Stochastics and Dynamics

Semilinear Fractional Stochastic Differential Equations driven by a γ-Hölder Continuous Signal with $\gamma > 2/3$*

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In this paper we use the techniques of fractional calculus to study the existence of a unique solution to semilinear fractional differential equation driven by a γ-Hölder continuous function θ with $\gamma \in \left(\frac{2}{3}, 1\right)$. Here, the initial condition is a function that may not be defined at zero and the involved integral with respect to θ is the extension of the Young integral [50] given by Zähle [51].

Keywords: Fractional Brownian motion; fractional derivatives and integrals; fractional differential equations; Hölder continuous functions; existence and uniqueness of solutions for stochastic differential equations; Young integral.

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1. Introduction

The fractional calculus has a wide range of applications in several areas of science and technology such as chemistry, engineering, mechanics, physics (see, for example, [8], [24], [27], [43] and references therein). In particular, we can mention chaotic dynamics and synchronization ([20], [22], [48], [49]), robotics [23], viscoelasticity [2], system identification [8], control ([8], [43]), analysis of electrode processes [25], Lorenz systems [20], systems with retards [10], quantic evolution of complex systems [28], stability ([37], [40], [47]), electromagnetic waves [23], quantitative finances [30], and many others. Lakshmikantham and Vatsala [29], and Podlubny [43], have nice

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surveys of basic properties of deterministic fractional differential equations. Junsheng *et al.* in [26] gave the solution for a linear case in terms of Mittag-Leffler function by the Adomian decomposition method. Matignon [37] and Radwan *et al.* [44] studied the stability for fractional linear equations. The non-linear case was treated, for example, by Li *et al.* [34], Martínez-Martínez [38] and Wen *et al.* [47], among others.

The book of Samko *et al.* [45], whose topics are fractional derivatives and integrals, is one of the fundamental tools for studying applications of fractional Brownian motion (fBm) via stochastic differential equations (see, for instance, Mishura [39], Nualart [41] or Zähle [51]). A fBm $B^H = \{B_t^H; t \geq 0\}$ with Hurst parameter $H \in (0,1)$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, is a zero-mean Gaussian process with covariance

$$\mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0.$$

In recent years, several authors have considered fBm because of its numerous properties such as stationary increments, self-similarity, Hölder's continuity, long-range dependence, etc (see, for example, Nualart [41]). It is well-known that $B^{1/2}$ is a Brownian motion and B^H is not a semimartingale for $H \neq 1/2$. Therefore, we cannot establish a calculus for fBm based on an integral introduced by using the classical Itô calculus, in general. So, it is necessary to apply another approaches to deal with integration with respect to B^H. In the literature, there are different definitions of integral with respect to fBm such as Stratonovich integral [1], Skorohod type integrals [41], Young integral [50], extensions of Young integral (see, for example, [7], [15], [21] and [51]), integration with respect to rough signals [36], etc. Consequently, several methods have been used by many researches to analyze the properties of stochastic differential equations driven by B^H. Among these authors we can mention Alòs *et al.* [1], Fiel *et al.* ([18], [17]), León *et al.* ([31], [32], [33]), Lin [35], Lyons [36], Nualart and Raşcanu [42] among others.

For $H > 1/2$, it is possible to consider Young integration to deal with fBm because B^H has γ-Hölder continuous paths for any $\gamma \in (0, H)$, with probability 1. Even we can study stochastic differential equations driven by fBm with either discontinuous coefficients (see Garzón *et al.* [19]), or coefficients with power type non-linearities (see León *et al.* [31]), by means of the extension of Young integral given by Zähle [51].

The purpose of this paper is to state the existence and uniqueness for the solution to semilinear fractional stochastic differential equations driven by B^H, which are Volterra type equations. These last equations with suitable kernels have been taking into account by Besalú *et al.* ([3], [4]), Coutin and Decreusefond [6], Decreusefond [9], Deya and Tindel ([11], [12]), Fan [16], and Wang and Yan [46]. In particular, Deya and Tindel [11] deal with the equation

$$y_t = a + \int_0^t (t-s)^{-\alpha} \Psi(y_s) dx_s, \qquad t \in [0,T], \tag{1.1}$$

where a is a constant in $\mathbb{R}^n$, the function Ψ is smooth enough, x is a Hölder continuous noise with values in $\mathbb{R}^m$ and the integral is a limit of Young type integrals. The existence and uniqueness for the solution is obtained by the fix-point theorem. Hence, it is necessary to state estimates for the involved integral. The main tool in this case is a variant of the rough path theory called algebraic integration. Consequently, the solution of (1.1) is Hölder continuous. The techniques established in [11] and [12] allowed the authors to work with non-linear rough heat equations [13].

The main goal of this article is to study the existence of a unique solution to the semilinear fractional stochastic differential equation of the form

$$X_t = \xi_t + \int_0^t (t-s)^{\beta-1}\ g(s, X_s)ds + \int_0^t (t-s)^{\alpha-1} f(s) X_s dB_s^H, \quad t \in [0, T]. \quad (1.2)$$

Here, $\alpha, \beta \in \left(\frac{2}{3}, 1\right)$, f is a Hölder continuous process (see Section 4), $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function that is Lipschitz continuous with linear growth on $\mathbb{R}$, uniformly on $\Omega \times [0, T]$, and $H > 2/3$. The stochastic integral with respect to B^H in (1.2) is the extension of the Young integral introduced by Zälhe [51]. In Propositions 2.1 and 2.2 we show that we can assume $f \equiv 1$ without loss of generality. Thus, due to equation (1.2) being interpreted pathwise and B^H having γ-Hölder continuous paths for any $\gamma < H$, we only need to analyze the equation

$$X_t = \xi_t + \int_0^t (t-s)^{\beta-1}\ g(s, X_s)ds + \int_0^t (t-s)^{\alpha-1} X_s d\theta_s, \quad t \in [0, T]. \quad (1.3)$$

Now g is a function on $[0, T] \times \mathbb{R}$ and θ is a γ-Hölder continuous function on $[0, T]$ with $\gamma > 2/3$. Unlike equation (1.1), the initial condition ξ in (1.3) is not a constant but a function and β could be different than α. Observe that the first component of x in (1.1) may be the identity function. That is, $x_t^{(1)} = t$, $t \in [0, T]$. Also, unlike equation (1.1), the solution of (1.3) may not be a Hölder continuous function and it may be $\pm\infty$ at zero. Indeed, we consider equation (1.3) for two different families of initial condition. Namely, either ξ is Hölder continuous, or ξ could be not defined at zero (see Hypotheses (H2) and (H3) below). Remember that the integral with respect to θ in (1.3) is the extension of the Young integral [51], which is given only by means of fractional derivatives. Therefore, we only apply properties of fractional derivatives and integrals established in [45] to state priori estimates for $t \longmapsto \int_0^t (t-s)^{\alpha-1} X_s d\theta_s$ in (1.3) (see Lemmas 3.1 and 3.3 below). In this way, we can also get our results throughout the fix-point theorem, although we cannot use algebraic integration as in [11] because the solution of (1.3) is not Hölder continuous for both cases considered in this paper for ξ (see Proposition 4.1 and 4.2).

We remark that it is natural to have functions as initial condition in (1.1) and (1.3). Indeed, suppose that we know the solution of (1.1) up to a time $T_0 < T$. In this case we need to analyze the equation

$$\begin{aligned} y_t &= a + \int_0^{T_0} (t-s)^{-\alpha}\Psi(y_s)dx_s + \int_{T_0}^t (t-s)^{-\alpha}\Psi(y_s)dx_s \\ &= \xi_t + \int_{T_0}^t (t-s)^{-\alpha}\Psi(y_s)dx_s, \qquad t \in [T_0, T], \end{aligned}$$

to know the solution y on $[0,T]$. But, in this equation, $\xi_t = a + \int_0^{T_0}(t-s)^{-\alpha}\Psi(y_s)dx_s$ is a function. We also remark that considering initial conditions that are Hölder continuous functions is a standard assumption in the study of delayed equations (see, for instance, Boufoussi *et al.* [5] and Diop and Garrido-Atienza [14]).

Finally, we would also like to point out that considering (H2) we deal with an equation in the sense of Riemann-Liouville and considering (H3) we deal with an equation in the sense of Caputo. For more information about this subject we refer to Podlubny [43] or Junsheng *et al.* in [26].

The organization of the paper is as follows. In Section 2 we introduce the framework of this article. In particular, in Section 2.2 we give the definition of the extension of the Young integral [51] and we state the basic tool to obtain priori estimates for equation (1.3). In Section 3, we establish the priori estimates, whose proofs are in Section Appendix A. The existence of the solution to (1.3) is showed in Section 4.

2. Preliminaries

This section is devoted to introduce the framework and the main tools that we use in this paper.

The space of all the μ-Hölder continuous functions on $[a,b]$ is denoted by $\mathcal{C}^\mu([a,b])$. Then, if $f \in \mathcal{C}^\mu([a,b])$, the norm of f is defined as follows

$$|||f|||_{\mu,[a,b]} := \|f\|_{\infty,[a,b]} + \|f\|_{\mu,[a,b]},$$

with

$$\|f\|_{\infty,[a,b]} := \sup_{a\le t\le b} |f(t)| \qquad \text{and} \qquad \|f\|_{\mu,[a,b]} := \sup_{a\le s<t\le b} \frac{|f(t)-f(s)|}{(t-s)^\mu}.$$

Let $\eta, \mu > 0$. We will need the equality

$$(t-s)^{\mu+\eta-1} = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu)\Gamma(\eta)} \int_s^t (t-r)^{\mu-1}(r-s)^{\eta-1}dr, \qquad 0 \le s \le t,$$

where Γ is the Gamma function. This equality is an immediate consequence of the relation between Γ and the Beta function B. That is,

$$\frac{\Gamma(\mu)\Gamma(\eta)}{\Gamma(\mu+\eta)} = B(\mu,\eta) = \int_0^1 (1-r)^{\mu-1}r^{\eta-1}dr.$$

Throughout the paper, C represents a generic constant whose value is not important and may change from line to line. When we want to indicate that C depends on some parameters $\eta_1, \dots, \eta_n$, we will use the notation $C_{\eta_1,\dots,\eta_n}$.

2.1. *Fractional derivatives and integrals*

Consider $0 \le a < b \le T$ and an $L^1([0,T])$-function f. For $\mu \in (0,1)$, the fractional integrals of f are defined as

$$I^\mu_{a+}(f)(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} f_r \, dr \quad \text{and} \quad I^\mu_{b-}(f)(t) = \frac{1}{\Gamma(\mu)} \int_t^b (r-t)^{\mu-1} f_r \, dr, \tag{2.1}$$

which are defined for almost all $t \in [a,b]$ due to Fubini theorem. For any $p \ge 1$, we denote by $I^\mu_{a+}(L^p)$ the image of $L^p([a,b])$ by I^μ_{a+}, and similarly for $I^\mu_{b-}(L^p)$. Sometimes we write $I^\mu_{a+}(L^p([a,b]))$ instead of $I^\mu_{a+}(L^p)$ if it is not clear the involved interval that we are considering. The inverse of operators I^μ_{a+} and I^μ_{b-} are called fractional derivatives, and are defined as follows. For $f \in I^\mu_{a+}(L^p)$ and almost all $t \in [a,b]$, we set

$$D^\mu_{a+} f_t = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\mu)} \left(\frac{f_t}{(t-a)^\mu} + \mu \int_a^{t-\varepsilon} \frac{f_t - f_r}{(t-r)^{1+\mu}} \, dr \right), \tag{2.2}$$

where we use the convention $f_r = 0$ on $[a,b]^c$. In the same way, for $f \in I^\mu_{b-}(L^p)$ and almost all $t \in [a,b]$, we set

$$D^\mu_{b-} f_t = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\mu)} \left(\frac{f_t}{(b-t)^\mu} + \mu \int_{t+\varepsilon}^b \frac{f_t - f_r}{(r-t)^{1+\mu}} \, dr \right). \tag{2.3}$$

By [45] (Theorem 13.2), we have that, for $p > 1$, $f \in I^\mu_{a+}(L^p)$ (resp. $f \in I^\mu_{b-}(L^p)$) if and only if $f \in L^p([a,b])$ and the limit in the right-hand side of (2.2) (resp. (2.3)) exists. In this case $f = I^\mu_{a+}(D^\mu_{a+} f)$ (resp. $f = I^\mu_{b-}(D^\mu_{b-} f)$). Note that if f belongs to $\mathcal{C}^{\mu+\varepsilon}([a,b])$, with $\varepsilon > 0$, then $D^\mu_{a+} f$ and $D^\mu_{b-} f$ given by (2.2) and (2.3), respectively, exist.

2.2. *Young integral*

In this section we introduce the extension of the Young integral defined by Zähle in [51].

Let g, f be two functions on an interval $[a,b]$ and $g^{b-}(r) = g^{b-}_r := g_r - g_{b-}$. We say that f is integrable with respect to g (the generalized fractional Lebesgue-Stieltjes integral) if and only if $D^\mu_{a+} f$ and $D^{1-\mu}_{b-} g^{b-}$ exist, and $(D^\mu_{a+} f) D^{1-\mu}_{b-} g^{b-} \in L^1([a,b])$. In this case we define the integral $\int_a^b f \, dg$ in the following way

$$\int_a^b f_r \, dg_r := \int_a^b (D^\mu_{a+} f)(r) D^{1-\mu}_{b-} g^{b-}(r) \, dr. \tag{2.4}$$

It is proven in [51] that this definition is independent of μ. It means, if there is $\gamma \in (0,1)$ such that $D^\gamma_{a+} f$ and $D^{1-\gamma}_{b-} g^{b-}$ exist, and $(D^\gamma_{a+} f) D^{1-\gamma}_{b-} g^{b-} \in L^1([a,b])$, then

$$\int_a^b (D^\mu_{a+} f)(r) D^{1-\mu}_{b-} g^{b-}(r) \, dr = \int_a^b (D^\gamma_{a+} f)(r) D^{1-\gamma}_{b-} g^{b-}(r) \, dr.$$

It is well-known that if $f \in \mathcal{C}^{\mu}([a,b])$ and $g \in \mathcal{C}^{\gamma}([a,b])$, with $\mu+\gamma>1$, then it can be checked that $\int_a^b f_r\, dg_r$ is well-defined, and that it coincides with the Young's integral defined as a limit of Riemann sums (see [51]), which makes this integral a main tool to interpret the meaning of solution to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter bigger than 1/2.

We will need the following auxiliary results later on.

Lemma 2.1. *Let $\gamma \in (0,1)$, $\alpha \in (1-\gamma,1)$, $\theta \in \mathcal{C}^{\gamma}([0,T])$ and $0 \le a < t \le T$. Then, the integral $\int_a^t (t-s)^{\alpha-1} d\theta_s$ is well-defined and, for $\varepsilon>0$ such that $\alpha+\gamma-\varepsilon>1$,*

$$\left|\int_a^t (t-s)^{\alpha-1} d\theta_s\right| \le C\|\theta\|_{\gamma,[a,t]}(t-a)^{\gamma+\alpha-\varepsilon-1},$$

where C is a constant depending on α, γ, ε and T.

Proof. In order to see that the last constant C only depends on α, γ, ε and T, we introduce the γ-Hölder continuous function

$$\eta_s(a,t) = \begin{cases} \theta_a,\ s \in [0,a], \\ \theta_s,\ s \in [a,t], \\ \theta_t,\ s \in [t,T]. \end{cases}$$

Now let $0<\varepsilon$ be such that $\alpha+\gamma-\varepsilon>1$, which implies that $\varepsilon<\alpha$. Then, for $r \in (0,T)$, we have

$$\begin{aligned} (t-r)^{\alpha-1} 1_{[0,t]}(r) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\varepsilon)\Gamma(\varepsilon)} \int_r^T (t-s)^{\varepsilon-1}(s-r)^{\alpha-\varepsilon-1} 1_{[0,t]}(s) ds \\ &= \frac{\Gamma(\alpha)}{\Gamma(\varepsilon)} I_{T-}^{\alpha-\varepsilon}\left((t-\cdot)^{\varepsilon-1} 1_{[0,t]}(\cdot)\right)(r) \end{aligned} \tag{2.5}$$

and

$$(t-r)^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\varepsilon)} I_{t-}^{\alpha-\varepsilon}\left((t-\cdot)^{\varepsilon-1}\right)(r), \tag{2.6}$$

where (2.5) and (2.6) follow from Definition (2.1). So, Samko et al. [45] (Corollary 1 of Theorem 11.5) implies that $(t-\cdot)^{\alpha-1}1_{[0,t]}(\cdot) \in I_{0+}^{\alpha-\varepsilon}\left(L^p([0,T])\right)$, for $1<p<\frac{1}{1-\varepsilon}$. Hence, using the Definitions (2.3) and (2.4) (see, for instance, inequality (12) in León et al. [31]), we can obtain, using straightforward calculations,

$$\left|D_{T-}^{1-\alpha+\varepsilon}\eta_s^{T-}(a,t)\right| \le C_{\alpha,\gamma,\varepsilon}||\theta||_{\gamma,[a,t]}(t-a)^{\gamma+\alpha-\varepsilon-1}, \quad s \in [0,T], \tag{2.7}$$

and

$$\begin{aligned} &\left|\int_0^T (t-s)^{\alpha-1} 1_{[0,t]}(s) d\eta_s(a,t)\right| \\ &= \left|\int_0^T \left(D_{0+}^{\alpha-\varepsilon}(t-\cdot)^{\alpha-1}1_{[0,t]}(\cdot)\right)(s)\left(D_{T-}^{1-\alpha+\varepsilon}\eta^{T-}(a,t)\right)(s) ds\right| \\ &\le C_{\alpha,\gamma,\varepsilon}\|\theta\|_{\gamma,[a,t]}(t-a)^{\gamma+\alpha-\varepsilon-1}\int_0^T \left|\left(D_{0+}^{\alpha-\varepsilon}(t-\cdot)^{\alpha-1}1_{[0,t]}(\cdot)\right)(s)\right| ds. \end{aligned} \tag{2.8}$$

Applying Samko et al. [45] (Corollary 1 of Theorem 11.5) again, together with (2.5), we are able to write, for $1 < p < \frac{1}{1-\varepsilon}$,

$$\begin{aligned}
&\int_0^T \left|\left(D_{0+}^{\alpha-\varepsilon}(t-\cdot)^{\alpha-1}1_{[0,t]}(\cdot)\right)(s)\right| ds \\
&\leq T^{1-\frac{1}{p}}\left(\int_0^T \left|\left(D_{0+}^{\alpha-\varepsilon}(t-\cdot)^{\alpha-1}1_{[0,t]}(\cdot)\right)(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\leq C_{T,\alpha,\varepsilon}T^{1-\frac{1}{p}}\left(\int_0^t (t-s)^{(\varepsilon-1)p}ds\right)^{\frac{1}{p}} \\
&\leq C_{T,\alpha,\varepsilon}T^{1-\frac{1}{p}}\left(t^{(\varepsilon-1)p+1}\right)^{\frac{1}{p}} \\
&\leq C_{T,\alpha,\varepsilon}T^{\varepsilon}.
\end{aligned}$$

Thus, from (2.8), we get

$$\left|\int_0^T (t-s)^{\alpha-1}1_{[0,t]}(s)d\eta_s(a,t)\right| \leq C_{\alpha,\gamma,T,\varepsilon}\|\theta\|_{\gamma,[a,t]}(t-a)^{\gamma+\alpha-\varepsilon-1}. \tag{2.9}$$

Finally, by Zähle [51] (Theorem 2.5), (2.3) and (2.6), we have

$$\begin{aligned}
&\int_0^T (t-s)^{\alpha-1}1_{[0,t]}(s)d\eta_s(a,t) \\
&= \int_0^a (t-s)^{\alpha-1}1_{[0,t]}(s)d\eta_s(a,t) + \int_a^T (t-s)^{\alpha-1}1_{[0,t]}(s)d\eta_s(a,t) \\
&= \int_a^T (t-s)^{\alpha-1}1_{[0,t]}(s)d\eta_s(a,t) \\
&= \int_0^T (t-s)^{\alpha-1}1_{[0,t]}(s)1_{[a,T]}(s)d\eta_s(a,t) = \int_a^t (t-s)^{\alpha-1}d\theta_s,
\end{aligned}$$

where we use that $\eta^{a-}(a,t) \equiv 0$ on $[0,a]$. We observe that we can see that $\int_a^t (t-s)^{\alpha-1}d\theta_s$ is well-defined proceeding as in (2.8) and (2.9) via (2.6) (instead of (2.5)). Consequently, the result follows from (2.9). □

Henceforth we utilize the notation $\tilde{\alpha} = \alpha + \beta - 1$.

Lemma 2.2. *Assume that* $1 < \alpha + \beta$. *Then, for* $x \in (a,b)$,

$$I_{a+}^{\tilde{\alpha}}\left((b-\cdot)^{-\beta}(\cdot-a)^{-\alpha}\right)(x) = (b-a)^{-\tilde{\alpha}}\frac{\Gamma(1-\alpha)}{\Gamma(\beta)}(x-a)^{\beta-1}(b-x)^{\alpha-1}.$$

Proof. The proof is an immediate consequence of Samko et al. [45] (equalities (1.74), (1.75) and (2.46)). □

Now, we can state the following consequence.

Corollary 2.1. *Let $\gamma \in (0,1)$ be such that $2-\gamma < \alpha+\beta$ and $\theta \in \mathcal{C}^\gamma([0,T])$. Then, for $0 \le a < b \le T$, we have*

$$\Xi := \left| \int_a^b (b-r)^{\alpha-1}(r-a)^{\beta-1} d\theta_r \right| \le C_{\tilde{\alpha},\gamma} \|\theta\|_\gamma \frac{\Gamma(\beta)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} (b-a)^{\tilde{\alpha}+\gamma-1}.$$

Proof. First we observe that, taking into account (2.3) and $2-\gamma < \alpha+\beta$, we can use the fact that

$$\left| \left(D_{b-}^{1-\tilde{\alpha}} \theta_{b-} \right)(s) \right| \le C_{\tilde{\alpha},\gamma} \|\theta\|_\gamma (b-s)^{\tilde{\alpha}+\gamma-1}, \quad s \in [a,b]. \tag{2.10}$$

Hence, Lemma 2.2 and (2.4) yield

$$\begin{aligned}
\Xi &= \left| \int_a^b \left(D_{a+}^{\tilde{\alpha}} ((b-\cdot)^{\alpha-1}(\cdot - a)^{\beta-1}) \right)(r) \left(D_{b-}^{1-\tilde{\alpha}} \theta_{b-} \right)(r) dr \right| \\
&\le C_{\tilde{\alpha},\gamma} \|\theta\|_\gamma (b-a)^{\tilde{\alpha}+\gamma-1} (b-a)^{\tilde{\alpha}} \frac{\Gamma(\beta)}{\Gamma(1-\alpha)} \int_a^b (b-r)^{-\beta}(r-a)^{-\alpha} dr \\
&= C_{\tilde{\alpha},\gamma} \|\theta\|_\gamma (b-a)^{2\tilde{\alpha}+\gamma-1} \frac{\Gamma(\beta)\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(1-\alpha)\Gamma(2-\alpha-\beta)} (b-a)^{-\tilde{\alpha}} \\
&= C_{\tilde{\alpha},\gamma} \|\theta\|_\gamma \frac{\Gamma(\beta)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} (b-a)^{\tilde{\alpha}+\gamma-1},
\end{aligned}$$

which implies that Ξ is well-defined and that the result is true. □

We will also need the following two propositions in order to study the existence of a unique solution to equation (1.2). The proofs of these results are given in Section Appendix A because they are so long and tedious.

Proposition 2.1. *Assume that $\gamma \in (0,1)$ is such that $2-\gamma < \alpha+\beta$. Let $0 \le a < t \le T$, $f \in \mathcal{C}^{\tilde{\alpha}+\eta}([a,t])$, for some $\eta > 0$, and $\theta \in \mathcal{C}^\gamma([a,t])$. Then,*

$$\int_a^t (t-r)^{\alpha-1}(r-a)^{\beta-1} f(r) d\theta_r = \int_a^t (t-r)^{\alpha-1}(r-a)^{\beta-1} d\tilde{\theta}_r,$$

with $\tilde{\theta} = \int_a^\cdot f(u) d\theta_u$.

Remark. Under these conditions we have $1-\gamma < \tilde{\alpha}$. If $\beta = \alpha$, then $1 - \frac{\gamma}{2} < \alpha$. Also note that $\tilde{\theta}$ is well-defined because $\tilde{\alpha}+\gamma > 1$.

Proposition 2.2. *Assume that $\gamma \in (1/2,1)$, $\alpha \in (1-\gamma,\gamma)$ and $0 \le a \le t \le T$. Let f be in $\mathcal{C}^\eta([a,t])$, for any $\eta < 1-\alpha$, and $\theta \in \mathcal{C}^\gamma([a,t])$. Then,*

$$\int_a^t (t-r)^{\alpha-1} f(r) d\theta_r = \int_a^t (t-r)^{\alpha-1} d\tilde{\theta}_r,$$

where $\tilde{\theta} = \int_a^\cdot f(u) d\theta_u$.

Remark. Note that, by Propositions 2.1 and 2.2, we can reduce the stochastic differential equation (1.2) to the case $f \equiv 1$. Indeed, by [32] (Theorem 2.5) and [51] (Theorem 4.2.1), we have that $\tilde{\theta} \in \mathcal{C}^\gamma([a,t])$.

3. Auxiliary results

The aim of this section is to state the tools needed to study the existence of a unique solution to equation (1.2) via the fix-point theorem. Namely, it consists in analyzing Hölder continuous properties of some involved integrals. As in Section 2.2, the proofs of the following results are developed in Section Appendix A.

We recall you that we are using the convention $\tilde{\alpha} = \alpha + \beta - 1$.

Lemma 3.1. *Let $\gamma \in (0,1)$, $\beta < \gamma$ and $\alpha > 1 - \frac{\gamma}{2}$ such that $2 - \gamma < \alpha + \beta$. Also let $\eta = (\tilde{\alpha} + \varepsilon) \vee (1 - \alpha)$, for $\varepsilon > 0$, $a \in [0,T)$ and $\theta \in \mathcal{C}^{\gamma}([a,T])$. Then, there exists ε small enough such that the function*

$$J_a(t) = (t-a)^{\alpha} \int_a^t (t-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t-}^{1-\tilde{\alpha}} \theta_{t-}\right)(r)dr, \quad t \in [a,T],$$

is η-Hölder continuous. Moreover, we have, for $a \le t_1 \le t_2 \le T$ and λ small enough,

$$|J_a(t_2) - J_a(t_1)| \le C\|\theta\|_{\gamma,[0,T]}(t_2 - t_1)^{\eta}(t_2 - a)^{\lambda},$$

where C only depends on T, α, β and γ. Furthermore, this constant is a non-decreasing function on T.

Remark. If $\beta = \alpha$, then $1 - \frac{\gamma}{2} < \alpha$ if and only if $2 - \gamma < \alpha + \beta$.

Now we state two similar results to Lemma 3.1.

Lemma 3.2. *Let $\alpha, \beta \in (1/2, 1)$ be such that $\beta \ge \alpha$. Also let $t \longmapsto t^{1-\alpha}\varrho_t$ be a continuous function on $[0,T]$ and g a function with linear growth. Then,*

$$\Xi(t) = t^{1-\alpha} \int_0^t (t-s)^{\beta-1} g(s, \varrho_s)ds, \quad t \in [0,T],$$

belongs to $\mathcal{C}^{\eta}([0,T])$ with $\eta = \min\{1-\alpha, \beta, 2\alpha - 1 + \varepsilon, \alpha + \beta - 1\}$, for some ε small enough.

Lemma 3.3. *Let $\gamma > 1/2$, $\alpha \in (1-\gamma, \gamma)$, $\theta \in \mathcal{C}^{\gamma}([0,T])$ and $\varrho \in \mathcal{C}^{\tilde{\eta}}([a,T])$ for some $\tilde{\eta} \ge 1 - \alpha$. Then, the function*

$$\Phi(t) = \int_a^t (t-s)^{\alpha-1} \varrho_s d\theta_s, \quad t \in [a,T],$$

is η-Hölder continuous, for any $\eta \in (0, \alpha + \gamma - 1)$, with norm independent of $a \in [0,T)$. Namely, for $a \le t_1 < t_2 \le \tilde{T} \le T$ and $\varepsilon < \alpha + \gamma - 1$,

$$|\Phi(t_2) - \Phi(t_1)| \le C_{\alpha,\gamma,\varepsilon,T} \|\tilde{\theta}\|_{\gamma,[a,\tilde{T}]}(t_2 - t_1)^{\gamma+\alpha-\varepsilon-1},$$

where $\tilde{\theta}_{\cdot} = \int_a^{\cdot} \varrho_s d\theta_s$.

4. Existence and uniqueness for the solution

In this section, we consider the existence of a unique solution to equation (1.3), when the initial condition satisfies either assumption **(H2)**, or assumption **(H3)** below. In order to be able to apply Propositions 2.1 and 2.2, and Lemmas 3.1-3.3 in our analysis, we include the following hypotheses:

(H1) Let $\gamma > 2/3$.

(H2) Let $\xi : [0,T] \to \mathbb{R}$ be a measurable function such that $t \mapsto t^{1-\alpha}\xi_t$ belongs to $\mathcal{C}^{2\alpha-1+\varepsilon}([0,T])$, for some ε small enough .

(H3) The function ξ belongs to $\mathcal{C}^{1-\alpha}([0,T])$.

Note that the function $\xi_t = t^{\alpha-1}\tilde{\xi}_t$, $t \in [0,T]$, satisfies **(H2)** if $\tilde{\xi} \in \mathcal{C}^{2\alpha-1+\varepsilon}([0,T])$, for some $\varepsilon > 0$. But, ξ is a discontinuous function at 0 if $\tilde{\xi}_0 \neq 0$.

Sometimes we write $\|f_t\|_{\eta,[0,T]}$ instead of $\|f\|_{\eta,[0,T]}$. We use this abuse of notation because we believe that the reader can identify the involve function f more easily. So, for instance, we change $\| \cdot^{1-\alpha}\, \xi_\cdot \|_{\eta,[0,T]}$ by $\|t^{1-\alpha}\xi_t\|_{\eta,[0,T]}$.

4.1. *Equation with initial condition satisfying (H2)*

Now we suppose that Hypotheses (H1) and (H2) are satisfied in this section. Under these assumptions, we study the existence and uniqueness for the solution to the equation

$$X_t = \xi_t + \int_0^t (t-s)^{\beta-1}\, g(s,X_s)ds + \int_0^t (t-s)^{\alpha-1} X_s f(s) d\theta_s, \quad t \in [0,T]. \quad (4.1)$$

Here, $f \in \mathcal{C}^{2\alpha-1+\varepsilon}([0,T])$. Under the conditions of Proposition 2.1, we only need to study the equation

$$X_t = \xi_t + \int_0^t (t-s)^{\beta-1}\, g(s,X_s)ds + \int_0^t (t-s)^{\alpha-1} X_s d\theta_s, \quad t \in [0,T]. \quad (4.2)$$

In this subsection, we suppose that $1 > \beta > \alpha$ and $\alpha \in (1-\frac{\gamma}{2}, 2/3)$. The function g satisfies the following:

(H4) $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with linear growth, uniformly in $[0,T]$.

Note that the conditions on α and β imply that η in Lemma 3.2 is equal to $2\alpha - 1 + \varepsilon$, for ε small enough. We will prove that (4.2) has a unique solution in the space

$$\mathcal{L}_T = \left\{\rho : [0,T] \to \mathbb{R} : (t \mapsto t^{1-\alpha}\rho_t) \in \mathcal{C}^{2\alpha-1+\tilde{\varepsilon}}([0,T]), \text{ for } \tilde{\varepsilon} < \varepsilon \right\}.$$

We recall that we are using the convention

$$|||\cdot|||_{\eta,[0,T]} = \|\cdot\|_{\eta,[0,T]} + \|\cdot\|_{\infty,[0,T]}.$$

On $\mathcal{L}_T$, we define, for $t \in [0,T]$,

$$\mathcal{M}(\rho)_t = \xi_t + \int_0^t (t-s)^{\beta-1}\ g(s,\rho_s)ds + \int_0^t (t-s)^{\alpha-1}\rho_s d\theta_s,$$

in order to apply the fixed-point theorem to equation (4.2).

Proposition 4.1. *Assume that $\gamma > \beta > \alpha$, $\alpha \in (1-\frac{\gamma}{2}, 2/3)$ and that Hypotheses **(H1)**, **(H2)** and **(H4)** hold. Let $\theta \in \mathcal{C}^\gamma([0,T])$. Then, equation (4.2) has a unique solution in the space $\mathcal{L}_T$.*

Proof. First of all, we have that $\mathcal{M}: \mathcal{L}_T \to \mathcal{L}_T$ due to Proposition 2.1, Lemmas 2.2, 3.1 and 3.2, and straightforward calculations. In order to apply fixed-point theorem, let $\rho^{(1)}, \rho^{(2)} \in \mathcal{L}_T$. Then, for $\eta = 2\alpha - 1 + \tilde{\varepsilon}$, we have

$$\begin{aligned}
&\left|\left|\left| t^{1-\alpha}\left(\mathcal{M}(\rho^{(1)})_t - \mathcal{M}(\rho^{(2)})_t\right)\right|\right|\right|_{\eta,[0,T]} \\
&\le (1+T^\eta)\left\| t^{1-\alpha}\left(\mathcal{M}(\rho^{(1)})_t - \mathcal{M}(\rho^{(2)})_t\right)\right\|_{\eta,[0,T]} \\
&\le (1+T^\eta)\left[\left\| t^{1-\alpha}\int_0^t (t-s)^{\beta-1}\left(g(s,\rho_s^{(1)}) - g(s,\rho_s^{(2)})\right)ds\right\|_{\eta,[0,T]}\right. \\
&\quad \left. + \left\| t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\left(s^{1-\alpha}\rho_s^{(1)} - s^{1-\alpha}\rho_s^{(2)}\right)d\theta_s\right\|_{\eta,[0,T]}\right] \\
&= (1+T^\eta)\left[\left\| t^{1-\alpha}\int_0^t (t-s)^{\beta-1}\left(g(s,\rho_s^{(1)}) - g(s,\rho_s^{(2)})\right)ds\right\|_{\eta,[0,T]}\right. \\
&\quad \left. + \left\| t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}d\tilde{\theta}_s\right\|_{\eta,[0,T]}\right],
\end{aligned}$$

where $\tilde{\theta}_\cdot = \int_0^\cdot \left(s^{1-\alpha}\rho_s^{(1)} - s^{1-\alpha}\rho_s^{(2)}\right)d\theta_s$ and, consequently, the last equality follows from Proposition 2.1. Thus, Lemmas 3.1 and 3.2, together with (A.39) below, and León and Tindel [32] (Theorem 2.5) yield, for $T_1 < T$,

$$\begin{aligned}
&\left|\left|\left| t^{1-\alpha}\left(\mathcal{M}(\rho^{(1)})_t - \mathcal{M}(\rho^{(2)})_t\right)\right|\right|\right|_{\eta,[0,T_1]} \\
&\quad \le C_{\alpha,\beta,\gamma,T,g}T_1^{\tilde{\varepsilon}}\left[\left\| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right\|_{\infty,[0,T_1]}\right. \\
&\quad \left. + \|\theta\|_{\gamma,[0,T]}\left\| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right\|_{\infty,[0,T_1]} + \|\theta\|_{\gamma,[0,T]}\left\| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right\|_{\eta,[0,T_1]}\right] \\
&\quad \le C_{\alpha,\beta,\gamma,T,g}T_1^{\tilde{\varepsilon}}\left[\left|\left|\left| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right|\right|\right|_{\eta,[0,T_1]}\right. \\
&\quad \left. + \|\theta\|_{\gamma,[0,T]}\left|\left|\left| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right|\right|\right|_{\eta,[0,T_1]}\right].
\end{aligned}$$

We can choose T_1 small enough such that $\mathcal{M}$ is a contraction. Hence, $\mathcal{M}$ has a fix point $X^{(1)}$, which is the solution to (4.2) on $[0,T_1]$.

On the other hand, we now introduce the space

$$\mathcal{L}_T^{(1)} = \Big\{\rho : [0,T] \to \mathbb{R} : (t \mapsto t^{1-\alpha}\rho_t) \in \mathcal{C}^{2\alpha-1+\tilde{\varepsilon}}([0,T])) \\ \text{for } \tilde{\varepsilon} < \varepsilon \text{ and } X^{(1)} = \rho \text{ on } [0,T_1]\Big\}.$$

Now choose $T_1 < \tilde{T}$ and $\rho^{(1)}, \rho^{(2)} \in \mathcal{L}_{\tilde{T}}^{(1)}$. Proceeding as before, for $\tilde{\eta}$ small enough, we get

$$\begin{aligned}
&\left|\left|\left| t^{1-\alpha}\left(\mathcal{M}(\rho^{(1)})_t - \mathcal{M}(\rho^{(2)})_t\right)\right|\right|\right|_{\eta,[0,\tilde{T}]} \\
&\leq C_{\alpha,\beta,\gamma,T,g}(1+\|\theta\|_{\gamma,[0,T]}) \left|\left|\left| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right|\right|\right|_{\eta-\tilde{\eta},[0,\tilde{T}]} \\
&= C_{\alpha,\beta,\gamma,T,g}(1+\|\theta\|_{\gamma,[0,T]}) \left|\left|\left| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right|\right|\right|_{\eta-\tilde{\eta},[T_1,\tilde{T}]} \\
&\leq C_{\alpha,\beta,\gamma,T,g}(\tilde{T}-T_1)^{\tilde{\eta}}(1+\|\theta\|_{\gamma,[0,T]}) \left|\left|\left| t^{1-\alpha}\left(\rho_t^{(1)} - \rho_t^{(2)}\right)\right|\right|\right|_{\eta,[0,\tilde{T}]}.
\end{aligned}$$

Note that the proof of Lemma 3.2 and Lemma 3.1 give that $C_{\alpha,\beta,\gamma,T,g}$ is independent of the points $\tilde{T}$ and T_1. In this way, we have shown that if equation (4.2) has a unique solution on the interval $[0,T_1]$, then it also has a unique solution on $[0,T_1+h]$, with $h = [2C_{\alpha,\beta,\gamma,T,g}(1+\|\theta\|_{\nu,[0,T]})]^{-\frac{1}{\tilde{\eta}}}$. Therefore, we can use induction to see that equation (4.2) has a unique solution on $[0,T]$. □

4.2. *Equation with initial condition satisfying (H3)*

Here we deal with Assumptions **(H1)**,**(H3)** and **(H4)**. Now, $1 > \beta \geq 1-\alpha$ and $\alpha \in \left(1-\frac{\gamma}{2},\gamma\right)$. Throughout this section, f in the equation (4.1) belongs to $\mathcal{C}^{1-\alpha}([0,T])$. Consequently, by Proposition 2.2, equations (4.1) and (4.2) have the same solutions.

Proposition 4.2. *Assume that $1 > \beta \geq 1-\alpha$, and $\alpha \in \left(1-\frac{\gamma}{2},\gamma\right)$, and that Hypotheses **(H1)**, **(H3)** and **(H4)** hold. Let $\theta \in \mathcal{C}^\gamma([0,T])$. Then, equation (4.2) has a unique solution in the space $\mathcal{C}^{1-\alpha}([0,T])$.*

Proof. From Proposition 2.2, Lemma 3.3 and $\beta \geq 1-\alpha$, we have that

$$\mathcal{M} : \mathcal{C}^{1-\alpha}([0,T]) \to \mathcal{C}^{1-\alpha}([0,T]).$$

Now suppose that equation (4.2) has a unique solution Y on an interval $[0,T_1]$, for some $T_1 < T$, which allows us to define the set

$$\mathcal{J}_{T_1,T} = \left\{\rho : [0,T] \longrightarrow \mathbb{R} : \rho \in \mathcal{C}^{1-\alpha}([0,T]); \rho = Y \text{ on } [0,T_1]\right\}.$$

Similarly to the proof of Proposition 4.1, choosing $T_1 < \tilde{T}$, we can prove

$$\begin{aligned}
\Theta(\tilde{T}) &:= \left|\left|\left|\left(\mathcal{M}(\rho^{(1)}) - \mathcal{M}(\rho^{(2)})\right)\right|\right|\right|_{\eta,[0,\tilde{T}]} \\
&\leq C_{\alpha,\beta,\gamma,T,g}\left(1+\|\theta\|_{\gamma,[0,T]}\right)\left(\tilde{T}-T_1\right)^{\tilde{\eta}} \left|\left|\left|\rho^{(1)} - \rho^{(2)}\right|\right|\right|_{\eta,[T_1,\tilde{T}]}.
\end{aligned}$$

Indeed, by Lemma 3.3 and $\beta \geq 1-\alpha$, $\mathcal{M} : \mathcal{J}_{T_1,\tilde{T}} \to \mathcal{J}_{T_1,\tilde{T}}$ and we can write, for $\eta = 1-\alpha$,

$$\begin{aligned}
\Theta(\tilde{T}) &\leq (1+T^{\eta}) \left\| \left(\mathcal{M}(\rho^{(1)}) - \mathcal{M}(\rho^{(2)})\right) \right\|_{\eta,[0,\tilde{T}]} \\
&\leq (1+T^{\eta}) \left[\left\| \int_0^t (t-s)^{\beta-1} \left(g(s,\rho_s^{(1)}) - g(s,\rho_s^{(2)})\right) ds \right\|_{\eta,[0,\tilde{T}]} \right. \\
&\quad \left. + \left\| \int_0^t (t-s)^{\alpha-1} \left(\rho_s^{(1)} - \rho_s^{(2)}\right) d\theta_s \right\|_{\eta,[0,\tilde{T}]} \right] \\
&\leq C_{\alpha,\beta,\gamma,T,g} \left[\left\| \rho^{(1)} - \rho^{(2)} \right\|_{\infty,[0,\tilde{T}]} + \|\theta\|_{\gamma,[0,T]} \left\| \rho^{(1)} - \rho^{(2)} \right\|_{\infty,[0,\tilde{T}]} \right. \\
&\quad \left. + \|\theta\|_{\gamma,[0,T]} \left\| \rho^{(1)} - \rho^{(2)} \right\|_{\eta-\tilde{\eta},[0,\tilde{T}]} \right] \\
&= C_{\alpha,\beta,\gamma,T,g} \left[\left\| \rho^{(1)} - \rho^{(2)} \right\|_{\infty,[T_1,\tilde{T}]} + \|\theta\|_{\gamma,[0,T]} \left\| \rho^{(1)} - \rho^{(2)} \right\|_{\infty,[T_1,\tilde{T}]} \right. \\
&\quad \left. + \|\theta\|_{\gamma,[0,T]} \left\| \rho^{(1)} - \rho^{(2)} \right\|_{\eta-\tilde{\eta},[T_1,\tilde{T}]} \right] \\
&\leq C_{\alpha,\beta,\gamma,T,g} \left(1 + \|\theta\|_{\gamma,[0,T]}\right) \left(\tilde{T} - T_1\right)^{\tilde{\eta}} \left\|\left| \rho^{(1)} - \rho^{(2)} \right|\right\|_{\eta,[T_1,\tilde{T}]} .
\end{aligned}$$

Thus, the proof is finished due to the fixed-point theorem. Indeed, for $T_1 = 0$, we define

$$\mathcal{J}_T = \left\{ \rho : [0,T] \longrightarrow \mathbb{R} : \rho \in \mathcal{C}^{1-\alpha}([0,T]); \rho_0 = \xi_0 \right\}.$$

□

Appendix A. Appendix

The purpose of this section is to give the missing proofs of this paper.

Proof of Proposition 2.1. Let $\varepsilon > 0$, then Samko et al. [45] (Theorem 3.6) yields

$$\left(r \longmapsto I_{a+}^{\tilde{\alpha}} \left((t+\varepsilon-\cdot)^{-\beta}(\cdot-(a-\varepsilon))^{-\alpha}\right)(r)\right) \in \mathcal{C}^{\tilde{\alpha}-\frac{1}{p}}([a,t]), \tag{A.1}$$

for any $p > 1/\tilde{\alpha}$. We choose $p > 1/\tilde{\alpha}$ such that $\gamma + \tilde{\alpha} - \frac{1}{p} > 1$. In this way, Fiel et al. [17] (Lemma 2.4) implies

$$\begin{aligned}
&\int_a^t (t-a)^{\tilde{\alpha}} I_{a+}^{\tilde{\alpha}} \left((t+\varepsilon-\cdot)^{-\beta}(\cdot-(a-\varepsilon))^{-\alpha}\right)(r) f(r) d\theta_r \\
&\qquad = \int_a^t (t-a)^{\tilde{\alpha}} I_{a+}^{\tilde{\alpha}} \left((t+\varepsilon-\cdot)^{-\beta}(\cdot-(a-\varepsilon))^{-\alpha}\right)(r) d\tilde{\theta}_r.
\end{aligned} \tag{A.2}$$

In order to show that the result holds, we will use Lemma 2.2 and let ε go to zero.

For the right-hand side of (A.2), we have that, for $\mu_{\alpha,\beta} = \frac{\Gamma(1-\alpha)}{\Gamma(\beta)}$,

$$
\begin{aligned}
&\Bigg| \int_a^t (t-a)^{\tilde{\alpha}} I_{a+}^{\tilde{\alpha}} \left((t+\varepsilon-\cdot)^{-\beta} (\cdot - (a-\varepsilon))^{-\alpha} \right)(r) d\tilde{\theta}_r \\
&\qquad - \mu_{\alpha,\beta} \int_a^t (t-r)^{\alpha-1} (r-a)^{\beta-1} d\tilde{\theta}_r \Bigg| \\
&= (t-a)^{\tilde{\alpha}} \Bigg| \int_a^t (t+\varepsilon-r)^{-\beta} (r-(a-\varepsilon))^{-\alpha} (D_{t-}^{1-\tilde{\alpha}} \tilde{\theta}_{t-})(r) dr \\
&\qquad - \int_a^t (t-r)^{-\beta} (r-a)^{-\alpha} (D_{t-}^{1-\tilde{\alpha}} \tilde{\theta}_{t-})(r) dr \Bigg| \\
&\leq C_{\tilde{\alpha},\gamma} (t-a)^{2\tilde{\alpha}+\gamma-1} \|\tilde{\theta}\|_{\gamma,[0,T]} \\
&\qquad \times \int_a^t \left| (t+\varepsilon-r)^{-\beta} (r-(a-\varepsilon))^{-\alpha} - (t-r)^{-\beta} (r-a)^{-\alpha} \right| dr, \qquad \text{(A.3)}
\end{aligned}
$$

which goes to zero, as $\varepsilon \to 0$, because of the dominated convergence theorem.

Now we deal with the left-hand side of (A.2). Towards this end, we utilize the conventions

$$
\Lambda^\varepsilon(r) = I_{a+}^{\tilde{\alpha}} \left((t+\varepsilon-\cdot)^{-\beta} (\cdot - (a-\varepsilon))^{-\alpha} \right)(r)
$$

and

$$
\Lambda(r) = I_{a+}^{\tilde{\alpha}} \left((t-\cdot)^{-\beta} (\cdot - a)^{-\alpha} \right)(r).
$$

Note that (A.1) and the fact that $f \in \mathcal{C}^{\tilde{\alpha}+\eta}([a,t])$ allow us to deduce, for $p \in (1, \frac{1}{\beta} \wedge \frac{1}{\alpha})$,

$$
\begin{aligned}
&D_{a+}^{\tilde{\alpha}} \left[\Lambda^\varepsilon(r) f(r) \right] \\
&= L^p - \lim_{\eta \downarrow 0} \frac{1}{\Gamma(1-\tilde{\alpha})} \left[\frac{\Lambda^\varepsilon(r) f(r)}{(r-a)^{\tilde{\alpha}}} + \tilde{\alpha} \int_a^{r-\eta} \frac{\Lambda^\varepsilon(r) f(r) - \Lambda^\varepsilon(u) f(u)}{(r-u)^{1+\tilde{\alpha}}} du \right] \\
&= L^p - \lim_{\eta \downarrow 0} \frac{f(r)}{\Gamma(1-\tilde{\alpha})} \left[\frac{\Lambda^\varepsilon(r)}{(r-a)^{\tilde{\alpha}}} + \tilde{\alpha} \int_a^{r-\eta} \frac{\Lambda^\varepsilon(r) - \Lambda^\varepsilon(u)}{(r-u)^{1+\tilde{\alpha}}} du \right] \\
&\quad + L^p - \lim_{\eta \downarrow 0} \frac{\tilde{\alpha}}{\Gamma(1-\tilde{\alpha})} \int_a^{r-\eta} \frac{f(r) - f(u)}{(r-u)^{1+\tilde{\alpha}}} \Lambda^\varepsilon(u) du \\
&= (t+\varepsilon-r)^{-\beta} (r-(a-\varepsilon))^{-\alpha} f(r) \\
&\quad + \frac{\tilde{\alpha}}{\Gamma(1-\tilde{\alpha})} \int_a^r \frac{f(r) - f(u)}{(r-u)^{1+\tilde{\alpha}}} \Lambda^\varepsilon(u) du. \qquad \text{(A.4)}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
D_{a+}^{\tilde{\alpha}} \left[\Lambda(r) f(r) \right] &= (t-r)^{-\beta} (r-a)^{-\alpha} f(r) \\
&\quad + \frac{\tilde{\alpha}}{\Gamma(1-\tilde{\alpha})} \int_a^r \frac{f(r) - f(u)}{(r-u)^{1+\tilde{\alpha}}} \Lambda(u) du. \qquad \text{(A.5)}
\end{aligned}
$$

Indeed, we only need to proceed as in (A.4) throughout the inequality

$$
\begin{aligned}
&(t-a)^{\tilde{\alpha}}\left|\int_a^r \frac{f(r)-f(u)}{(r-u)^{1+\tilde{\alpha}}} I_{a+}^{\tilde{\alpha}}\left((t-\cdot)^{-\beta}(\cdot-a)^{-\alpha}\right)(u)du\right| \\
&\quad\le \frac{\Gamma(1-\alpha)}{\Gamma(\beta)}\int_a^r \left|\frac{f(r)-f(u)}{(r-u)^{1+\tilde{\alpha}}}\right|(t-u)^{\alpha-1}(u-a)^{\beta-1}du \\
&\quad\le C_{\alpha,\beta}\|f\|_{\tilde{\alpha}+\eta}\int_a^r (r-u)^{\eta-1}(t-u)^{\alpha-1}(u-a)^{\beta-1}du \\
&\quad\le C_{\alpha,\beta}\|f\|_{\tilde{\alpha}+\eta}(t-r)^{\alpha-1}\int_a^r (r-u)^{\eta-1}(u-a)^{\beta-1}du \\
&\quad= C_{\alpha,\beta}\|f\|_{\tilde{\alpha}+\eta}(t-r)^{\alpha-1}\frac{\Gamma(\eta)\Gamma(\beta)}{\Gamma(\eta+\beta)}(r-a)^{\eta+\beta-1},
\end{aligned}
$$

where the first inequality follows from Lemma 2.2 and the last term belongs to $L^p([a,t])$, for $p<\frac{1}{1-\alpha}\wedge\frac{1}{1-\beta}$. Hence, Definition (2.4) implies

$$
\begin{aligned}
&\int_a^t (t-a)^{\tilde{\alpha}} I_{a+}^{\tilde{\alpha}}\left((t+\varepsilon-\cdot)^{-\beta}(\cdot-(a-\varepsilon))^{-\alpha}\right)(r)f(r)d\theta_r \\
&\quad= (t-a)^{\tilde{\alpha}}\int_a^t (t+\varepsilon-r)^{-\beta}(r-(a-\varepsilon))^{-\alpha}f(r)(D_{t-}^{1-\tilde{\alpha}}\theta_{t-})(r)dr \qquad \text{(A.6)} \\
&\qquad+(t-a)^{\tilde{\alpha}}\frac{\tilde{\alpha}}{\Gamma(1-\tilde{\alpha})}\int_a^t\int_a^r \frac{f(r)-f(u)}{(r-u)^{1+\tilde{\alpha}}}\Lambda^{\varepsilon}(u)du(D_{t-}^{1-\tilde{\alpha}}\theta_{t-})(r)dr.
\end{aligned}
$$

We also have that

$$
\begin{aligned}
&\int_a^r \left|\frac{f(r)-f(u)}{(r-u)^{1+\tilde{\alpha}}}\left[\Lambda^{\varepsilon}(u)-I_{a+}^{\tilde{\alpha}}\left((t-\cdot)^{-\beta}(\cdot-a)^{-\alpha}\right)(u)\right]\right|du \\
&\quad\le \frac{\|f\|_{\tilde{\alpha}+\eta}}{\Gamma(\tilde{\alpha})}\int_a^r (r-u)^{\eta-1} \\
&\qquad\times\int_a^u \frac{\left|(t-w)^{-\beta}(w-a)^{-\alpha}-(t+\varepsilon-w)^{-\beta}(w+\varepsilon-a)^{-\alpha}\right|}{(u-w)^{1-\tilde{\alpha}}}dwdu.
\end{aligned}
$$

Since $\left|(t-w)^{-\beta}(w-a)^{-\alpha}-(t+\varepsilon-w)^{-\beta}(w+\varepsilon-a)^{-\alpha}\right|\longrightarrow 0$ when $\varepsilon\to 0$ (a.s), it is bounded by $2\left|(t-w)^{-\beta}(w-a)^{-\alpha}\right|$ and

$$
\begin{aligned}
\int_a^u (t-w)^{-\beta}(w-a)^{-\alpha}(u-w)^{\tilde{\alpha}-1}dw &\le (t-u)^{-\beta}\int_a^u (w-a)^{-\alpha}(u-w)^{\tilde{\alpha}-1}dw \\
&= (t-u)^{-\beta}\frac{\Gamma(\tilde{\alpha})\Gamma(1-\alpha)}{\Gamma(\tilde{\alpha}+1-\alpha)}(u-a)^{\tilde{\alpha}-\alpha}.
\end{aligned}
$$

Then

$$
\int_a^u \frac{\left|(t-w)^{-\beta}(w-a)^{-\alpha}-(t+\varepsilon-w)^{-\beta}(w+\varepsilon-a)^{-\alpha}\right|}{(u-w)^{1-\tilde{\alpha}}}dw\longrightarrow 0,
$$

as $\varepsilon \to 0$, and it is bounded by $C_{\alpha,\beta}(t-u)^{-\beta}(u-a)^{\tilde{\alpha}-\alpha}$. Thus,

$$\begin{aligned}\int_a^r (r-u)^{\eta-1}(t-u)^{-\beta}(u-a)^{\tilde{\alpha}-\alpha}du &\le (t-r)^{-\beta}\int_a^r (r-u)^{\eta-1}(u-a)^{\tilde{\alpha}-\alpha}du \\ &= \frac{\Gamma(\eta)\Gamma(\tilde{\alpha}-\alpha+1)}{\Gamma(\eta+1+\tilde{\alpha}-\alpha)}(t-r)^{-\beta}(r-a)^{\eta+\tilde{\alpha}-\alpha} \in L^1.\end{aligned}$$

All these last calculations applied to (A.6) lead us to write

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_a^t (t-a)^{\tilde{\alpha}} I_{a+}^{\tilde{\alpha}}\left((t+\varepsilon-\cdot)^{-\beta}(\cdot-(a-\varepsilon))^{-\alpha}\right)(r)f(r)d\theta_r \\ &\qquad = \frac{\Gamma(1-\alpha)}{\Gamma(\beta)}\int_a^t (t-r)^{\alpha-1}(r-a)^{\beta-1}f(r)d\theta_r.\end{aligned}$$

Finally, putting together this limit, (A.2) and (A.3) we obtain the result. □

Proof of Proposition 2.2. Let $\tilde{\varepsilon} > 0$ be such that $\alpha+\gamma-\tilde{\varepsilon}-1 > 0$ and $\gamma-\alpha-\tilde{\varepsilon} > 0$. Also let $\tilde{\eta} > 0$ be such that $\gamma-\alpha-\tilde{\varepsilon}-\tilde{\eta} > 0$. Note that $\tilde{\eta} < 1-\alpha$. Set $\varepsilon := \alpha+\gamma-\tilde{\varepsilon}-1$, which is less than α, and $\eta = 1-\alpha-\tilde{\eta}$.

From (2.6) and [45] (Corollary 1 of Theorem 11.5), we know that

$$(t-r)^{\alpha-1} = I_{a+}^{\alpha-\varepsilon}(h)(r), \quad r \in [a,t],$$

where $h \in L^p([a,t])$ and $1 < p < \frac{1}{1-\varepsilon}$. Now choose a sequence $\{h_n : n \in \mathbb{N}\} \subset L^p([a,t])$ of bounded functions converging to h in $L^p([a,t])$. Therefore, as in (A.2), we are able to establish

$$\int_a^t I_{a+}^{\alpha-\varepsilon}(h_n)(r)f(r)d\theta_r = \int_a^t I_{a+}^{\alpha-\varepsilon}(h_n)(r)d\tilde{\theta}_r. \tag{A.7}$$

Proceeding as in (A.3), we can see that the right-hand side of (A.7) goes to $\int_a^t I_{a+}^{\alpha-\varepsilon}(h)(r)d\tilde{\theta}_r$, as $n \uparrow \infty$.

Finally, we can study the convergence of the left-hand side of (A.7) as in (A.4) by noting that the Fubini theorem and the fact that $f \in \mathcal{C}^\eta([a,t])$ imply that, for $r \in [a,t]$,

$$\begin{aligned}&\int_a^r \frac{|f(r)-f(u)|}{(r-u)^{1+\alpha-\varepsilon}}\left|I_{a+}^{\alpha-\varepsilon}(h_n-h)(u)\right|du \\ &\le C\int_a^r (r-u)^{\varepsilon-2\alpha-\tilde{\eta}}\left|I_{a+}^{\alpha-\varepsilon}(h_n-h)(u)\right|du \\ &\le C\int_a^r\int_a^u (r-u)^{\varepsilon-2\alpha-\tilde{\eta}}(u-s)^{\alpha-\varepsilon-1}\left|h_n(s)-h(s)\right|dsdu \\ &= C\int_a^r |h_n(s)-h(s)|\int_s^r (r-u)^{\varepsilon-2\alpha-\tilde{\eta}}(u-s)^{\alpha-\varepsilon-1}duds \\ &= C\int_a^r |h_n(s)-h(s)|(r-s)^{-\alpha-\tilde{\eta}}ds = CI_{a+}^{1-\alpha-\tilde{\eta}}(|h_n-h|)(r),\end{aligned}$$

which goes to zero in $L^p([a,t])$. Thus the proof is complete. □

Proof of Lemma 3.1. We divide the proof into two steps.
Step 1: Here we consider the case that $\eta = \tilde{\alpha}+\varepsilon$. We have that, for $a \le t_1 < t_2 \le T$,

$$|J_a(t_2) - J_a(t_1)| \le I_1 + I_2, \tag{A.8}$$

with

$$\begin{aligned} I_1 &= |(t_2-a)^\alpha - (t_1-a)^\alpha| \left| \int_a^{t_2} (t_2-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \right|, \\ I_2 &= (t_1-a)^\alpha \bigg| \int_a^{t_2} (t_2-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \\ &\qquad - \int_a^{t_1} (t_1-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) dr \bigg|. \end{aligned}$$

We first study I_1. We can get, for $\varepsilon > 0$ such that $\tilde{\alpha}+\varepsilon < \alpha$ and $\beta + \varepsilon < \gamma$,

$$\begin{aligned} I_1 &\le |(t_2-a)^\alpha - (t_1-a)^\alpha|^{\frac{\tilde{\alpha}+\varepsilon}{\alpha}} |(t_2-a)^\alpha - (t_1-a)^\alpha|^{1-\frac{\tilde{\alpha}+\varepsilon}{\alpha}} \\ &\quad \times \left| \int_a^{t_2} (t_2-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \right| \\ &\le C_{\alpha,\beta,\gamma} (t_2-t_1)^{\tilde{\alpha}+\varepsilon} (t_2-a)^{\alpha-\tilde{\alpha}-\varepsilon} \|\theta\|_{\gamma,[0,T]} \int_a^{t_2} (t_2-r)^{\tilde{\alpha}+\gamma-1-\beta} (r-a)^{-\alpha} dr \\ &\le C_{\alpha,\beta,\gamma} (t_2-t_1)^{\tilde{\alpha}+\varepsilon} (t_2-a)^{\alpha-\tilde{\alpha}-\varepsilon} \|\theta\|_{\gamma,[0,T]} \\ &\quad \times \frac{\Gamma(\tilde{\alpha}+\gamma-\beta)\Gamma(1-\alpha)}{\Gamma(\gamma)} (t_2-a)^{\tilde{\alpha}+\gamma-\beta-\alpha} \\ &\le C_{\alpha,\beta,\gamma} (t_2-t_1)^{\tilde{\alpha}+\varepsilon} (t_2-a)^{\gamma-\beta-\varepsilon} \|\theta\|_{\gamma,[0,T]}. \end{aligned} \tag{A.9}$$

The triangle inequality yields

$$I_2 \le I_{2,1} + I_{2,2}, \tag{A.10}$$

where

$$\begin{aligned} I_{2,1} &= (t_1-a)^\alpha \bigg| \int_a^{t_1} (t_2-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \\ &\qquad - \int_a^{t_1} (t_1-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) dr \bigg|, \\ I_{2,2} &= (t_1-a)^\alpha \left| \int_{t_1}^{t_2} (t_2-r)^{-\beta}(r-a)^{-\alpha} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \right|. \end{aligned}$$

For $I_{2,2}$ we choose $\lambda > 0$ such that $\lambda < \alpha$ and $\gamma - \beta - \lambda > 0$. Thus,

$$\begin{aligned}
I_{2,2} &= (t_1 - a)^{\alpha} \left| \int_{t_1}^{t_2} (t_2 - r)^{-\beta} (r-a)^{-\lambda} (r-a)^{-\alpha+\lambda} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \right| \\
&\le (t_1 - a)^{\lambda} \left| \int_{t_1}^{t_2} (t_2 - r)^{-\beta} (r-a)^{-\lambda} \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) dr \right| \\
&\le C_{\tilde{\alpha},\gamma} \|\theta\|_{\gamma,[0,T]} (t_1 - a)^{\lambda} \int_{t_1}^{t_2} (t_2 - r)^{\tilde{\alpha}+\gamma-1-\beta} (r-a)^{-\lambda} dr \\
&\le C_{\tilde{\alpha},\gamma} \|\theta\|_{\gamma,[0,T]} (t_1 - a)^{\lambda} \int_{t_1}^{t_2} (t_2 - r)^{\tilde{\alpha}+\gamma-1-\beta} (r-t_1)^{-\lambda} dr \\
&\le C_{\tilde{\alpha},\gamma} \|\theta\|_{\gamma,[0,T]} (t_1 - a)^{\lambda} \frac{\Gamma(\tilde{\alpha}+\gamma-\beta)\Gamma(1-\lambda)}{\Gamma(\tilde{\alpha}+\gamma-\beta+1-\lambda)} (t_2 - t_1)^{\tilde{\alpha}+\gamma-\beta-\lambda} \\
&\le C_{\alpha,\beta,\gamma} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} (t_2 - a)^{\lambda} \|\theta\|_{\gamma,[0,T]}.
\end{aligned} \tag{A.11}$$

We continue with the analysis of (A.10) by means of the next calculus:

$$I_{2,1} \le I_{2,1,1} + I_{2,1,2}, \tag{A.12}$$

with

$$\begin{aligned}
I_{2,1,1} &= (t_1 - a)^{\alpha} \int_a^{t_1} (t_2 - r)^{-\beta} (r-a)^{-\alpha} \left| \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) - \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) \right| dr, \\
I_{2,1,2} &= (t_1 - a)^{\alpha} \int_a^{t_1} (r-a)^{-\alpha} \left| (t_2 - r)^{-\beta} - (t_1 - r)^{-\beta} \right| \left| \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) \right| dr.
\end{aligned}$$

Using the mean value theorem, for $\varepsilon > 0$ such that $\gamma - \beta - \varepsilon > 0$ and $1 - \tilde{\alpha} - \varepsilon > 0$, we have

$$\begin{aligned}
I_{2,1,2} &\le (t_1 - a)^{\alpha} \int_a^{t_1} (r-a)^{-\alpha} \left| (t_2 - r)^{-\beta} - (t_1 - r)^{-\beta} \right|^{\tilde{\alpha}+\varepsilon} \\
&\quad \times \left| (t_2 - r)^{-\beta} - (t_1 - r)^{-\beta} \right|^{1-\tilde{\alpha}-\varepsilon} \left| \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) \right| dr \\
&\le \beta^{\tilde{\alpha}+\varepsilon} (t_1 - a)^{\alpha} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} \int_a^{t_1} (r-a)^{-\alpha} (t_1 - r)^{-(\beta+1)(\tilde{\alpha}+\varepsilon)} \\
&\quad \times (t_1 - r)^{-\beta(1-\tilde{\alpha}-\varepsilon)} \left| \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) \right| dr \\
&\le C_{\alpha,\beta,\gamma,\varepsilon} (t_1 - a)^{\alpha} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} \|\theta\|_{\gamma,[0,T]} \\
&\quad \times \int_a^{t_1} (r-a)^{-\alpha} (t_1 - r)^{(-\beta-1)(\tilde{\alpha}+\varepsilon)-\beta(1-\tilde{\alpha}-\varepsilon)+\tilde{\alpha}+\gamma-1} dr \\
&\le C_{\alpha,\beta,\gamma,\varepsilon} (t_1 - a)^{\alpha} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} \|\theta\|_{\gamma,[0,T]} \int_a^{t_1} (r-a)^{-\alpha} (t_1 - r)^{\gamma-\beta-\varepsilon-1} dr \\
&\le C_{\alpha,\beta,\gamma,\varepsilon} (t_1 - a)^{\alpha} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} \|\theta\|_{\gamma,[0,T]} (t_1 - a)^{\gamma-\beta-\alpha-\varepsilon} \\
&\le C_{\alpha,\beta,\gamma,\varepsilon} \|\theta\|_{\gamma,[0,T]} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} (t_1 - a)^{\gamma-\beta-\varepsilon}.
\end{aligned} \tag{A.13}$$

In order to deal with $I_{2,1,1}$, we need to analyze the following term

$$\left| \left(D_{t_2-}^{1-\tilde{\alpha}} \theta_{t_2-} \right)(r) - \left(D_{t_1-}^{1-\tilde{\alpha}} \theta_{t_1-} \right)(r) \right| \le J_1 + J_2, \tag{A.14}$$

with

$$J_1 = \frac{1}{\Gamma(\tilde{\alpha})} \left| \frac{\theta_r - \theta_{t_2}}{(t_2 - r)^{1-\tilde{\alpha}}} - \frac{\theta_r - \theta_{t_1}}{(t_1 - r)^{1-\tilde{\alpha}}} \right|,$$
$$J_2 = \frac{1-\tilde{\alpha}}{\Gamma(\tilde{\alpha})} \left| \int_r^{t_2} \frac{\theta_r - \theta_s}{(s-r)^{2-\tilde{\alpha}}} ds - \int_r^{t_1} \frac{\theta_r - \theta_s}{(s-r)^{2-\tilde{\alpha}}} ds \right|.$$

The mean value theorem implies that

$$\begin{aligned} J_2 &= \frac{1-\tilde{\alpha}}{\Gamma(\tilde{\alpha})} \left| \int_{t_1}^{t_2} \frac{\theta_r - \theta_s}{(s-r)^{2-\tilde{\alpha}}} ds \right| \\ &\leq C_{\alpha,\beta,\gamma} \|\theta\|_{\gamma,[0,T]} \left| \int_{t_1}^{t_2} (s-r)^{\tilde{\alpha}-2+\gamma} ds \right| \\ &\leq C_{\alpha,\beta,\gamma} \|\theta\|_{\gamma,[0,T]} \left| (t_2 - r)^{\tilde{\alpha}-1+\gamma} - (t_1 - r)^{\tilde{\alpha}-1+\gamma} \right| \\ &\leq C_{\alpha,\beta,\gamma} \|\theta\|_{\gamma,[0,T]} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} \left| (t_1 - r)^{\tilde{\alpha}-2+\gamma} \right|^{\tilde{\alpha}+\varepsilon} \left| (t_2 - r)^{\tilde{\alpha}-1+\gamma} \right|^{1-\tilde{\alpha}-\varepsilon} \end{aligned} \tag{A.15}$$

Now we study J_1. We can establish

$$\begin{aligned} J_1 &= \frac{1}{\Gamma(\tilde{\alpha})} \left| \frac{(t_1 - r)^{1-\tilde{\alpha}} (\theta_r - \theta_{t_2}) - (t_2 - r)^{1-\tilde{\alpha}} (\theta_r - \theta_{t_1})}{(t_2 - r)^{1-\tilde{\alpha}} (t_1 - r)^{1-\tilde{\alpha}}} \right| \\ &\leq J_{1,1} + J_{1,2}. \end{aligned} \tag{A.16}$$

Here

$$J_{1,1} = C_{\alpha,\beta} \left| \frac{(t_1 - r)^{1-\tilde{\alpha}} - (t_2 - r)^{1-\tilde{\alpha}}}{(t_2 - r)^{1-\tilde{\alpha}} (t_1 - r)^{1-\tilde{\alpha}}} \right| |\theta_r - \theta_{t_2}|,$$
$$J_{1,2} = C_{\alpha,\beta} \frac{1}{(t_1 - r)^{1-\tilde{\alpha}}} |(\theta_r - \theta_{t_2}) - (\theta_r - \theta_{t_1})|.$$

Grouping correctly we have

$$\begin{aligned} J_{1,2} &\leq C_{\alpha,\beta} (t_1 - r)^{\tilde{\alpha}-1} |(\theta_r - \theta_{t_2}) - (\theta_r - \theta_{t_1})|^{\frac{\tilde{\alpha}+\varepsilon}{\gamma}} |(\theta_r - \theta_{t_2}) - (\theta_r - \theta_{t_1})|^{1-\frac{\tilde{\alpha}+\varepsilon}{\gamma}} \\ &\leq C_{\alpha,\beta,\gamma} \|\theta\|_{\gamma,[0,T]}^{1-\frac{\tilde{\alpha}+\varepsilon}{\gamma}} (t_1 - r)^{\tilde{\alpha}-1} |\theta_{t_2} - \theta_{t_1}|^{\frac{\tilde{\alpha}+\varepsilon}{\gamma}} |(t_2 - r)^{\gamma}|^{1-\frac{\tilde{\alpha}+\varepsilon}{\gamma}} \\ &\leq C_{\alpha,\beta,\gamma} \|\theta\|_{\gamma,[0,T]} (t_1 - r)^{\tilde{\alpha}-1} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} (t_2 - r)^{\gamma-\tilde{\alpha}-\varepsilon}. \end{aligned} \tag{A.17}$$

Using the mean value theorem again, we get

$$\begin{aligned} J_{1,1} &\leq C_{\alpha,\beta} \|\theta\|_{\gamma,[0,T]} (t_2 - r)^{\tilde{\alpha}-1+\gamma} (t_1 - r)^{\tilde{\alpha}-1} \left| (t_1 - r)^{1-\tilde{\alpha}} - (t_2 - r)^{1-\tilde{\alpha}} \right| \\ &= C_{\alpha,\beta} \|\theta\|_{\gamma,[0,T]} (t_2 - r)^{\tilde{\alpha}-1+\gamma} (t_1 - r)^{\tilde{\alpha}-1} \left| (t_1 - r)^{1-\tilde{\alpha}} - (t_2 - r)^{1-\tilde{\alpha}} \right|^{\tilde{\alpha}+\varepsilon} \\ &\quad \times \left| (t_1 - r)^{1-\tilde{\alpha}} - (t_2 - r)^{1-\tilde{\alpha}} \right|^{1-\tilde{\alpha}-\varepsilon} \\ &\leq C_{\alpha,\beta} \|\theta\|_{\gamma,[0,T]} (t_2 - r)^{\tilde{\alpha}-1+\gamma} (t_1 - r)^{\tilde{\alpha}-1} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} (t_1 - r)^{-\tilde{\alpha}(\tilde{\alpha}+\varepsilon)} \\ &\quad \times (t_2 - r)^{(1-\tilde{\alpha})(1-\tilde{\alpha}-\varepsilon)} \\ &= C_{\alpha,\beta} \|\theta\|_{\gamma,[0,T]} (t_2 - t_1)^{\tilde{\alpha}+\varepsilon} (t_2 - r)^{\gamma-\tilde{\alpha}+\tilde{\alpha}^2+\varepsilon(\tilde{\alpha}-1)} (t_1 - r)^{\tilde{\alpha}-\tilde{\alpha}^2-\varepsilon\tilde{\alpha}-1}. \end{aligned} \tag{A.18}$$

Then, putting (A.15), (A.16), (A.17) and (A.18) in $I_{2,1,1}$, we obtain

$$I_{2,1,1} \leq (t_1 - a)^{\alpha} \int_a^{t_1} (t_2 - r)^{-\beta} (r - a)^{-\alpha} [J_{1,1} + J_{1,2} + J_2] \, dr = \tilde{J}_{1,1} + \tilde{J}_{1,2} + \tilde{J}_2, \tag{A.19}$$

with

$$\begin{aligned}
\tilde{J}_{1,1} &= C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha} \\
&\quad\times\int_a^{t_1}(r-a)^{-\alpha}(t_2-r)^{\gamma-\tilde{\alpha}+\tilde{\alpha}^2-\beta+\varepsilon(\tilde{\alpha}-1)}(t_1-r)^{\tilde{\alpha}-\tilde{\alpha}^2-\varepsilon\tilde{\alpha}-1}dr, \\
\tilde{J}_{1,2} &= C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha} \\
&\quad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\tilde{\alpha}-1}(t_2-r)^{\gamma-\tilde{\alpha}-\beta-\varepsilon}dr, \\
\tilde{J}_{2} &= C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha} \\
&\quad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{(\tilde{\alpha}-2+\gamma)(\tilde{\alpha}+\varepsilon)}(t_2-r)^{(\tilde{\alpha}-1+\gamma)(1-\tilde{\alpha}-\varepsilon)-\beta}dr.
\end{aligned}$$

We now analyze these terms depending on the sign of the exponent of (t_2-r). We first study $\tilde{J}_{1,1}$. Then, if $\gamma-\tilde{\alpha}+\tilde{\alpha}^2-\beta\geq 0$, for $\varepsilon>0$ small enough, we establish

$$\begin{aligned}
\tilde{J}_{1,1} &\leq C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha} \\
&\quad\times T^{\gamma-\tilde{\alpha}+\tilde{\alpha}^2-\beta+\varepsilon(\tilde{\alpha}-1)}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\tilde{\alpha}-\tilde{\alpha}^2-\varepsilon\tilde{\alpha}-1}dr \\
&\leq C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\tilde{\alpha}-\tilde{\alpha}^2-\varepsilon\tilde{\alpha}}\ T^{\gamma-\tilde{\alpha}+\tilde{\alpha}^2-\beta+\varepsilon(\tilde{\alpha}-1)}. \qquad (A.20)
\end{aligned}$$

If $\gamma-\tilde{\alpha}+\tilde{\alpha}^2-\beta<0$, then

$$\begin{aligned}
\tilde{J}_{1,1} &\leq C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha} \\
&\quad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\gamma-\beta-1-\varepsilon}dr \\
&\leq C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\gamma-\beta-\varepsilon}. \qquad (A.21)
\end{aligned}$$

For $\tilde{J}_{1,2}$, if $\gamma-\tilde{\alpha}-\beta\leq 0$, we then obtain

$$\begin{aligned}
\tilde{J}_{1,2} &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\gamma-\beta-\varepsilon-1}dr \\
&\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\gamma-\beta-\varepsilon}. \qquad (A.22)
\end{aligned}$$

If $\gamma-\tilde{\alpha}-\beta>0$, then

$$\begin{aligned}
\tilde{J}_{1,2} &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}T^{\gamma-\tilde{\alpha}-\beta-\varepsilon}(t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\tilde{\alpha}-1}dr \\
&\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\tilde{\alpha}}T^{\gamma-\tilde{\alpha}-\beta-\varepsilon}. \qquad (A.23)
\end{aligned}$$

The last term $\tilde{J}_2$, if $(\tilde{\alpha}-1+\gamma)(1-\tilde{\alpha})-\beta>0$, can be estimate by

$$\begin{aligned}
\tilde{J}_{2} &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha}T^{(\tilde{\alpha}-1+\gamma)(1-\tilde{\alpha}-\varepsilon)-\beta} \qquad (A.24) \\
&\quad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{(\tilde{\alpha}-2+\gamma)(\tilde{\alpha}+\varepsilon)}dr \\
&\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{(\tilde{\alpha}-2+\gamma)(\tilde{\alpha}+\varepsilon)+1}T^{(\tilde{\alpha}-1+\gamma)(1-\tilde{\alpha}-\varepsilon)-\beta} \qquad (A.25)
\end{aligned}$$

Here $(\tilde{\alpha}-2+\gamma)\tilde{\alpha}+1$ has to be positive and this is true because $\gamma+\tilde{\alpha}-2>-1$ and $\tilde{\alpha}<1$.

If $(\tilde{\alpha}-1+\gamma)(1-\tilde{\alpha})-\beta\le 0$, then

$$\begin{aligned}\tilde{J}_2 &\le C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\alpha}\\ &\quad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{(\tilde{\alpha}-2+\gamma)(\tilde{\alpha}+\varepsilon)+(\tilde{\alpha}-1+\gamma)(1-\tilde{\alpha}-\varepsilon)-\beta}dr\\ &\le C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{\tilde{\alpha}+\varepsilon}(t_1-a)^{\gamma-\beta-\varepsilon}. \end{aligned}\tag{A.26}$$

Finally, putting together (A.8)-(A.13) and (A.19)-(A.26) we obtain the desired result for $\eta=\tilde{\alpha}+\varepsilon$, for ε small enough.

Step 2: Now we study the case $\eta=1-\alpha$. We first study I_1 in (A.8) as follows

$$\begin{aligned}I_1 &\le \left|(t_2-a)^{\alpha}-(t_1-a)^{\alpha}\right|^{\frac{1-\alpha}{\alpha}}\left|(t_2-a)^{\alpha}-(t_1-a)^{\alpha}\right|^{1-\frac{1-\alpha}{\alpha}}\\ &\quad\times\left|\int_a^{t_2}(t_2-r)^{-\beta}(r-a)^{-\alpha}\left(D_{t_2-}^{1-\tilde{\alpha}}\theta_{t_2-}\right)(r)dr\right|\\ &\le C_{\alpha,\beta,\gamma}(t_2-t_1)^{1-\alpha}(t_2-a)^{2\alpha-1}\|\theta\|_{\gamma,[0,T]}\int_a^{t_2}(t_2-r)^{\tilde{\alpha}+\gamma-1-\beta}(r-a)^{-\alpha}dr\\ &\le C_{\alpha,\beta,\gamma}(t_2-t_1)^{1-\alpha}(t_2-a)^{2\alpha+\gamma-2}\|\theta\|_{\gamma,[0,T]}. \end{aligned}\tag{A.27}$$

In (A.10), we introduce $I_{2,2}$. We can see that

$$I_{2,2}\le C_{\alpha,\beta,\gamma}(t_2-t_1)^{1-\alpha}(t_2-a)^{2\alpha+\gamma-2}\|\theta\|_{\gamma,[0,T]}.\tag{A.28}$$

That is, $\lambda=2\alpha+\gamma-2$ in the inequality before the last one in (A.11).

We now deal with $I_{2,1,2}$ in (A.13)

$$\begin{aligned}I_{2,1,2} &= (t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}\left|(t_2-r)^{-\beta}-(t_1-r)^{-\beta}\right|^{1-\alpha}\\ &\quad\times\left|(t_2-r)^{-\beta}-(t_1-r)^{-\beta}\right|^{\alpha}\left|\left(D_{t_1-}^{1-\tilde{\alpha}}\theta_{t_1-}\right)(r)\right|dr\\ &\le C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_1-a)^{\alpha}(t_2-t_1)^{1-\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{-(\beta+1)(1-\alpha)}\\ &\quad\times(t_1-r)^{-\beta\alpha}(t_1-r)^{\tilde{\alpha}+\gamma-1}dr\\ &\le C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_1-a)^{\alpha}(t_2-t_1)^{1-\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{2\alpha+\gamma-3}dr\\ &\le C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-a)^{2\alpha+\gamma-2}. \end{aligned}\tag{A.29}$$

Proceeding as in (A.15), we can obtain

$$J_2\le C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-r)^{(1-\alpha)(\tilde{\alpha}+\gamma-2)}(t_2-r)^{\alpha(\tilde{\alpha}-1+\gamma)}.\tag{A.30}$$

It can happen two things. The first one is that $\alpha(\tilde{\alpha}-1+\gamma)-\beta \geq 0$. In this case,

$$\begin{aligned}\tilde{J}_2 &= (t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_2-r)^{-\beta}J_2\,dr\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-a)^{\alpha}\;T^{\alpha(\tilde{\alpha}-1+\gamma)-\beta}\\ &\qquad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{(1-\alpha)(\tilde{\alpha}+\gamma-2)}dr\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\;T^{\alpha(\tilde{\alpha}-1+\gamma)-\beta}\;(t_1-a)^{(1-\alpha)(\tilde{\alpha}+\gamma-2)+1},\end{aligned}\tag{A.31}$$

and the exponent is positive because $\alpha+\beta+\gamma-3>-1$ and $1-\alpha<1$. For the other case, it means $\alpha(\tilde{\alpha}-1+\gamma)-\beta<0$, we deduce

$$\begin{aligned}\tilde{J}_2 &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-a)^{\alpha}\\ &\qquad\times\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{(1-\alpha)(\tilde{\alpha}+\gamma-2)+\alpha(\tilde{\alpha}-1+\gamma)-\beta}dr\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\;(t_1-a)^{2\alpha+\gamma-2}.\end{aligned}\tag{A.32}$$

Concerning (A.17), we get

$$\begin{aligned}J_{1,2} &\leq C_{\alpha,\beta}(t_1-r)^{\tilde{\alpha}-1}\left|(\theta_r-\theta_{t_2})-(\theta_r-\theta_{t_1})\right|^{\frac{1-\alpha}{\gamma}}\left|(\theta_r-\theta_{t_2})-(\theta_r-\theta_{t_1})\right|^{1-\frac{1-\alpha}{\gamma}}\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_1-r)^{\tilde{\alpha}-1}(t_2-t_1)^{1-\alpha}(t_2-r)^{\gamma+\alpha-1}.\end{aligned}\tag{A.33}$$

Here, first we can also suppose that $\gamma+\alpha-1-\beta\geq 0$ to obtain

$$\begin{aligned}\tilde{J}_{1,2} &= (t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_2-r)^{-\beta}J_{1,2}\,dr\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\;T^{\gamma+\alpha-1-\beta}\;(t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\tilde{\alpha}-1}dr\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\;T^{\gamma+\alpha-1-\beta}\;(t_1-a)^{\tilde{\alpha}}.\end{aligned}\tag{A.34}$$

Now we assume that $\gamma+\alpha-1-\beta<0$ to calculate

$$\begin{aligned}\tilde{J}_{1,2} &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\;(t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\tilde{\alpha}-1+\gamma+\alpha-1-\beta}dr\\ &\leq C_{\alpha,\beta,\gamma}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\;(t_1-a)^{2\alpha+\gamma-2}.\end{aligned}\tag{A.35}$$

For $J_{1,1}$ we have

$$\begin{aligned}\tilde{J}_{1,1} &\leq C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-r)^{\tilde{\alpha}-1+\gamma}(t_1-r)^{\tilde{\alpha}-1}\left|(t_1-r)^{1-\tilde{\alpha}}-(t_2-r)^{1-\tilde{\alpha}}\right|^{1-\alpha}\\ &\qquad\times\left|(t_1-r)^{1-\tilde{\alpha}}-(t_2-r)^{1-\tilde{\alpha}}\right|^{\alpha}\\ &\leq C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-r)^{\tilde{\alpha}-1+\gamma}(t_1-r)^{\tilde{\alpha}-1}(t_2-t_1)^{1-\alpha}(t_1-r)^{-\tilde{\alpha}(1-\alpha)}(t_2-r)^{\alpha(1-\tilde{\alpha})}\\ &= C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_2-r)^{\tilde{\alpha}-1+\gamma+\alpha-\alpha\tilde{\alpha}}(t_1-r)^{\alpha\tilde{\alpha}-1}.\end{aligned}\tag{A.36}$$

If $2\alpha+\gamma-\alpha\tilde{\alpha}-2<0$, then

$$\begin{aligned}\tilde{J}_{1,1} &= (t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_2-r)^{-\beta}J_{1,1}\ dr\\ &\le C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-a)^{\alpha}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\tilde{\alpha}-2+\gamma+\alpha-\beta}dr\\ &\le C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-a)^{2\alpha+\gamma-2}. \end{aligned}\tag{A.37}$$

If $2\alpha+\gamma-\alpha\tilde{\alpha}-2\ge 0$, then we are able to write

$$\begin{aligned}\tilde{J}_{1,1} &\le C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}(t_1-a)^{\alpha}\ T^{2\alpha+\gamma-\alpha\tilde{\alpha}-2}\int_a^{t_1}(r-a)^{-\alpha}(t_1-r)^{\alpha\tilde{\alpha}-1}dr\\ &\le C_{\alpha,\beta}\|\theta\|_{\gamma,[0,T]}(t_2-t_1)^{1-\alpha}\ T^{2\alpha+\gamma-\alpha\tilde{\alpha}-2}\ (t_1-a)^{\alpha\tilde{\alpha}}. \end{aligned}\tag{A.38}$$

Finally, putting (A.27)-(A.38) in (A.8) we obtain the desired result for $\eta=1-\alpha$, which, together with Step 1, implies that the proof is complete. □

Proof of Lemma 3.2. In order to prove the result we will see that, for $0\le t_1<t_2\le T$ and $\varepsilon>0$ small enough,

$$\begin{aligned}&|\Xi(t_2)-\Xi(t_1)|\\ &\le C_{\alpha,\beta,g}(t_2-t_1)^{1-\alpha}\ \left[T^{\beta}+T^{\alpha+\beta-1}\|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]}\right]+C_{\alpha,\beta,g}(t_2-t_1)^{\beta}\ T^{1-\alpha}\\ &\quad+C_{\alpha,\beta,g}(t_2-t_1)^{2\alpha-1+\varepsilon}\ T^{1+\beta-2\alpha-\varepsilon}\|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]}\\ &\quad+C_{\alpha,\beta,g}(t_2-t_1)^{\alpha+\beta-1}\ T^{1-\alpha}\|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]}, \end{aligned}\tag{A.39}$$

holds. To do so, we decompose the following difference into three parts:

$$|\Xi(t_2)-\Xi(t_1)|\le I_1+I_2+I_3,\tag{A.40}$$

where

$$\begin{aligned}I_1 &= \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right|\int_0^{t_2}(t_2-s)^{\beta-1}\,|g(s,\varrho_s)|\,ds,\\ I_2 &= t_1^{1-\alpha}\int_0^{t_1}\left|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right|\,|g(s,\varrho_s)|\,ds,\\ I_3 &= t_1^{1-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,|g(s,\varrho_s)|\,ds. \end{aligned}$$

The hypothesis on ϱ and the fact that g has linear growth imply

$$\begin{aligned}I_1 &\le C_g(t_2-t_1)^{1-\alpha}\int_0^{t_2}(t_2-s)^{\beta-1}(1+|\varrho_s|)ds\\ &= C_g(t_2-t_1)^{1-\alpha}\left[\int_0^{t_2}(t_2-s)^{\beta-1}ds+\int_0^{t_2}(t_2-s)^{\beta-1}s^{\alpha-1}\left|s^{1-\alpha}\varrho_s\right|ds\right]\\ &\le C_g(t_2-t_1)^{1-\alpha}\left[\frac{1}{\beta}t_2^{\beta}+\|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]}\int_0^{t_2}(t_2-s)^{\beta-1}s^{\alpha-1}ds\right]\\ &\le C_g(t_2-t_1)^{1-\alpha}\left[\frac{1}{\beta}T^{\beta}+\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}T^{\alpha+\beta-1}\|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]}\right]. \end{aligned}\tag{A.41}$$

Using that g has linear growth again, we get

$$I_2 \le C_g(I_{2,1} + I_{2,2}), \tag{A.42}$$

with

$$I_{2,1} = t_1^{1-\alpha} \int_0^{t_1} \left|(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right| ds,$$
$$I_{2,2} = t_1^{1-\alpha} \int_0^{t_1} \left|(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right| |\varrho_s|\, ds.$$

The first term can be bounded by

$$I_{2,1} \le t_1^{1-\alpha} \frac{1}{\beta} \left[t_2^{\beta} - t_1^{\beta} + (t_2 - t_1)^{\beta}\right] \le 2\frac{1}{\beta} T^{1-\alpha}(t_2 - t_1)^{\beta}. \tag{A.43}$$

To deal with the other term, we apply the mean value theorem and the fact that $2\alpha - \beta < 1$ in order to see that there exists $\varepsilon > 0$ small enough such that

$$\begin{aligned}
I_{2,2} &\le t_1^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} \int_0^{t_1} \left|(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right| s^{\alpha-1} ds \\
&= t_1^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} \int_0^{t_1} \left|(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right|^{2\alpha-1+\varepsilon} \\
&\qquad \times \left|(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right|^{2-2\alpha-\varepsilon} s^{\alpha-1}\, ds \\
&\le C_{\alpha,\beta}\, t_1^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} (t_2 - t_1)^{2\alpha-1+\varepsilon} \\
&\qquad \times \int_0^{t_1} (t_1 - s)^{(\beta-2)(2\alpha-1+\varepsilon)+(\beta-1)(2-2\alpha-\varepsilon)} s^{\alpha-1}\, ds \\
&= C_{\alpha,\beta}\, t_1^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} (t_2 - t_1)^{2\alpha-1+\varepsilon} \int_0^{t_1} (t_1 - s)^{\beta-2\alpha-\varepsilon} s^{\alpha-1}\, ds \\
&\le C_{\alpha,\beta}\, t_1^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} (t_2 - t_1)^{2\alpha-1+\varepsilon} \frac{\Gamma(\beta - 2\alpha - \varepsilon + 1)\Gamma(\alpha)}{\Gamma(\beta - \alpha - \varepsilon + 1)} t_1^{\beta-\alpha-\varepsilon} \\
&\le C_{\alpha,\beta} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} (t_2 - t_1)^{2\alpha-1+\varepsilon} T^{1+\beta-2\alpha-\varepsilon}.
\end{aligned} \tag{A.44}$$

For the last term, we have

$$\begin{aligned}
I_3 &\le C_g t_1^{1-\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1}(1 + |\varrho_s|) ds \\
&\le C_g t_1^{1-\alpha} \frac{1}{\beta}(t_2 - t_1)^{\beta} + C_g t_1^{1-\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1}(s - t_1)^{\alpha-1} \|s^{1-\alpha}\varrho_s\|_{\infty,[0,T]} ds \\
&\le C_g t_1^{1-\alpha} \frac{1}{\beta}(t_2 - t_1)^{\beta} + C_g t_1^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}\, (t_2 - t_1)^{\alpha+\beta-1} \\
&\le C_g \left[T_1^{1-\alpha} \frac{1}{\beta}(t_2 - t_1)^{\beta} + T^{1-\alpha} \|t^{1-\alpha}\varrho_t\|_{\infty,[0,T]} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}\, (t_2 - t_1)^{\alpha+\beta-1}\right]
\end{aligned} \tag{A.45}$$

Consequently, (A.40)-(A.45) yield that (A.39) is true and therefore the proof is complete. □

Proof of Lemma 3.3. From Proposition 2.2, for $0 \le a < t \le T$, we have

$$\int_a^t (t-s)^{\alpha-1} \varrho_s d\theta_s = \int_a^t (t-s)^{\alpha-1} d\tilde{\theta}_s,$$

where $\tilde{\theta}_t = \int_a^t \varrho_s d\theta_s$. In the remaining of this proof, we utilize the notation introduced in the proof of Lemma 2.1.

Now we fix $t_1, t_2 \in [a, \tilde{T}]$, $t_1 < t_2$ and $\tilde{T} \in [a, T]$. Therefore, Zähle (Theorem 2.5) and Lemma 2.1 imply

$$|\Phi(t_2) - \Phi(t_1)| = \left| \int_a^{t_2} (t_2 - s)^{\alpha-1} d\tilde{\theta}_s - \int_a^{t_2} (t_1 - s)^{\alpha-1} d\tilde{\theta}_s \right| \le H_1 + H_2, \quad \text{(A.46)}$$

with

$$H_1 = \left| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} d\tilde{\theta}_s \right|,$$
$$H_2 = \left| \int_a^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] d\tilde{\theta}_s \right|.$$

We have already estimate H_1 in Lemma 2.1. Thus, we can establish

$$H_1 \le C_{\alpha,\varepsilon,T} \|\tilde{\theta}\|_{\gamma,[a,\tilde{T}]} (t_2 - t_1)^{\gamma+\alpha-\varepsilon-1}, \quad \text{(A.47)}$$

where $\varepsilon < \alpha + \gamma - 1$.

Changing θ by $\tilde{\theta}$ in the definition of $\eta(a, t_1)$ (see proof of Lemma 2.1) and taking $\tilde{\varepsilon}$ close to $\alpha + \gamma - 1$ such that $\tilde{\varepsilon} < \alpha + \gamma - 1$, we have, by inequality (2.7) and [45] (Corollary 1 of Theorem 11.5),

$$\begin{aligned}
H_2 &= \left| \int_a^{t_1} \left[(t_2 - s)^{\alpha-1} \mathbf{1}_{[0,t_2]}(s) - (t_1 - s)^{\alpha-1} \mathbf{1}_{[0,t_1]}(s) \right] d\tilde{\theta}_s \right| \\
&= \left| \int_0^{T} \left[(t_2 - s)^{\alpha-1} \mathbf{1}_{[0,t_2]}(s) - (t_1 - s)^{\alpha-1} \mathbf{1}_{[0,t_1]}(s) \right] d\eta_s(a, t_1) \right| \\
&\le C_{\alpha,\gamma,\tilde{\varepsilon}} \|\tilde{\theta}\|_{\gamma,[a,t_1]} (t_1 - a)^{\gamma+\alpha-\tilde{\varepsilon}-1} \\
&\qquad \times \left[\int_0^T \left| D_{0+}^{\alpha-\tilde{\varepsilon}} \left((t_2 - s)^{\alpha-1} \mathbf{1}_{[0,t_2]}(s) - (t_1 - s)^{\alpha-1} \mathbf{1}_{[0,t_1]}(s) \right) \right| ds \right] \\
&\le C_{\alpha,\gamma,\tilde{\varepsilon},T} \|\tilde{\theta}\|_{\gamma,[a,\tilde{T}]} (t_1 - a)^{\gamma+\alpha-\tilde{\varepsilon}-1} T^{1-1/p} \\
&\qquad \times \left[\int_0^T \left| (t_2 - s)^{\tilde{\varepsilon}-1} \mathbf{1}_{[0,t_2]}(s) - (t_1 - s)^{\tilde{\varepsilon}-1} \mathbf{1}_{[0,t_1]}(s) \right|^p ds \right]^{1/p} \\
&\le C_{\alpha,\gamma,\tilde{\varepsilon},T} \|\tilde{\theta}\|_{\gamma,[a,\tilde{T}]} (t_1 - a)^{\gamma+\alpha-\tilde{\varepsilon}-1} T^{1-1/p} \left(H_{2,1} + H_{2,2} \right), \qquad \text{(A.48)}
\end{aligned}$$

with $1 < p < \frac{1}{1-\tilde{\varepsilon}}$,

$$H_{2,1} = \left[\int_{t_1}^{t_2} (t_2 - s)^{(\tilde{\varepsilon}-1)p} ds \right]^{1/p},$$

and

$$H_{2,2} = \left[\int_0^{t_1} \left| (t_2 - s)^{\tilde{\varepsilon}-1} - (t_1 - s)^{\tilde{\varepsilon}-1} \right|^p ds \right]^{1/p}.$$

It is not difficult to check that

$$H_{2,1} \leq C_{\tilde{\varepsilon}} \left| t_2 - t_1 \right|^{\tilde{\varepsilon}+\frac{1}{p}-1} \tag{A.49}$$

and the mean value theorem, for $\delta \in (0, \tilde{\varepsilon})$ and $p < (1 + \delta - \tilde{\varepsilon})^{-1}$, yields

$$\begin{aligned} H_{2,2} &= \left[\int_0^{t_1} \left| (t_2 - s)^{\tilde{\varepsilon}-1} - (t_1 - s)^{\tilde{\varepsilon}-1} \right|^{p\delta} \left| (t_2 - s)^{\tilde{\varepsilon}-1} - (t_1 - s)^{\tilde{\varepsilon}-1} \right|^{p(1-\delta)} ds \right]^{1/p} \\ &\leq C_{\tilde{\varepsilon}} (t_2 - t_1)^{\delta} \left[\int_0^{t_1} (t_1 - s)^{p(\tilde{\varepsilon}-\delta-1)} ds \right]^{1/p} \\ &\leq C_{\tilde{\varepsilon}} (t_2 - t_1)^{\delta} t_1^{\tilde{\varepsilon}-\delta-1+\frac{1}{p}}. \end{aligned}$$

We can choose p close enough to 1 such that $\tilde{\varepsilon} - \delta - 1 + \frac{1}{p} > 0$. Thus, (A.46)-(A.49), together with last inequality, allow us to finish the proof. □

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