)$ is absolutely convergent, due to condition $\sum_{n=1} n!\, \|f_n\|_2^2 < +\infty$. Assume there exist a sequence $\{\omega^n,\ n \in \mathbb{N}\} \subset \mathcal{H}$ such that $P - \lim_{n\to\infty} \Phi(\omega^n) = F$ and, moreover, for any $h \in \mathcal{H}$, $n \in \mathbb{N}$, there exists an absolutely continuous transformation $T_n^h : \Omega \to \Omega$ such that $P - \lim_{n\to\infty} F \circ T_n^h = \Phi(h)$. If, in addition $F^\varepsilon \in \mathbb{D}^\infty$ and $\|\det \Gamma_{F^\varepsilon}^{-1}\|_p < +\infty$, $\forall p \in (1, \infty)$, Theorem 3.41 in [1] establishes the following characterization for the points of positive density for F^ε:

$$\{p^\varepsilon(y) > 0\} = \{y : \exists\, h \in \mathcal{H} : \Phi(h) = y \quad \text{and} \quad D\Phi(h)\ \text{surjective}\}.$$

Assume the family $\{F^\varepsilon,\ \varepsilon \in (0,1]\}$ possess the aproximating property described before and is uniformly non-degenerate. Then, for $y = E(F)$, $p^\varepsilon(y) > 0$. Indeed, $\Phi(0) = E(F) = y$ and, for any $k \in \mathcal{H}$,

$$D\,\Phi(0)\,(k) = \int_T f_1(s)\; k(s)\; d\mu(s).$$

Thus, since $\sigma^2 := \det\,\Big(\mathrm{Cov}(I_1(f_1))\Big) > 0$, $D\,\Phi(h)$ is surjective.

3 Applications

We devote this section to study two examples where Theorem 2.4 can be applied

3.1 A hyperbolic stochastic partial differential equation

Let $T = [0,1]^2$ and $\{W_{s,t},\ (s,t) \in T\}$ be a Wiener sheet. Consider the hyperbolic stochastic partial differential equation

$$\frac{\partial^2 X_{s,t}}{\partial s\,\partial t} = a_3(X_{s,t},\ s,t)\ \dot{W}_{s,t} + a_4(X_{s,t},\ s,t) + a_1(s,t)\ \frac{\partial X_{s,t}}{\partial s} + a_2(s,t)\ \frac{\partial X_{s,t}}{\partial t}\,. \tag{3.1.1}$$

with deterministic initial condition $X_{s,t} = x_0$ if $(s,t) \in T$, $s \cdot t = 0$. We refer to [6], and specially to [11] for results on this equation used in this section. Here we will deal with the particular situation made precise in the following set of assumptions

(H1) $a_i : T \to \mathbb{R}$, $i = 1,2$ are differentiable and bounded with bounded first partial derivatives.

(H2) $a_i : \mathbb{R} \times T \to \mathbb{R}$, $i = 3,4$ are *linear* in the space variable, that means

$$a_i(x,\ s,t) = a_{i1}(s,t)x + a_{i2}(s,t)\,.$$

In addition, a_{31}, a_{32}, a_{41} and a_{42} are supposed to be continuous.
A solution of (3.1.1) is a stochastic process $\{X_{s,t},\ (s,t) \in T\}$ satisfying

$$X_{s,t} = x_0 + \int_{R_{s,t}} \gamma_{s,t}\ (u,v)\ \{a_3\,(X_{u,v},\ u,v)\,d\,W_{u,v} + a_4(X_{u,v},\ u,v)\,du\,dv\}\,. \tag{3.1.2}$$

where $R_{s,t}$ denotes the rectangle $[0,s] \times [0,t]$ and $\gamma_{s,t}(u,v)$ is the Green function associated with the second order differential operator

$$\mathcal{L} f(s,t) = \frac{\partial^2 f(s,t)}{\partial s\, \partial t} - a_1(s,t)\, \frac{\partial f(s,t)}{\partial t} - a_2(s,t)\, \frac{\partial f(s,t)}{\partial s}\,.$$

Here are some properties of the Green function; their proofs are given in [11]:

Boundedness:

$$\sup_{(s,t)\in T}\ \sup_{(u,v)\leq(s,t)} |\gamma_{s,t}(u,v)| \leq C\,.$$

Lipschitz property:

$$\sup_{(s,t)\in T} |\gamma_{s,t}(\overline{u},\overline{v}) - \gamma_{s,t}(u,v)| \leq C\left\{|\overline{u}-u| + |\overline{v}-v|\right\},\ (\overline{u},\overline{v}),(u,v) \leq (s,t)\,.$$

$$\sup_{(u,v)\in T} |\gamma_{\overline{s},\overline{t}}(u,v) - \gamma_{s,t}(u,v)| \leq C\left\{|\overline{s}-s| + |\overline{t}-t|\right\},\ (\overline{s},\overline{t}),(s,t) \geq (u,v)\,.$$

Positivity:

$$\gamma_{s,t}(s,v) = \exp\left(\int_v^t a_2(s,w)\,dw\right),\quad 0 \leq v \leq t\,.$$

$$\gamma_{s,t}(u,t) = \exp\left(\int_u^s a_1(r,t)\,dr\right),\quad 0 \leq u \leq s\,.$$

Theorem 2.1 in [11] proves the existence and uniqueness of a continuous and adapted process $\{X_{s,t}, (s,t) \in T\}$ bounded in L^p, for any $p \geq 2$. Moreover, $X_{s,t} \in \mathbb{D}^\infty$, $\forall (s,t) \in T$. For any $\varepsilon \in (0,1]$ set

$$X^\varepsilon_{s,t} = x_0 + \int_{R_{s,t}} \gamma_{s,t}(u,v)\left\{\varepsilon\, a_3(X^\varepsilon_{u,v},u,v)\ dW_{u,v} + a_4(X^\varepsilon_{u,v},u,v)\,du\,dv\right\} \tag{3.1.3}$$

and, for any $h \in \mathcal{H}$, the Cameron-Martin space associated to $\{W_{s,t}, (s,t) \in T\}$,

$$S^h_{s,t} = x_0 + \int_{R_{s,t}} \gamma_{s,t}(u,v)\left\{a_3(S^h_{u,v},u,v)\ dh_{u,v} + a_4(S^h_{u,v},u,v)\,du\,dv\right\}.$$

Proposition 3.1.1 *Assume* (H1) *and* (H2). *For any* $z \in T$, $z = (s,t)$, $st \neq 0$, *let*

$$X_z = E\,X_z + \sum_{n=1}^{\infty} I_n(f_n)$$

be the Wiener-Chaos decomposition of the solution of (3.1.2) *at* $z = (s,t)$. *Then, for any* $\varepsilon \in (0,1]$,

$$X_z^\varepsilon = E\,X_z + \sum_{n=1}^{\infty} \varepsilon^n\, I_n(f_n)\,.$$

Proof: By a result proved in [12], $f_n(\alpha) = \frac{1}{n!}\, E\,(D_\alpha^n\, X_z)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$. Thus, if $X_z^\varepsilon = E\,X_z^\varepsilon + \sum_{n=1}^{\infty} I_n(f_n^\varepsilon)$, it suffices to prove $E\,X_z = E\,X_z^\varepsilon$ and $E\,(D_\alpha^n\, X_z^\varepsilon) = \varepsilon^n\, E(D_\alpha^n\, X_z)$, $n \geq 1$.
Taking expectations in (3.1.2), (3.1.3) by uniqueness of solution we immediately obtain

$$E\,X_z = E\,X_z^\varepsilon = S_z^0\,.$$

Fix $N \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_N \in R_z$. Denote by $\underline{\alpha}$ the vector $(\alpha_1, \ldots, \alpha_N)$; set $\alpha^i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_N)$, $N \geq 2$, $\sup \underline{\alpha} = \alpha_1 \vee \ldots \vee \alpha_N$. The particular form of the coefficients a_i, $i = 3,4$ and the rules of Malliavin Calculus yield the following expressions for $N \geq 2$.

$$\begin{aligned} D_{\underline{\alpha}}^N\, X_z &= \sum_{i=1}^{N} a_{3,1}(\alpha_i)\gamma_z(\alpha_i)\; D_{\alpha^i}^{N-1}\; X_{\alpha_i} \\ &\quad + \int_{[\sup \underline{\alpha}, z]} \gamma_z(\eta)\, [a_{3,1}\,(\eta)\, D_{\underline{\alpha}}^N\, X_\eta\, dW_\eta + a_{4,1}\,(\eta)\; D_{\underline{\alpha}}^N\; X_\eta\, d\eta]\,. \end{aligned}$$

$$\begin{aligned} D_{\underline{\alpha}}^N\, X_z^\varepsilon &= \sum_{i=1}^{N} \varepsilon\, a_{3,1}(\alpha_i)\gamma_z(\alpha_i)\; D_{\alpha^i}^{N-1}\; X_{\alpha_i}^\varepsilon \\ &\quad + \int_{[\sup \underline{\alpha}, z]} \gamma_z(\eta)\, [\varepsilon\, a_{3,1}\,(\eta)\, D_{\underline{\alpha}}^N\, X_\eta^\varepsilon\, dW_\eta + a_{4,1}\,(\eta)\; D_{\underline{\alpha}}^N\; X_\eta^\varepsilon\, d\eta]\,. \end{aligned}$$

Let $U_{\underline{\alpha}}^N(z)$, $N \geq 1$, be the solution of the equation

$$U_{\underline{\alpha}}^N(z) = 1 + \int_{[\sup \underline{\alpha}, z]} \gamma_z(\eta)\; a_{4,1}(\eta)\; U_{\underline{\alpha}}^N(\eta)\, d\eta\,.$$

Then, clearly

$$E\,(D_{\underline{\alpha}}^N\, X_z) = \Big(\sum_{i=1}^{N} a_{3,1}(\alpha_i)\,\gamma_z(\alpha_i)\; E(D_{\alpha^i}^{N-1}\, X_{\alpha_i})\Big) U_{\underline{\alpha}}^N(z)\,.$$

$$E\,(D^N_{\underline{\alpha}}\,X^\varepsilon_z) = \Big(\sum_{i=1}^{N}\,\varepsilon\,a_{3.1}(\alpha_i)\,\gamma_z(\alpha_i)\;E(D^{N-1}_{\alpha^i}\,X^\varepsilon_{\alpha_i})\Big)U^N_{\underline{\alpha}}(z)\,. \qquad (3.1.4)$$

For $N = 1$,

$$E\,(D_\alpha\,X_z) = \gamma_z(\alpha)\,[a_{3.1}(\alpha)\;E\,X_\alpha + a_{3.2}(\alpha)]\;U^1_\alpha\,(z)\,.$$

$$E\,(D_\alpha\,X^\varepsilon_z) = \varepsilon\,\gamma_z(\alpha)\,[a_{3.1}(\alpha)\;E\,X^\varepsilon_\alpha + a_{3.2}(\alpha)]\;U^1_\alpha\,(z)\,.$$

Thus, $E\,(D_\alpha\,X^\varepsilon_z) = \varepsilon\,E(D_\alpha\,X_z)$, because $E\,X_\alpha = E\,X^\varepsilon_\alpha$. This fact and (3.1.4) allow to finish the proof using a recurrent argument. $\square$

In the sequel we fix $z \in T$ not belonging to the axes. The following notation will be used. Set

$$\hat{X}^\varepsilon_z = \frac{X^\varepsilon_z - S^0_z}{\varepsilon}\,,\;\; X^\varepsilon_j(z) = \frac{d^j}{d\,\varepsilon^j}\;X^\varepsilon_z\,,\;\; \hat{X}^\varepsilon_j(z) = \frac{d^j}{d\,\varepsilon^j}\;\hat{X}^\varepsilon_z\,,\; j \in \mathbb{N}.$$

By Corollary 2.2 applied to $F = X_z$ we know that these derivatives exist. One easily checks

$$\begin{aligned} X^\varepsilon_1(z) = \int_{R_z} \gamma_z(\eta)\;\Big(&(a_{3.1}(\eta)\;X^\varepsilon_\eta + a_{3.2}(\eta)\,)\,d\,W_\eta + \varepsilon\,a_{3.1}(\eta)\;X^\varepsilon_1(\eta)\,d\,W_\eta \\ &+ a_{4.1}(\eta)\;X^\varepsilon_1(\eta)\,d\eta\Big), \end{aligned} \qquad (3.1.5)$$

$$\begin{aligned} X^\varepsilon_j(z) = \int_{R_z} \gamma_z(\eta)\;(&j\,a_{3.1}(\eta)\;X^\varepsilon_{j-1}(\eta)\,d\,W_\eta + \varepsilon\,a_{3.1}(\eta)\;X^\varepsilon_j(\eta)\,d\,W_\eta \\ &+ a_{4.1}(\eta)\;X^\varepsilon_j(\eta)\,d\eta)\,,\quad j \ge 2\,. \end{aligned} \qquad (3.1.6)$$

Let $X^0_j(z) = \lim_{\varepsilon\downarrow 0}\;X^\varepsilon_j(z)$, $j \ge 1$. Then $X^0_j(z)$ satisfies the following stochastic differential equations

$$X^0_1(z) = \int_{R_z} \gamma_z(\eta)\;\Big(\,(a_{3.1}(\eta)\;S^0_\eta + a_{3.2}(\eta)\,)\,d\,W_\eta + a_{4.1}(\eta)\;X^0_1(\eta)\,d\,\eta\Big)\,. \qquad (3.1.7)$$

$$X^0_j(z) = \int_{R_z} \gamma_z(\eta)\;\Big(j\,a_{3.1}(\eta)\;X^0_{j-1}(\eta)\,d\,W_\eta + a_{4.1}(\eta)\;X^0_j(\eta)\,d\,\eta\Big)\,. \qquad (3.1.8)$$

Lemma 3.1.2 *We assume* (H1) *and* (H2). *Then*

$$\hat{X}^\varepsilon_j(z) = \frac{1}{j+1}\;\Big[X^0_{j+1}(z) + \varepsilon\int_0^1 (1 - \xi^{j+1})\;X^{\xi\varepsilon}_{j+2}(z)\;d\xi\Big]\,, \qquad (3.1.9)$$

$j \in \mathbb{Z}^+$, *where, by convention,* $\hat{X}^\varepsilon_0(z) = \hat{X}^\varepsilon_z$.

Proof: For $j = 0$ the identity (3.1.9) follows from a Taylor development of X_z^ε around $\varepsilon = 0$, taking into account that $X_z^0 = S_z^0$. An easy recurrent argument establishes (3.1.9) for any $j \geq 1$. □

In the next Proposition we check assumption (2.12) of Theorem 2.4 for $\hat{F}^\varepsilon = \hat{X}_z^\varepsilon$.

Remark. We know that there exists a version of $\{\hat{X}_j^\varepsilon(z),\ z \in T\}$ which is continuous in ε. From the previous Lemma and (2.4) we obtain

$$I_n(f_n) = \frac{X_n^0(z)}{n!}\,, \quad n \geq 1\,.$$

Proposition 3.1.3 *Suppose* (H1) *and* (H2) *are satisfied. For any* $j \in \mathbb{Z}^+$, $k \in \mathbb{N}$, $p \in (1, \infty)$,

$$\sup_{0<\varepsilon\leq 1} \left\| \frac{d^j}{d\,\varepsilon^j}\, \hat{X}_z^\varepsilon \right\|_{k,p} \leq C\,.$$

Proof: Due to (3.1.9) the proof follows from the following facts:

$$\sup_{z\in T} \|X_j^0(z)\|_{k,p} \leq C\,, \tag{3.1.10}$$

$$\sup_{0<\varepsilon\leq 1}\ \sup_{z\in T} E\left(\,|X_j^\varepsilon(z)\,|^p\right) \leq C\,, \tag{3.1.11}$$

$$\sup_{0<\varepsilon\leq 1}\ \sup_{z\in T}\ \sup_{\underline{\alpha}:\,\sup\underline{\alpha}<z} E(\,|D_{\underline{\alpha}}^k\, X_j^\varepsilon(z)\,|^p) \leq C\,, \tag{3.1.12}$$

for any $j,\ k \in \mathbb{N}$, $p \in (1, \infty)$ and some positive constant C.
From the remark following Lemma 3.1.2 one clearly gets $X_j^0(z) \in \mathcal{H}_j$, for any $j \in \mathbb{N}$. This yields (3.1.10).
We know (see [11])

$$\sup_{0<\varepsilon\leq 1}\ \sup_{z\in T} E\left(\,|X_z^\varepsilon|^p\right) \leq C\,, \quad p \in (1, \infty)\,.$$

Then, the standard arguments based on Burkholder's, Hölder's and Gronwall's inequalities applied to the equations (3.1.5) and (3.1.6) yield (3.1.11) by a recurrent argument.
Finally, for the proof of (3.1.12) we first write the equations satisfied by $D_{\underline{\alpha}}^k\, X_j^\varepsilon(z)$, $j \in \mathbb{N}$; this can be done using (3.1.5), (3.1.6) and the rules of Malliavin Calculus. Then we proceed as for the proof of (3.1.11). This estimate allows to use the recurrent argument which is needed. □

We finish the study of this example by checking the uniform non-degeneracy property. We need the following additional assumptions on the coefficients.

(H3) $|a_{3j}(s,t) - a_{3j}(s,' t')| \leq C\{|s-s'| + |t-t'|\}$,

$j = 1,2$, $(s,t), (s',t') \in T$.

(H4) $\sup_{t\in[0,1]} |a_{4j}(s,t) - a_{4j}(s',t)| \leq C|s-s'|$, $j = 1,2$, $(s,s') \in T$

$\sup_{(s,t)\in T} |\partial_1^2 a_{3j}(s,t)| \leq C$, $j = 1,2$,

(H5) $a_{31}(0,t)x_0 + a_{32}(0,t) \neq 0$, $t \neq 0$,

(H6) $a_{31}(0,v)x_0 + a_{32}(0,v) = 0$, $\forall v \in (0,t]$,

$\partial_1 a_{31}(0,t)\, x_0 + \partial_1 a_{32}(0,t) + a_{31}(0,t) \int_0^t \gamma_{0,t}\,(0,w)\,(a_{41}(0,w)\,x_0$

$+a_{42}(0,w))dw \neq 0$,

for some positive constant C and where ∂_1 means the derivative with respect to the variable s.

Proposition 3.5 in [11] establishes $X_{s,t} \in \mathbb{D}^\infty$ under (H1) and (H2), for any $(s,t) \in T$. Thus $X_{s,t}^\varepsilon$ also belongs to $\mathbb{D}^\infty$, for every $\varepsilon \in (0,1]$, $(s,t) \in T$.

Proposition 3.1.4 *Let $z = (s,t) \in T$, $s{\cdot}t \neq 0$ be fixed. One of the following set of conditions implies $\|\Gamma_{X_z^\varepsilon}^{-1}\|_p \leq C\,\varepsilon^{-2}$, for some positive constant C and every $\varepsilon \in (0,1]$, $p \in (1,\infty)$*

(a) (H1) *to* (H3) *and* (H5),

(b) (H1) *to* (H4) *and* (H6).

Proof. It suffices to check that the inverse of the random variable $\varepsilon^{-2}\int_{R_z} |D_\alpha X_z^\varepsilon|^2\, d\alpha$ has moments of any order. Consider the stochastic differential equation

$$Y_z^\varepsilon(\alpha) = \gamma_z(\alpha) + \int_{(\alpha,z]} \gamma_z(\eta)\, Y_\eta^\varepsilon(\alpha)\,\{\varepsilon\, a_{3,1}(\eta)\, dW_\eta + a_{4,1}(\eta)\, d\eta\}, \quad 0 \leq \alpha \leq z.$$

Then, $D_\alpha X_z^\varepsilon = \varepsilon\, a_3(X_\alpha^\varepsilon, \alpha)\, Y_z^\varepsilon(\alpha)$. Consequently we need to show

$$P\left\{\int_{R_z} \left(a_3(X_\alpha^\varepsilon, \alpha)\, Y_z^\varepsilon(\alpha)\right)^2 d\alpha \leq \eta\right\} \leq \eta^p,$$

for any $p \in (1,\infty)$ and $\eta \leq \eta_0$.

This has been proved in Propositions 3.6 and 3.7 in [11]. We point out that, although assumptions (H3) and (H4) in this reference are stronger, they can be relaxed to the situation of our statement. □

Proposition 3.1.5 *Suppose* (H1) *to* (H3) *are satisfied. Either under* (H5) *or* (H4) *and* (H6), $E(|X_1^0(z)|^2) > 0$, *for any* $z = (s,t) \in T$, $s \cdot t \neq 0$.

Proof. We will show $\Gamma_z = \int_{R_z} |D_\alpha X_1^0(z)|^2 \, d\alpha > 0$. Since $X_1^0(z)$ is Gaussian this is equivalent to the statement. For any $0 \le \alpha \le z$, let

$$J_z(\alpha) = \gamma_z(\alpha) + \int_{(\alpha,z]} \gamma_z(\eta)\, a_{4,1}(\eta)\, J_\eta(\alpha)\, d\eta .$$

Then, $D_\alpha X_1^0(z) = 1_{\{\alpha \le z\}}\, a_3(S_\alpha^0, \alpha)\, J_z(\alpha)$. For any $\varepsilon, \beta, \delta > 0$ set $C^z_{\beta,\delta}(\varepsilon) = (0, \varepsilon^\beta) \times (t - \varepsilon^\delta, t)$, and $\alpha = (r, w)$.

Assume (H5) and define

$$B_1(\varepsilon) = \int_{C^z_{1,1}(\varepsilon)} a_3(x_0, 0, t)^2\, \gamma_{s,t}(r,t)^2 \, dr\, dw .$$

$$B_2(\varepsilon) = \int_{C^z_{1,1}(\varepsilon)} \Big(a_3(S^0_{r,w}, r, w)\, J_{s,t}(r,w) - a_3(x_0, 0, t)\, \gamma_{s,t}(r,t) \Big)^2 \, dr\, dw .$$

Clearly

$$\Gamma_z \ge \frac{1}{2}\, B_1(\varepsilon) - B_2(\varepsilon) . \tag{3.1.13}$$

Since $\gamma_{s,t}(r,t) > 0$, $B_1(\varepsilon) \ge C\, \varepsilon^2$. The following properties can be easily checked:

$$\sup_{z \in T} |S_z^0| \le C , \tag{3.1.14}$$

$$|S^0_{s,t} - x_0| \le C\, |s \cdot t| , \tag{3.1.15}$$

$$\sup_{z \in T} \sup_{\alpha \le z} |J_z(\alpha)| \le C ,$$

$$|J_z(\alpha) - \gamma_z(\alpha)| \le C\, \varepsilon^\delta, \quad \alpha \in C^z_{\beta,\delta}(\varepsilon), \tag{3.1.16}$$

for some positive constant C. Then, (H2), (H3) and (3.1.14) yield

$$\Big| a_3\,(S^0_{r,w}, r, w)\; J_{s,t}(r,w) - a_3\,(x_0, 0, t)\, \gamma_{s,t}(r,t) \Big|$$
$$\le C\, \{\, |J_{s,t}(r,w) - \gamma_{s,t}(r,t)\,| + |S^0_{r,w} - x_0| + |r| + |t - w|\, \} .$$

Therefore, by (3.1.15), (3.1.16) and the Lipschitz property of γ, we obtain $B_2(\varepsilon) \le C\, \varepsilon^4$. Thus, from (3.1.13), $\Gamma_z \ge C\, (\varepsilon^2 - \varepsilon^4) > 0$, for ε small enough.

In the sequel we assume (H4) and (H6). For any $\varepsilon, \beta, \delta > 0$ set

$$B_1(\varepsilon, \beta, \delta) = \frac{1}{2} \int_{C^z_{\beta,\delta}(\varepsilon)} \Big(\gamma_z(r, w)\; a_3(S^0_{r,w}, r, w)\Big)^2 dr\, dw\,.$$

$$B_2(\varepsilon, \beta, \delta) = \int_{C^z_{\beta,\delta}(\varepsilon)} \Big(a_3(S^0_{r,w}, r, w)\; (J_z(r, w) - \gamma_z(r, w))\Big)^2 dr\, dw\,.$$

Clearly $\Gamma_z \geq B_1(\varepsilon, \beta, \delta) - B_2(\varepsilon, \beta, \delta)$ and, by (3.1.14) and (3.1.16), $B_2(\varepsilon, \beta, \delta) \leq C\, \varepsilon^{\beta+3\delta}$.
Let

$$B_{11}(\varepsilon, \beta, \delta) = \frac{1}{4} \int_{C^z_{\beta,\delta}(\varepsilon)} \Big(a_3(S^0_{r,w}, r, w)\; \gamma_z(r, t)\Big)^2 dr\, dw\,.$$

$$B_{12}(\varepsilon, \beta, \delta) = \frac{1}{2} \int_{C^z_{\beta,\delta}(\varepsilon)} \Big(a_3(S^0_{r,w}, r, w)\; (\gamma_z(r, w) - \gamma_z(r, t))\Big)^2 dr\, dw\,.$$

We have $B_1(\varepsilon, \beta, \delta) \geq B_{11}(\varepsilon, \beta, \delta) - B_{12}(\varepsilon, \beta, \delta)$.
The Lipschitz property of γ yields

$$B_{12}(\varepsilon, \beta, \delta) \leq C\, \varepsilon^{\beta+3\delta}\,. \tag{3.1.17}$$

Taylor's formula implies, for some $\overline{r} \in (0, r)$,

$$a_{31}(r, w)\, S^0_{r,w} + a_{32}(r, w) = a_{31}(0, w)\, S^0_{r,w} + a_{32}(0, w)$$

$$+ \Big(\partial_1 a_{31}(0, w) S^0_{r,w} + \partial_1 a_{32}(0, w)\Big) r + \frac{1}{2}\Big(\partial_1^2 a_{31}(\overline{r}, w) S^0_{r,w} + \partial_1^2 a_{32}(\overline{r}, w)\Big) r^2.$$

Thus, by (H6)

$$a_{31}(r, w)\, S^0_{r,w} + a_{32}(r, w) = a_{31}(0, w)\, \Big(S^0_{r,w} - x_0\Big) + r\, \partial_1 a_{31}(0, w)\, \Big(S^0_{r,w} - x_0\Big)$$

$$+ r\Big(\partial_1 a_{31}(0, w) x_0 + \partial_1 a_{32}(0, w)\Big) + \frac{1}{2} r^2 \Big(\partial_1^2 a_{31}(\overline{r}, w) S^0_{r,w} + \partial_1^2 a_{32}(\overline{r}, w)\Big).$$

In addition,

$$\begin{aligned} a_{31}(0, w)(S^0_{r,w} - x_0) &= a_{31}(0, w) \int_0^r \int_0^w \gamma_{r,w}(u, v) a_4(S^0_{u,v}, u, v) du\, dv \\ &= a_{31}(0, w)\, r \int_0^w \gamma_{0,t}(0, v)\, a_4(x_0, 0, v) dv \\ &\quad + a_{31}(0, w) \int_0^r \int_0^w \Big(\gamma_{r,w}(u, v) a_4(S^0_{u,v}, u, v) \\ &\quad - \gamma_{0,t}(0, v) a_4(x_0, u, v)\Big) du\, dv\,. \end{aligned}$$

Let $a_3(S^0_{r,w}, r, w) = \sum_{j=1}^4 L^j_{r,w}$ with

$$L^1_{r,w} = a_{31}(0,w)\int_0^r\int_0^w \Big\{\gamma_{r,w}(u,v)\Big(a_{41}(u,v)S^0_{u,v} + a_{42}(u,v)\Big) - \gamma_{0,t}(0,v)\Big(a_{41}(0,v)x_0 + a_{42}(0,v)\Big)\Big\}du\,dv\,.$$

$$L^2_{r,w} = r\,\partial_1\, a_{31}(0,w)(S^0_{r,w} - x_0)\,.$$

$$L^3_{r,w} = \frac{1}{2}\, r^2\Big(\partial_1^2\, a_{31}(\bar{r},w)\; S^0_{r,w} + \partial_1^2\, a_{32}(\bar{r},w)\Big).$$

$$L^4_{r,w} = r\Bigg(\partial_1 a_{31}(0,w)\,x_0 + \partial_1 a_{32}(0,w) + a_{31}(0,w)\int_0^w \gamma_{0,t}(0,v)\Big(a_{41}(0,v)x_0 + a_{42}(0,v)\Big)dv\Bigg).$$

Define

$$B_{111}(\varepsilon,\beta,\delta) = \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)\;(L^4_{r,w})^2 dr\,dw\,.$$

$$B_{112}(\varepsilon,\beta,\delta) = \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)\;\Big\{(L^1_{r,w})^2 + (L^2_{r,w})^2 + (L^3_{r,w})^2\Big\}dr\,dw\,.$$

Then $8\,B_{11}(\varepsilon,\beta,\delta) \geq B_{111}(\varepsilon,\beta,\delta) - 8B_{112}(\varepsilon,\beta,\delta)$. Moreover, $B_{111}(\varepsilon,\beta,\delta) \geq B_{1111}(\varepsilon,\beta,\delta) - B_{1112}(\varepsilon,\beta,\delta)$ where

$$B_{1111}(\varepsilon,\beta,\delta) = \frac{1}{2}\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)\; r^2\Big\{\partial_1 a_{31}(0,t)x_0 + \partial_1 a_{32}(0,t) + a_{31}(0,t)\int_0^t \gamma_{0,t}(0,v)a_4(x_0,0,v)dv\Big\}^2 dr\,dw\,,$$

$$B_{1112}(\varepsilon,\beta,\delta) = \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)\; r^2\Big\{(\partial_1 a_{31}(0,w) - \partial_1 a_{31}(0,t))x_0 + (\partial_1 a_{32}(0,w) - \partial_1 a_{32}(0,t)) + (a_{31}(0,w) - (a_{31}(0,t))\int_0^w \gamma_{0,t}(0,v)\; a_4(x_0,0,v)\,dv + a_{31}(0,t)\int_w^t \gamma_{0,t}(0,v)\; a_4(x_0,0,v)\,dv\Big\}^2 dr\,dw\,.$$

Property (3.1.14) and assumptions (H2), (H4) ensure

$$\Big|\gamma_{r,w}(u,v)\Big(a_{41}(u,v)S^0_{u,v} + a_{42}(u,v)\Big) - \gamma_{0,t}(0,v)\Big(a_{41}(0,v)x_0 + a_{42}(0,v)\Big)\Big| \leq C\Big\{\,|r| + |t-w| + |u| + |S^0_{u,v} - x_0|\,\Big\}\,.$$

Thus, (3.1.15) yields

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)(L^1_{r,w})^2 dr\, dw \le C\,(\varepsilon^{3\beta+3\delta} + \varepsilon^{5\beta+\delta}) \tag{3.1.18}$$

and

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)\ r^2\ \partial_1\, a_{31}(0,w)^2(S^0_{r,w} - x_0)^2 dr\, dw \le C\,\varepsilon^{5\beta+\delta}\,. \tag{3.1.19}$$

By (H4) and (3.1.14),

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r,t)\ (L^3_{r,w})^2\ dr\, dw \le C\,\varepsilon^{5\beta+\delta}. \tag{3.1.20}$$

Consequently, (3.1.18) to (3.1.20) give

$$B_{112}(\varepsilon,\beta,\delta) \le C\,(\varepsilon^{3\beta+3\delta} + \varepsilon^{5\beta+\delta})\,. \tag{3.1.21}$$

By (H6) and the positivity of $\gamma_{s,t}(u,t)$

$$B_{1111}(\varepsilon,\beta,\delta) \ge C\,\varepsilon^{3\beta+\delta}\,. \tag{3.1.22}$$

Finally, by (H2) to (H4)

$$B_{1112}(\varepsilon,\beta,\delta) \le C \int_{C^z_{\beta,\delta}(\varepsilon)} r^2\ |t-w|^2\, dr\, dw \le C\,\varepsilon^{3\beta+3\delta}\,. \tag{3.1.23}$$

Therefore, putting together the estimate for $B_2(\varepsilon,\beta,\delta)$, (3.1.17) and (3.1.21) to (3.1.23), we obtain

$$\Gamma_z \ge C\,\varepsilon^{3\beta+\delta} - C\,(\varepsilon^{\beta+3\delta}\ \varepsilon^{3\beta+3\delta} + \varepsilon^{5\beta+\delta})\,.$$

This clearly yields $\Gamma_z > 0$ by choosing ε small enough and $\beta < \delta$. The proof of the Proposition is complete. □

Propositions 3.1.1, 3.1.3, 3.1.4, 3.1.5 establish all the necessary ingredients to apply Theorem 2.4 to the family $\{X^\varepsilon_z,\ \varepsilon \in (0,1]\}$ defined by (3.1.3) with $z = (s,t) \in T,\ s \cdot t \ne 0$.

3.2 An Itô equation on the plane

Consider a one-dimensional Wiener process $\{W_{s,t}, (s,t) \in T\}$, $T = [0,1]^2$, vector fields $A(x) = \binom{x_1}{x_2}$, $A_0(x) = \binom{x_2}{x_1}$ and the stochastic differential equation on $\mathbb{R}^2$,

$$Z_z = x_0 + \int_{R_z} [A(Z_\eta)\, dW_\eta + A_0(Z_\eta)\, d\eta], \quad z \in T. \tag{3.2.1}$$

with initial condition $x_0 = \binom{1}{0}$. Let $\{Z_z^\varepsilon, z \in T\}$ be the solution of

$$Z_z^\varepsilon = x_0 + \int_{R_z} [\varepsilon\, A(Z_\eta^\varepsilon)\, dW_\eta + A_0(Z_\eta^\varepsilon)\, d\eta]$$

and $\{\Psi(z), z \in T\}$ be given by

$$\Psi(z) = x_0 + \int_{R_z} A_0(\Psi(\eta)) d\eta .$$

In the sequel z will be a fixed point in T not on the axis. The analogue of Proposition 3.1.1 for the solution of (3.2.1) can be proved by the same arguments, due to the linearity of the coefficients A and A_0. Thus,

$$Z_z^\varepsilon = E\, Z_z + \sum_{n=1}^{\infty} \varepsilon^n\, I_n(f_n),$$

where $Z_z = E\, Z_z + \sum_{n=1}^{\infty} I_n(f_n)$ is the Wiener-Chaos decomposition of the L^2-functional Z_z.

Let $\hat{Z}_z^\varepsilon = \frac{Z_z^\varepsilon - \Psi(z)}{\varepsilon}$. Following the ideas of the proof of Proposition 3.1.3 we obtain

$$\sup_{0<\varepsilon\leq 1} \left\| \frac{d^j}{d\,\varepsilon^j}\, \hat{Z}_z^\varepsilon \right\|_{k,p} \leq C,$$

for any $j \in Z^+$, $k \in \mathbb{N}$, $p \in (1,\infty)$.

In [10] we have proved $Z_z \in \mathbb{D}^\infty$ and $\|\Gamma_{Z_z}^{-1}\|_p \leq C$, for any $p \in [1,\infty)$. By considering the coefficient εA instead of A we also have $Z_z^\varepsilon \in \mathbb{D}^\infty$ and $\|\Gamma_{Z_z^\varepsilon}^{-1}\|_p \leq C_\varepsilon$, $p \in [1,\infty)$, for some constant C_ε depending on $\varepsilon \in (0,1]$.

Proposition 3.2.1 *Let $I_1(f_1)$ be the gaussian component in the Wiener-Chaos decomposition of Z_z. Then,* $\sigma^2 := \det\left(\mathrm{Cov}\,(I_1(f_1))\right) > 0$.

Proof. Since $Z_z \in \mathbb{D}^\infty$, $f_1(\alpha) = E\,(D_\alpha\, Z_z)$, $\alpha \le z$. So we will first give an explicit expression for $E\,(D_\alpha\, Z_z)$.
Consider the continuous functions defined on T given by the series

$$m(z) = \sum_{k=0}^{\infty} \frac{(s\,t)^{2k}}{(\,(2k)!)^2}\,, \quad n(z) = \sum_{k=0}^{\infty} \frac{(s\,t)^{2k+1}}{(\,(2k+1)!)^2}\,, \tag{3.2.2}$$

$z = (s,t)$.
Set $Z_z = (X_z, Y_z)$. The mean vector $\{E\,Z_z,\ z \in T\}$ satisfies the deterministic equation

$$\left.\begin{aligned} E\,X_z &= 1 + \int_{R_z} E\,Y_\eta\, d_\eta \\ E\,Y_z &= \int_{R_z} E\,X_\eta\, d_\eta \end{aligned}\right\}$$

Consequently,

$$\left.\begin{aligned} E\,X_z &= 1 + \int_{R_z}\int_{R_{\eta_1}} E\,X_{\eta_2}\, d\eta_2\, d\eta_1 \\ E\,Y_z &= \int_{R_z} \left[1 + \int_{R_{\eta_1}} E\,Y_{\eta_2}\, d\eta_2\right] d\eta_1 \end{aligned}\right\}$$

and therefore, $E\,X_z = m(z)$, $E\,Y_z = n(z)$.
The Malliavin derivative of Z_z satisfies the stochastic differential equation

$$D_\alpha\, Z_z = A\,(Z_\alpha) + \int_{(\alpha,z]} [\nabla A\,(Z_\eta)\; D_\alpha\, Z_\eta\, dW_\eta + \nabla A_0(Z_\eta)\, D_\alpha\, Z_\eta\, d\eta]\,. \tag{3.2.3}$$

Consequently,

$$\left.\begin{aligned} E\,(D_\alpha\, X_z) &= m(\alpha) + \int_{(\alpha,z]} E\,(D_\alpha\, Y_\eta)\, d\eta \\ E\,(D_\alpha\, Y_z) &= n(\alpha) + \int_{(\alpha,z]} E\,(D_\alpha\, X_\eta)\, d\eta \end{aligned}\right\} \tag{3.2.4}$$

From (3.2.4) we easily obtain

$$\left.\begin{aligned} E\,(D_\alpha\, X_z) &= m(\alpha)\; m(z-\alpha) + n(\alpha)\; n(z-\alpha) \\ E\,(D_\alpha\, Y_z) &= n(\alpha)\; m(z-\alpha) + m(\alpha)\; n(z-\alpha)\,. \end{aligned}\right\}$$

where $z - \alpha = (s - \alpha_1,\ t - \alpha_2)$ for $z = (s,t)$, $\alpha = (\alpha_1, \alpha_2)$. Therefore,

$$f_1(\alpha) = \Big(m(\alpha)\; m(z-\alpha) + n(\alpha)\; n(z-\alpha)\,,\ n(\alpha)\; m(z-\alpha) + m(\alpha)\; n(z-\alpha)\Big)$$

and

$$\sigma^2 = \left(\int_T \Big(m(\alpha)\ m(z-\alpha) + n(\alpha)\ n(z-\alpha) \Big)^2 d\alpha \right)$$
$$\times \left(\int_T \Big(n(\alpha)\ m(z-\alpha) + m(\alpha)\ n(z-\alpha) \Big)^2 d\alpha \right)$$
$$- \left(\int_T \Big(m(\alpha)\ m(z-\alpha) + n(\alpha)\ n(z-\alpha) \Big) \right.$$
$$\left. \times \Big(n(\alpha)\ m(z-\alpha) + m(\alpha)\ n(z-\alpha) \Big) d\alpha \right)^2 .$$

Assume $\sigma^2 = 0$. Schwarz's inequality yields the existence of $\lambda \in \mathbb{R} : f_1^1(\alpha) = \lambda\, f_1^2(\alpha)$, for any $\alpha \in R_z$. This leads to contradiction. Indeed, assume for simplicity $z = (1,1)$. For $\alpha = (\frac{1}{2}, \frac{1}{2})$ we have $m(\alpha) = m(z-\alpha)$, $n(\alpha) = n(z-\alpha)$, therefore

$$m^2\Big(\frac{1}{2}, \frac{1}{2}\Big) + n^2\Big(\frac{1}{2}, \frac{1}{2}\Big) = 2\,\lambda\, n\Big(\frac{1}{2}, \frac{1}{2}\Big)\ m\Big(\frac{1}{2}, \frac{1}{2}\Big). \tag{3.2.5}$$

On the other hand, $n(\frac{1}{4}, 1) = n(\frac{1}{2}, \frac{1}{2})$, $m(\frac{1}{4}, 1) = m(\frac{1}{2}, \frac{1}{2})$, $m((1,1) - (\frac{1}{4}, 1)) = 1$, $n((1,1) - (\frac{1}{4}, 1)) = 0$. Consequently

$$m\Big(\frac{1}{2}, \frac{1}{2}\Big) = \lambda\, n\Big(\frac{1}{2}, \frac{1}{2}\Big). \tag{3.2.6}$$

(3.2.5) and (3.2.6) ensure $m(\frac{1}{2}, \frac{1}{2}) = n(\frac{1}{2}, \frac{1}{2})$ which is impossible. In fact, from (3.2.2) we clearly have $n(\frac{1}{2}, \frac{1}{2}) < m(\frac{1}{2}, \frac{1}{2})$. $\square$

We close this section checking $\|\Gamma_{Z_z^\varepsilon}^{-1}\|_p \le C\,\varepsilon^{-2}$ for any $\varepsilon \in (0,1]$, $p \in (1,\infty)$ and some positive constant C. Clearly, it suffices to show

$$\sup_{\varepsilon\in(0,1]} E(|\det \gamma_\varepsilon^{-1}|^p) \le C, \quad p \in (1,\infty),$$

with $\gamma_\varepsilon = \varepsilon^{-2}\,\Gamma_{Z_z^\varepsilon}$. This property will follow from the following fact

$$\sup_{\varepsilon\in(0,1]} \sup_{|v|=1} P\{ v^* \gamma_\varepsilon\, v \le \eta \} \le C(p)\,\eta^p, \tag{3.2.7}$$

for any $p \in (1,\infty)$ and η small enough.
Using (3.2.3) we easily obtain

$$\gamma_\varepsilon^{ij} = \int_{R_z} \xi_k^{\varepsilon,i}(z,r)\ A^k(Z_r^\varepsilon)\ \xi_{k'}^{\varepsilon,j}(z,r)\ A^{k'}(Z_r^\varepsilon)\,dr, \quad 1 \le i,j \le 2.$$

where $\{\xi^\varepsilon(z,r),\ 0 \le r \le z\}$ is a $\mathbb{R}^2 \otimes \mathbb{R}^2$-valued process solution to the stochastic differential equation

$$\xi^\varepsilon(z,r) = I + \int_{(r,z]} \{\, \varepsilon \nabla A(Z^\varepsilon_\eta)\, \xi^\varepsilon(\eta,r)\, dW_\eta + \nabla A_0(Z^\varepsilon_\eta)\, \xi^\varepsilon(\eta,r)\, d\eta \,\}\,.$$

Then, as in [10], the proof of (3.2.7) is reduced to that of

$$\sup_{\varepsilon\in(0,1]} \sup_{|r|=1} P\Big\{ \int_0^s |v_i\, A^i(Z^\varepsilon_{\sigma t})\,|^2\, d\sigma \le \eta \Big\} \le C(p)\, \eta^p\,.$$

Let $\mathcal{D} = \{A,\ A_0^\nabla A\}$. Clearly the span of $\mathcal{D}$ at $x_0 = \binom{1}{0}$ is $\mathbb{R}^2$. Consequently, there exists $R > 0,\ c > 0$ such that

$$\sum_{V\in\mathcal{D}} \Big(v_i\, V^i(y)\Big)^2 \ge c\,, \tag{3.2.8}$$

for any $|v| = 1$ and $y \in B_R(x_0)$.
Let $S^\varepsilon = \inf\{\sigma \ge 0\ : \sup_{\xi\le\sigma,\tau\le t}\ |Z^\varepsilon_{\xi\tau} - x_0| \ge R\} \wedge s$. Then

$$P\Big\{ \int_0^{S^\varepsilon} |v_i\, A^i(Z^\varepsilon_{\sigma t})|^2\, d\sigma < \eta \Big\} \le p_1^\varepsilon(\eta) + p_2^\varepsilon(\eta) + p_3^\varepsilon(\eta)$$

with

$$\begin{aligned} p_1^\varepsilon(\eta) &= P\Big\{ \int_0^{S^\varepsilon} |v_i\, A^i(Z^\varepsilon_{\sigma t})|^2\, d\sigma < \eta,\ \int_0^{S^\varepsilon} |v_i (A_0^\nabla\, A)^i\, (Z^\varepsilon_{\sigma t})|^2\, d\sigma < \eta^\alpha, \\ &\qquad S^\varepsilon \ge \eta^\beta \Big\}, \\ p_2^\varepsilon(\eta) &= P\,\{S^\varepsilon < \eta^\beta\}\,, \\ p_3^\varepsilon(\eta) &= P\Big\{ \int_0^{S^\varepsilon} |v_i\, A^i(Z^\varepsilon_{\sigma t})|^2\, d\sigma < \eta,\ \int_0^{S^\varepsilon} |v_i (A_0^\nabla\, A)^i\, (Z^\varepsilon_{\sigma t})|^2\, d\sigma \ge \eta^\alpha \Big\}. \end{aligned}$$

where $0 < \beta < \alpha < 1$.
Property (3.2.8) and the choice of β, α yields $p_1^\varepsilon(\eta) = 0$ for η sufficiently small. Chebychev's inequality and Burkholder's and Hölder's inequalities ensure $\sup_{\varepsilon\in(0,1]}\ p_2(\eta) \le C\eta^{\beta q/2}$. The term $p_3^\varepsilon(\eta)$ demands a careful analysis. This has been done in [10] (pg. 15) and corresponds to the term A_2 is this reference with $V = A$, $X_{\sigma\cdot} = Z^\varepsilon_{\sigma\cdot}$, $\varepsilon^{m(j-1)} = \eta$, $\alpha = \frac{m(j)}{m(j-1)}$. As a hint for the reader, we point out that span $\Big(A(x_0),\ A_0^\nabla\ A(x_0)\Big) = \mathbb{R}^2$ implies the validity of assumption (H2) of Theorem 2.2 in [10]. Indeed, using the notation in this article,

$$A(x_0) = A^1(x_0),\quad A_0^\nabla A(x_0) = \Big[\int_0^1 (A_0 * A)\ (\tau, 1)\, d\tau\Big]\,(x_0)\,.$$

Since all the estimates in the above mentioned proof can be obtained uniformly in the paramater ε, we conclude. $\square$

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