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MÀSTER DE MATEMÀTICA AVANÇADA

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

MINIMAL DISCREPANCY POINTS ON THE SPHERE

Author: Pol Ribera Baraut

Supervisor: Dr. Jordi Marzo Sánchez
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Abstract

Let $\mathbb{S}^k = \{x \in \mathbb{R}^{k+1};\ \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^{k+1}$ and consider the normalized surface area measure σ^*. It is well known that a set of n points $x_1, \dots, x_n \in \mathbb{S}^k$ is asymptotically uniformly distributed, i.e., the probability measure $\frac{1}{n}\sum_{j=1}^n \delta_{x_j}$ converges in the weak-* topology to σ^*, if and only if the spherical cap discrepancy of the set $P = \{x_1, \dots, x_n\}$, defined as

$$\mathbb{D}_n(P) = \sup_{C(x,t)\subset\mathbb{S}^k} \left|\operatorname{card}(P \cap C(x,t)) - n\sigma^*(C(x,t))\right|,$$

where

$$C(x,t) = \{y \in \mathbb{S}^k;\ \langle x, y\rangle \le t\}$$

is a spherical cap on $\mathbb{S}^k$ with $x \in \mathbb{S}^k$ and $-1 \le t \le 1$, converges to zero when $n \to \infty$. It is therefore natural to consider the velocity of this convergence as a measure of the distribution of the sets P.

In a couple of papers from 1984, J. Beck established the following results, which give the best bounds known up to now, [5, 6]:

- There exist n-element sets of points $P \subset \mathbb{S}^k$ such that

$$\mathbb{D}_n(P) \lesssim n^{\frac{1}{2}-\frac{1}{2k}}\sqrt{\log n}.$$

- For any n-element set of points $P \subset \mathbb{S}^k$

$$\mathbb{D}_n(P) \gtrsim n^{\frac{1}{2}-\frac{1}{2k}}.$$

It is not known if any of these bounds is sharp.

The lower bound uses Fourier analysis and the upper bound some random configurations in regular area partitions of the sphere. The main objective of this master thesis is to study J. Beck's work and the "almost tight" examples obtained through determinantal point processes [9].

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Chapter 1

Introduction

Consider the following problem: which set of n points on the k-dimensional unit sphere $\mathbb{S}^k$ is better distributed? It seems to be clear that, in $\mathbb{S}^1$, it can be the set of vertices of a regular n-gon, but what happens if $k > 1$?

One way to study the distribution is to study the spherical cap discrepancy function, defined as follows: given an n-element set of points P on $\mathbb{S}^k$, its spherical cap discrepancy is defined as

$$\mathbb{D}_n(P) = \sup_{C(x,t)\subset\mathbb{S}^k} \big|Z(P,x,t) - n\sigma^*(C(x,t))\big|,$$

where

$$C(x,t) = \{y \in \mathbb{S}^k;\ \langle x, y\rangle \leq t\}$$

is a spherical cap on $\mathbb{S}^k$, σ^* the normalized surface area on $\mathbb{S}^k$ and $Z(P,x,t)$ denotes the number of points of P that lie in $C(x,t)$. So, to solve the problem, we must find the set of points P on $\mathbb{S}^k$ with minimum spherical cap discrepancy. Unfortunately, this is not possible in practice so one alternative is to study the asymptotic behaviour of this function as we increase the number of points.

In 1984, the Hungarian mathematician Jószef Beck, in a couple of papers, [5, 6], gave the best bounds known up to now for the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$:

- There exist n-element sets of points $P \subset \mathbb{S}^k$ such that

$$\mathbb{D}_n(P) \lesssim n^{\frac{1}{2}-\frac{1}{2k}}\sqrt{\log n}.$$

- For any n-element set of points $P \subset \mathbb{S}^k$

$$\mathbb{D}_n(P) \gtrsim n^{\frac{1}{2}-\frac{1}{2k}}.$$

It is not known if any of these bounds is sharp. These results are presented in the text as Theorem 3.2.1 and Theorem 2.2.4. The lower bound result is an immediate consequence of another result, due to J. Beck, concerning the so-called L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$, defined by: given an n-element set of points P on $\mathbb{S}^k$, its L^2-discrepancy is defined as

$$\mathbb{D}_n(P)_2 = \int_{-1}^{1} \left(\frac{1}{\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \big(Z(P,x,t) - n\sigma^*(C(x,t))\big)^2 \, d\sigma(x) \right) dt.$$

We will study also an upper bound concerning the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$, due to K. B. Stolarsky, proved in 1973. The results are the following:

- There exist n-element sets of points $P \subset \mathbb{S}^k$ such that
$$\mathbb{D}_n(P)_2 \lesssim n^{1-\frac{1}{k}}.$$
- For any n-element set of points $P \subset \mathbb{S}^k$
$$\mathbb{D}_n(P)_2 \gtrsim n^{1-\frac{1}{k}}.$$

It is known that the sets of minimal spherical cap discrepancy have L^2-discrepancy of order $n^{1-\frac{1}{k}}$. These results are presented in the text as Theorem 3.2.2 and Theorem 2.2.3. To prove these results we will use Stolarky's Invariance Principle:

- For any n-element set of points $P \subset \mathbb{S}^k$
$$\sum_{\substack{1 \le i < j \le n \\ x_i, x_j \in P}} \|x_i - x_j\| + \mathbb{D}_n(P)_2 = \frac{n^2}{2\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \|x_0 - x\| \, d\sigma(x),$$
where $x_0 = (1, 0, \dots, 0) \in \mathbb{R}^{k+1}$.

From this formula we deduce that the problem of minimising the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$ is equivalent to the problem of maximising the sum of all Euclidean distances between the points of an n-element set on $\mathbb{S}^k$.

These are the main results of the thesis. The main objective is to study J. Beck's work, to study the proofs of these results and the techniques used in them, and to get upper bounds for the "almost tight" examples obtained through determinantal point processes.

Chapter 2 is entirely dedicated to study the lower bound. The proof uses Fourier analysis techniques. It is based on what is called J. Beck's amplification method. Basically, the discrepancy of an n-element set of points on $\mathbb{S}^k$ can we written as a convolution of two functions, a geometric part and a measure part. Using Plancherel's identity, we can separate the geometric part of this convolution from the measure part, and study them separately. To study all this in detail, we will recall some basic notions on Fourier analysis and special function to provide the necessary background.

Chapter 3 is entirely dedicated to study the upper bound. The proof is based in a technique called jittered sampling. Jittered sampling is a way of producing well-separated semi-random sets of points. It applies in many different geometric settings but we will use it on spheres. Independent and uniformly distributed random sets behave very bad, in terms of well-distribution, while constructing sets with good properties is a very difficult task. Jittered sampling consists on taking a partition of our space into pieces with equal volume and almost equal diameter and then taking a point uniformly random in each of the pieces, independently of the others. In our setting, we will use regular area partitions of $\mathbb{S}^k$. This type of partitions were used by K. B. Stolarsky in 1973, but he only affirmed the existence, without giving a construction nor a proof. In 1984, J. Beck quote K. B. Stolarsky and since then many other mathematicians keep quoting them without having the existence ensured. It was not until 2002 that U. Feige and G. Schechtman gave a complete construction.

To get upper bounds for the discrepancy one needs to construct examples. J. Beck used the jittered sampling but it has been recently proved that one can use other random configurations, [9], given by the so-called determinantal point processes. In Chapter 4 we will introduce the concept of determinantal point process. We will follow [4]. The notion of determinantal point process was first introduced by Odile Macchi, a French physicist and mathematician, in 1975, as a way to model fermions in quantum mechanics. This type of point processes also arise surprisingly often in random matrix theory and combinatorics. We will see, following [2], that jittered sampling is a special case of determinantal point process. This means that J. Beck's approach on the proof of the upper bound can also be understood as the new approaches that use determinantal point process. These new methodologies open the problem of finding the optimal determinantal point process to try to obtain better upper bounds that the ones given by J. Beck. Finally, we will study the spherical ensemble, an example of determinantal point process in $\mathbb{S}^2$, and we will see that the behaviour of its spherical cap discrepancy attains the best upper bound known up to now studied in the previous chapters.

Chapter 2

Lower bounds

In this chapter we will present the best lower bounds known up to now for the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$ and for the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$. Both results, due to J. Beck, are presented in [7]. The main result is Theorem 2.2.3, which is a result concerning the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$. For this result we will follow the proof presented in [6]. Theorem 2.2.4, which is the result concerning the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$, is an immediate consequence of Theorem 2.2.3. The main techniques used in the proof of Theorem 2.2.3 are results on Fourier analysis and asymptotic properties of special functions, so before starting we will introduce some basic notions on Fourier analysis and some background on special functions: Gamma function, Beta function and Bessel function.

Let $\mathbb{S}^k = \{x \in \mathbb{R}^{k+1};\ \|x\| = 1\}$ be the *k-dimensional unit sphere*, with $\|\cdot\|$ the usual Euclidean distance. Given $x \in \mathbb{S}^k$ and $-1 \le t \le 1$, we denote as $C(x,t)$ the spherical cap $\{y \in \mathbb{S}^k;\ \langle x, y\rangle \le t\}$, where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product, and given an n-element set of points P on $\mathbb{S}^k$, we denote by $Z(P,x,t)$ the number of points of P that lie in the spherical cap $C(x,t)$. We denote by σ the surface area measure on $\mathbb{S}^k$ and by σ^* the normalized surface area measure on $\mathbb{S}^k$. Given a spherical cap $C(x,t)$ we denote its normalized surface area $\sigma^*(C(x,t))$ by $\sigma^*(t)$.

Definition 2.0.1. Let P be an n-element set of points on $\mathbb{S}^k$. We define the *spherical cap discrepancy of P* by

$$\mathbb{D}_n(P) = \sup_{C(x,t)\subset\mathbb{S}^k} \big|Z(P,x,t) - n\sigma^*(t)\big|.$$

We define the *L^2-discrepancy of P* by

$$\mathbb{D}_n(P)_2 = \int_{-1}^{1} \left(\frac{1}{\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \big(Z(P,x,t) - n\sigma^*(t)\big)^2 \, d\sigma(x) \right) dt.$$

Given an n-element set of points P on $\mathbb{S}^k$, the objective is to prove the existence of functions $f(n)$ and $g(n)$ such that $f(n) \lesssim \mathbb{D}_n(P) \lesssim g(n)$ (also $\mathbb{D}_n(P)_2$), where $\lesssim$ ($\gtrsim$) denote smaller (bigger) than something times a constant only depending on k.

2.1 Preliminaries

2.1.1 Basic notions on Fourier analysis

Let us start this section by introducing some basic notions and results on Fourier analysis that will be very useful in the proof of Theorem 2.2.3. We will present all these results without proofs.

Definition 2.1.1. Given a function $f \in L^1(\mathbb{R}^N)$, the *Fourier transform of* f is

$$\widehat{f}(t) = \frac{1}{(\sqrt{2\pi})^N} \int_{\mathbb{R}^N} e^{-i\langle x,t\rangle} f(x)\, dx, \quad t \in \mathbb{R}^N.$$

Definition 2.1.2. Given two functions $f, g \in L^1(\mathbb{R}^N)$, the *convolution of* f *and* g is the function

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\, dy, \quad x \in \mathbb{R}^N.$$

Basic properties concerning the Fourier transform and the convolution operators are the following:

Proposition 2.1.3. *Given two functions $f, g \in L^1(\mathbb{R}^N)$ we have that*

(i) $\widehat{f * g} = \widehat{f}\widehat{g}$,

(ii) $\widehat{fg} = \widehat{f} * \widehat{g}$.

An important result on this field is the well-known Plancherel's identity.

Theorem 2.1.4. *(Plancherel's identity). Given a function $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ we have that*

$$\int_{\mathbb{R}^N} |f(x)|^2 dx = \int_{\mathbb{R}^N} |\widehat{f}(t)|^2\, dt.$$

The proofs of these results can be found in any introductory book in Fourier analysis.

2.1.2 The Gamma function and the Beta function

The Gamma function and its relation with the Beta function will appear in the proof of Theorem 2.2.3, so now we will present these functions and their most important properties.

Definition 2.1.5. Given $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ we define the *Gamma function* by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx.$$

The Gamma function is one of the extensions of the factorial function and this integral converges absolutely under the hypothesis $\mathrm{Re}(z) > 0$.

It is easy to see that $\Gamma(1) = 1$ and using integration by parts one can see that $\Gamma(z+1) = z\Gamma(z)$, therefore, $\Gamma(n) = (n-1)!$ for all non-negative integers. One interesting value of this function is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Definition 2.1.6. Given $x, y \in \mathbb{C}$ with $\mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$, we define the *Beta function* by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

The Beta function, also known as Euler integral of the first type, is a symmetric function, and one of its most important properties is its relation with the Gamma function.

Property 2.1.7. *(Relation between the Beta function and the Gamma function). Given $x, y \in \mathbb{C}$ with $Re(x) > 0,\ Re(y) > 0$, we have the identity*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. We have that

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \left(\int_0^\infty e^{-u}u^{x-1}\,du\right)\left(\int_0^\infty e^{-v}v^{y-1}\,dv\right)\\ &= \int_0^\infty\int_0^\infty e^{-u-v}u^{x-1}v^{y-1}\,dudv.\end{aligned}$$

Now, using the change of variables $u = zt,\ v = z(1-t)$, we get that the absolute value of the Jacobian is z, therefore

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty\int_0^1 e^{-z}(zt)^{x-1}(z(1-t))^{y-1}|J(z,t)|\,dtdz\\ &= \left(\int_0^\infty e^{-z}z^{x+y-1}\,dz\right)\left(\int_0^1 t^{x-1}(1-t)^{y-1}\,dt\right)\\ &= \Gamma(x+y)B(x,y),\end{aligned}$$

as we wanted to see. □

With these results one can prove the following important property of the Gamma function, due to Legendre, which is called Legendre's duplication formula.

Property 2.1.8. *(Legendre's duplication formula). If $z \in \mathbb{C}$ with $Re(z) > 0$, then*

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

Proof. By Property 2.1.7 applied to $x = y = z$, we have that

$$\begin{aligned}\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} &= B(z,z)\\ &= \int_0^1 t^{z-1}(1-t)^{z-1}\,dt.\end{aligned}$$

Using the change $t = \frac{1+u}{2}$,

$$\begin{aligned}\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} &= \frac{1}{2}\int_{-1}^{1}\left(\frac{1+u}{2}\right)^{z-1}\left(\frac{1-u}{2}\right)^{z-1} du\\ &= \frac{1}{2^{2z-1}}\int_{-1}^{1}(1-u^2)^{z-1}\,du.\end{aligned}$$

On the other hand, using the change $t = u^2$ in the Beta function, we obtain

$$B(x,y) = \int_0^1 u^{2x-2}(1-u^2)^{y-1}2u\,du.$$

Taking $x = \frac{1}{2}$ and $y = z$,

$$B\left(\frac{1}{2}, z\right) = 2\int_0^1 (1-u^2)^{z-1}\,du.$$

Hence,

$$\begin{aligned}2^{2z-1}\Gamma(z)\Gamma(z) &= \Gamma(2z)B\left(\frac{1}{2}, z\right)\\ &= \Gamma(2z)\frac{\Gamma(z)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)},\end{aligned}$$

and using that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we obtain the desired identity. □

2.1.3 The Bessel function

We are going to introduce the Bessel function, different expressions of it and an asymptotic expansion, due to Hankel (1869), that will be crucial in the proof of Theorem 2.2.3.

Definition 2.1.9. Given a real or complex variable ν and $z \in \mathbb{C}$ we define the *Bessel function* $J_\nu(z)$ by the series

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\Gamma(\nu+s+1)}\left(\frac{z}{2}\right)^{2s}.$$

Using the M-test one can see that this series converges uniformly on compact sets in the planes of ν and z. Next result provides us the expression of the Bessel function that we are going to use.

Proposition 2.1.10. *If $Re(\nu) > -\frac{1}{2}$, then*

$$J_\nu(z) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

From this expression, using Euler's formula and some easy computations, we obtain

$$J_\nu(z) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_{-1}^{1} \cos(zt)(1-t^2)^{\nu-\frac{1}{2}}\,dt, \quad \mathrm{Re}(\nu) > -\frac{1}{2}. \tag{2.1.1}$$

Proof. Let us start by manipulating the integral in the formula:

$$\begin{aligned}
\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt &= \int_{-1}^{1}\sum_{n=0}^{\infty}\frac{(iz)^n t^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\,dt \\
&= \sum_{n=0}^{\infty}\frac{(iz)^n}{n!}\int_{-1}^{1} t^n(1-t^2)^{\nu-\frac{1}{2}}\,dt \\
&= \sum_{k=0}^{\infty}\frac{(iz)^{2k}}{(2k)!}\int_{-1}^{1} t^{2k}(1-t^2)^{\nu-\frac{1}{2}}\,dt,
\end{aligned}$$

using the well-known Dominated Convergence Theorem and

$$\int_{-1}^{1} t^{2k+1}(1-t^2)^{\nu-\frac{1}{2}}\,dt = 0.$$

Now, using the change $u = t^2$,

$$\begin{aligned}
\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt &= \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}\left(2\int_0^1 u^k(1-u)^{\nu-\frac{1}{2}}\frac{du}{2\sqrt{u}}\right) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}\left(\int_0^1 u^{k-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}\,du\right) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}B\left(k+\frac{1}{2},\nu+\frac{1}{2}\right),
\end{aligned}$$

since $\mathrm{Re}(\nu) > -\frac{1}{2}$, where B is the Beta function. By Property 2.1.7,

$$\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+k+1)}. \tag{2.1.2}$$

Now, using Legendre's duplication formula, Property 2.1.8, we can write

$$\begin{aligned}\Gamma\left(k+\frac{1}{2}\right) &= \frac{\sqrt{\pi}\,\Gamma(2k)}{2^{2k-1}\,\Gamma(k)}\\ &= \frac{\sqrt{\pi}\,\Gamma(2k+1)}{2^{2k}\,\Gamma(k+1)}\\ &= \frac{\sqrt{\pi}(2k)!}{2^{2k}k!}.\end{aligned}$$

The result follows combining last expression and (2.1.2). □

The next result is an asymptotic expansion of the Bessel function. We will present it without proof, since it is quite involve and the methods used are out of the scope of this thesis. The proof can be found in [8], page 133.

Given two functions f and g, the notation $f(x) \sim g(x)$ means that the quotient $\frac{f(x)}{g(x)}$ goes to 1, as x goes to infinity. In this case we say that g is an *asymptotic expansion* of f.

Proposition 2.1.11. *(Hankel). If $x \to \infty$, then*

$$\begin{aligned}J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}\Bigg(&\cos\left(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\sum_{j=0}^{\infty}(-1)^j\frac{A_{2j}(\nu)}{x^{2j}}\\ &-\sin\left(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\sum_{j=0}^{\infty}(-1)^j\frac{A_{2j+1}(\nu)}{x^{2j+1}}\Bigg),\end{aligned}$$

where

$$A_j(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdot\ldots\cdot(4\nu^2-(2j-1)^2)}{j!8^j}.$$

2.2 Main results

We are going to study J. Beck's paper about discrepancies on the sphere, [6]. This paper starts with some historical background and results on the area and then focuses on proving our Theorem 2.2.3, which establishes a lower bound for the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$. First we are going to use this result to prove Theorem 2.2.4 and then we are going to study in detail the proof of the main result in this section, Theorem 2.2.3. The techniques used in the proof are basically Fourier analysis. We are going to follow also [7], J. Beck's and W. W. L. Chen's book to see different approaches at some parts of the proof of Theorem 2.2.3. We will start by Stolarsky's invariance principle (1973), Theorem 2.2.1, due to K. B. Stolarsky, which asserts that the sum of Euclidean distances between points on $\mathbb{S}^k$ plus the its L^2-discrepancy is a constant times the square of the number of points.

Let $U(r)$ denote the surface of the $(k+1)$-dimensional Euclidean ball centered at 0 and of radius $r \geq 0$, i.e., $U(r) = \{x \in \mathbb{R}^{k+1};\ \|x\| = r\}$ and denote by $\mathbb{S}^k$ the case $r = 1$. Let P be a set of n points $z_1, \dots, z_n \in \mathbb{S}^k$, with $n \geq 2$. We define

$$S(n,k,P) = \sum_{1 \leq i < j \leq n} \|z_i - z_j\|,$$

and

$$S(n,k) = \max_{\mathrm{card}(P)=n} S(n,k,P),$$

where $\mathrm{card}(P)$ denotes the cardinality of P. Observe that this maximum is achieved since $S(n,k,\cdot)$ is a continuous function defined on a compact domain. We denote the set of points on $\mathbb{S}^k$ that attains the maximum as P^*.

The next result due to K. B. Stolarsky is the starting point of this thesis.

Theorem 2.2.1. *(Stolarsky's invariance principle).*

$$S(n,k,P) + \int_{-1}^{1} \left(\frac{1}{\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \big(Z(P,x,t) - n\sigma^*(t)\big)^2 \, d\sigma(x) \right) dt = \frac{n^2}{2\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \|x_0 - x\| \, d\sigma(x),$$

where $x_0 = (1,0,\dots,0) \in \mathbb{R}^{k+1}$.

Observe that the right hand-side of Theorem 2.2.1 is a constant times n^2, i.e., if we set

$$c_0(k) = \frac{1}{2\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \|x_0 - x\| \, d\sigma(x),$$

the right hand-side of Theorem 2.2.1 is $c_0(k)n^2$. These constants can be explicitly calculated, for example, $c_0(1) = \frac{2}{\pi}$ or $c_0(2) = \frac{3}{2}$.

Observe also that $c_0(k)n^2 - S(n,k)$ is strictly positive. This is an immediate consequence of Theorem 2.2.1, for P^*, and the fact that we can enlarge a little bit a given spherical cap preserving the number of points of P^* that are in it.

Remark 2.2.2. As a consequence of Stolarky's invariance principle, the set of points on $\mathbb{S}^k$ that maximizes the sum of the Euclidean distances between them is a set of minimal L^2-discrepancy, and thus, is a set of minimal spherical cap discrepancy.

Let us state the main result of this section, Theorem 2.2.3, and let us use it to prove the result concerning the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$, Theorem 2.2.4.

Theorem 2.2.3. *(J. Beck).*

$$c_1(k)n^{1-\frac{1}{k}} < c_0(k)n^2 - S(n,k).$$

Theorem 2.2.4. *(J. Beck). Given an n-element set of points P on the unit sphere $\mathbb{S}^k$, there exists a spherical cap $C(x,t) = \{y \in \mathbb{S}^k;\ \langle x,y\rangle \leq t\} \subset \mathbb{S}^k$, with $x \in \mathbb{S}^k$ and $-1 \leq t \leq 1$, such that*

$$\big|Z(P,x,t) - n\sigma^*(t)\big| > c_2(k)n^{\frac{1}{2}-\frac{1}{2k}}.$$

Proof. (Theorem 2.2.4). Let P be an n-element set of points on $\mathbb{S}^k$. By Theorem 2.2.3, we have that

$$\int_{-1}^{1}\left(\frac{1}{\sigma(\mathbb{S}^k)}\int_{\mathbb{S}^k}\big(Z(P,x,t)-n\sigma^*(t)\big)^2\,d\sigma(x)\right)dt \geq c_0(k)n^2 - S(n,k) > c_1(k)n^{1-\frac{1}{k}}.$$

Now, there exist $x\in\mathbb{S}^k$ and $-1\leq t\leq 1$ such that

$$\big(Z(P,x,t)-n\sigma^*(t)\big)^2 > c_1(k)n^{1-\frac{1}{k}},$$

and therefore,

$$\big|Z(P,x,t)-n\sigma^*(t)\big| > c_2(k)n^{\frac{1}{2}-\frac{1}{2k}},$$

because if

$$\big(Z(P,x,t)-n\sigma^*(t)\big)^2 \leq c_1(k)n^{1-\frac{1}{k}}$$

for all $x\in\mathbb{S}^k$ and $-1\leq t\leq 1$, then we get

$$c_0(k)n^2 - S(n,k) \leq c_1(k)n^{1-\frac{1}{k}},$$

and this contradicts Theorem 2.2.3. □

Once we know that, given an n-element set of points P on $\mathbb{S}^k$, there exists a spherical cap satisfying such an inequality, taking the supremum over all spherical caps on $\mathbb{S}^k$ we get

$$\mathbb{D}_n(P) \gtrsim n^{\frac{1}{2}-\frac{1}{2k}},$$

which is the best lower bound, known up to know, for the spherical cap discrepancy of an n-element set of points P on $\mathbb{S}^k$.

Proof. (Theorem 2.2.3). Let r be such that $\sigma\big(U(r)\big)$ is equal to n, i.e.,

$$r=\left(\frac{\Gamma\left(\frac{k+1}{2}\right)}{(2\pi)^{\frac{k+1}{2}}}\right)^{\frac{1}{k}} n^{\frac{1}{k}} = c_3(k)^{\frac{1}{k}}n^{\frac{1}{k}} = c_4(k)n^{\frac{1}{k}}.$$

Let $P=\{z_1,\dots,z_n\}$ be an n-element set of points on $\mathbb{S}^k$ and let $\tilde{P}=\{\tilde{z}_1,\dots,\tilde{z}_n\}$ be an n-element set of points on $U(r)$. Let us introduce the following measures. For any $H\subset\mathbb{R}^{k+1}$ denote by

$$Z_0(H)=\sum_{z_j\in H}1$$

and

$$Z_0^r(H)=\sum_{\tilde{z}_j\in H}1$$

the number of points of P (and $\tilde{P}$, respectively) that lie in H. We can write also these measures in terms of the characteristic function of H,

$$Z_0(H)=\sum_{j=1}^{n}\chi_H(z_j),\quad Z_0^r(H)=\sum_{j=1}^{n}\chi_H(\tilde{z}_j).$$

For any Lebesgue measurable set $H \subset \mathbb{R}^{k+1}$ denote by σ_0 the normalized surface area of the intersection $H \cap \mathbb{S}^k$, i.e.,

$$\sigma_0(H) = n\sigma^*(H \cap \mathbb{S}^k) = n\frac{\sigma(H \cap \mathbb{S}^k)}{\sigma(\mathbb{S}^k)},$$

and for σ_0^r the surface area of the intersection $H \cap U(r)$, i.e.,

$$\sigma_0^r(H) = \sigma(H \cap U(r)).$$

It is not difficult to see that the conditions of the definition of measure are satisfied: non-negativity, null empty set and countable additivity.

Let $B_q = \{x \in \mathbb{R}^{k+1};\ \|x\| \leq q\}$ be the $(k+1)$-dimensional ball of radius q. Consider the functions

$$F_q = \chi_{B_q} * (dZ_0 - d\sigma_0)$$

and

$$\tilde{F}_q = \chi_{B_q} * (dZ_0^r - d\sigma_0^r)$$

where $*$ denotes the convolution operator. We can rewrite these functions as

$$\begin{aligned}
F_q(x) &= \int_{R^{k+1}} \chi_{B_q}(x-y)(dZ_0 - d\sigma_0)(y)\\
&= \int_{x+B_q} (dZ_0 - d\sigma_0)(y)\\
&= (Z_0 - \sigma_0)(x + B_q)\\
&= \sum_{z_j \in x+B_q} 1 - n\sigma^*((x+B_q) \cap \mathbb{S}^k)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{F}_q &= \int_{R^{k+1}} \chi_{B_q}(x-y)(dZ_0^r - d\sigma_0^r)(y)\\
&= \int_{x+B_q} (dZ_0^r - d\sigma_0^r)(y)\\
&= (Z_0^r - \sigma_0^r)(x + B_q)\\
&= \sum_{\tilde{z}_j \in x+B_q} 1 - \sigma((x+B_q) \cap U(r)),
\end{aligned}$$

since $B_q = -B_q$, where $x + B_q$ denotes the translated image of the ball B_q by the vector x.

The next step is a key step in this proof. We are going to see that

$$\int_{\delta(k)}^{\frac{1}{2}} \left(\frac{1}{r^{k+1}} \int_{\mathbb{R}^{k+1}} \tilde{F}_{\alpha r}^2(x)\, dx\right) d\alpha \lesssim \int_{-1}^{1} \left(\frac{1}{\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \left(Z(P,x,t) - n\sigma^*(t)\right)^2 d\sigma(x)\right) dt, \tag{2.2.1}$$

where $\delta(k) > 0$ is a small constant depending only on k that we will specify later, at the end of the proof. In fact, what we are going to see is that

$$\int_1^2 \int_{\mathbb{R}^{k+1}} F_s^2(x)\, dx ds \lesssim \int_{-1}^1 \left(\frac{1}{\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \left(Z(P,x,t) - n\sigma^*(t)\right)^2 d\sigma(x) \right) dt, \quad (2.2.2)$$

and then we are going to pass from the left hand-side of (2.2.1) to the left hand-side of (2.2.2). Recall that

$$F_s(x) = \sum_{z_j \in x+B_s} 1 - n\sigma^*((x+B_s)\cap \mathbb{S}^k).$$

Observe that $(x+B_s)\cap\mathbb{S}^k$ is a spherical cap $C(v,t)\subset\mathbb{S}^k$ with $t = t(\|x\|, s)$ given by the expression

$$t = t(\|x\|, s) = -\frac{\langle x, y\rangle}{\|x\|},$$

where $y \in \mathbb{S}^k$ such that $\|x - y\| = s$, and

$$v = -\frac{x}{\|x\|},$$

whenever $-1+s < \|x\| < 1+s$ (draw the case $k=1$ to clarify). We have to see that t is well-defined. If we take another $y' \in \mathbb{S}^k$ with $\|x-y'\| = s$, then

$$\begin{aligned} \|x-y\| = \|x-y'\| &\Rightarrow \langle x-y, x-y\rangle = \langle x-y', x-y'\rangle \\ &\Rightarrow \langle x, y\rangle = \langle x, y'\rangle, \end{aligned}$$

so t does not depend on the choice of y.

We are going to restrict all possible radius to $1 < s < 2$ because it is what we need in (2.2.2). Given $x \in \mathbb{R}^{k+1}$ we can write $x = \rho w$ with $\rho \geq 0$ and $w \in \mathbb{S}^k$, so $\rho = \|x\|$. With this notation, $t = t(\rho, s)$ and $y = -w$, we get

$$F_s(\rho w) = Z(P, -w, t(\rho,s)) - n\sigma^*(t(\rho,s)).$$

Now,

$$\begin{aligned} \int_1^2 \int_{\mathbb{R}^{k+1}} F_s^2(x)\, dx ds &= \int_1^2 \int_{s-1}^{s+1} \rho^k \int_{\mathbb{S}^k} F_s^2(\rho w)\, d\sigma(w) d\rho ds \\ &= \int_1^2 \int_{s-1}^{s+1} \rho^k \int_{\mathbb{S}^k} \left(Z(P,-w,t(\rho,s)) - n\sigma^*(t(\rho,s))\right)^2 d\sigma(w) d\rho ds \\ &= \int_1^2 \int_{s-1}^{s+1} \rho^k \int_{\mathbb{S}^k} \left(Z(P,w,t(\rho,s)) - n\sigma^*(t(\rho,s))\right)^2 d\sigma(w) d\rho ds. \end{aligned} \quad (2.2.3)$$

Let us see the form of $t = t(\rho, s)$. Since

$$\begin{aligned} C(v,t) &= \{y \in \mathbb{S}^k;\ \langle v, y\rangle \leq t\} \\ &= \{y \in \mathbb{S}^k;\ \|v - y\| \geq \arccos t\}, \end{aligned}$$

and by some geometrical arguments, we get

$$\left(\sqrt{1-t^2}\right)^2 + (\rho + t^2)^2 = s^2,$$

i.e.,

$$t = t(\rho, s) = \frac{s^2 - 1 - \rho^2}{2\rho}.$$

Recall that we have the restrictions $1 < s < 2$ and $s - 1 < \rho < s + 1$. We are going to see that the function

$$\frac{\partial t}{\partial \rho}(\rho, s) = \frac{1 - s^2 - \rho^2}{2\rho^2}$$

has no extreme points in the region A defined as

$$A = \{(\rho, s);\ 1 < s < 2, s - 1 < \rho < s + 1\}.$$

To simplify the notation, we write $f = \frac{\partial t}{\partial \rho}$. Since $s > 1$, we can easily chech that

$$\frac{\partial f}{\partial s}(\rho, s) = -\frac{s}{\rho^2} \neq 0,$$

$$f(\rho, 1) = -\frac{1}{2} \neq 0, \quad f(\rho, 2) = -\frac{3 + \rho^2}{2\rho^2} < 0,$$

$$f(s-1, s) = \frac{s(1-s)}{(s-1)^2} < 0, \quad f(s+1, s) = -\frac{s}{s+1} < 0.$$

Hence, we conclude that

$$\frac{\partial t}{\partial \rho}(\rho, s) < 0 \ \text{ in } \ A,$$

therefore

$$\inf_{\substack{1<s<2 \\ s-1<\rho<s+1}} \left| \frac{\partial t}{\partial \rho}(\rho, s) \right| \geq c > 0,$$

for some constant c. Going back to (2.2.3),

$$\begin{aligned}\int_1^2 \int_{\mathbb{R}^{k+1}} F_s^2(x)\, dx ds &\leq \int_1^2 (s+1)^k \int_{s-1}^{s+1} \int_{\mathbb{S}^k} \left(Z(P, w, t(\rho, s) - n\sigma^*(t(\rho, s)))\right)^2 d\sigma(w) d\rho ds \\ &\leq 3^k \int_1^2 \int_{s-1}^{s+1} \int_{\mathbb{S}^k} \left(Z(P, w, t(\rho, s) - n\sigma^*(t(\rho, s)))\right)^2 \frac{\frac{\partial t}{\partial \rho}(\rho, s)}{\frac{\partial t}{\partial \rho}(\rho, s)} d\sigma(w) d\rho ds.\end{aligned}$$

Applying the change of variables $t(\rho, s) = T$ with

$$t(s+1, s) = -1, \quad t(s-1, s) = 1, \quad \frac{\partial t}{\partial \rho} d\rho = dT$$

and that $\frac{\partial t}{\partial \rho}(\rho, s) = \Psi(T, s) < 0$ with

$$\inf_{\substack{1<s<2 \\ -1<T<1}} |\Psi(T, s)| \geq c > 0,$$

we obtain

$$\int_1^2 \int_{\mathbb{R}^{k+1}} F_s^2(x)\, dx ds \leq 3^k \int_1^2 \int_1^{-1} \int_{\mathbb{S}^k} \big(Z(P,w,T) - n\sigma^*(T)\big)^2 \frac{1}{\Psi(T,s)}\, d\sigma(w) dT ds$$
$$\leq \frac{3^k \sigma(\mathbb{S}^k)}{c} \int_{-1}^1 \left(\frac{1}{\sigma(\mathbb{S}^k)} \int_{\mathbb{S}^k} \big(Z(P,w,T) - n\sigma^(T)\big)^2\, d\sigma(w) \right) dT$$

so we have seen (2.2.2). Using that for all $q \geq 0$,

$$F_s(x) = \tilde{F}_{sq}(qx), \tag{2.2.4}$$

we can get

$$\int_{\delta(k)}^{\frac{1}{2}} \left(\frac{1}{r^{k+1}} \int_{\mathbb{R}^{k+1}} \tilde{F}^2_{\alpha r}(x)\, dx \right) d\alpha = \delta(k) \int_1^{\frac{1}{2\delta(k)}} \left(\frac{1}{r^{k+1}} \int_{\mathbb{R}^{k+1}} \tilde{F}^2_{\delta(k) r s}(x)\, dx \right) ds$$
$$= \delta(k)^{k+2} \int_1^{\frac{1}{2\delta(k)}} \left(\int_{\mathbb{R}^{k+1}} \tilde{F}_{\delta(k) r s}(\delta(k) r y)\, dy \right) ds$$
$$= \delta(k)^{k+2} \int_1^{\frac{1}{2\delta(k)}} \int_{\mathbb{R}^{k+1}} F_s^2(y)\, dy ds,$$

where in the first equality we have done the change of variables $\alpha = \delta(k)s$ and in the second one we have done the change of variables $\delta(k)ry = x$.

Since $\delta(k) > 0$ is going to be a very small constant depending only k, and since we have some freedom to adjust it, if $\delta(k) < \frac{1}{4}$, then we have that all the spherical caps that can be formed as an intersection of balls of radius in $\big(1, \frac{1}{2\delta(k)}\big)$ with $\mathbb{S}^k$ can be also formed with balls of radius in $(1, 2)$, so the integrals

$$\int_1^2 \int_{\mathbb{R}^{k+1}} F_s^2(y)\, dy ds \quad \text{and} \quad \int_1^{\frac{1}{2\delta(k)}} \int_{\mathbb{R}^{k+1}} F_s^2(y)\, dy ds$$

are essentially the same. So, we have seen (2.2.1).

Recall that now we only have to see that

$$\int_{\delta(k)}^{\frac{1}{2}} \left(\int_{\mathbb{R}^{k+1}} \tilde{F}^2_{\alpha r}(x)\, dx \right) d\alpha \gtrsim n^{1-\frac{1}{k}} r^{k+1} = c_5(k) n^2, \tag{2.2.5}$$

because of the choice of r. We are going to use the Fourier transform techniques studied before. From this point on, we are only going to use $\tilde{F}$, Z_0^r and σ_0^r, so to simplify the notation we are not going to write the r nor the $\sim$ in the notation. By Proposition 2.1.3

$$\widehat{F}_q = \widehat{\chi_{B_q} * (dZ_0 - d\sigma_0)} = \widehat{\chi_{B_q}}(\widehat{dZ_0 - d\sigma_0}), \tag{2.2.6}$$

and by Plancherel's identity, Theorem 2.1.4,

$$\int_{\mathbb{R}^{k+1}} F_q^2(x)\,dx = \int_{\mathbb{R}^{k+1}} |\widehat{F}_q(t)|^2\,dt, \tag{2.2.7}$$

hence,

$$\int_{\mathbb{R}^{k+1}} F_q^2(x)\,dx = \int_{\mathbb{R}^{k+1}} |\widehat{\chi_{B_q}}(t)|^2 |\widehat{(dZ_0 - d\sigma_0)}(t)|^2\,dt. \tag{2.2.8}$$

We will denote $\Phi(t) = \widehat{(dZ_0 - d\sigma_0)}(t)$ to simplify the notation.

We want to, somehow, split (2.2.8) into two parts, one involving $\widehat{\chi_{B_q}}$ (geometric part) and the other Φ (measure part). Let us start proving that for a large enough constant $c_6(k) > 0$, that we will determine later,

$$\int_{\|t\|<c_6(k)} |\Phi(t)|^2\,dt \gtrsim n. \tag{2.2.9}$$

The geometric background of (2.2.9) is the apparently trivial fact that any spherical cap on $U(r)$ with area $\frac{1}{2}$ has discrepancy $\geq \frac{1}{2}$. Let

$$\begin{aligned} h(x) &= \frac{1}{(2\pi)^{k+1}} \prod_{j=1}^{k+1} \left(2\frac{\sin(bx_j)}{x_j}\right)^2 \\ &= \left(\frac{1}{(\sqrt{2\pi})^{k+1}} \prod_{j=1}^{k+1} \left(2\frac{\sin(bx_j)}{x_j}\right)\right)^2, \end{aligned}$$

where the parameter b will be fixed later. By Proposition 2.1.3 we know that $\widehat{h}$ will be the convolution of the Fourier transform of

$$\frac{1}{(\sqrt{2\pi})^{k+1}} \prod_{j=1}^{k+1} \left(2\frac{\sin(bx_j)}{x_j}\right)$$

with itself. Observe that

$$\begin{aligned} \widehat{\chi_{[-b,b]^{k+1}}}(t) &= \frac{1}{(\sqrt{2\pi})^{k+1}} \int_{\mathbb{R}^{k+1}} e^{-i\langle x,t\rangle} \chi_{[-b,b]^{k+1}}(x)\,dx \\ &= \frac{1}{(\sqrt{2\pi})^{k+1}} \prod_{j=1}^{k+1} \left(\int_{-b}^{b} e^{-ix_jt_j}\,dx_j\right) \\ &= \frac{1}{(\sqrt{2\pi})^{k+1}} \prod_{j=1}^{k+1} \left(2\frac{\sin(bt_j)}{t_j}\right), \end{aligned}$$

using Euler's formula and that $\cos(x)$ is an even function. Hence, the Fourier transform $\widehat{h}$ of h is the convolution of the characteristic function of the cube $[-b,b]^{k+1}$ with itself, i.e.,

$$\begin{aligned}\widehat{h}(t) &= (\chi_{[-b,b]^{k+1}} * \chi_{[-b,b]^{k+1}})(t)\\ &= \prod_{j=1}^{k+1}\left(\int_{\mathbb{R}} \chi_{[-b+t_j,b+t_j]\cap[-b,b]}(x_j)\,dx_j\right)\\ &= \prod_{j=1}^{k+1}\left(2b-|t_j|\right)^{+},\end{aligned}$$

where $(y)^{+} = y$ if $y > 0$ and 0 otherwise. Let $H(x)$ be the convolution defined by

$$H(x) = (h * (dZ_0 - d\sigma_0))(x).$$

By (2.1.3),

$$\widehat{H}(t) = \widehat{h}(t)\widehat{(dZ_0 - d\sigma_0)}(t)$$

$$= \widehat{h}(t)\Phi(t),$$

and by Plancherel's identity, Theorem 2.1.4, and the expression above

$$\begin{aligned}\int_{\mathbb{R}^{k+1}} H^2(x)\,dx &= \int_{\mathbb{R}^{k+1}} |\widehat{H}(t)|^2\,dt\\ &= \int_{\mathbb{R}^{k+1}} |\widehat{h}(t)|^2|\Phi(t)|^2\,dt\\ &= \int_{\mathbb{R}^{k+1}} \left(\prod_{j=1}^{k+1}\left(2b-|t_j|\right)^{+}\right)^2 |\Phi(t)|^2\,dt,\end{aligned}$$

and hence

$$\int_{\mathbb{R}^{k+1}} H^2(x)\,dx \le (2b)^{2(k+1)} \int_{\|t\|<2b\sqrt{k+1}} |\Phi(t)|^2\,dt, \tag{2.2.10}$$

because the product inside the integral is different from zero if and only if $|t_j| < 2b$ for all j, hence $\|t\| = \sqrt{t_1^2 + \ldots t_{k+1}^2} < 2b\sqrt{k+1}$, and $\left(2b-|t_j|\right)^{+} \le 2b$ for all j.

On the other hand, since $4\sin^2(x) > x^2$ whenever $x \in [-1,1]$, we have that

$$h(x) = \frac{1}{(2\pi)^{k+1}}\prod_{j=1}^{k+1}\left(2\frac{\sin(bx_j)}{x_j}\right)^2 > \frac{1}{(2\pi)^{k+1}} b^{2(k+1)}$$

whenever $x \in \left[-\frac{1}{b},\frac{1}{b}\right]^{k+1}$.

Given a point $x \in \left[-\frac{1}{b}, \frac{1}{b}\right]^{k+1}$ and $y \in \mathbb{S}^k$ we have that

$$h(x-y) = \frac{1}{(2\pi)^{k+1}} \prod_{j=1}^{k+1} \left(2\frac{\sin(b(x_j - y_j))}{x_j - y_j} \right)^2.$$

Observe that the difference in the denominator may cancel. In this case there is no problem because the sine in the numerator fixes it. However, since we have to take bounds, we need to control de differences $x_j - y_j$ for all j, and there are a lot of differences cases. We are going to cover the case where only one of the differences $x_j - y_j$ is very small (can be 0). Observe,

$$\begin{aligned}
\int_{U(r)} h(x-y)\, d\sigma(y) &= \int_{U(r)} \frac{1}{(2\pi)^{k+1}} \prod_{j=1}^{k+1} \left(2\frac{\sin(b(x_j - y_j))}{x_j - y_j} \right)^2 d\sigma(y) \\
&\leq \int_{U(r)} \frac{4^{k+1} b^{2(k+1)}}{(2\pi)^{k+1}} \prod_{j=1}^{k+1} \left(\frac{\sin(b|x_j - y_j|)}{b|x_j - y_j|} \right)^2 d\sigma(y) \\
&\leq \left(\frac{2}{\pi}\right)^{k+1} b^{2(k+1)} \int_{U(r)} \prod_{j=1}^{k} \left(\frac{1}{b\left|\frac{1}{b} - \frac{r}{\sqrt{k+1}}\right|} \right)^2 d\sigma(y) \\
&= \left(\frac{2}{\pi}\right)^{k+1} b^{2(k+1)} \int_{U(r)} \prod_{j=1}^{k} \left(\frac{1}{\left(1 - \frac{br}{\sqrt{k+1}}\right)^2} \right) d\sigma(y).
\end{aligned}$$

The small difference is the one that we remove using the sine in the numerator, bounding the fraction by 1. Then, we have k terms remaining and we use that the nearest point on $U(r)$ to a point in $\left[-\frac{1}{b}, \frac{1}{b}\right]^{k+1}$ satisfies

$$|x_j - y_j| = \left| \frac{1}{b} - \frac{r}{\sqrt{k+1}} \right|,$$

for all $j = 1, \dots, k+1$. For $b > \frac{2}{r}$ we have that

$$(1 - br)^2 \geq 1 - 2br + b^2 r^2 \geq b^2 r^2 - br > 0.$$

We can make this assumption because r goes to infinity as n goes to infinity so, since b is going to be a number like 60, this condition is satisfied. Therefore,

$$\begin{aligned}
\int_{U(r)} h(x-y)\, d\sigma(y) &\leq \left(\frac{2}{\pi}\right)^{k+1} b^{2(k+1)} \int_{U(r)} \prod_{j=1}^{k} \left(\frac{1}{\left(1 - \frac{br}{\sqrt{k+1}}\right)^2} \right) d\sigma(y) \\
&\leq \left(\frac{2}{\pi}\right)^{k+1} b^{2(k+1)} \int_{U(r)} \prod_{j=1}^{k} \left(\frac{1}{br(br-1)} \right) d\sigma(y) \\
&= \left(\frac{2}{\pi}\right)^{k+1} b^{2(k+1)} \left(\frac{1}{br}\right)^{k} \int_{U(r)} d\sigma(y) \\
&\leq \frac{2^{k+1}}{\pi} b^{2(k+1)} \left(\frac{1}{b}\right)^{k} = c_7(k) b^{k+2}.
\end{aligned}$$

because $\sigma(U(r)) = 1$ and $r \geq \frac{1}{\pi}$. In all the other cases, taking into account the number of small differences, we get similar bounds. Taking b large enough satisfying all the required properties ($b = 60$, for example), we get

$$h(x - z_j) - \int_{U(r)} h(x-y)\,d\sigma(y) \geq \frac{1}{(2\pi)^{k+1}} b^{2(k+1)} - c_7(k) b^{k+1} \geq 1, \tag{2.2.11}$$

whenever

$$z_j \in x + \left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1},$$

in all the cases. Now, since clearly $h(x)$ is a positive function and using (2.2.11),

$$\begin{aligned} |H(x)| &= |(h * (dP_0 - d\sigma_0))(x)| \\ &= \left| \int_{\mathbb{R}^{k+1}} h(x-y)(dP_0 - d\sigma_0)(y) \right| \\ &= \left| \int_{\mathbb{R}^{k+1}} h(x-y)\,dP_0(y) - \int_{\mathbb{R}^{k+1}} h(x-y)\,d\sigma_0(y) \right| \\ &= \left| \sum_{z_j \in P} h(x - z_j) - \int_{U(r)} h(x-y)\,d\sigma(y) \right| \\ &\geq \sum_{z_j \in x + \left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1}} h(x - z_j) - \int_{U(r)} h(x-y)\,d\sigma(y) \\ &\geq \sum_{z_j \in x + \left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1}} 1 = \sum_{j=1}^{n} \chi_{x + \left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1}}(z_j) = v(x), \end{aligned}$$

where $v(x)$ denotes the number of points $z_j \in P$ that lie in the translated cube $x + \left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1}$. Finally, since $H^2(x) \geq v^2(x) \geq v(x)$, we have

$$\int_{R^{k+1}} H^2(x)\,dx \geq \int_{\mathbb{R}^{k+1}} v(x)\,dx \gtrsim \sum_{z_j \in P} 1 = c_8(k) n, \tag{2.2.12}$$

because

$$\begin{aligned} \int_{\mathbb{R}^{k+1}} v(x)\,dx &= \int_{\mathbb{R}^{k+1}} \sum_{j=1}^{n} \chi_{x + \left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1}}(z_j)\,dx \\ &= \sum_{j=1}^{n} \int_{\left[-\frac{1}{60}, \frac{1}{60}\right]^{k+1}} dy \\ &= n \left(\frac{2}{60}\right)^{k+1}. \end{aligned}$$

Taking $b = 60$, and $c_6(k) = 120\sqrt{k+1}$, and using (2.2.10) and (2.2.12), (2.2.9) follows:

$$\begin{aligned}\int_{\|t\|<120\sqrt{k+1}} |\widehat{\Phi}(t)|^2\, dt &\geq \frac{1}{120^{2(k+1)}}\int_{\mathbb{R}^{k+1}} H^2(x)\, dx \\ &\geq \frac{1}{120^{2(k+1)}}\left(\frac{2}{60}\right)^{k+1} n \\ &= c_9(k)n,\end{aligned}$$

as we wanted to see.

Up to this point we have studied one of the two integrands in (2.2.8). Now, we will also study the other, $\widehat{\chi}_{B_q}(t)$, in the ball $\|t\| < c_6(k) = 120\sqrt{k+1}$. Let $\tau = \|t\|$ and $g(\tau, q) = \widehat{\chi}_{B_q}(t)$ to make the notation lighter. So,

$$\begin{aligned}g(\tau, q) &= \frac{1}{(\sqrt{2\pi})^{k+1}}\int_{\mathbb{R}^{k+1}} e^{-i\langle x,t\rangle}\chi_{B_q}(x)\, dx \\ &= \frac{1}{(\sqrt{2\pi})^{k+1}}\int_{B_q} e^{-i\langle x,t\rangle}\, dx.\end{aligned}$$

Doing the linear change of variables given by the orthogonal matrix taking t to $(0,\dots,0,\tau)$ (it exists because the two vectors have same modulus), and using that for $x \in B_q$, $\sqrt{x_1^2+\ldots+x_k^2} \leq \sqrt{q^2 - x_{k+1}^2}$, we have

$$\begin{aligned}g(\tau, q) &= \frac{1}{(\sqrt{2\pi})^{k+1}}\int_{B_q} e^{-i\tau x_{k+1}}\, dx_1\ldots dx_k dx_{k+1} \\ &= \frac{1}{(\sqrt{2\pi})^{k+1}}\int_{-q}^{q} e^{-i\tau y}\left(\int_{B^k_{\sqrt{q^2-y^2}}} dx_1\ldots dx_k\right) dy \\ &= \frac{1}{(\sqrt{2\pi})^{k+1}}\frac{\sqrt{\pi}^k}{\Gamma\left(\frac{k}{2}+1\right)}\int_{-q}^{q} e^{-i\tau y}(q^2-y^2)^{\frac{k}{2}}\, dy \\ &= c_{10}(k)\int_{-q}^{q} e^{-i\tau y}(q^2-y^2)^{\frac{k}{2}}\, dy.\end{aligned}$$

Applying the change of variables $y = qu$ and making some easy computations we get

$$g(\tau, q) = c_{10}(k)q^{k+1}\int_{-1}^{1}\big(\cos(\tau q u)\big)(1-u^2)^{\frac{k}{2}}\, du. \tag{2.2.13}$$

By (2.1.1), and continuing in (2.2.13), since $\frac{k+1}{2} > -\frac{1}{2}$ because $k \geq 1$, we have that

$$\begin{aligned}g(\tau, q) &= c_{10}(k)q^{k+1}\frac{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+1\right)}{(q\tau)^{\frac{k+1}{2}}}2^{\frac{k+1}{2}}J_{\frac{1}{2}(k+1)}(q\tau) \\ &= c_{11}(k)\left(\frac{q}{\tau}\right)^{\frac{k+1}{2}}J_{\frac{1}{2}(k+1)}(q\tau). \end{aligned} \tag{2.2.14}$$

By Hankel's expansion, Proposition 2.1.11, we see that if x is big enough $J_{\frac{1}{2}(k+1)}(x)$ has essentially the form of $x^{-1/2}\cos\left(x - \frac{1}{4}k\pi - \frac{1}{2}\pi\right)$. Also, if x is small enough, then

$$\int_{-1}^{1} \cos(xu)(1-u^2)^{\frac{k}{2}}\,du$$

almost equals

$$\begin{aligned}\int_{-1}^{1} (1-u^2)^{\frac{k}{2}}\,du &= \int_{0}^{1} t^{-\frac{1}{2}}(1-t)^{\frac{k}{2}}\,dt \\ &= B\left(\frac{1}{2}, \frac{k}{2}+1\right) \\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k+3}{2}\right)} \\ &= \frac{2\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+1\right)}{(k+1)\Gamma\left(\frac{k+1}{2}\right)},\end{aligned}$$

using Property 2.1.7. Therefore, if x is small enough, then by (2.1.1), $J_{\frac{1}{2}(k+1)}(x)$ almost equals

$$\frac{x^{\frac{1}{2}(k+1)}}{2^{\frac{1}{2}(k-1)}(k+1)\Gamma\left(\frac{k+1}{2}\right)} = c_{12}(k)x^{\frac{1}{2}(k+1)}.$$

Now, using (2.2.14), given $\varepsilon > 0$, there exists $c_{13}(k,\varepsilon) > 0$ such that

$$\left|\frac{\frac{1}{c_{11}(k)}\left(\frac{\tau}{q}\right)^{\frac{1}{2}(k+1)} g(\tau,q)}{\frac{1}{(q\tau)^{\frac{1}{2}}}\cos\left(q\tau - \frac{1}{4}k\pi - \frac{1}{2}\pi\right)} - 1\right| < \varepsilon, \quad \text{whenever } q\tau > c_{13}(k,\varepsilon),$$

therefore,

$$\left|g(\tau,q) - c_{14}(k)\frac{q^{\frac{k}{2}}}{\tau^{\frac{k}{2}+1}}\cos\left(q\tau - \frac{1}{4}k\pi - \frac{1}{2}\pi\right)\right| < \varepsilon c_{14}(k)\frac{q^{\frac{k}{2}}}{\tau^{\frac{k}{2}+1}}, \tag{2.2.15}$$

whenever $q\tau > c_{13}(k,\varepsilon)$. Also, there exists $c_{15}(k,\varepsilon) > 0$ such that

$$\left|\frac{\frac{1}{c_{11}(k)}\left(\frac{\tau}{q}\right)^{\frac{1}{2}(k+1)} g(\tau,q)}{c_{12}(k)(q\tau)^{\frac{1}{2}(k+1)}} - 1\right| < \varepsilon \quad \text{whenever } 0 < q\tau < c_{15}(k,\varepsilon),$$

hence,

$$\left|g(\tau,q) - c_{16}(k)q^{k+1}\right| < \varepsilon c_{16}(k)q^{k+1}, \tag{2.2.16}$$

whenever $0 < q\tau < c_{15}(k,\varepsilon)$. We are going to use this bounds to see that the quadratic average

$$\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau,\alpha r)\,d\alpha$$

is uniformly large in the following sense: if $\delta(k) > 0$ is small enough, depending only on k, then

$$\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau, \alpha r)\, d\alpha \gtrsim \min\left\{\frac{r^k}{\tau^{k+2}}, r^{2k+2}\right\} \gtrsim r^k = c_{17}(k)n, \tag{2.2.17}$$

for all $0 \leq \tau = |t| < c_6(k)$.

By (2.2.15), for $q\tau > c_{13}(k, \varepsilon)$, we have that

$$g(\tau, q) > \left(\cos\left(q\tau - \frac{1}{4}k\pi - \frac{1}{2}\pi\right) - \varepsilon\right) c_{14}(k) \frac{q^{\frac{k}{2}}}{\tau^{\frac{k}{2}+1}},$$

therefore, for $q\tau > c_{13}(k, \varepsilon)$,

$$g^2(\tau, q) > c_{18}(k) \frac{q^k}{\tau^{k+2}}. \tag{2.2.18}$$

By (2.2.16), for $0 < q\tau < c_{15}(k, \varepsilon)$, we have that

$$g(\tau, q) > (1 - \varepsilon) c_{16}(k) q^{k+1},$$

therefore, for $0 < q\tau < c_{15}(k, \varepsilon)$,

$$g^2(\tau, q) > c_{19}(k) q^{2k+2}. \tag{2.2.19}$$

In the integral (2.2.17) we have $g^2(\tau, \alpha r)$, where $\alpha \in (\delta(k), \frac{1}{2})$, $r = c_4(k) n^{\frac{1}{k}}$ and $0 \leq \tau < c_6(k)$, thus

$$\alpha r \tau \in \left(\delta(k) r\tau, \frac{r\tau}{2}\right)$$

when $\tau \neq 0$ or $\alpha r\tau = 0$ when $\tau = 0$. Now we have different cases:

1. $\alpha r \tau = 0$.
2. $\left(\delta(k) r\tau, \frac{r\tau}{2}\right) \subset (0, c_{15}(k, \varepsilon))$.
3. $\left(\delta(k) r\tau, \frac{r\tau}{2}\right) \subset (c_{13}(k, \varepsilon), \infty)$.
4. $\left(\delta(k) r\tau, \frac{r\tau}{2}\right) \subset (c_{15}(k, \varepsilon), c_{13}(k, \varepsilon))$.

Let us start by case 1. We are going to use that

$$\lim_{x \to 0^+} \frac{J_\nu(x)}{x^\nu} = c(\nu),$$

where $c(\nu) > 0$ is a constant depending only on ν. By (2.2.14),

$$\begin{aligned}
\lim_{\tau \to 0} g(\tau, q) &= \lim_{\tau \to 0} c_{11}(k) \left(\frac{q}{\tau}\right)^{\frac{k+1}{2}} J_{\frac{1}{2}(k+1)}(q\tau) \\
&= c_{11}(k) \lim_{\tau \to 0} \left(\frac{q}{\tau}\right)^{\frac{k+1}{2}} \frac{J_{\frac{1}{2}(k+1)}(q\tau)}{(q\tau)^{\frac{k+1}{2}}} (q\tau)^{\frac{k+1}{2}} \\
&= c_{11}(k) q^{k+1} \lim_{\tau \to 0} \frac{J_{\frac{1}{2}(k+1)}(q\tau)}{(q\tau)^{\frac{k+1}{2}}} \\
&= c_{20}(k) q^{k+1}.
\end{aligned}$$

So,

$$\begin{aligned}\int_{\delta(k)}^{\frac{1}{2}} g^2(0,\alpha r)\,d\alpha &= \int_{\delta(k)}^{\frac{1}{2}} c_{20}(k)^2(\alpha r)^{2k+2}\,d\alpha\\ &= c_{20}(k)^2 r^{2k+2}\left(\frac{\left(\frac{1}{2}\right)^{2k+3}-(\delta(k))^{2k+3}}{2k+3}\right)\\ &= c_{21}(k) r^{2k+2}.\end{aligned}$$

Cases 2 and 3 are direct applications of (2.2.19) and (2.2.18). Let us see them.

$$\begin{aligned}\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau,\alpha r)\,d\alpha &> \int_{\delta(k)}^{\frac{1}{2}} c_{19}(k)(\alpha r)^{2k+2}\,d\alpha\\ &= c_{19}(k) r^{2k+2}\left(\frac{\left(\frac{1}{2}\right)^{2k+3}-(\delta(k))^{2k+3}}{2k+3}\right)\\ &= c_{22}(k) r^{2k+2},\end{aligned}$$

and

$$\begin{aligned}\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau,\alpha r)\,d\alpha &> \int_{\delta(k)}^{\frac{1}{2}} c_{18}(k)\frac{(\alpha r)^k}{\tau^{k+2}}\,d\alpha\\ &= c_{18}(k)\frac{r^k}{\tau^{k+2}}\left(\frac{\left(\frac{1}{2}\right)^{k+1}-(\delta(k))^{k+1}}{k+1}\right)\\ &= c_{23}(k)\frac{r^k}{\tau^{k+2}},\end{aligned}$$

Finally, let us study case 4. We have that

$$c_{15}(k,\varepsilon) < \alpha r\tau < c_{13}(k,\varepsilon),$$

thus

$$\frac{c_{15}(k,\varepsilon)}{\alpha c_4(k)} < n^{\frac{1}{k}}\tau < \frac{c_{13}(k,\varepsilon)}{\alpha c_4(k)},$$

and from this we conclude that $n^{\frac{1}{k}}\tau$ behaves like a constant depending on k. Hence, τ behaves like $c_{24}(k)n^{-\frac{1}{k}}$. Therefore,

$$\begin{aligned}\int_{\delta(k)}^{\frac{1}{2}} g^2\big(c_{24}(k)n^{-\frac{1}{k}}, c_4(k)n^{\frac{1}{k}}\alpha\big)\,d\alpha &= \int_{\delta(k)}^{\frac{1}{2}}\left(\frac{c_4(k)n^{\frac{1}{k}}\alpha}{c_{24}(k)n^{-\frac{1}{k}}}\right)^{k+1} J^2_{\frac{1}{2}(k+1)}(c_4(k)c_{24}(k)\alpha)\,d\alpha\\ &\geq n^{\frac{2k+2}{k}} J^2_{\frac{1}{2}(k+1)}(c_4(k)c_{24}(k)\delta(k))\left(\frac{c_4(k)}{c_{24}(k)}\right)^{k+1}\left(\frac{\left(\frac{1}{2}\right)^{k+2}-(\delta(k))^{k+2}}{k+2}\right)\\ &= c_{25}(k)n^{\frac{2k+2}{k}} = c_{26}(k)r^{2k+2},\end{aligned}$$

and we have covered all cases, so we get the first inequality in (2.2.17). To second inequality in (2.2.17) follows immediately using that $\tau < c_6(k)$ and $r \geq c_4(k)$.

If the interval $\left(\delta(k)r\tau, \frac{r\tau}{2}\right)$ is not entirely contained in one of the intervals $(0, c_{15}(k,\varepsilon))$, $(c_{15}(k,\varepsilon), c_{13}(k,\varepsilon))$ or $(c_{13}(k,\varepsilon,\infty)$, we integrate in the contained region to obtain the desired inequality as in the other cases.

We are in position to finish the proof. Recall that by (2.2.1), it is enough to see (2.2.5). By Plancherel's identity (Theorem 2.1.4), (2.2.8) and Tonelli's Theorem,

$$\begin{aligned}\int_{\delta(k)}^{\frac{1}{2}}\int_{\mathbb{R}^{k+1}} F_{\alpha r}^2(x)\,dxd\alpha &= \int_{\delta(k)}^{\frac{1}{2}}\int_{\mathbb{R}^{k+1}} |\widehat{F}_{\alpha r}^2(t)|^2\,dtd\alpha \\ &= \int_{\delta(k)}^{\frac{1}{2}}\int_{\mathbb{R}^{k+1}} |\widehat{\chi}_{B_{\alpha r}}(t)|^2|\Phi(t)|^2\,dtd\alpha \\ &= \int_{\mathbb{R}^{k+1}} |\Phi(t)|^2\left(\int_{\delta(k)}^{\frac{1}{2}} |\widehat{\chi}_{B_{\alpha r}}(t)|^2\,d\alpha\right)dt.\end{aligned}$$

Since $g(\tau,\alpha r) = \widehat{\chi}_{B_{\alpha r}}(t)$, with $\tau = \|t\|$, and using (2.2.9) and (2.2.17),

$$\begin{aligned}\int_{\delta(k)}^{\frac{1}{2}}\int_{\mathbb{R}^{k+1}} F_{\alpha r}^2(x)\,dxd\alpha &= \int_{\mathbb{R}^{k+1}} |\Phi(t)|^2\left(\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau,\alpha r)\,d\alpha\right)dt \\ &\geq \int_{\tau<c_6(k)} |\Phi(t)|^2\left(\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau,\alpha r)\,d\alpha\right)dt \\ &\geq \left(\int_{\tau<c_6(k)} |\Phi(t)|^2\,dt\right)\min_{0\leq\tau<c_6(k)}\int_{\delta(k)}^{\frac{1}{2}} g^2(\tau,\alpha r)\,d\alpha \gtrsim n^2,\end{aligned}$$

and (2.2.5) follows, so we have proved the desired lower bound. □

As we have seen, this proof is based on the fact that we can write the spherical cap discrepancy of an n-element set of points P on $\mathbb{S}^k$ as a convolution of two functions. This method is called *Beck's amplification method*. Thanks to this, using Fourier analysis seems a good idea since, by Plancherel's identity, Theorem 2.1.4, we can separate the geometric component and the measure component of this convolution, as we have done in (2.2.6), and we can study them separately. In [1], there is an extensive overview of this method in full generality.

Chapter 3

Upper bounds

In this chapter we will present the best upper bound known up to now for the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$ and a sharp upper bound for the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$. Both results, the first due to J. Beck and the second due to K. B. Stolarsky, are presented in [7]. Unlike what happens with the lower bounds, the result concerning the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$, Theorem 3.2.1 (1984), is not a consequence of a result concerning the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$. Theorem 3.2.2, due to K. B. Stolarsky, was proved in 1973, some years earlier than the work of J. Beck, but the strategies and techniques used in their proofs are almost the same. We will study the proof of Theorem 3.2.1 presented in [7] complementing it with a more general proof presented in [5].

The techniques we use are probabilistic. The first one, called jittered sampling, will be presented in Subsection 3.1.3 and, roughly speaking, it allows us to take an n-element set of points on $\mathbb{S}^k$ in a semi-random way. Independent and uniformly distributed random sets behave really bad, in terms of well-distribution, while trying to construct sets of points is extremely hard. On the other hand, the sets taken using this technique have reasonable good properties related with the spherical cap discrepancy. Jittered sampling is based on regular area partitions, presented in Subsection 3.1.1. Basically, these are partitions of $\mathbb{S}^k$ into pieces that have equal surface area and their diameter is uniformly controlled. This technique can also be applied in cubes, balls, rectangles, etc. The second one is the combination of the Bernstein-Chernoff's inequality, Lemma 3.1.7, and an approximation of the family of all spherical caps on $\mathbb{S}^k$ by a finite subfamily with certain controlled cardinality.

3.1 Preliminaries

3.1.1 Regular area partitions

We start the section studying regular area partitions of $\mathbb{S}^k$. We are going to see the main definitions and a result concerning the existence of such partitions.

Definition 3.1.1. We say that $\{S_i\}_{i=1}^n \subset \mathbb{S}^k$ is a *partition of* $\mathbb{S}^k$ if

$$\mathbb{S}^k = \bigcup_{i=1}^n S_i,$$

with $\sigma(S_i \cap S_j) = 0$ for all $i \neq j$.

Definition 3.1.2. Let $\{S_i\}_{i=1}^n \subset \mathbb{S}^k$ be a partition of $\mathbb{S}^k$. We call it an *equal-area partition of* $\mathbb{S}^k$ if

$$\sigma(S_i) = \frac{\sigma(\mathbb{S}^k)}{n},$$

for all $i = 1, \ldots, n$.

Definition 3.1.3. Let $\{S_i\}_{i=1}^n \subset \mathbb{S}^k$ be an equal-area partition of $\mathbb{S}^k$. We call it a *regular area partition*, with constant $c(k) > 0$, if

$$\operatorname{diam}(S_i) \leq c(k) n^{-\frac{1}{k}}$$

for all $i = 1, \ldots, n$,

It is clear that in the k-dimensional unit cube there exist regular area partitions, for example, take disjoint squares of side-length $n^{-\frac{1}{k}}$, but it is not so clear in the case of $\mathbb{S}^k$.

Remark 3.1.4. The isodiametric inequality tells us that spherical caps have the biggest surface area among all the sets on $\mathbb{S}^k$ of a given diameter. Hence, for a set $A \subset \mathbb{S}^k$ with

$$\operatorname{diam}(A) = \operatorname{diam}(C(x,t)),$$

where $C(x,t)$ is a certain spherical cap on $\mathbb{S}^k$, with $x \in \mathbb{S}^k$ and $-1 \leq t \leq 1$, we have that

$$\sigma(A) \leq \sigma(C(x,t)) = c(k) \operatorname{diam}(C(x,t))^k = c(k)\operatorname{diam}(A)^k,$$

with $c(k) > 0$ a constant depending on k. As a consequence, for the sets $\{S_i\}_{i=1}^n \subset \mathbb{S}^k$ of a regular area partition we have

$$\frac{\sigma(\mathbb{S}^k)}{n} = \sigma(S_i) \leq c(k)\operatorname{diam}(S_i)^k,$$

i.e.,

$$\operatorname{diam}(S_i) \geq c'(k) n^{-\frac{1}{k}}.$$

We have seen that for these type of partitions the diameter of the sets is not only controlled by above by $n^{-\frac{1}{k}}$ but we know that is of order $n^{-\frac{1}{k}}$. So the sets of this partition are quite regular (we may even thing that too regular to exist).

The next result, due to P. Leopardi, ensures the existence of regular area partitions of $\mathbb{S}^k$ and gives an explicit value for the constant. The proof can be found in [11].

Theorem 3.1.5. *(P. Leopardi). For all $n \in \mathbb{N}$ there exist regular area partitions $\{S_i\}_{i=1}^n$ of $\mathbb{S}^k$ with constant $c_{27}(k)$ given by*

$$c_{27}(k) = 8\left(k\frac{\sigma(\mathbb{S}^k)}{\sigma(\mathbb{S}^{k-1})}\right)^{\frac{1}{k}}.$$

The history of these partitions, as D. Bilyk and M. T. Lacey comment in [2], is curious. In 1973, in one of his works, Stolarsky asserts that for $k \geq 2$, there exist regular area partitions of $\mathbb{S}^k$, without giving a construction nor a proof. Later, in 1984, J. Beck and W. W. L. Chen, in [7], quote Stolarsky. Bourgain and Lindenstrauss that worked also on these problems, quote J. Beck and W. W. L. Chen in 1988. Finally, in 2002 U. Feige and G. Schechtman gave a complete construction of a regular area partition of $\mathbb{S}^k$. In 2009, P. Leopardi found the value for the constant stated in Theorem 3.1.5.

3.1.2 Concentration inequalities

Unlike the lower bound results, which use Fourier transform techniques, the techniques used in the upper bound results are probabilistic. In this preliminary subsection we are going to study an inequality, due to Bernstein and Chernoff, that plays a key role in the proof of Theorem 3.2.1 and also in the proof of Theorem 4.3.2 in Section 4.3. The word concentration points out the fact that these inequalities measure the probability of a random variable to be far from its expectation, i.e., measure its concentration.

Lemma 3.1.6. *(Markov's inequality). If X is a non-negative random variable and $a > 0$, then*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Proof. Consider the event $\{X \geq a\}$. Clearly, since $\mathbb{E}(\cdot)$ is a monotone increasing function, whenever $X \geq a$,

$$\mathbb{E}\big(a\mathbb{1}_{\{X\geq a\}}\big) = \mathbb{E}(a) \leq \mathbb{E}(X),$$

and

$$\mathbb{E}\big(a\mathbb{1}_{\{X\geq a\}}\big) = a\mathbb{P}(X \geq a).$$

Since $a > 0$, we can divide by a on both sides and we obtain the desired inequality. □

With this result we can now study the important concentration inequality.

Lemma 3.1.7. *(Bernstein-Chernoff's inequality). Let $\xi_1, \dots, \xi_m$ be independent random variables with $|\xi_i| \leq 1$ for $i = 1, \dots, m$. Let*

$$\beta = \sum_{i=1}^{m} \mathbb{E}\big((\xi_i - \mathbb{E}(\xi_i))^2\big).$$

Then,

$$\mathbb{P}\left(\left|\sum_{i=1}^{m}(\xi_i - \mathbb{E}(\xi_i))\right| \geq \gamma\right) \leq \begin{cases} 2e^{-\frac{\gamma}{4}} & \text{if } \gamma \geq \beta, \\ 2e^{-\frac{\gamma^2}{4\beta}} & \text{if } \gamma \leq \beta. \end{cases}$$

Proof. Without loss of generality we can assume that the random variables are centered, i.e., $\mathbb{E}(\xi_i) = 0$ for $i = 1, \dots, m$. This is because of the linearity of the expectation. Set

$$X = \sum_{i=1}^{m} \xi_i.$$

We are only going to prove that $\mathbb{P}(X \geq \gamma)$ is as stated because following the same steps we can prove that $\mathbb{P}(X \leq -\gamma)$ also satisfies the required inequalities.

Using Lemma 3.1.6,

$$\mathbb{P}(X \geq \gamma) = \mathbb{P}(e^{yX} \geq e^{y\gamma}) \leq e^{-y\gamma}\mathbb{E}(e^{yX}), \tag{3.1.1}$$

with y satisfying $0 < y \leq 1$ that we will determine later. Since X is a finite sum of independent random variables

$$\mathbb{E}(e^{yX}) = \prod_{i=1}^{m} \mathbb{E}(e^{y\xi_i}). \tag{3.1.2}$$

Let us study the term $\mathbb{E}(e^{y\xi_i})$ separately. By the exponential's series expansion, and since $|\xi_i| \leq 1$, we have

$$\begin{aligned}
\mathbb{E}(e^{y\xi_i}) &= \mathbb{E}\left(\sum_{n=0}^{\infty} \frac{y^n \xi_i^n}{n!}\right) \\
&= 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + \mathbb{E}\left(\sum_{n=3}^{\infty} \frac{y^n \xi_i^n}{n!}\right) \\
&\leq 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + \mathbb{E}\left(\sum_{n=3}^{\infty} \frac{y^n |\xi_i|^n}{n!}\right) \\
&\leq 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + \mathbb{E}\left(\sum_{n=3}^{\infty} \frac{y^n |\xi_i|^2}{n!}\right) \\
&= 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + y^3\mathbb{E}(\xi_i^2) \sum_{n=0}^{\infty} \frac{y^n}{(n+3)!}
\end{aligned}$$

By induction one can easily prove that $(n+3)! \geq 6 \cdot 3^n$. Continuing the computations we get

$$\begin{aligned}
\mathbb{E}(e^{y\xi_i}) &\leq 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + \frac{y^3\mathbb{E}(\xi_i^2)}{6} \sum_{n=0}^{\infty} \frac{y^n}{3^n} \\
&= 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + \frac{y^3\mathbb{E}(\xi_i^2)}{6}\left(\frac{1}{1 - \frac{y}{3}}\right) \\
&= 1 + \frac{y^2\mathbb{E}(\xi_i^2)}{2} + \frac{y^3\mathbb{E}(\xi_i^2)}{6 - 2y}.
\end{aligned}$$

Using that $1+x<e^x$, for $x>0$, and applying these last computations to (3.1.2) we get

$$\begin{aligned}\mathbb{E}(e^{yX}) &\leq \prod_{i=1}^{m}\left(1+\frac{y^2\mathbb{E}(\xi_i^2)}{2}+\frac{y^3\mathbb{E}(\xi_i^2)}{6-2y}\right)\\ &\leq e^{\frac{y^2}{2}\left(1+\frac{y}{3-y}\right)\sum_{i=1}^{m}\mathbb{E}(\xi_i^2)}.\end{aligned}$$

Using this in (3.1.1), we obtain

$$\mathbb{P}(X\geq\gamma)\leq e^{\frac{y^2\beta}{2}\left(1+\frac{y}{3-y}\right)-y\gamma},$$

because $\beta=\sum_{i=1}^{m}\mathbb{E}(\xi_i)$. Hence, if $\gamma\geq\beta$, let $y=1$ to conclude

$$\mathbb{P}(X\geq\gamma)\leq e^{\frac{3\beta}{4}-\gamma}\leq e^{-\frac{\gamma}{4}},$$

and if $\gamma\leq\beta$, let $y=\frac{\gamma}{\beta}$ to conclude

$$\mathbb{P}(X\geq\gamma)\leq e^{\frac{\gamma^2}{2\beta}\left(1+\frac{\gamma}{3\beta-\gamma}\right)-\frac{\gamma^2}{\beta}}\leq e^{-\frac{\gamma^2}{4\beta}}.$$

□

3.1.3 Jittered sampling

As we will see in Section 3.2, both upper bound results, for the spherical discrepancy of an n-element set of points on $\mathbb{S}^k$, Theorem 3.2.1, and for the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$, Theorem 3.2.2, are based in a technique involving regular area partitions of $\mathbb{S}^k$ called jittered sampling. We are going to follow [2].

Basically, jittered sampling is a way of producing semi-random sets of points. In our case, we are going to use this technique in generating sets of points on $\mathbb{S}^k$ but it applies in many geometric different settings: balls, rotated rectangles, cubes, spherical caps, etc. The general structure of the jittered sampling technique is the following: we first divide our original manifold in n regions of equal volume and almost equal diameter, then we take a point uniformly random in each of these pieces, independently of the others. In our setting, we use regular area partitions of $\mathbb{S}^k$, where instead of volume we use surface area (normalized) and the diameter of the pieces behaves like the number of pieces to the power $-\frac{1}{k}$ times a constant depending only on k.

Intuitively, this construction guarantees that the set of points which results is well distributed in the sense that there are no large gaps. In many situations this distribution yields nearly optimal discrepancy. This is consistent with the fact that purely random constructions are far from optimal and deterministic sets are very hard to construct, in terms of well-distribution.

As we will see in Section 4.2, following the proof presented in [3], this technique can be viewed as a determinantal process with a specific kernel.

3.2 Main results

We are going to study [5], J. Beck's paper about upper bounds in the theory of irregularities of distribution. Concretely, we are going to study Theorem 2, page 117. In [5] there is a more general version than the one presented in this thesis, but since we are only studying discrepancies on $\mathbb{S}^k$, we are going to complement the study with [7], J. Beck's and W. W. L. Chen's book, because it contains a spherical version of this result, which is the following result:

Theorem 3.2.1. *(J. Beck). For an arbitrary integer $n \geq 2$, there exists an n-element set of points P on the unit k-dimensional sphere $\mathbb{S}^k$ such that, for any spherical cap $C(x,t) = \{y \in \mathbb{S}^k;\, \langle x, y\rangle \leq t\} \subset \mathbb{S}^k$ with $x \in \mathbb{S}^k$ and $-1 \leq t \leq 1$, we have*

$$\left|Z(P,x,t) - n\sigma^*(t)\right| < c_{28}(k) n^{\frac{1}{2}-\frac{1}{2k}} \sqrt{\log n}.$$

Proof. We use the jittered sampling technique presented in Subsection 3.1.3. Let us consider a regular area partition $\{S_i\}_{i=1}^n$ of $\mathbb{S}^k$ with constant $c_{27}(k)$. We know its existence by Theorem 3.1.5. Let us associate with each S_i, for $i = 1, \dots, n$, a uniformly distributed random point $\xi_i \in S_i$, i.e., these random points, or random variables, $\{\xi_i\}_{i=1}^n$ satisfy

$$\mathbb{P}(\xi_i \in H) = \frac{\sigma(H \cap S_i)}{\sigma(S_i)}, \tag{3.2.1}$$

for every measurable set $H \subset \mathbb{S}^k$ and for every $i = 1, \dots, n$. We can also assume that $\{\xi_i\}_{i=1}^n$ are independent.

Let us prove the existence of $\{\xi_i\}_{i=1}^n$. For every $i = 1, \dots, n$, defining

$$\mathcal{F}_i = \{H \cap S_i\,;\, H \subset \mathbb{S}^k \text{ measurable}\}$$

and

$$\mathbb{P}_i(H \cap S_i) = \frac{\sigma(H \cap S_i)}{\sigma(\mathbb{S}^k)},$$

we have that each triple $(S_i, \mathcal{F}_i, \mathbb{P}_i)$ is a probability space and the identity function $\xi_i : S_i \to S_i$ is a random point having the desired distribution. However, we cannot ensure the independence. But there exist independent copies of $\{\xi_i\}_{i=1}^n$, that we will denote also by $\{\xi_i\}_{i=1}^n$, which can be defined in the probability space

$$(S_1 \times \dots \times S_n, \mathcal{F}, \mathbb{P})$$

where

$$\mathcal{F} = \{(H_1 \cap S_1, \dots, H_n \cap S_n)\,;\, H_1, \dots, H_n \subset \mathbb{S}^k \text{ measurable}\}$$

and

$$\mathbb{P}(H_1 \cap S_1, \dots, H_n \cap S_n) = \mathbb{P}_1(H_1 \cap S_1) \cdot \ldots \cdot \mathbb{P}_n(H_n \cap S_n),$$

as the projections $\xi_i : S_1 \times \dots \times S_n \to \mathbb{S}^k$, $\xi_i(s_1, \dots, s_n) = s_i$, for all $i = 1, \dots, n$.

Let us fix a spherical cap $C = C(x,t) \subset \mathbb{S}^k$, with $x \in \mathbb{S}^k$ and $-1 \leq t \leq 1$. We denote by $\tilde{C}$ the union of all S_i contained in C, i.e.,

$$\tilde{C} = \bigcup_{S_i \subset C} S_i.$$

Clearly, $\tilde{C}$ contains as much ξ_i as S_i contained in C. So we need to study the discrepancy of $C \setminus \tilde{C}$. Denote by $\mathcal{I}(C) \subset \{1, \dots, n\}$ the set of indices such that, if $i \in \mathcal{I}(C)$, then

$$C \cap S_i \neq \emptyset \quad \text{and} \quad (\mathbb{S}^k \setminus C) \cap S_i \neq \emptyset.$$

We can easily check that

$$C \setminus \tilde{C} = \bigcup_{i \in \mathcal{I}(C)} (S_i \cap C). \tag{3.2.2}$$

Note that, in fact, it is a disjoint union since the sets $\{S_i\}_{i=1}^n$ are pairwise disjoint. We claim that

$$\operatorname{card}(\mathcal{I}(C)) \leq c_{29}(k) n^{1-\frac{1}{k}}, \tag{3.2.3}$$

To see this observe that, for each $i \in \mathcal{I}(C)$, S_i intersects the boundary of C. Because of the properties of this regular area partition, (see Definition 3.1.3), each S_i is contained in the $(c_{27}(k)n^{-\frac{1}{k}})$-neighbourhood of this boundary, which means that the whole union is in there.

The surface area of a b-neighbourhood of a $(k-1)$-dimensional sphere inside $\mathbb{S}^k$ is bounded above by the surface area of the b-neighbourhood of the equator of $\mathbb{S}^k$, which at the same time is bounded by a constant $c_{30}(k)$ times b. Therefore, since all S_i have the same surface area, we conclude that

$$\operatorname{card}(\mathcal{I}(C)) \frac{\sigma(\mathbb{S}^k)}{n} \leq \frac{c_{30}(k)}{n^{\frac{1}{k}}},$$

with $c_{29}(k) = c_{27}(k)c_{30}(k)$, as we have claimed.

To visualize this a bit more, we can think in the case $k = 2$, so we have $\mathbb{S}^2 \subset \mathbb{R}^3$. All S_i with $i \in \mathcal{I}(C)$ are included in a $(c_{27}(2)n^{-\frac{1}{2}})$-neighbourhood around the boundary of the spherical cap, which in this case is a spherical zone with high $h = 2c_{27}(2)n^{-\frac{1}{2}}$. Recall that in this case the boundary is $r\mathbb{S}^1$, with $0 \leq r \leq 1$. The surface area of this spherical zone is $4r\pi h$, with r the radius of the boundary of the spherical cap. This value is maximized in the hemisphere, when $r = 1$.

Let us define the random variables η_i, $i \in \mathcal{I}(C)$, as follows

$$\eta_i = \begin{cases} 1 & \text{if } \xi_i \in C \cap S_i, \\ 0 & \text{if } \xi_i \notin C \cap S_i. \end{cases}$$

Then,

$$\sum_{\xi_i \in C} 1 - n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)} = \sum_{\xi_i \in C, S_i \subset C} 1 - n\frac{\sigma(\tilde{C})}{\sigma(\mathbb{S}^k)} + \sum_{\xi_i \in C, S_i \not\subset C} 1 - n\frac{\sigma(C \setminus \tilde{C})}{\sigma(\mathbb{S}^k)}. \tag{3.2.4}$$

Observe that the difference of the first two terms in (3.2.4) is zero. Using that $\tilde{C}$ is the union of the disjoint sets S_i contained in C, and that all of them have the same measure, $\frac{\sigma(\mathbb{S}^k)}{n}$,

$$\begin{aligned}\sum_{\xi_i\in C, S_i\subset C} 1 - n\frac{\sigma(\tilde{C})}{\sigma(\mathbb{S}^k)} &= \sum_{\xi_i\in C, S_i\subset C} 1 - \frac{n}{\sigma(\mathbb{S}^k)}\sum_{S_i\subset C}\sigma(S_i)\\ &= \sum_{\eta_i\in C, S_i\subset C} 1 - \sum_{S_i\subset C} 1 = 0.\end{aligned}$$

Back to (3.2.4), using (3.2.2), the definition of the random variables $\{\eta_i\}_{i\in\mathcal{I}(C)}$ and the equation (3.2.1),

$$\begin{aligned}\sum_{\xi_i\in C} 1 - n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)} &= \sum_{i\in\mathcal{I}(C)} \eta_i - \frac{n}{\sigma(\mathbb{S}^k)}\sigma\Big(\bigcup_{i\in\mathcal{I}(C)}(S_i\cap C)\Big)\\ &= \sum_{i\in\mathcal{I}(C)}\left(\eta_i - \frac{\sigma(S_i\cap C)}{\sigma(S_i)}\right)\\ &= \sum_{i\in\mathcal{I}(C)}\big(\eta_i - \mathbb{P}(\xi_i\in S_i\cap C)\big).\end{aligned}$$

Since $\{\eta_i\}_{i\in\mathcal{I}(C)}$ take values 0 or 1, its expectation is $\mathbb{E}(\eta_i) = \mathbb{P}(\xi_i \in S_i\cap C)$, therefore

$$\sum_{\xi_i\in C} 1 - n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)} = \sum_{i\in\mathcal{I}(C)}(\eta_i - \mathbb{E}(\eta_i)). \tag{3.2.5}$$

Now, we are going to apply Bernstein-Chernoff's inequality (Lemma 3.1.7). Let $\gamma = c_{31}(k)n^{\frac{1}{2}-\frac{1}{2k}}\sqrt{\log n}$, where the constant $c_{31}(k) > 0$ will be specified later. Recall that the β appearing in Lemma 3.1.7 satisfies $\beta \le \mathrm{card}(\mathcal{I}(C)) \le c_{29}(k)n^{1-\frac{1}{k}}$ since

$$\beta = \sum_{i\in\mathcal{I}(C)}\mathbb{E}\big((\eta_i - \mathbb{E}(\eta_i))^2\big)$$

and using (3.2.3). Therefore, applying Lemma 3.1.7 to (3.2.5), since $\gamma\le\beta$

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{\xi_i\in C} 1 - n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)}\right| \ge c_{31}(k)n^{\frac{1}{2}-\frac{1}{2k}}\sqrt{\log n}\right) &= \mathbb{P}\left(\left|\sum_{i\in\mathcal{I}(C)}(\eta_i-\mathbb{E}(\eta_i))\right|\ge\gamma\right)\\ &\le e^{-\frac{(c_{31}(k)n^{\frac{1}{2}-\frac{1}{2k}}\sqrt{\log n})^2}{4c_{29}(k)n^{1-\frac{1}{k}}}}\\ &= \frac{1}{n^{c_{32}(k)}},\end{aligned}$$

with $c_{32}(k) = \frac{c_{31}(k)^2}{4c_{29}(k)} \to \infty$ as $c_{31}(k)\to\infty$.

The strategy we have followed, proving the result for a fixed spherical cap, has one big problem: the family of the spherical caps on $\mathbb{S}^k$ is an uncountable family of sets. A priori it is not clear that we can pass the bounds obtained for the discrepancy of a fixed spherical cap to bounds for the discrepancy of the whole family of the spherical caps on $\mathbb{S}^k$, and to preserve the order of the discrepancy.

But, it can be seen that there exists a finite subfamily $\mathcal{S}$ of the family of all spherical caps on $\mathbb{S}^k$, with cardinality $n^{c_{33}(k)}$, where $c_{33}(k) > 0$ is a constant depending only on k, satisfying that, given a spherical cap $C' = C(x', t')$ on $\mathbb{S}^k$, there exist $A, B \in \mathcal{S}$ with

$$A \subset C' \subset B \quad \text{and} \quad \frac{\sigma(B \setminus A)}{\sigma(\mathbb{S}^k)} < \frac{1}{n}.$$

If $A, B \in \mathcal{S}$ are as before, with respect to a spherical cap $C' = C(x', t')$ on $\mathbb{S}^k$, we have that

$$\begin{aligned} |Z(P, x', t') - n\sigma^*(t')| &\le \left| \sum_{\xi_i \in B} 1 - n\frac{\sigma(B)}{\sigma(\mathbb{S}^k)} \right| + \left| n\frac{\sigma(B \setminus C')}{\sigma(\mathbb{S}^k)} \right| \\ &< \left| \sum_{\xi_i \in B} 1 - n\frac{\sigma(B)}{\sigma(\mathbb{S}^k)} \right| + n\frac{1}{n} \\ &\le \max\left\{ \left| \sum_{\xi_i \in B} 1 - n\frac{\sigma(B)}{\sigma(\mathbb{S}^k)} \right|, \left| \sum_{\xi_i \in A} 1 - n\frac{\sigma(A)}{\sigma(\mathbb{S}^k)} \right| \right\} + 1. \end{aligned}$$

This tells us that the discrepancy of the family of the spherical caps on $\mathbb{S}^k$ is of the same order as the discrepancy of the subfamily $\mathcal{S}$, and hence, that we can restrict ourselves to spherical caps in $\mathcal{S}$. So, using this reduction, we can bound the spherical cap discrepancy of the union of the sets of $\mathcal{S}$ by

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{\xi_i \in C} 1 - n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)} \right| \ge c_{31}(k) n^{\frac{1}{2} - \frac{1}{2k}} \sqrt{\log n}, \text{ for some } C \in \mathcal{S} \right) &\le \operatorname{card}(\mathcal{S}) \frac{1}{n^{c_{32}(k)}} \\ &\le \frac{1}{n^{c_{32}(k) - c_{33}(k)}} \\ &\le \frac{1}{2}, \end{aligned}$$

if $c_{32}(k) > c_{33}(k)$, and recall that in the expression of $c_{32}(k)$ it appears $c_{31}(k)$ for which we have some freedom in adjusting it, and this finishes the proof because we have seen

$$\mathbb{P}\left(\left| \sum_{\xi_i \in C} 1 - n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)} \right| \gtrsim n^{\frac{1}{2} - \frac{1}{2k}} \sqrt{\log n} \right) \le \frac{1}{2},$$

so the probability with the reverse inequality is bigger than $\frac{1}{2}$, which ensures the existence of such an n-element set of points on $\mathbb{S}^k$. $\square$

As an immediate consequence, taking the supremum over all spherical caps on $\mathbb{S}^k$, we have that

$$\mathbb{D}_n(P) \lesssim n^{\frac{1}{2} - \frac{1}{2k}} \sqrt{\log n},$$

which is the best upper bound, known up to now, for the spherical cap discrepancy of an n-element set of points P on $\mathbb{S}^k$.

We have studied two results concerning lower bounds, due to J. Beck, Theorem 2.2.3 and Theorem 2.2.4, in Section 2.2. The first one, Theorem 2.2.4, gives a lower bound for the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$ and is an immediate consequence of the second one, Theorem 2.2.3, which is the main result concerning lower bounds. Its starting point is Stolarsky's Invariance Principle, Theorem 2.2.1, and it gives a lower bound for the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$.

Unlike before, the best known upper bound for the spherical cap discrepancy of an n-element set of points on $\mathbb{S}^k$, Theorem 3.2.1, it is not a consequence of a result concerning the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$, but such a result exists. K. B. Stolarsky, in a work previous to all these results due to J. Beck, in 1973, established the best upper bound for this L^2-average and it is known that the bound he gave is sharp.

Theorem 3.2.2. *(K. B. Stolarsky).*

$$c_0(k)n^2 - S(n,k) < c_{34}(k)n^{1-\frac{1}{k}}.$$

Proof. We have to find an n-element set P on $\mathbb{S}^k$ such that

$$\int_{-1}^{1}\left(\frac{1}{\sigma(\mathbb{S}^k)}\int_{\mathbb{S}^k}\big(Z(P,x,t)-n\sigma^*(t)\big)^2\,d\sigma(x)\right)dt \le c_{34}(k)n^{1-\frac{1}{k}}, \tag{3.2.6}$$

because

$$\begin{aligned} c_0(k)n^2 - S(n,k) &\le c_0(k)n^2 - S(n,k,P) \\ &= \int_{-1}^{1}\left(\frac{1}{\sigma(\mathbb{S}^k)}\int_{\mathbb{S}^k}\big(Z(P,x,t)-n\sigma^*(t)\big)^2\,d\sigma(x)\right)dt. \end{aligned}$$

We follow exactly the same strategy, step by step, as the proof of Theorem 3.2.1, until the moment before applying Lemma 3.1.7, equation (3.2.5). Basically, we use the jittered sampling technique presented in Subsection 3.1.3 and then, defining random variables for a subcollection of sets of the partition with certain properties, we find out that

$$\sum_{\xi_i\in C} 1 - n\frac{\sigma(C)}{\sigma(S^k)} = \sum_{i\in\mathcal{I}(C)}\big(\eta_i-\mathbb{E}(\eta_i)\big).$$

Observe that

$$\begin{aligned}\mathbb{E}\Big(\Big(\sum_{i\in\mathcal{I}(C)}(\eta_i-\mathbb{E}(\eta_i))\Big)^2\Big)&=\mathbb{E}\Big(\sum_{i_1\in\mathcal{I}(C)}\sum_{i_2\in\mathcal{I}(C)}(\eta_{i_1}-\mathbb{E}(\eta_{i_1}))(\eta_{i_2}-\mathbb{E}(\eta_{i_2}))\Big)\\&=\sum_{i_1\in\mathcal{I}(C)}\sum_{i_2\in\mathcal{I}(C)}\mathbb{E}\Big((\eta_{i_1}-\mathbb{E}(\eta_{i_1}))(\eta_{i_2}-\mathbb{E}(\eta_{i_2}))\Big)\\&=\sum_{i_1\in\mathcal{I}(C)}\sum_{i_2\in\mathcal{I}(C)}\mathbb{E}\Big((\eta_{i_1}-\mathbb{E}(\eta_{i_1}))\Big)\mathbb{E}\Big((\eta_{i_2}-\mathbb{E}(\eta_{i_2}))\Big)\\&=\sum_{i\in\mathcal{I}(C)}\mathbb{E}\Big((\eta_i-\mathbb{E}(\eta_i))^2\Big),\end{aligned}$$

just using the linearity of the expectation and that, since $\{\eta_i\}_{i\in\mathcal{I}(C)}$ are independent random variables, in the last equality only remain the terms where $i_1=i_2$. Hence,

$$\begin{aligned}\mathbb{E}\Big(\Big(\sum_{\xi_i\in C}1-n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)}\Big)^2\Big)&=\mathbb{E}\Big(\Big(\sum_{i\in\mathcal{I}(C)}(\eta_i-\mathbb{E}(\eta_i))\Big)^2\Big)\\&=\sum_{i\in\mathcal{I}(C)}\mathbb{E}\Big((\eta_i-\mathbb{E}(\eta_i))^2\Big)\\&\le\sum_{i\in\mathcal{I}(C)}\mathbb{E}(\eta_i^2)\\&\le\sum_{i\in\mathcal{I}(C)}1=\operatorname{card}(\mathcal{I}(C)),\end{aligned}$$

using these lasts expressions and that $|\eta_i|\le 1$. By equation (3.2.3) of the proof of Theorem 3.2.1, we get

$$\mathbb{E}\Big(\Big(\sum_{\xi_i\in C}1-n\frac{\sigma(C)}{\sigma(\mathbb{S}^k)}\Big)^2\Big)\le c_{29}(k)n^{1-\frac{1}{k}}. \tag{3.2.7}$$

Finally, by (3.2.7) and Tonelli's Theorem,

$$\mathbb{E}\Big(\int_{-1}^{1}\Big(\frac{1}{\sigma(\mathbb{S}^k)}\Big|\sum_{\xi_i\in C}1-n\frac{\sigma(C(x,t))}{\sigma(\mathbb{S}^k)}\Big|^2 d\sigma(x)\Big)\,dt\Big)\le c_{35}(k)n^{1-\frac{1}{k}}.$$

As a consequence, there exists an n-element set of points on $\mathbb{S}^k$ satisfying (3.2.6), as we wanted to see. □

Combining Theorem 2.2.3 and Theorem 3.2.1 we have that the L^2-discrepancy of an n-element set of points P on $\mathbb{S}^k$ satisfies

$$c_1(k)n^{1-\frac{1}{k}}<\mathbb{D}_n(P)_2<c_{34}(k)n^{1-\frac{1}{k}}.$$

As a remark, observe that the cases $k=1$ and $k\ge 2$ are completely different. While in the case $k=1$ the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$ remains bounded as $n\to\infty$, in the case $k\ge 2$ the L^2-discrepancy of an n-element set of points on $\mathbb{S}^k$ goes to infinity as $n\to\infty$ with polynomial speed.

Chapter 4

Determinantal point processes

In this chapter we will introduce the notion of determinantal point process, first introduced by Odile Macchi in 1975, we will see that the point process given by the jittered sampling technique, explained in Section 4.2, can be viewed as a determinantal point process and finally we will study a result due to K. Alishahi and M. Zamani, presented in [9], that gives a construction of a specific determinantal point process which achieves the bounds presented in the previous chapters, in $\mathbb{S}^2$. We will start by presenting the basic definitions and concepts related with the determinantal point processes, in full generality, to provide the necessary background for the next sections. To do so, we will follow [4]. To see that the point process given by jittered sampling technique is a determinantal one we will follow [3].

4.1 Basic concepts and setting

Let us start this introductory section with some definitions.

Definition 4.1.1. We say that Λ is a *locally compact Polish space* if it is a topological space which admits a topology induced by a complete and separable metric.

Definition 4.1.2. We say that μ is a *Radon measure* on Λ if it is a Borel measure which is finite on compact sets.

The examples that would have bigger relevance in this thesis are the following:

1. Let Λ be an open subset of $\mathbb{R}^k$ and μ the k-dimensional Lebesgue measure restricted to Λ.

2. Let Λ be a finite or countable set and μ such that it assigns unit mass to each element of Λ (the counting measure on Λ).

Definition 4.1.3. A *point process* $\mathcal{X}$ on Λ is a random integer-valued positive Radon measure on Λ. If $\mathcal{X}$ almost surely (a.s.) assigns at most measure 1 to singletons, it is a *simple* point process.

In this case, $\mathcal{X}$ can be identified with a random discrete subset of Λ. We denote as $\mathcal{X}(D)$ the number of points of this set that lie in $D \subset \Lambda$.

One way to describe the distribution of a point process is through its joint intensities (also known as correlation functions).

Definition 4.1.4. Let $\mathcal{X}$ be a simple point process. The *joint intensities* of a point process $\mathcal{X}$ with respect to μ are functions (if any exist) $\rho_n : \Lambda^n \to [0, \infty)$, for $n \geq 1$ such that for any family of pairwise disjoint sets $D_1, \dots, D_n$ of Λ

$$\mathbb{E}\left[\prod_{i=1}^{n} \mathcal{X}(D_i)\right] = \int_{D_1 \times \ldots \times D_n} \rho_n(x_1, \dots, x_n)\, d\mu(x_1) \cdot \ldots \cdot d\mu_(x_n).$$

In addition, we shall require that $\rho_n(x_1, \dots, x_n)$ vanishes if $x_i = x_j$ for some $i \neq j$.

If Λ is finite and μ is the counting measure on Λ, then for distinct $x_1, \dots, x_n$ the quantity $\rho_n(x_1, \dots, x_n)$ is just the probability that $x_1, \dots, x_n \in \mathcal{X}$.

Let $\mathbb{K} : \Lambda^2 \to \mathbb{C}$ be a measurable function.

Definition 4.1.5. A point process $\mathcal{X}$ on Λ is said to be a *determinantal point process* with kernel $\mathbb{K}$ if it is simple and its joint intensities with respect to the measure μ satisfy

$$\rho_n(x_1, \dots, x_n) = \det\left(\mathbb{K}(x_i, x_j)_{1 \leq i,j \leq n}\right),$$

for every $n \geq 1$ and $x_1, \dots, x_n \in \Lambda$.

Observe that we need $\mathbb{K}(x_i, x_j)$ to be well defined for every pair (x_i, x_j). Also, in order to have a definition which makes sense, we need $\det\left(\mathbb{K}(x_i, x_j)_{1 \leq i,j \leq n}\right)$ to be locally integrable on Λ^n. Both of these problems disappear when $\mathbb{K}$ is continuous.

The notion of determinantal point process was first introduced by Odile Macchi, in 1975, as a way of model fermions in quantum mechanics. These type of point processes arise surprisingly often in random matrix theory, combinatorics and physics.

A typical example of determinantal point process is the *circular unitary ensemble (CUE)*. The points on this process are the set of eigenvalues of a random unitary matrix sampled from the Haar measure on the group of $n \times n$ unitary matrices.

The *Haar measure* is the unique Borel probability measure on the group of $n \times n$ unitary matrices that is invariant under left multiplication by unitary matrices. It is also invariant under right multiplication by unitary matrices and under inversion. The result, due to Weyl and Dyson, is the following:

Theorem 4.1.6. *(Weyl, Dyson). Given an $n \times n$ unitary matrix U, let $\{e^{i\theta_j}; 1 \leq j \leq n\}$ be the set of its eigenvalues. The counting measure of eigenvalues is a determinantal point process on $\mathbb{S}^1$ with kernel*

$$\mathbb{K}(e^{i\theta}, e^{i\phi}) = \frac{1}{2\pi} \sum_{l=0}^{n-1} e^{il\theta - il\phi},$$

with respect to the Lebesgue measure on $\mathbb{S}^1$ *(with total measure* 2π*). Equivalently, the vector of eigenvalues, in uniform random order, has density*

$$\frac{1}{n!(2\pi)^n} \prod_{j<l} \|e^{i\theta_j} - e^{i\theta_l}\|^2,$$

with respect to the Lebesgue measure on $(\mathbb{S}^1)^n$*.*

4.2 Jittered sampling as a determinantal point process

In this section we are going to see that the point process given by the jittered sampling technique explained in Section 4.2 can be viewed as a determinantal point process. We have applied the jittered sampling technique to a regular area partition of $\mathbb{S}^k$ but, as we have mentioned before, this is a more general technique that applies in many different geometric settings, so this proof is going to be in a Polish space Λ.

Consider a partition $A = \{A_i\}_{i=1}^n$ of the space Λ into pairwise disjoint measurable sets of equal measure, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$, $\mu(A_i) = \frac{1}{n}$ and $\mu(\Lambda) = 1$. Define the projection operator

$$p_A(f)(x) = \sum_{i=1}^{n} \frac{\chi_{A_i}(x)}{\mu(A_i)} \int_{A_i} f(y)\, d\mu(y) = \int_{\Lambda} \mathbb{K}_A(x,y) f(y)\, d\mu(y)$$

to the space of measurable functions with respect to the finite σ-algebra generated by A. The kernel of this operator is given by

$$\mathbb{K}_A(x,y) = \sum_{i=1}^{n} \frac{\chi_{A_i}(x)\chi_{A_i}(y)}{\mu(A_i)}.$$

The determinantal point process defined by the projection kernel $\mathbb{K}_A$, $\mathcal{X}_n^A$, is then equal to the jittered sampling process associated to the partition A, which can be seen by computing

$$\mathbb{E}(\mathcal{X}_n^A(A_1) \cdot \ldots \cdot \mathcal{X}_n^A(A_n)) = \int_{A_1} \cdot \ldots \cdot \int_{A_n} \det\left(\mathbb{K}(x_i, x_j)_{1\leq i,j\leq n}\right) d\mu(x_1) \cdot \ldots \cdot d\mu(x_n).$$

If we expand the determinant, taking into account that $\mathbb{K}_A(x_i, x_j) = 0$ whenever $i \neq j$ with $x_i \in A_i$, $x_j \in A_j$ (because the sets in A are pairwise disjoint), we have that

$$\mathbb{E}(\mathcal{X}_n^A(A_1)\cdot\ldots\cdot\mathcal{X}_n^A(A_n)) = \sum_{\pi\in\mathbb{S}_n} \mathrm{sgn}(\pi) \int_{A_1} \cdot\ldots\cdot \int_{A_n} \prod_{i=1}^{n} \mathbb{K}_A(x_i, x_{\pi(i)})\, d\mu(x_1)\cdot\ldots\cdot d\mu(x_n)$$

$$= \prod_{i=1}^{n} \int_{A_i} \mathbb{K}_A(x_i, x_i)\, d\mu(x_i) = 1, \tag{4.2.1}$$

where $\mathbb{S}_n$ denotes the symmetric group and $\mathrm{sgn}(\pi)$ denotes the sign of the permutation π, using the definition of $\mathbb{K}_A$, because for all permutations, except for $\pi = id$, the term $\mathbb{K}_A(x_i, x_{\pi(i)})$ vanishes.

We know, by [4], that $\mathcal{X}_n^A$ samples n points almost surely. Thus, the product $\mathcal{X}_n^A(A_1) \cdot \ldots \cdot \mathcal{X}_n^A(A_n)$ is either 0 or 1, almost surely. By equation (4.2.1), it is equal to 1, almost surely, which means that the process $\mathcal{X}_n^A$ samples exactly one point per set of the partition A. Moreover,

$$\begin{aligned}
\mathbb{E}(\mathcal{X}_n^A(D)) &= \int_D \mathbb{K}(x,x)\, d\mu(x) \\
&= \prod_{i=1}^{n} \int_D \frac{\chi_{A_i}(x)^2}{\mu(A_i)}\, d\mu(x) \\
&= \sum_{i=1}^{n} \frac{\mu(A_i \cap D)}{\mu(A_i)} \\
&= n\mu(D),
\end{aligned}$$

because of the properties of the partition A. If $D \subset A_i$, then

$$\mathbb{E}(\mathcal{X}_n^A(D)) = \frac{\mu(D)}{\mu(A_i)}$$

because only one of the terms $\mu(A_i \cap D)$ survives thanks to the disjointness of the sets of the partition A. This means that the sample point chosen from A_i is distributed with measure $\mu_i = \frac{\mu}{\mu(A_i)}$ on A_i.

4.3 Spherical ensemble

In this section we are going to study a result, due to K. Alishahi and M. Zamani, presented in [9], concerning a specific determinantal point process called *spherical ensemble*. We are going to see the construction of such a process, in $\mathbb{S}^2$, providing a small piece of code that generates an image of such a process, and then we are going to enter into the proof of Theorem 4.3.2, and as we are going to see, using some assumptions, is very similar to the proof of Theorem 3.2.1, presented in Section 3.2.

Given an n-element set of points $P = \{x_1, \ldots, x_n\}$ on $\mathbb{S}^2$ we have defined its spherical cap discrepancy as

$$\mathbb{D}_n^2(P) = \sup_{C(x,t) \subset \mathbb{S}^2} \left| Z(P,x,t) - \frac{n\sigma(C(x,t))}{4\pi} \right|,$$

where $Z(P,x,t)$ denotes the number of points of P that lie in the spherical cap $C(x,t)$, with $x \in \mathbb{S}^2$ and $-1 \leq t \leq 1$, and σ is the surface area measure on $\mathbb{S}^2$, so $\sigma(\mathbb{S}^2) = 4\pi$. We denote the spherical cap discrepancy of an n-element set of points in $\mathbb{S}^2$ by $\mathbb{D}_n^2$.

In Section 2.2, Theorem 2.2.4, we have seen that for any n-element set of points P on $\mathbb{S}^2$ we have

$$cn^{\frac{1}{4}} \leq \mathbb{D}_n^2(P),$$

where $c > 0$ is a numerical constant that does not depend on n. Also, in Section 3.2, Theorem 3.2.1, we have seen that for any $n \geq 1$ there exists an n-element set of points P, with cardinality n, on $\mathbb{S}^2$ such that

$$\mathbb{D}_n^2(P) \leq Cn^{\frac{1}{4}}\sqrt{\log n},$$

where $C > 0$ is a numerical constant that does not depend on n.

4.3.1 Construction

Let us see how the spherical ensemble is constructed. We are going to construct a point process $\mathcal{X}^{(n)} = \sum_{i=1}^n \delta_{P_i}$, so we must determine $\{P_1, \ldots, P_n\}$. Let A_n and B_n be two independent $n \times n$ random matrices with independent and identically distributed standard complex Gaussian entries.

Definition 4.3.1. A complex-valued random variable Z follows a *standard complex Gaussian distribution*, and it is denoted by $Z \sim N_{\mathbb{C}}(0,1)$, if its density function is

$$f_Z(z) = \frac{1}{\pi} e^{-|z|^2}, \quad z \in \mathbb{C},$$

with respect to the Lebesgue measure.

Let us denote as $\{\lambda_1, \ldots, \lambda_n\}$ the set of eigenvalues of $A_n^{-1} B_n$. We can consider this set of eigenvalues as a (simple) random point process on $\mathbb{C}$. We can describe it using the joint intensities $\rho_k^{(n)}$, $1 \leq k \leq n$, with respect to the measure

$$d\mu(z) = \frac{n}{\pi(1+|z|^2)^{n+1}}\, dz$$

where dz denotes the Lebesgue measure on $\mathbb{C}$.

M. Krishnapur showed, in [10], that this random point process is a determinantal point process on the complex plane with kernel

$$\mathbb{K}^{(n)}(z,w) = (1+z\bar{w})^{n-1}$$

with respect to the measure $d\mu(z)$, i.e.,

$$\rho_k^{(n)}(z_1, \ldots, z_k) = \det\left(\mathbb{K}^{(n)}(z_i, z_j)\right)_{1 \leq i,j \leq k}$$

for every $k \geq 1$ and $z_1, \ldots, z_k \in \mathbb{C}$.

Let us consider h the stereographic projection of the sphere $\mathbb{S}^2$ from the north pole $(0,0,1)$ onto $\mathbb{C}$, seen as the two-dimensional plane $\{(t_1, t_2, 0);\, t_1, t_2 \in \mathbb{R}\}$, i.e., for $(x_1, x_2, x_3) \in \mathbb{S}^2$

$$h(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$

Now, let $P_i = h^{-1}(\lambda_1)$, for $i = 1, \ldots, n$. As K. Alishahi and M. Zamani comment in [9], it can be seen that the vector $(P_1, \ldots, P_n)$, in uniform random order, has the joint density

$$c \prod_{i<j} \|p_i - p_j\|^2,$$

with respect to the Lebesgue measure on $(\mathbb{S}^2)^n$, where $c > 0$ is a numerical constant that does not depend on n.

As we have seen, the spherical ensemble is quite simple to construct. Let us see a figure of a spherical ensemble of 500 points.

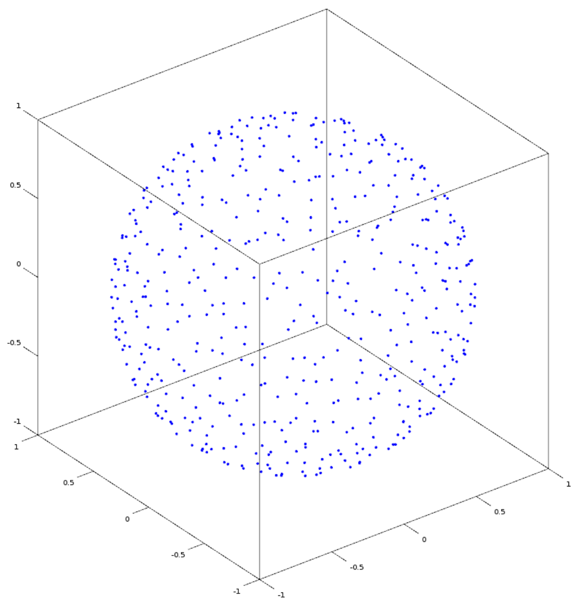

Figure 4.1: Spherical ensemble of 500 points.

It has been obtained with the following piece of code, that can run in Matlab or Octave.

```
% We want 500 points.
n=500;
% Standard deviation of the real and imaginary parts.
c=1/sqrt(2)
```

```
% Eigenvalues of the matrix M^{-1}*N.
A=eig(inv(c*randn(n)+i*c*randn(n))*(c*randn(n)+i*c*randn(n)));
% Inverse of the stereographic projection.
B=[2*real(A)./(1+A.*conj(A)),2*imag(A)./(1+A.*conj(A)),
(-1+A.*conj(A))./(1+A.*conj(A))];
% Plot.
plot3( B(:,1), B(:,2), B(:,3),'.');
axis square
```

Let us study the result, presented in [9], which describes the behaviour of the spherical cap discrepancy of a concrete type of set of points, which is a determinantal point process, the spherical ensemble, as we increase the number of points.

4.3.2 Spherical cap discrepancy of the spherical ensemble

Given two functions f and g, the notation $f(x) = O(g(x))$ means that there exist a constant $C > 0$ and x_0 such that $|f(x)| \leq Cg(x)$ for all $x \geq x_0$.

Theorem 4.3.2. *Consider the point process $\mathcal{X}^{(n)} = \sum_{i=1}^{n} \delta_{P_i}$ where the points $\{P_1, \dots, P_n\}$ are the ones given by the spherical ensemble. For every $M > 0$ independent of n, we have*

$$\mathbb{D}_n^2(\{P_1, \dots, P_n\}) = O(n^{\frac{1}{4}}\sqrt{\log n})$$

with probability at least $1 - \frac{1}{n^M}$.

Let $C = C(x,t)$ be a spherical cap on $\mathbb{S}^2$, with $x \in \mathbb{S}^2$ and $-1 \leq t \leq 1$. Recall that $\mathcal{X}^{(n)}(C)$ denotes the number of points of the point process $\mathcal{X}^{(n)}$ that lie in C, following the notation established at the beginning of this chapter. We are going to assume the following, proved in [9]:

1. For any spherical cap $C = C(x,t)$ on $\mathbb{S}^2$, with $x \in \mathbb{S}^2$ and $-1 \leq t \leq 1$, the random variable $\mathcal{X}^{(n)}(C)$ has the same distribution as $\sum_k \eta_k$ where η_k are independent Bernoulli random variables.

2. The expected value of $\mathcal{X}^{(n)}(C)$ is equal to $n\frac{\sigma(C)}{4\pi}$.

3. There exists an absolute constant $c > 0$ such that

$$\mathrm{Var}(\mathcal{X}^{(n)}(C)) \leq c\sqrt{n}.$$

In [9], Alishani and Zamani specify the value $\mathbb{P}(\eta_k = 1)$, but for our purposes we do not need this value. Let us pass to study the proof of Theorem 4.3.2.

Proof. Using the first assumption above and Bernstein-Chernoff's inequality, Lemma 3.1.7, presented in Section 3.1.2, we have

$$\mathbb{P}(|\mathcal{X}^{(n)}(C) - \mathbb{E}(\mathcal{X}^{(n)}(C))| \geq t) \leq 2e^{-\min\left(\frac{t^2}{4\mathrm{Var}(\mathcal{X}^{(n)}(C))}, \frac{t}{4}\right)},$$

and using the third assumption we get

$$\mathbb{P}(|\mathcal{X}^{(n)}(C) - \mathbb{E}(\mathcal{X}^{(n)}(C))| \geq t) \leq 2e^{-\min\left(\frac{t^2}{4c\sqrt{n}}, \frac{t}{4}\right)}.$$

Now, taking $t = O(n^{\frac{1}{4}}\sqrt{\log n})$ we have

$$\mathbb{P}\big(|\mathcal{X}^{(n)}(C) - \mathbb{E}(\mathcal{X}^{(n)}(C))| \geq O(n^{\frac{1}{4}}\sqrt{\log n})\big) \leq 2e^{-\min\left(\frac{O(\log n)}{4c}, \frac{O(n^{\frac{1}{4}}\sqrt{\log n})}{4}\right)}.$$

Since the exponential term goes (quickly) to 0 as $n \to \infty$, because $\log n$ and $n^{\frac{1}{4}}\sqrt{\log n}$ go to infinity as $n \to \infty$, for any $M > 0$ we have

$$\mathcal{X}^{(n)}(C) = \mathbb{E}(\mathcal{X}^{(n)}(C)) + O(n^{\frac{1}{4}}\sqrt{\log n}) \tag{4.3.1}$$

with probability at least $1 - \frac{1}{n^M}$, where the implied constant in $O(\cdot)$ does not depend on C.

As we have argued previously, in the proof of Theorem 3.2.1, there exists a subfamily $\mathcal{S}^2$ of the family of all spherical caps on $\mathbb{S}^2$, with cardinality $n^{c_{40}}$, where $c_{40} > 0$ is an absolute constant, satisfying that, given a spherical cap $C' \subset \mathbb{S}^2$, there exists $A, B \in \mathcal{S}$ with

$$A \subset C' \subset B \quad \text{and} \quad \sigma(B \setminus A) < \frac{4\pi}{n}.$$

If $A, B \in \mathcal{S}^2$ are as before, with respect to the spherical cap C' on $\mathbb{S}^2$, we have that

$$\begin{aligned}
|\mathcal{X}^{(n)}(C') - \mathbb{E}(\mathcal{X}^{(n)}(C'))| &\leq \left|\mathcal{X}^{(n)}(B) - \frac{n\sigma(C')}{4\pi}\right| \\
&\leq \left|\mathcal{X}^{(n)}(B) - n\frac{\sigma(B)}{4\pi}\right| + \left|n\frac{\sigma(B \setminus C')}{4\pi}\right| \\
&< |\mathcal{X}^{(n)}(B) - \mathbb{E}(\mathcal{X}^{(n)}(B))| + n\frac{1}{n} \\
&\leq \max\left\{|\mathcal{X}^{(n)}(B) - \mathbb{E}(\mathcal{X}^{(n)}(B))|, |\mathcal{X}^{(n)}(A) - \mathbb{E}(\mathcal{X}^{(n)}(A))|\right\} + 1,
\end{aligned}$$

using the second assumption. This implies that the discrepancy of the family of the spherical caps on $\mathbb{S}^2$ is of the same order as the discrepancy of the subfamily $\mathcal{S}^2$, and hence, we can restrict ourselves to spherical caps in $\mathcal{S}^2$. Using this reduction in the union bound we get that (4.3.1) holds uniformly in C, as we wanted to see. $\square$

As we have seen, the structure of this proof is the same as the proof of Theorem 3.2.1. We see, using Bernstein-Chernoff's inequality, Lemma 3.1.7, that the probability of the complementary statement is really small for a fixed spherical cap and then we pass this estimation to the whole family using an approximation family of certain cardinality.

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