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PRODUCT LOGIC AND THE DEDUCTION THEOREM

by

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Abstract

In this paper we prove the following negative result: Product Logic [9] does not have the Deduction Theorem, that is, there is no binary defined connective in the language of Product Logic such that the Deduction Theorem is satisfied with respect to it. We prove this theorem mainly by using algebraic methods: we prove that Product Logic is algebraizable, that the variety of Product Algebras is its equivalent quasivariety semantics and that this variety has no equationally definable principal congruences.

1 Introduction

Product Logic [9] (ΠL, for short) arises in the context of the studies of fuzzy logics (in the narrow sense), in other words, in the context of the studies of logical systems appropriate for a formalization of approximate reasoning [8]. It is defined semantically by interpreting the connectives of the language on the unit interval $[0, 1]$ in the following way: conjunction is interpreted by the ordinary product of real numbers, implication is interpreted as the residuated implication with respect to product, and the constant is interpreted by the real number 0. Product logic is one of the three logics based on the main

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t-norms (see for example [12] and [13]): Lukasiewicz's t-norm ($t(x, y) = \max(0, x + y - 1)$), Gödel's t-norm ($t(x, y) = \min(x, y)$) and Product t-norm ($t(x, y) = x \cdot y$).

A complete axiom system for Product Logic has been obtained in [9], where the quasivariety of Product Algebras is mainly studied with regard to the completeness result.

In this paper we show that the class of Product Algebras is a variety of residuated lattices (Definition 4) and is also a variety of bounded hoops (Theorem 7). This variety is generated by the unit interval Product algebra (Theorem 10). This is similar to the case of the variety of MV-algebras, which is generated by the unit interval MV-algebra.

Product algebras have a very strong link to Product Logic: the variety of Product algebras is the equivalent quasivariety semantics of Product logic (Corollary 12). This means, roughly speaking, that there is an interpretation of the formulas of Product Logic as equations and an interpretation of equations as formulas such that they are essentially inverse to each other.

The main result of this paper is that Product Logic does not have the Deduction Theorem (Theorem 17), that is, there is no binary defined connective $\eta(x, y)$ in the language of Product Logic such that the following equivalence holds:

For all $\Gamma \subseteq Fm$ and $\varphi, \psi \in Fm$,

$$\Gamma, \varphi \vdash_{\Pi L} \psi \Leftrightarrow \Gamma \vdash_{\Pi L} \eta(\varphi, \psi)$$

This means, roughly speaking, that the deduction relation $\varphi \vdash_{\Pi L} \psi$ cannot be "recorded" by using a formula in two variables $\eta(x, y)$. This is similar to the case of the infinite-valued Łukasiewicz Logic, which does not possess the Deduction Theorem either.

2 Preliminaries

This section contains the basic known definitions and results about deductive systems which will be used in this paper. We use papers [2] and [4] as the main references.

By a propositional language we mean a set $\mathcal{L}$ of propositional connectives. The $\mathcal{L}$-formulas are built in the usual way from the propositional variables by using the connectives of $\mathcal{L}$. We denote by $Fm_{\mathcal{L}}$the set of all $\mathcal{L}$-formulas.

A deductive system S over $\mathcal{L}$ is a pair $S = \langle \mathcal{L}, \vdash_S \rangle$ where $\vdash_S$ is a finitary consequence relation over $Fm_{\mathcal{L}}$. That is, $\vdash_S$ is a relation between subsets of $Fm_{\mathcal{L}}$ and elements of $Fm_{\mathcal{L}}$ that satisfies the following conditions for all Γ, $\Delta \subseteq Fm_{\mathcal{L}}$ and φ, $\psi \in Fm_{\mathcal{L}}$:

(i) $\varphi \in \Gamma$ implies $\Gamma \vdash_S \varphi$,

(ii) $\Gamma \vdash_S \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_S \varphi$,

(iii) $\Gamma \vdash_S \varphi$ and $\Delta \vdash_S \psi$ for every $\psi \in \Gamma$ implies $\Delta \vdash_S \varphi$.

In addition, $\vdash_S$ is finitary in the sense

(iv) $\Gamma \vdash_S \varphi$ implies $\Gamma' \vdash_S \varphi$ for some finite $\Gamma' \subseteq \Gamma$,

and it is structural in the sense

(v) $\Gamma \vdash_S \varphi$ implies $\sigma(\Gamma) \vdash_S \sigma(\varphi)$ for every substitution $\sigma : \mathbf{Fm} \to \mathbf{Fm}$.

A set of axioms and inference rules over $\mathcal{L}$ defines a deductive system S in the usual way: for $\Gamma \subseteq Fm_{\mathcal{L}}$ and $\varphi \in Fm_{\mathcal{L}}$, $\Gamma \vdash_S \varphi$ iff φ is contained in the smallest set of $\mathcal{L}$-formulas that includes Γ, together with all the substitution instances of the axioms of S, and is closed under direct derivability by the inference rules of S.

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra and $F \subseteq A$. The Leibniz congruence $\Omega_{\mathbf{A}}F$ is defined by: ([2, Definition 1.4])

$$\Omega_{\mathbf{A}}F = \Big\{(a,b) \in A^2 : \varphi^{\mathbf{A}}(a, c_0, ..., c_{k-1}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, c_0, ..., c_{k-1}) \in F \\ \text{for all } \varphi(p, q_0, ..., q_{k-1}) \in Fm_{\mathcal{L}} \text{ and all } c_0, ..., c_{k-1} \in A\,\Big\}$$

$\Omega_{\mathbf{A}}F$ can be characterized as the largest congruence of $\mathbf{A}$ compatible with F (i.e., the largest congruence θ such that $a \in F$ and $(a,b) \in \theta$ imply $b \in F$, for all $a, b \in A$). The function $\Omega_{\mathbf{A}}$ with domain the set of all subsets of A is called the Leibniz operator on $\mathbf{A}$.

Let S be a deductive system. A matrix model of S is a pair $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra, $F \subseteq A$ and if $\Gamma \vdash_S \varphi$, then, for every $h \in Hom(Fm_{\mathcal{L}}, \mathbf{A})$, $h(\Gamma) \subseteq F$ implies $h(\varphi) \in F$. Then we say that F is an S-filter. Thus F is an S-filter iff F contains all the interpretations of the logical axioms of S and is closed under each inference rule of S.

Let S be a deductive system and K a class of $\mathcal{L}$-algebras. K is an equivalent algebraic semantics for S (in the sense of [2]) with defining equations $\{\delta_i(p) \approx \epsilon_i(p) : i < n\}$ and equivalence formulas $\{\Delta_j(p,q) : j < m\}$ iff:

(i) For every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \vdash_S \varphi \Leftrightarrow \{\{\delta_i(\psi) \approx \epsilon_i(\psi) : i < n\} : \psi \in \Gamma\} \vDash_K \{\delta_i(\varphi) \approx \epsilon_i(\varphi) : i < n\}$$

(ii) For every $\varphi \approx \psi \in Fm^2_{\mathcal{L}}$,

$$\varphi \approx \psi \dashv\models_K \{\delta_i(\varphi\Delta_j\psi) \approx \epsilon_i(\varphi\Delta_j\psi) : i < n,\ j < n\}$$

As a consequence of (i) and (ii), it follows that (see [2, Corollary 2.9]):

(iii) For every $\Gamma \cup \{\varphi \approx \psi\} \subseteq Fm^2_{\mathcal{L}}$,

$$\Gamma \models_K \varphi \approx \psi \Leftrightarrow \{\xi\Delta\eta : \xi \approx \eta \in \Gamma\} \vdash_S \varphi\Delta\psi,$$

(iv) For every $\vartheta \in Fm_{\mathcal{L}}$, $\quad \vartheta \dashv\vdash_S \delta(\vartheta)\Delta\epsilon(\vartheta)$.

A deductive system is said to be algebraizable if it has an equivalent algebraic semantics. For an algebraizable deductive system there is a unique equivalent quasivariety semantics. It is called the equivalent quasivariety semantics of the system (see [2, Theorem 2.15]).

The following theorem is a very useful characterization of the equivalent quasivariety semantics of an algebraizable deductive system.

Theorem 1 *([2, **Theorem 5.1**]) Let S be a deductive system and K a quasivariety. The following are equivalent:*

(i) S is algebraizable with equivalent algebraic semantics K.

(ii) For every algebra $\mathbf{A}$ the Leibniz operator $\Omega_{\mathbf{A}}$ is an isomorphism between the lattices of S-filters and K-congruences of $\mathbf{A}$ (θ is a K-congruence if $\mathbf{A}/\theta \in K$).

A deductive system $S = \langle \mathcal{L}, \vdash_S \rangle$ has the deduction-detachment theorem (DDT, for short) (see [4]) if there exists a finite set $E(p,q) = \{\eta_0(p,q), \ldots, \eta_{m-1}(p,q)\}$ of formulas in two variables such that, for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$, we have

$$\Gamma \cup \{\varphi\} \vdash_S \psi \ \text{ iff } \ \Gamma \vdash_S E(\varphi, \psi).$$

Here $\Gamma \vdash_S E(\varphi,\psi)$ is an abbreviation for the conjunction of the assertions $\Gamma \vdash_S \eta_i(\varphi,\psi)$, $i < m$. If S has the DDT with respect to $E(\varphi,\psi)$, we call $E(\varphi,\psi)$ a deduction-detachment set for S. We can see the finite set $E(\varphi,\psi)$ as collectively behaving as an implication connective.

A quasivariety K of $\mathcal{L}$-algebras has equationally definable principal relative congruences (EDPRC, for short) (see [4] and [5]) if there are 4-ary terms (i.e., $\mathcal{L}$-formulas)

$$\eta_{i,0}(p_0, p_1, q_0, q_1), \ \ \eta_{i,1}(p_0, p_1, q_0, q_1), \ i < m,$$

such that, for all $\mathbf{A} \in K$ and all $a, b, c, d \in A$,

$$c \equiv d \ (\Theta_{\mathbf{A}}(a,b)) \Longleftrightarrow \eta_{i,0}^{\mathbf{A}}(a,b,c,d) = \eta_{i,1}^{\mathbf{A}}(a,b,c,d), \ i < m,$$

where $\Theta_{\mathbf{A}}(a,b)$ denotes the principal K-congruence of $\mathbf{A}$ generated by the pair (a,b).

There is the following connection between EDPRC and the DDT.

Theorem 2 *([4, **Theorem VI.1.3**]) Let S be an algebraizable deductive system with equivalent quasivariety semantics K. Then S has the DDT iff K has EDPRC.*

3 Residuated lattices, Product algebras and bounded hoops

In this section we will define the variety of Product algebras (Definition 4) as a variety of residuated lattices and prove that Product algebras can also be defined as a variety of bounded hoops (Theorem 7).

Let us recall the definition of a residuated lattice.

A **residuated lattice** is an algebra $\mathbf{A} = \langle A, \odot, \rightarrow, \wedge, \vee, 0, 1\rangle$ of type $(2,2,2,2,0,0)$ such that:

(i) $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice, where 1 and 0 are the maximum and minimum elements, respectively, that is; for all $x, y, z \in A$,

*rl*1. $x \wedge y = y \wedge x$
*rl*2. $x \vee y = y \vee x$
*rl*3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
*rl*4. $x \vee (y \vee z) = (x \vee y) \vee z$
*rl*5. $x = x \wedge (x \vee y)$
*rl*6. $x = x \vee (x \wedge y)$
*rl*7. $x \wedge 1 = x$
*rl*8. $x \wedge 0 = 0$

(ii) $\langle A, \odot, 1\rangle$ is a commutative monoid, that is; for all $x, y, z \in A$,

*rl*9. $(x \odot y) \odot z = x \odot (y \odot z)$
*rl*10. $x \odot y = y \odot x$
*rl*11. $1 \odot x = x$

(iii) Residuation, that is, for all for all $x, y, z \in A$,

*rl*12. $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$

Residuated lattices have been studied by many authors authors, see for example Krull [11], Dilworth and Ward [6], Ward [16], Pavelka [14] and Zlatoš [17].

In this section we will deal with the special class of residuated lattices where the meet $\wedge$ and join $\vee$ are defined by means of $\odot$ and $\rightarrow$ in the following way:

$$\begin{aligned} x \wedge y &= x \odot (x \rightarrow y), \\ x \vee y &= ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x) \end{aligned}$$

Let us recall the definition of Product algebras.

A **Product algebra** ([9, Definition 2]) is an algebra $\mathbf{A} = \langle A, \odot, \rightarrow, 0 \rangle$ of type $(2, 2, 0)$ such that if we define the binary operations $\wedge$ and $\vee$, the unary operation $\neg$ and the constant 1 in the following way

$$\begin{aligned} 1 &= 0 \rightarrow 0, \\ \neg x &= x \rightarrow 0, \\ x \wedge y &= x \odot (x \rightarrow y), \\ x \vee y &= ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x), \end{aligned}$$

then for all $x, y, z \in A$ the following conditions are satisfied:

- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice ($x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$), 1 and 0 are the maximum and minimum elements, respectively, that is: $x \wedge 1 = x$ and $x \wedge 0 = 0$.

- $\langle A, \odot, 1 \rangle$ is a commutative monoid, that is:
  1. $(x \odot y) \odot z = x \odot (y \odot z)$
  2. $x \odot y = y \odot x$
  3. $1 \odot x = x$

- $\langle \odot, \rightarrow \rangle$ is an adjoint couple on A, that is:
  1. $x \leq y$ implies $(x \odot z) \leq (y \odot z)$
  2. $x \leq y$ implies $(y \rightarrow z) \leq (x \rightarrow z)$
  3. $x \leq y$ implies $(z \rightarrow x) \leq (z \rightarrow y)$
  4. $x \leq (y \rightarrow z)$ iff $(x \odot y) \leq z$

- $(x \rightarrow y) \vee (y \rightarrow x) = 1$

- $(\neg\neg z \odot ((x \odot z) \rightarrow (y \odot z))) \rightarrow (x \rightarrow y) = 1$

- $x \wedge \neg x = 0$

- $(x \odot (y \vee z)) \to ((x \odot y) \vee (x \odot z)) = 1$
- $((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge z)) = 1$

The next theorem shows that the definition of Product algebra can be simplified.

Theorem 3 *Let* $\mathbf{A} = \langle A, \odot, \to, \wedge, \vee, 0, 1\rangle$ *be a residuated lattice such that the binary operations* $\wedge$ *and* $\vee$ *are defined in the following way*

$$\begin{aligned} x \wedge y &= x \odot (x \to y), \\ x \vee y &= ((x \to y) \to y) \wedge ((y \to x) \to x), \end{aligned}$$

then the following properties hold:

(1) $x \leq y$ *iff* $x \to y = 1$

(2) $x \leq y$ *implies* $(x \odot z) \leq (y \odot z)$

(3) $x \leq y$ *implies* $(y \to z) \leq (x \to z)$

(4) $x \leq y$ *implies* $(z \to x) \leq (z \to y)$

(5) $(x \to y) \vee (y \to x) = 1$

(6) $(x \odot (y \vee z)) \to ((x \odot y) \vee (x \odot z)) = 1$

(7) $((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge z)) = 1$

Proof.

The results (1), (2), (3) and (4) are well known for arbitrary residuated lattices (see for example [17]).

(1) $x \leq y$ iff $1 \odot x \leq y$ iff $1 \leq x \to y$ iff $x \to y = 1$

(2) From $y \odot z \leq y \odot z$ by applying the property $(rl12)$ we have $y \leq z \to (y \odot z)$. If $x \leq y$ then $x \leq z \to (y \odot z)$, by using the property $(rl12)$ we obtain $x \odot z \leq y \odot z$.

(3) If $x \leq y$ then by using (2) and the definition of $\wedge$ we obtain $(y \to z) \odot x \leq (y \to z) \odot y \leq z$ and by applying $(rl12)$ we have $(y \to z) \leq (x \to z)$

(4) By using the definition of $\wedge$ we have $(z \to x) \odot z \leq x$. If $x \leq y$ then $(z \to x) \odot z \leq y$ and by applying the property $(rl12)$ we have $(z \to x) \leq (z \to y)$.

(5) By applying the property ($rl12$) and the definition of $\wedge$ we obtain $x \leq (x \rightarrow y) \rightarrow x$ and by using the property (3) we have $((x \rightarrow y) \rightarrow x) \rightarrow y \leq x \rightarrow y$. Moreover, by using the property ($rl12$) and the definition of $\wedge$ we have $x \rightarrow y \leq ((x \rightarrow y) \rightarrow x) \rightarrow x$. Hence, $((x \rightarrow y) \rightarrow x) \rightarrow y \leq ((x \rightarrow y) \rightarrow x) \rightarrow x$. By using the property ($rl12$) we have $(((x \rightarrow y) \rightarrow x) \rightarrow y) \odot ((x \rightarrow y) \rightarrow x) \leq x$, and applying the commutativity of $\wedge$ we obtain $(y \rightarrow ((x \rightarrow y) \rightarrow x)) \odot y \leq x$, by ($rl12$) we have $(y \rightarrow ((x \rightarrow y) \rightarrow x)) \leq y \rightarrow x$, by ($rl12$) and commutativity of $\odot$ we obtain $((x \rightarrow y) \rightarrow (y \rightarrow x)) \leq y \rightarrow x$, that is $((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = 1$. Hence, by the definition of $\vee$, we get $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

(6) From $x \odot y \leq (x \odot y) \vee (x \odot z)$ and $x \odot z \leq (x \odot y) \vee (x \odot z)$ by applying the property ($rl12$) we have $y \leq x \rightarrow ((x \odot y) \vee (x \odot z))$ and $z \leq x \rightarrow ((x \odot y) \vee (x \odot z))$. Hence, we obtain $y \vee z \leq x \rightarrow ((x \odot y) \vee (x \odot z))$ and by using ($rl12$), we get $(x \odot (y \vee z)) \rightarrow ((x \odot y) \vee (x \odot z)) = 1$.

(7) By the property (5) we have $(x \odot y) \wedge (x \odot z) = ((x \odot y) \wedge (x \odot z)) \odot ((y \rightarrow z) \vee (z \rightarrow y))$ and by using (6) we obtain $(x \odot y) \wedge (x \odot z) = (((x \odot y) \wedge (x \odot z)) \odot (y \rightarrow z)) \vee (((x \odot y) \wedge (x \odot z)) \odot (z \rightarrow y))$. By the definition of $\wedge$ we have $(x \odot y) \wedge (x \odot z) \leq (x \odot y \odot (y \rightarrow z)) \vee (x \odot z \odot (z \rightarrow y)) \leq x \odot (y \wedge z)$. So $((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge z)) = 1$. $\square$

Next we give an equivalent definition of Product algebras.

Definition 4 *A Product algebra is an algebra* $\mathbf{A} = \langle A, \odot, \rightarrow, 0 \rangle$ *of type* $(2, 2, 0)$ *such that if we define two binary operations* $\wedge$ *and* $\vee$ *as in Theorem 3, a unary operation* $\neg$ *as* $\neg x = x \rightarrow 0$ *and a constant* 1 *as* $1 = 0 \rightarrow 0$*, then* $\langle A, \odot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ *is a residuated lattice which satisfies the following equations:*

- $(\neg\neg z \odot (x \odot z \rightarrow y \odot z)) \rightarrow (x \rightarrow y) = 1$
- $x \wedge \neg x = 0$

By using Theorem 3 it is clear that Definition 4 is equivalent to the definition of Product Algebra given above, where Product Algebras are defined by means of a set of equations and quasiequations. As a consequence of our definition we have that Product Algebras are an equationally definable class, and so they form a variety.

We will show in Theorem 7 that the variety of Product algebras is also a variety of bounded hoops.

Let us recall the definition of hoop [5]. Let $\mathbf{A} = \langle A, \odot, \rightarrow, 1\rangle$ be an algebra of type $(2,2,0)$, $\mathbf{A}$ is a hoop if for all $x, y, z \in A$ the following equations are satisfied [5]:

$$\begin{array}{ll}
(i) & 1 \odot x = x \odot 1 = x \\
(ii) & x \odot y = y \odot x \\
(iii) & (x \rightarrow y) \odot x = (y \rightarrow x) \odot y \\
(iv) & (x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) \\
(v) & x \rightarrow x = 1
\end{array}$$

Hoops can also be defined as algebras $\mathbf{A} = \langle A, \odot, \rightarrow\rangle$ of type $(2,2)$ satisfying the following equations, for all $x, y, z \in A$:

$$\begin{array}{ll}
(H1) & (y \rightarrow y) \odot x = x \odot (y \rightarrow y) = x \\
(H2) & x \odot y = y \odot x \\
(H3) & (x \rightarrow y) \odot x = (y \rightarrow x) \odot y \\
(H4) & (x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) \\
(H5) & x \rightarrow x = y \rightarrow y
\end{array}$$

It is easy to check that hoops in this sense are definitionally equivalent to hoops in the sense of [5].

A hoop is n-potent [5] if it satisfies the identity $x^n = x^{n-1}$ for some n, $0 < n < \omega$, where x^k is defined recursively in the standard way: $x^k = x \odot x^{k-1}$.

A bounded hoop is an algebra $\mathbf{A} = \langle A, \odot, \rightarrow, 0\rangle$ of type $(2,2,0)$ such that $\langle A, \odot, \rightarrow\rangle$ is a hoop and for all $x, y \in A$ the following equation is satisfied

$$(H6) \quad 0 \rightarrow x = y \rightarrow y$$

i.e., 0 is the lower bound of $\mathbf{A}$.

Now we will show that the class of Product algebras $\langle A, \odot, \rightarrow, 0\rangle$ is a variety of bounded hoops. To prove this result we will use the following lemmas.

Lemma 5 *Let $\mathbf{A} = \langle A, \odot, \rightarrow, 0\rangle$ be a Product algebra, then $\mathbf{A}$ is a bounded hoop.*

Proof.
The properties $(H1)$-$(H6)$ follow immediately from Definition 4 and Theorem 3. □

Lemma 6 *(Cf. [7]) Let $\langle A, \odot, \rightarrow, 0\rangle$ be a bounded hoop. Define the operations $\wedge$, $\vee$ and the constant 1 as in Definition 4. Then $\mathbf{A} = \langle A, \odot, \rightarrow, \wedge, \vee, 0, 1\rangle$ is a residuated lattice iff the operation $\vee$ is associative.*

Proof.

i) Obviously if $\mathbf{A}$ is a residuated lattice, then $\vee$ is associative.

ii) Suppose that $\vee$ is associative, we only have to show that $\wedge$ is the infimum operation and $\vee$ is the supremum operation.

As hoops are left-commutative monoids [5], then by [5, Lemma 1.3], we have that $\wedge$ is the infimum.

Let us show that $\vee$ is the supremum operation. We first show $x \vee y$ is an upper bound of x, y. By applying that $\wedge$ is the infimum and the definition of $\vee$, we have that $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x) \leq (x \rightarrow y) \rightarrow y$. So $x \rightarrow (x \vee y) \leq x \rightarrow ((x \rightarrow y) \rightarrow y) = (x \odot (x \rightarrow y)) \rightarrow y$, and by applying $(H2)$, $(H4)$ and $(H5)$ we have $x \rightarrow (x \vee y) \leq ((x \rightarrow y) \odot x) \rightarrow y = (x \rightarrow y) \rightarrow (x \rightarrow y) = 1$. So $x \leq x \vee y$. It can be obtained that $y \leq x \vee y$ in a similar way.

Now we have to show that $x \vee y$ is the least upper bound of x, y, let $z \in A$ such that $x \leq z$ and $y \leq z$

Let us start by showing the following property: for all $x, y \in A$, $x \leq y$ iff $x \vee y = y$. By the definition of $\vee$ we have $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. If $x \leq y$, that is $x \rightarrow y = 1$, then $x \vee y = (1 \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. By using $(H1)$ and the definition of $\wedge$ we obtain $x \vee y = (1 \odot (1 \rightarrow y)) \wedge ((y \rightarrow x) \rightarrow x) = y \wedge ((y \rightarrow x) \rightarrow x)$ and by applying the definition of $\wedge$ and the property $(H4)$ we have $x \vee y = y \odot (y \rightarrow ((y \rightarrow x) \rightarrow x)) = y \odot ((y \odot (y \rightarrow x)) \rightarrow x)$. By using the definition of $\wedge$ and that it is the infimum we have $x \vee y = y \odot ((y \wedge x) \rightarrow x) = y \odot 1$. Hence, by applying $(H1)$ we obtain $x \vee y = y$.

If $x \leq z$ then we have $x \vee z = z$ and $y \vee y = y$. Thus $z \vee y = (x \vee z) \vee (y \vee y)$. By using the associativity and commutativity of $\vee$ we obtain $z \vee y = (x \vee y) \vee (z \vee y)$. Hence, by using that $\vee$ is an upper bound we have $x \vee y \leq z \vee y$, and if $y \leq z$ we have $z \vee y \leq z$. So $x \vee y \leq z$. $\square$

Let us show that the class of Product algebras is also a variety of bounded hoops.

Theorem 7 *Let* $\mathbf{A} = \langle A, \odot, \to, 0 \rangle$ *be an algebra of type* $(2,2,0)$. *Then* $\mathbf{A}$ *is a Product algebra iff* $\mathbf{A}$ *is a bounded hoop which satisfies the following equations (where the new operations are defined as in Definition 4):*

(i) $x \vee (y \vee z) = (x \vee y) \vee z$,

(ii) $(\neg\neg z \odot (x \odot z \to y \odot z)) \to (x \to y) = 1$,

(iii) $x \wedge \neg x = 0$.

Proof.

a) If $\mathbf{A}$ is a Product algebra, then by Lemma 5, $\mathbf{A}$ is a bounded hoop, and by Definition 4, $\mathbf{A}$ satisfies the properties (i), (ii) and (iii).

b) Suppose that $\mathbf{A}$ is a bounded hoop which satisfies the properties (i) – (iii), then by Lemma 6 $\langle A, \odot, \to, \wedge, \vee, 0, 1 \rangle$ is a residuated lattice, so as it satisfies the properties (ii) and (iii), it is a Product algebra. □

According with Theorem 7, we give below a set of axioms for Product algebras (where the new operations are defined as in Definition 4). Product algebras $\langle A, \odot, \to, 0 \rangle$ can be defined as a variety of bounded hoops by means of the following equations:

Π1. $1 \odot x = x \odot 1 = x$,

Π2. $x \odot y = y \odot x$,

Π3. $x \to x = 1$,

Π4. $(x \to y) \odot x = (y \to x) \odot y$,

Π5. $(x \odot y) \to z = x \to (y \to z)$,

Π6. $0 \to x = 1$,

Π7. $(x \vee y) \vee z = x \vee (y \vee z)$,

Π8. $(\neg\neg z \odot ((x \odot z) \to (y \odot z))) \to (x \to y) = 1$,

Π9. $x \wedge \neg x = 0$,

We will finish this section by applying some of the results of [9] to show that the variety of Product algebras is generated by the unit interval Product algebra.

Recall that the unit interval Product algebra is the algebra $\langle [0,1], \odot, \rightarrow, 0 \rangle$ of type $(2,2,0)$ where $[0,1] \subset \mathbb{R}$ and the connectives $\odot$, $\rightarrow$ are defined as follows:

$$
\begin{aligned}
x \odot y &= x \cdot y \\
x \rightarrow y &= \begin{cases} y/x, & \text{if } y < x \\ 1, & \text{if } x \leq y \end{cases}
\end{aligned}
$$

for any $x, y \in [0,1]$, where $\cdot$ denotes the product of real numbers and $/$ denotes the quotient of real numbers.

Theorem 8 *([9, Corollary 4]) Each Product algebra is a subdirect product of linearly ordered Product algebras.*

Theorem 9 *([9, Theorem 3]) If an identity $\tau = \sigma$, in the language of Product algebras, is valid in the unit interval algebra then it is valid in all linearly ordered product algebras.*

Now we are ready to prove the following result.

Theorem 10 *The variety of Product algebras is generated by the unit interval Product algebra.*

Proof.
The variety generated by the unit interval Product algebra is the class of $\mathcal{L}$-algebras satisfying the equations which are valid in the unit interval Product algebra. By theorems 8 and 9 every Product algebra belongs to this class. As the unit interval algebra is a Product algebra, every algebra in this class is a Product algebra. $\square$

4 Product logic and the Deduction Theorem

Let $\mathcal{L} = \{\odot, \rightarrow, 0\}$ be a propositional language of type $(2,2,0)$. Define the new connectives $\wedge, \vee$ and $\neg$ in the following way:

$$
\begin{aligned}
\neg x &= x \rightarrow 0 \\
x \wedge y &= x \odot (x \rightarrow y) \\
x \vee y &= ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)
\end{aligned}
$$

The **Product logic** ΠL ([9, Definition 1]) is the deductive system over $\mathcal{L}$ defined by the axioms:

$(A1)$ $\quad \varphi \rightarrow (\psi \rightarrow \varphi)$

(A2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A3) $0 \to \varphi$

(A4) $(\varphi \odot \psi) \to (\psi \odot \varphi)$

(A5) $(\varphi \odot (\psi \odot \chi)) \to ((\varphi \odot \psi) \odot \chi)$

(A6) $((\varphi \odot \psi) \odot \chi) \to (\varphi \odot (\psi \odot \chi))$

(A7) $((\varphi \odot \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(A8) $(\varphi \to (\psi \to \chi)) \to ((\varphi \odot \psi) \to \chi)$

(A9) $(\varphi \to \psi) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$

(A10) $\neg\neg\chi \to (((\varphi \odot \chi) \to (\psi \odot \chi)) \to (\varphi \to \psi))$

(A11) $(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi)))$

(A12) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

(A13) $(\varphi \to \psi) \vee (\psi \to \varphi)$

(A14) $(\varphi \wedge \neg\varphi) \to 0$

and the modus ponens rule:

$$\varphi, \varphi \to \psi \vdash \psi$$

Recall the following theorems of ΠL [9, lemmas 3 and 4]:

(t1) $\varphi \to \varphi$

(t2) $(\varphi \to \psi) \to ((\chi \to \gamma) \to ((\varphi \odot \chi) \to (\psi \odot \gamma)))$

(t3) $\varphi \to (1 \odot \varphi)$

(t4) $(\varphi \odot \psi) \to \psi$

(t5) $((\varphi \to \psi) \odot \varphi) \to ((\psi \to \varphi) \odot \psi)$

(t6) $(\varphi \vee (\psi \vee \chi)) \to ((\varphi \vee \psi) \vee \chi)$

(t7) $((\varphi \vee \psi) \vee \chi) \to (\varphi \vee (\psi \vee \chi))$

(t8) $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$, where $\varphi,\ \psi,\ \chi,\ \gamma \in Fm_{\mathcal{L}}$.

Now we will show that the variety of Product algebras is the equivalent quasivariety semantics for Product Logic. The proof follows the lines of the proof that the variety of Wajsberg algebras (or MV-algebras) is the equivalent quasivariety semantics for the infinite-valued Łukasiewicz logic [15].

Let $\mathbf{A}$ be an arbitrary algebra of type $(2,2,0)$. Denote by $\mathcal{F}_{\Pi}$ the family of all ΠL-filters of $\mathbf{A}$, and by $Con_{\Pi}(\mathbf{A})$ the family of all product congruence relations of $\mathbf{A}$ (i.e. the congruence relations θ of $\mathbf{A}$ such that the quotient $\mathbf{A}/\theta$ is a Product algebra). Then we have

Theorem 11 *Let* $\mathbf{A} = \langle A, \odot, \rightarrow, 0\rangle$ *be an algebra of type* $(2,2,0)$*. The map*

$$\begin{array}{rcl} \Theta_{\mathbf{A}} : \mathcal{F}_{\Pi} & \longrightarrow & Con_{\Pi}(\mathbf{A}) \\ F & \longmapsto & \Theta_{\mathbf{A}}F = \left\{(a,b) \in A^2 : a \rightarrow b,\ b \rightarrow a \in F\right\} \end{array}$$

is an order isomorphism, whose inverse is:

$$\begin{array}{rcl} H_{\mathbf{A}} : Con_{\Pi}(\mathbf{A}) & \longrightarrow & \mathcal{F}_{\Pi} \\ \theta & \longmapsto & H_{\mathbf{A}}\theta = \{a \in A : (a, a \rightarrow a) \in \theta\} \end{array}$$

Proof.

Let F be a ΠL-filter of $\mathbf{A}$. By using $(t1)$, $(A2)$, $(t8)$ and modus ponens, then it is easy to see that $\Theta_{\mathbf{A}}F$ is an equivalence relation on $\mathbf{A}$. By using $(A2)$ and $(t8)$ it is easy to check that $\Theta_{\mathbf{A}}F$ satisfies that substitution property relative to $\rightarrow$, and by using $(t2)$ it is easy to check that $\Theta_{\mathbf{A}}F$ satisfies that substitution property relative to $\odot$. Thus $\Theta_{\mathbf{A}}F$ is a congruence of $\mathbf{A}$.

Now we will show that $\mathbf{A}/\Theta_{\mathbf{A}}F$ is a Product algebra, that is, it satisfies the equations $\Pi1 - \Pi9$, stated in Section 3 as an axiomatization of Product algebras. If $a,\ b \in F$ then, by $(A1)$ and modus ponens we have: $(a,b) \in \Theta_{\mathbf{A}}F$. If $a \in F$ and $b \notin F$, then by modus ponens $a \rightarrow b \notin F$, so $(a,b) \notin \Theta_{\mathbf{A}}F$. Thus F is an equivalence class and $(a \rightarrow a)/\Theta_{\mathbf{A}}F = F$, so the equation $\Pi3$ holds. It is easy to check that equation $\Pi2$ holds by using axiom $(A4)$, equation $\Pi1$ by using theorems $(t3)$ and $(t4)$, equation $\Pi5$ by using axioms $(A7)$ and $(A8)$, equation $\Pi6$ by using axiom $(A3)$, equation $\Pi7$ by using theorems $(t6)$ and $(t7)$, equation $\Pi8$ by using axiom $(A10)$ and equation $\Pi9$ by using axiom $(A14)$. So $\mathbf{A}/\Theta_{\mathbf{A}}F$ is a Product algebra and thus $\Theta_{\mathbf{A}}F \in Con_{\Pi}(\mathbf{A})$. It is easy to see that if F and F' are ΠL-filters of $\mathbf{A}$ and $F \subseteq F'$, then $\Theta_{\mathbf{A}}F \subseteq \Theta_{\mathbf{A}}F'$.

Let $\theta \in Con_{\Pi}(\mathbf{A})$, then by using the properties of Product algebras it is easy to see that $H_{\mathbf{A}}\theta$ is a ΠL-filter of $\mathbf{A}$. Moreover, if $\theta,\ \theta' \in Con_{\Pi}(\mathbf{A})$, then $\theta \subseteq \theta'$ implies $H_{\mathbf{A}}\theta \subseteq H_{\mathbf{A}}\theta'$. Moreover, $(a,b) \in \Theta_{\mathbf{A}}(H_{\mathbf{A}}\theta)$ iff $a \to b,\ b \to a \in (H_{\mathbf{A}}\theta)$ iff $a/\theta = b/\theta$ iff $(a,b) \in \theta$, so $\Theta_{\mathbf{A}}(H_{\mathbf{A}}\theta) = \theta$. On the other hand it is easy to see that $H_{\mathbf{A}}(\Theta_{\mathbf{A}}F) = F$. Thus both $H_{\mathbf{A}}$ and $\Theta_{\mathbf{A}}$ are one to one and order preserving and so they are order isomorphisms. $\square$

Corollary 12 *For any algebra $\mathbf{A} = \langle A, \odot, \to, 0\rangle$ of type $(2,2,0)$, the Leibniz operator $\Omega_{\mathbf{A}}$ is an isomorphism from $\mathcal{F}_\Pi$ onto $Con_\Pi(\mathbf{A})$. Hence the Product logic is algebraizable and the variety of Product algebras is its equivalent quasivariety semantics.*

Proof.

For any $F \in \mathcal{F}_\Pi$, $\Theta_{\mathbf{A}}F$ is defined elementarily over the matrix $(\mathbf{A}, F)$ without equality, is a congruence on $\mathbf{A}$ and is compatible with F, so by [2, Theorem 1.6], $\Theta_{\mathbf{A}}F = \Omega_{\mathbf{A}}(F)$. Thus by [2, Theorem 5.1] and Theorem 11 we have that the Product logic is algebraizable with equivalent quasivariety semantics the variety of Product algebras. $\square$

Now we will show that the deduction-detachment theorem fails for ΠL. Recall that the deduction theorem with respect to the connective $\to$ fails for ΠL [9], in other words, the set $E(p,q) = \{p \to q\}$ is not a deduction-detachment set for ΠL.

Recall also that ΠL satisfies the following weak version of the deduction theorem [9]:

Let $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$. Then,

$$\Gamma \cup \{\varphi\} \vdash_{\Pi L} \psi \text{ iff } \Gamma \vdash_{\Pi L} \overbrace{\varphi \odot \cdots \odot \varphi}^{n \text{ times}} \to \psi \text{ for some natural number } n.$$

This weak version of the deduction theorem is called the "local deduction theorem" in [3].

To show that the deduction-detachment theorem fails for ΠL we will use the followings results.

Theorem 13 *The variety of Product algebras is not n-potent.*

Proof.

As the variety of Product algebras is generated by the unit interval algebra, the result follows from the fact that the unit interval Product algebra does not satisfy any identity of the form : $x^n = x^{n-1}$. $\square$

Theorem 14 *[5, Corollary 5.5] A variety of hoops has EDPRC iff it is n-potent for some n.*

Theorem 15 *A variety of bounded hoops has EDPRC iff its is n-potent for some n.*

Proof.
As bounded hoops are hoops with normal multiplicative operators [5, pg. 559], it is easy to see that a proof completely analogous to that of Theorem 14 applies. □

Theorem 16 *The variety of Product algebras does not have EDPRC.*

Proof.
As the class of Product algebras is a variety of bounded hoops, this result follows from Theorems 15, and 13. □

Theorem 17 *Product logic ΠL does not have the deduction-detachment theorem (DDT).*

Proof.
If ΠL had the DDT then as the class of Product algebras is its equivalent variety semantics, then by Theorem 2 this variety would have EDPRC, a contradiction. □

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