Consistency, weak fairness, and the Shapley value

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Abstract

The Shapley value (Shapley, 1953) has been axiomatically characterized from different points of view. van den Brink (2001) proposes a characterization by means of efficiency, fairness and the null player property. In this paper, we characterize the family of single-valued solutions obtained by relaxing fairness into weak fairness. To point out the Shapley value, we impose the additional axiom of weak self consistency and strengthen the null player property into the dummy player property. Remarkably, impossibility results emerge when replacing self consistency by a large set of α-consistency properties à Thomson (1990).

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1. Introduction

Probably the most relevant single-valued solution for cooperative games with transferable utility (games, hereafter) is the *Shapley value* (Shapley, 1953). Many characterizations of this solution, including his original axiomatic approach, use the principle that if a player contributes zero to all coalitions, then she must receive a zero payoff: the *null player property*. Various authors have proposed alternative foundations of the Shapley value imposing the *null player property*. Particularly, van den Brink (2001) interprets the Shapley value as the unique solution satisfying, additionally, *efficiency*, imposing that all the gains from cooperation are distributed among the players, and *fairness*, a property inspired by Myerson's (1977) fairness. For single-valued solutions, *fairness* essentially imposes that if a game suffers an impact consisting in adding another game in which two players are symmetric, then their payoffs should change by the same amount. If we measure the relevance of a player in terms of her marginal contributions to all coalitions,

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fairness is a quite natural requirement since adding such a game does not change the contributions of symmetric players.

In this paper, we study what solutions emerge when weakening *fairness* into *weak fairness* (van den Brink et al., 2016),[1] combined again with *efficiency* and either the *null player property* or the *dummy player property*, which states that if a player contributes only her individual worth to all coalitions then she must receive her individual worth. *Weak fairness*, a property very much related to *strong aggregate monotonicity* (Arin, 2013), can be viewed as a solidarity axiom in the sense that if only the worth of the grand coalition varies, while the worth of all other coalitions remain unchanged, then players' payoffs should be affected equally.[2]

Another different principle used from Hart and Mas-Colell (1989) to interpret the Shapley value is *self consistency*. Consistency is an outstanding relational property widely used in the axiomatic analysis of solutions imposing that an original agreement should be reconfirmed in the underlying reduced game when some agents leave.[3] Calleja and Llerena (2019) impose *self consistency* and *fairness* together with *covariance*, a classical invariance property with respect to changes in scale and origin, to characterize the Shapley value. In this work, we impose *weak self consistency*, that is, *self consistency* when only one or two agents stay, to select the Shapley value from the set of solutions satisfying *efficiency*, *weak fairness* and the *dummy player property*.[4] This characterization has the flavour of Hart and Mas-Colell (1989), van den Brink (2001) and Calleja and Llerena (2019) but it uses substantially weaker versions of *fairness* and *self consistency*.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we characterize the family of single-valued solutions satisfying *efficiency*, *weak fairness* and either the *dummy player property* or the *null player property*. Remarkably, these characterizations can be extended to any domain of games. In Section 4, we provide a new axiomatization of the Shapley value by means of *weak self consistency*, *weak fairness* and the *dummy player property*. These properties still characterize the Shapley value on the domain of convex games. Interestingly, we show that incompatibilities emerge when replacing self consistency by a huge class of consistency properties that includes, as particular cases, *max consistency* (Davis and Maschler, 1965), *complement consistency* (Moulin, 1985) and *projected consistency* (Funaki, 1998).

[1] These authors impose *weak fairness*, together with other properties, to characterize all convex combinations of the equal division solution and the center of imputations (Driessen and Funaki, 1991).

[2] *Strong aggregate monotonicity* imposes that if only the grand coalition become richer then all players are affected equally and are strictly better off.

[3] See Thomson (2012) for an essay on the consistency property.

[4] The *dummy player property* does not imply (is not implied by) *covariance*.

2. Preliminaries

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. A **coalition** is a non-empty finite subset of $\mathbb{N}$ and let $\mathcal{N}$ denote the set of all coalitions of $\mathbb{N}$. Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ and $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. A **transferable utility coalitional game** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a real number $v(S)$, representing what S can achieve by agreeing to cooperate, with the convention $v(\emptyset) = 0$. For simplicity of notation, and if no confusion arises, we write $v(i), v(ij), \ldots$ instead of $v(\{i\}), v(\{i,j\}), \ldots$. By Γ we denote the class of all games.

Given $N \in \mathcal{N}$, the **unanimity game** (N, u_N) associated to N is defined as $u_N(N) = 1$ and $u_N(S) = 0$ otherwise. Given a game (N, v) and $\emptyset \neq N' \subset N$, the **subgame** $(N', v_{|N'})$ is defined as $v_{|N'}(S) = v(S)$ for all $S \subseteq N'$. For any two games (N, v), (N, w), and $\alpha \in \mathbb{R}$, we define the game $(N, v + w)$ as $(v + w)(S) = v(S) + w(S)$, and the game $(N, \alpha \cdot v)$ as $(\alpha \cdot v)(S) = \alpha \cdot v(S)$, for all $S \subseteq N$.

Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. The vector $e_N \in \mathbb{R}^N$ is defined as $e_{N,i} = 1$ for all $i \in N$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} = (x_i)_{i \in T} \in \mathbb{R}^T$.

Let $N \in \mathcal{N}$. The **preimputation set** of (N, v) consists of the efficient payoff vectors, that is, $X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$, and the **core** is the set of preimputations where each coalition gets at least its worth, that is $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S) \; \forall \, S \subseteq N\}$. A game (N, v) is **balanced** if it has a non-empty core, and it is **(strictly) convex** if $v(S \cup T) + v(S \cap T) \; (>) \geq v(S) + v(T)$ for all $S, T \subseteq N$. For $t \in \mathbb{R}$ and any game (N, v), the game (N, v^t) is defined as $v^t(S) = v(S)$ for all $S \subset N$ and $v^t(N) = v(N) + t$. Player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subseteq N \setminus \{i\}$, and is called a **null player** in (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. We say that players i and j are **symmetric** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **single-valued solution** on $\Gamma' \subseteq \Gamma$ is a function $\sigma : \Gamma' \to \bigcup\limits_{N \in \mathcal{N}} \mathbb{R}^N$ that associates with each game $(N, v) \in \Gamma'$ an $|N|$-dimensional real vector $\sigma(N, v)$. The **Shapley value**, Sh, is defined by

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! \, (|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S) \right) \text{ for all } i \in N.$$

With any preimputation $x \in X(N, v)$ we associate the vector of all excesses $e(S, x) = v(S) - x(S)$, $\emptyset \neq S \subset N$, the components of which are non-increasingly ordered. The

prenucleolus (Schmeidler, 1969), ν, is the preimputation that minimizes with respect to the lexicographic order[5] the vector of excesses over the set of preimputations.

3. Efficiency, weak fairness and dummy or null player property

van den Brink (2001) characterizes the Shapley value on the domain of all games making use of *fairness* together with *efficiency* and the *null player property.*

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Efficiency** (**E**) if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$ it holds that $\sigma(N, v) \in X(N, v)$;
- **Null player property** (**NP**) if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $i \in N$, if i is a null player in (N, v), then $\sigma_i(N, v) = 0$;
- **Fairness** (**F**) if for all $N \in \mathcal{N}$, all $(N, v), (N, v') \in \Gamma'$ with $(N, v + v') \in \Gamma'$ and all $i, j \in N$ such that i and j are symmetric in (N, v'), we have $\sigma_i(N, v+v') - \sigma_i(N, v) = \sigma_j(N, v + v') - \sigma_j(N, v)$.

In this section, we investigate what single-valued solutions appear when weakening *fairness* into *weak fairness* combined with *efficiency* and either the *dummy player property* or the *null player property.*

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Weak Fairness** (**wF**) if for all $N \in \mathcal{N}$, all $(N, v), (N, v') \in \Gamma'$ such that $v(S) = v'(S)$ for all $S \subset N$ and all $i, j \in N$, we have $\sigma_i(N, v') - \sigma_i(N, v) = \sigma_j(N, v') - \sigma_j(N, v)$;
- **Dummy player property** (**DP**) if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $i \in N$, if i is a dummy player in (N, v) then $\sigma_i(N, v) = v(i)$.

Weak fairness is a solidarity axiom that plays an important role in the paper. It imposes that if the game suffers an impact that changes only the worth of the grand coalition, while all other coalitions remain the same, then, and due to all players are equally responsible of it, they should be treated equally. By taking *weak fairness* together with *efficiency* and the *dummy player property* a large family of single-valued solutions emerge. In order to describe such a family for any domain of games, we first introduce some concepts. In the remainder of this section we deal with a fixed player set N and, consequently, a game (N, v) is described by its characteristic function v and Γ denotes the set of all games with player set N.

[5]Given two vectors $x, y \in \mathbb{R}^N$, we say that $x \leq_{lex} y$ if either $x = y$, or $x_1 < y_1$ or there exists $k \in \{2, \ldots, |N|\}$ such that $x_i = y_i$ for all $1 \leq i \leq k-1$ and $x_k < y_k$.

Given $v \in \Gamma$, a player $i \in N$ is called a **potential dummy player** in v if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subset N \setminus \{i\}$. By $D(v)$ and $PD(v)$ we denote the set of dummies and potential dummies in v, respectively. Notice that $D(v) \subseteq PD(v)$ and, moreover, any player $i \in PD(v) \backslash D(v)$ will become a dummy player in game $w \in \Gamma$ with $w(S) = v(S)$ for all $S \subset N$ and $w(N) = v(i) + v(N \backslash \{i\})$. Moreover, Lemma 3 in Calleja et al. (2012) states that if $i \in D(v)$ and $j \in PD(v)$, $i \neq j$, then $j \in D(v)$ and $v(N) = \sum_{i \in PD(v)} v(i) + v(N \backslash PD(v))$. Hence, either $D(v) = \emptyset$ or $D(v) \neq \emptyset$ and thus $D(v) = PD(v)$.

Let $\Gamma' \subseteq \Gamma$ be a certain domain of games with player set N. We next define the **equivalence relation** $\mathcal{R}$ on Γ' as follows: for all $v, w \in \Gamma'$

$$v \mathcal{R} w \text{ if and only if } v(S) = w(S) \text{ for all } S \subset N.$$

The **equivalence class** containing $v \in \Gamma'$ is denoted by $[v] = \{w \in \Gamma' : w \mathcal{R} v\}$. Let $\Gamma' \diagup \mathcal{R} = \{[v] : v \in \Gamma'\}$ be the **quotient set**. For every equivalence class $[v] \in \Gamma' \diagup \mathcal{R}$ we fix a **representative element**, denoted by v_*. If there is $w \in [v]$ such that $D(w) \neq \emptyset$, by Lemma 3 in Calleja et al. (2012), w is unique and then choose $v_* = w$. Otherwise, choose an arbitrary $v_* \in [v]$. It is worth to mention that although $v \in \Gamma'$ and $PD(v) \neq \emptyset$, there might be no $w \in [v]$ such that $D(w) \neq \emptyset$.

Let Γ'_* stand for a **set of representative games**, one for each equivalence class. Observe that the set of representative games might not be unique. Any $v \in [v_*]$ can be expressed as

$$v = v_* + (v(N) - v_*(N)) \cdot u_N.$$

Moreover, $PD(v) = PD(v_*)$ and $v_*(N) = \sum_{i \in PD(v)} v(i) + v(N \backslash PD(v))$, whenever $D(v_*) \neq \emptyset$.

Definition 1. *Let $\Gamma' \subseteq \Gamma$ and Γ'_* be a set of representative games. A dummy-adapted Γ'_*-selection is a function $F : \Gamma'_* \to \mathbb{R}^N$ such that $\sum_{i \in N} F_i(v_*) = v_*(N)$ and $F_i(v_*) = v_*(i)$ for all $i \in D(v_*)$.*

Let $\Gamma' \subseteq \Gamma$ and Γ'_* be a set of representative games. A dummy-adapted Γ'_*-selection associates an efficient vector to every representative game of Γ'_*, with the particularity that if dummy players appear they receive exactly their individual worth. Let $\mathcal{F}_D(\Gamma'_*)$ denote the class of dummy-adapted Γ'_*-selections. Given $F \in \mathcal{F}_D(\Gamma'_*)$, we can define the associated single-valued solution σ^F as follows: for all $v \in \Gamma'$,

$$\sigma^F(v) = F(v_*) + \frac{v(N) - v_*(N)}{|N|} \cdot e_N, \tag{1}$$

being $v_* \in \Gamma'_*$ such that $v \in [v_*]$.

The interpretation of σ^F is as follows: given $v \in \Gamma'$, let $v_* \in \Gamma'_*$ be such that $v \in [v_*]$, σ^F first distributes $v_*(N)$ among players according to F, and then it distributes what is left of the gains of cooperation equally. Geometrically, σ^F is the set of straight lines (one for every element of $\Gamma' \diagup \mathcal{R}$) going through $F(v_*)$ with direction vector $\frac{e_N}{|N|}$.

Let us illustrate the calculation of σ^F solutions by an example.

Example 1. *Let $N = \{1,2,3,4\}$ and $\Gamma' = \Gamma$. We choose the set of representative games Γ_* as follows. For a given game $v \in \Gamma$, if $PD(v) = \emptyset$ then take $v_* \in [v]$ with $v_*(S) = v(S)$ for all $S \subset N$ and $v_*(N) = 0$. While if $PD(v) \neq \emptyset$ then take $v_* \in [v]$ with $v_*(S) = v(S)$ for all $S \subset N$ and $v_*(N) = \sum_{i \in PD(v)} v(i) + v(N \setminus PD(v))$. Note that if $PD(v) = \emptyset$ we have some freedom to choose $v_*(N)$. On the other hand, if $PD(v) \neq \emptyset$ then v_* is unique, $D(v_*) \neq \emptyset$ and $D(v_*) = PD(v_*) = PD(v)$.*

Since both the Shapley value and the prenucleolus satisfy efficiency *and the* dummy player property*, two examples of dummy-adapted Γ_*-selections, $F, G \in \mathcal{F}_D(\Gamma_*)$, are: $F(v_*) = Sh(v_*)$ and $G(v_*) = \nu(v_*)$, for all $v_* \in \Gamma_*$.*

Now we take the game $v(12) = v(13) = v(14) = v(123) = v(124) = v(134) = 1$, $v(N) = 2$, and $v(S) = 0$ otherwise. Since $PD(v) = \emptyset$, as a representative element of $[v]$ we choose the game v_*, being $v_*(N) = 0$ and $v_*(S) = v(S)$ for any other coalition $S \subset N$. Then,

$$\sigma^F(v) = Sh(v_*) + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}\right) + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

while

$$\sigma^G(v) = \nu(v_*) + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{3}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}\right) + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{5}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Clearly, $F(v_*)$ and $G(v_*)$ propose efficient allocations of $v_*(N)$. In a second stage, to obtain $\sigma^F(v)$ and $\sigma^G(v)$, the difference $v(N) - v_*(N)$ is distributed equally among the players.

Now consider the game $w(1) = w(23) = w(123) = 0$, $w(234) = 3$, $w(N) = 4$ and $w(S) = 1$ otherwise. Since $PD(w) = \{1\}$, the representative element $w_* \in [w]$ is given by $w_*(N) = w(1) + w(234) = 3$ and $w_*(S) = w(S)$ for any other coalition $S \subset N$. Hence,

$$\sigma^F(w) = Sh(w_*) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(0, \frac{5}{6}, \frac{5}{6}, \frac{4}{3}\right) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}, \frac{13}{12}, \frac{13}{12}, \frac{19}{12}\right)$$

while

$$\sigma^G(w) = \nu(w_*) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = (0,1,1,1) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right).$$

Here, $F(w_*)$ and $G(w_*)$ are efficient allocations of $w_*(N)$ such that player 1, which is a dummy player in w_*, receives its individual worth $w_*(1) = w(1) = 0$. In a second stage, to obtain $\sigma^F(w)$ and $\sigma^G(w)$, we distribute equally $w(N) - w_*(N)$.

To finish, observe that $\sigma^F(v) = Sh(v)$ *and* $\sigma^F(w) = Sh(w)$ *due to the Shapley value, in addition to* efficiency *and the* dummy player property*, satisfies* weak fairness*. However, although* $\sigma^G(w) = \nu(w)$ *we have that* $\sigma^G(v) \neq \nu(v)$*, confirming that the prenucleolus does not satisfy* weak fairness*.*

Now, we are in a position to obtain our characterization result.

Theorem 1. *Let* $\Gamma' \subseteq \Gamma$ *and* Γ'_* *be a set of representative games. A single-valued solution* σ *on* Γ' *satisfies efficiency, weak fairness and the dummy player property if and only if there exists a dummy-adapted* Γ'_*-*selection* $F \in \mathcal{F}_D(\Gamma'_*)$ *such that* $\sigma = \sigma^F$*.*

Proof Let σ be a single-valued solution on Γ' and $F \in \mathcal{F}_D(\Gamma'_*)$ such that $\sigma = \sigma^F$. Then, **E** follows directly from $F \in \mathcal{F}_D(\Gamma'_*)$. To check **DP**, let $v \in \Gamma'$ be with $D(v) \neq \emptyset$ and let $v_* \in \Gamma'_*$ be such that $v \in [v_*]$, then $v = v_*$. Consequently, for all $i \in D(v)$,

$$\sigma_i(v) = \sigma_i^F(v) = F_i(v_*) = v_*(i) = v(i),$$

where the last but one equality follows from $F \in \mathcal{F}_D(\Gamma'_*)$. To check **wF**, let $v, w \in \Gamma'$ be such that $v(S) = w(S)$ for all $S \subset N$. Hence, $v, w \in [v_*]$ being $v_* \in \Gamma'_*$ and, for all $i \in N$,

$$\begin{aligned} \sigma_i(w) - \sigma_i(v) &= F_i(v_*) + \tfrac{w(N)-v_*(N)}{|N|} - F_i(v_*) - \tfrac{v(N)-v_*(N)}{|N|} \\ &= \tfrac{w(N)-v(N)}{|N|}. \end{aligned}$$

To show the reverse implication, let σ be a single-valued solution on Γ' satisfying **E**, **wF** and **DP**. Define $F : \Gamma'_* \to \mathbb{R}^N$ as $F(v_*) = \sigma(v_*)$ for all $v_* \in \Gamma'_*$. By **E** and **DP**, it follows directly that $F \in \mathcal{F}_D(\Gamma'_*)$. To finish, let us show that $\sigma = \sigma^F$ with $F \in \mathcal{F}_D(\Gamma'_*)$ as defined above. Let $v \in \Gamma'$, $v \in [v_*]$ being $v_* \in \Gamma'_*$. Since $v = v_* + (v(N) - v_*(N)) \cdot u_N$, by **E** and **wF**,

$$\begin{aligned} \sigma(v) &= \sigma(v_*) + \tfrac{v(N)-v_*(N)}{|N|} \cdot e_N \\ &= F(v_*) + \tfrac{v(N)-v_*(N)}{|N|} \cdot e_N \\ &= \sigma^F(v). \end{aligned}$$

□

Remarkably, a solution satisfying *efficiency*, *weak fairness*, and the *dummy player property* must be described in two stages. Given a game v, we first search the representative element $v_* \in [v]$ and distribute efficiently $v_*(N)$ taking into account that if $D(v_*) \neq \emptyset$, each dummy player should receive her individual worth. In a second stage, the difference $v(N) - v_*(N)$ is distributed equally among the players. The first stage is a direct consequence of *efficiency* and the *dummy player property*, and the second stage is a direct consequence of *efficiency* and *weak fairness.*

It is natural to ask for the consequences of weakening the *dummy player property* into the *null player property* in Theorem 1. It is not difficult to extend the notion of potential dummy player to potential null player in order to characterize the family of single-valued solutions satisfying *efficiency*, *weak fairness* and the *null player property*.

Let $v \in \Gamma$, a player $i \in N$ is called a **potential null player** in v if $v(S \cup \{i\}) = v(S)$ for all $S \subset N \setminus \{i\}$. Let $N(v)$ and $PN(v)$ be the set of null and potential null players in v, respectively. Clearly, $N(v) \subseteq PN(v)$ and, moreover, from Lemma 3 in Calleja et al. (2012) either $N(v) = \emptyset$ or $N(v) \neq \emptyset$ and thus $N(v) = PN(v)$ with $v(N) = v(N \setminus PN(v))$.

Let $\Gamma' \subseteq \Gamma$ be a certain domain of games with player set N. For every equivalence class $[v] \in \Gamma' \diagup \mathcal{R}$ we choose a **representative element** $v_\bullet$ as follows: if there is $w \in [v]$ such that $N(w) \neq \emptyset$, w is unique, and then choose $v_\bullet = w$; otherwise, choose an arbitrary $v_\bullet \in [v]$. Let $\Gamma'_\bullet$ stand for a set of representative games. Now, we can define a null-adapted $\Gamma'_\bullet-$selection analogously to Definition 1. Let $\mathcal{F}_N(\Gamma'_\bullet)$ denote the class of null-adapted $\Gamma'_\bullet-$selections.

Theorem 2. *Let $\Gamma' \subseteq \Gamma$ and $\Gamma'_\bullet$ be a set of representative games. A single-valued solution σ on Γ' satisfies efficiency, weak fairness and the null player property if and only if there exists a null-adapted $\Gamma'_\bullet$-selection $F \in \mathcal{F}_N(\Gamma'_\bullet)$ such that $\sigma = \sigma^F$.*

Note that both Theorem 1 and Theorem 2 are stated for any domain $\Gamma' \subseteq \Gamma$. The properties in both theorems are non-redundant on Γ. The **equal division solution**, ED, defined by $ED_i(v) = \frac{v(N)}{|N|}$ for all $v \in \Gamma$ and all $i \in N$ meets all properties but neither the *dummy player property* nor the *null player property*. The single-valued solution ρ defined as $\rho_i(v) = v(i)$ for all $v \in \Gamma$ and all $i \in N$ meets all properties but *efficiency*. Let π be a permutation on N, the **marginal contribution solution** relative to π, mc^π, defined as $mc_i^\pi(v) = v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\})$ for all $v \in \Gamma$ and all $i \in N$ meets all properties but *weak fairness*. On the contrary, there are domains $\Gamma' \subset \Gamma$ where the properties are redundant. For instance, on the domain of assignment games, Γ_A, (Shapley and Shubik, 1972) *weak fairness* does not apply since if $v \in \Gamma_A$ and $t \in \mathbb{R}$, $t \neq 0$, then $v^t \notin \Gamma_A$. Hence, *weak fairness* is redundant in both theorems. Similarly, on the domain of strictly convex games, Γ_{SC}, the *dummy player property* (and also the *null player property*) does not apply since for all $v \in \Gamma_{SC}$, $D(v) = \emptyset$ (and thus $N(v) = \emptyset$). Note that, furthermore, $PD(v) = \emptyset$ (and thus $PN(v) = \emptyset$). Hence, the *dummy player property* is redundant in Theorem 1 and the *null player property* in Theorem 2. However, observe that the solutions proposed above show the independence of the properties in Theorem 1 (Theorem 2) for domains $\Gamma' \subseteq \Gamma$ that are rich enough, that is, domains that contain a game(s) for which ED, ρ, and mc^π does not satisfy the *dummy player property* (the *null player property*), *efficiency*, and *weak fairness*, respectively.

4. Consistency, weak fairness and dummy or null player property

Consistency is an internal stability requirement that relates the solution of a game to the solution of a reduced game that appears when some agents leave. The different ways in which the coalitions of the remaining agents are evaluated give rise to different notions of reduced game. Particularly interesting are the reduced games introduced by Davis and Maschler (1965), Moulin (1985) and Funaki (1998). All these instances can be included in a general definition making use of the concept of **admissible subgroup correspondence** introduced by Thomson (1990).

Definition 2. *An admissible subgroup correspondence* $\alpha : \mathcal{N} \to \mathcal{N}$ *is a correspondence that associates with each* $N \in \mathcal{N}$ *a non-empty list* $\alpha(N)$ *of coalitions of* N*.*

We denote by $\mathcal{A}$ the set of all admissible subgroup correspondences. Examples of admissible subgroup correspondences can be given by taking into consideration several aspects of coordination between players: communication, hierarchies, geographical areas, law requirements, or the size of the subgroups.

Given $\alpha \in \mathcal{A}$, we introduce the associated α-**max reduced game**.

Definition 3. *Let* $\alpha \in \mathcal{A}$ *and* (N, v) *be a game,* $\emptyset \neq N' \subset N$ *and* $x \in \mathbb{R}^N$*. The* α*-max reduced game relative to* N' *at* x *is the game* $\left(N', r^{N'}_{\alpha,x}(v)\right)$ *defined by*

$$r^{N'}_{\alpha,x}(v)(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ \max\limits_{Q \in \alpha(N \setminus N')} \{v(R \cup Q) - x(Q)\} & \text{if } \emptyset \neq R \subset N', \\ v(N) - x(N \setminus N') & \text{if } R = N'. \end{cases} \quad (2)$$

The interpretation of the α-max reduced game is as in Davis and Maschler (1965) but here the options of members in N' to cooperate with members in $N \setminus N'$ are restricted by the admissible subgroup correspondence α. That is, in the α-max reduced game (relative to N' at x) the worth of a coalition $R \subset N'$ is determined under the assumption that R can choose the best partners in $\alpha(N \setminus N')$. The **Davis** and **Maschler reduced game** is a particular case when $\alpha(N) = 2^N$ for all $N \in \mathcal{N}$. Other well-known reduced games can also be obtained by taking a suitable admissible subgroup correspondence. For instance, the **complement reduced game** proposed by Moulin (1985) is defined by $\alpha(N) = \{N\}$ for all $N \in \mathcal{N}$, or the **projected reduced game** (Funaki, 1998) by $\alpha(N) = \{\emptyset\}$ for all $N \in \mathcal{N}$. The above reduction operations will be denoted by α_{DM}, α_M and α_P, respectively.

Given $\alpha \in \mathcal{A}$, a single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- α-**Consistency** *(*α-**CO***)* if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $\emptyset \neq N' \subset N$ we have $\left(N', r^{N'}_{\alpha,x}(v)\right) \in \Gamma'$ and $x_{|N'} = \sigma\left(N', r^{N'}_{\alpha,x}(v)\right)$ where $x = \sigma(N, v)$.

The consistency principle states that in the corresponding α-max reduced game the original agreement should be reconfirmed. α-**weak consistency** (α-**wCO**) imposes this internal stability requirement for reduced games with at most two players.

Surprisingly, there is no admissible subgroup correspondence $\alpha \in \mathcal{A}$ for which the properties in Theorem 2, *efficiency*, *weak fairness* and the *null player property*, are compatible with α-*weak consistency*.

Theorem 3. *Let $\alpha \in \mathcal{A}$. There is no single-valued solution on the domain of all games that satisfies efficiency, α-weak consistency, weak fairness and the null player property.*

Proof Let $\alpha \in \mathcal{A}$ and suppose, on the contrary, that there exists a single-valued solution σ satisfying **E**, **NP**, **wF** and α-**wCO** on Γ. Let (N, v) be a game with set of players $N = \{1, 2, 3\}$ and characteristic function $v(i) = 0$ for all $i \in N$, $v(12) = v(123) = 1$ and $v(13) = v(23) = 0$. Let $x = \sigma(N, v)$ and $z = \sigma(N, v^{1.5})$. Since player 3 is a null player in the game (N, v), by **NP**

$$x_3 = 0, \tag{3}$$

and **wF** together with **E** lead to

$$z_3 = x_3 + \frac{1}{2} = \frac{1}{2}. \tag{4}$$

Denote $N' = \{1, 2\}$ and $N'' = \{2, 3\}$. Note that for a single player $i \in N$, $\alpha(i)$ can only be $\alpha_{DM}(i)$, $\alpha_P(i)$ or $\alpha_M(i)$. We distinguish three cases:

Case 1: $\alpha(N \setminus N') = \alpha(3) = \alpha_{DM}(3) = \{\emptyset, \{3\}\}$. By definition of α_{DM}-max reduced game, $r^{N'}_{\alpha_{DM},z}(v^{1.5})(1) = r^{N'}_{\alpha_{DM},z}(v^{1.5})(2) = 0$ and $r^{N'}_{\alpha_{DM},z}(v^{1.5})(12) = 2.5 - \frac{1}{2} = 2$. Let us denote $w_1 = r^{N'}_{\alpha_{DM},z}(v^{1.5})$. Note that $(w_1)^{-2}(12) = 0$. Hence, players 1 and 2 are null players in the game $(N', (w_1)^{-2})$. By **NP**, **wF** and **E** we receive $\sigma(N', w_1) = (0, 0) + (1, 1) = (1, 1)$. Finally, by α-**wCO** and (4) we obtain

$$z = \left(1, 1, \frac{1}{2}\right).$$

Let us now consider the following subcases:

(A) $\alpha(N \setminus N'') = \alpha_{DM}(1) = \{\emptyset, \{1\}\}$. By definition of the α_{DM}-max reduced game, $r^{N''}_{\alpha_{DM},z}(v^{1.5})(2) = r^{N''}_{\alpha_{DM},z}(v^{1.5})(3) = 0$ and $r^{N''}_{\alpha_{DM},z}(v^{1.5})(23) = 2.5-1 = 1.5$. Let us denote $w_2 = r^{N''}_{\alpha_{DM},z}(v^{1.5})$. Note that $(w_2)^{-1.5}(23) = 0$. Hence, players 2 and 3 are null players in the game $(N'', (w_2)^{-1.5})$. By **NP**, **wF** and **E** we receive $\sigma(N'', w_2) = (0, 0) + (\frac{3}{4}, \frac{3}{4}) = (\frac{3}{4}, \frac{3}{4})$. By α-**wCO**, $\sigma_{|\{2,3\}}(N, v^{1.5}) = \left(\frac{3}{4}, \frac{3}{4}\right) \neq \left(1, \frac{1}{2}\right) = z_{|\{2,3\}}$, a contradiction.

(B) $\alpha(N \setminus N'') = \alpha_M(1) = \{\{1\}\}$. By definition of the α_M-max reduced game, $r^{N''}_{\alpha_M,z}(v^{1.5})(2) = 0$, $r^{N''}_{\alpha_M,z}(v^{1.5})(3) = -1$ and $r^{N''}_{\alpha_M,z}(v^{1.5})(23) = 2.5 - 1 = 1.5$. Let us denote $w_3 = r^{N''}_{\alpha_M,z}(v^{1.5})$. Note that $(w_3)^{-2.5}(23) = -1$. Hence, player 2 is a null player in the game $(N'', (w_3)^{-2.5})$. By **NP**, **wF** and **E** we receive $\sigma(N'', w_3) = (0, -1) + (\frac{5}{4}, \frac{5}{4}) = (\frac{5}{4}, \frac{1}{4})$. By α-**wCO**, $\sigma_{|\{2,3\}}(N, v^{1.5}) = \left(\frac{5}{4}, \frac{1}{4}\right) \neq \left(1, \frac{1}{2}\right) = z_{|\{2,3\}}$, a contradiction.

(C) $\alpha(N \setminus N'') = \alpha_P(1) = \{\emptyset\}$. By definition of the α_P-max reduced game, $r^{N''}_{\alpha_P,z}(v^{1.5})(2) = r^{N''}_{\alpha_P,z}(v^{1.5})(3) = 0$ and $r^{N''}_{\alpha_P,z}(v^{1.5})(23) = 2.5 - 1 = 1.5$. Note that $w_2 = r^{N''}_{\alpha_P,z}(v^{1.5})$. Hence, players 2 and 3 are null players in the game $(N'', (w_2)^{-1.5})$. By **NP**, **wF** and **E** we receive $\sigma(N'', w_2) = (0,0) + (\frac{3}{4}, \frac{3}{4}) = (\frac{3}{4}, \frac{3}{4})$. By α-**wCO**, $\sigma_{|\{2,3\}}(N, v^{1.5}) = \left(\frac{3}{4}, \frac{3}{4}\right) \neq \left(1, \frac{1}{2}\right) = z_{|\{2,3\}}$, a contradiction.

Case 2: $\alpha(N \setminus N') = \alpha(3) = \alpha_P(3) = \{\emptyset\}$. By definition of α_P-max reduced game, $r^{N'}_{\alpha_P,z}(v^{1.5})(1) = r^{N'}_{\alpha_P,z}(v^{1.5})(2) = 0$ and $r^{N'}_{\alpha_P,z}(v^{1.5})(12) = 2.5 - \frac{1}{2} = 2$. Note that $w_1 = r^{N'}_{\alpha_P,z}(v^{1.5})$. Hence players 1 and 2 are null players in the game $\left(N', (w_1)^{-2}\right)$. By **NP**, **wF** and **E** we receive $\sigma(N', w_1) = (0,0) + (1,1) = (1,1)$. Finally, by α-**wCO** and (4) we obtain

$$z = \left(1, 1, \frac{1}{2}\right).$$

At this point, we may consider the subcases **(A)**, **(B)** and **(C)** displayed in Case 1 and following the same arguments we get a contradiction.

Case 3: $\alpha(N \setminus N') = \alpha(3) = \alpha_M(3) = \{\{3\}\}$. By definition of α_M-max reduced game, $r^{N'}_{\alpha_M,x}(v)(1) = r^{N'}_{\alpha_M,x}(v)(2) = 0$ and $r^{N'}_{\alpha_M,x}(v)(12) = 1 - 0 = 1$. Let us denote $w_4 = r^{N'}_{\alpha_M,x}(v)$. Note that $w_4^{-1}(12) = 0$. Hence, players 1 and 2 are null players in the game $\left(N', w_4^{-1}\right)$. By **NP**, **wF** and **E** we receive $\sigma(N', w_4) = (0,0) + (\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$. Now, by α-**wCO** and (3) we obtain $x = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Finally, by **wF** and **E** we receive

$$x + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(1, 1, \frac{1}{2}\right) = z.$$

As before, we may consider the subcases **(A)**, **(B)** and **(C)**, and the same arguments applied here lead to a contradiction.

□

Theorem 3 deserves some intuitive explanation. Apart from the Shapley value, there are many other well-known single-valued solutions that satisfy *efficiency*, the *dummy player property* and *weak fairness* for two-person games. For instances, the **center of imputations** (CI) (Driessen and Funaki, 1991), the **equal sharing of non separable costs** (ESNC) (Moulin, 1985) and the prenucleolus. However, when these solutions are extended to games with more than two players through *α-max consistency*, $\alpha \in \{\alpha_P, \alpha_M, \alpha_{DM}\}$, they lose some of the aforementioned properties: both CI and ESNC fall to satisfy the *dummy player property*, and the prenucleolus does not meet *weak fairness*. What is remarkable (as shown Theorem 3) is that, independently of the *α-max consistency* properties we work with, the family of single-valued solutions that emerge does not satisfy *efficiency*, the *dummy player property* and *weak fairness* for an arbitrary set of players N, even for three-player games.

In Theorem 3 we use the characterization result presented in Theorem 2. In fact, whenever we have to compute the solution for a game (N, v) we simply find first the representative element $v_* \in [v]$ that happens to be unique because we always work with games with potential null players. In (N, v_*) we impose the *dummy player property* and *efficiency* and to return to game (N, v) we use *efficiency* and *weak fairness*. This procedure is pointed out by the definition of σ^F, being F a null-adapted selection.

Let us observe that the *dummy player property* and *α-weak consistency*, $\alpha \in \mathcal{A}$, imply *efficiency*. To show it, let $(\{i\}, v)$ be a one-player game. By the *dummy player property*, $\sigma(\{i\}, v) = v(i)$, which means that σ satisfies efficiency for one-player games. Now, it is not difficult to check that efficiency for one player games together with *α-weak consistency* imply *efficiency* (see, for instance, the proof of Proposition 1 in Calleja and Llerena, 2019). This observation leads to the following result.

Corollary 1. *Let $\alpha \in \mathcal{A}$. There is no single-valued solution on the domain of all games that satisfies α-weak consistency, weak fairness and the dummy player property.*

Since the Shapley value satisfies *weak fairness* and the *dummy player property*, Corollary 1 has the following immediate consequence.

Corollary 2. *Let $\alpha \in \mathcal{A}$. On the domain of all games, the Shapley value does not satisfy α-weak consistency.*

To show that each of the properties in Theorem 3 and Corollary 1 is logically independent of the remaining properties first we need to fix $\alpha \in \mathcal{A}$. In case $\alpha = \alpha_{DM}$, the equal division solution meets all properties but the *null player property* (and thus the *dummy player property*), the Shapley value satisfies all properties but *α_{DM}-weak consistency*, the prenucleolus meets all properties but *weak fairness* (see Example 1) and the **zero solution** ($z(N, v) = (0, \ldots, 0) \in \mathbb{R}^N$ for all $(N, v) \in \Gamma$) satisfies all properties but *efficiency* and the *dummy player property*. Let $\alpha \in \mathcal{A}, \alpha \neq \alpha_{DM}$. It can be easily checked that the equal division solution, the Shapley value and the zero solution show the independence of the *null player property*, *α-weak consistency* and *efficiency* in Theorem 3, respectively. While the equal division solution and the Shapley value, show the independence of the *dummy player property* and *α-weak consistency* in Corollary 1, respectively. However, it is open if *weak fairness* is redundant in both impossibility results.

Although, among the families of single-valued solutions characterized in Theorems 1 and 2 there is no one satisfying *α-weak consistency* for any $\alpha \in \mathcal{A}$, the Shapley value is uniquely selected when considering *self consistency* (Hart and Mas-Colell, 1989).

Definition 4. *Let σ be a single-valued solution, $N \in \mathcal{N}$, $(N, v) \in \Gamma$, and $\emptyset \neq N' \subset N$.*

*The **self reduced game** relative to N' at σ is the game $\left(N', r_\sigma^{N'}(v)\right)$ defined by*

$$r_\sigma^{N'}(v)(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ v(R \cup N'') - \sum_{i \in N''} \sigma_i(R \cup N'', v_{|R \cup N''}) & \text{if } \emptyset \neq R \subseteq N', \end{cases}$$

where $N'' = N \setminus N'$.

In the self reduced game (relative to N' at σ), the worth of a coalition $R \subseteq N'$ is determined under the assumption that R joins all members of $N'' = N \setminus N'$, provided they are paid according to σ in the subgame associated to $R \cup N''$.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Self consistency** (**SC**) if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $\emptyset \neq N' \subset N$, we have $\left(N', r_\sigma^{N'}(v)\right) \in \Gamma'$ and $\sigma(N, v)_{|N'} = \sigma\left(N', r_\sigma^{N'}(v)\right)$.

Self consistency has been used to characterize the Shapley value from Hart and Mas-Colell (1989). **Weak self consistency** (**wSC**) requires consistency for reduced games with at most two players. It turns out that imposing *weak self consistency*, in addition to those properties in the statement of Theorem 1, provides a new axiomatic interpretation of the Shapley value on the domain of all games. However, these properties are redundant, what allows us to drop *efficiency* leading to the following characterization.

Theorem 4. *The Shapley value is the unique single-valued solution on Γ that satisfies weak self consistency, weak fairness and the dummy player property.*

Proof. Clearly the Shapley value satisfies **wSC**, **wF** and **DP.** Let σ be a single-valued solution on Γ satisfying these properties. To prove that $\sigma = Sh$, we will use an induction argument on the number of players. First, we show that **wSC** and **DP** imply **E**. Let $(\{i\}, v)$ be a one-player game, since player i is a dummy player, by **DP** $\sigma(\{i\}, v) = v(i)$, which means that σ satisfies efficiency for one-player games. It is well known that **E** for one player games together with **wSC** imply **E** (see, for instance, the proof of Lemma 8.3.8 in Peleg and Sudhölter, 2007). Thus, from **E**, $\sigma = Sh$ for one-player games. Now, let $N = \{i, j\} \in \mathcal{N}$ and $(N, v) \in \Gamma$. If $v(N) = v(i) + v(j)$, then by **DP** it follows directly that $\sigma(N, v) = (v(i), v(j))$. If $v(N) \neq v(i) + v(j)$, consider the associated game (N, v') defined as follows: $v'(k) = v(k)$ for all $k \in N$, and $v'(N) = v(i) + v(j)$. Clearly $v = v' + (v(N) - v(i) - v(j)) \cdot u_N$. Then, by **wF, DP** and **E** we obtain $\sigma_k(N, v) = v(k) + \frac{1}{2}(v(N) - v(i) - v(j))$ for all $k \in N$. Thus, $\sigma = Sh$ for two-player games.

Induction hypothesis: Fix $t \geq 2$, let $\sigma(N, v) = Sh(N, v)$ for all $(N, v) \in \Gamma$ with $|N| \leq t$. From this point, following literally the proof of Theorem 8.3.6 in Peleg and Sudhölter (2007) we obtain $\sigma(N, v) = Sh(N, v)$. □

The properties in Theorem 4 are non-redundant on Γ. The equal division solution meets all properties but the *dummy player property*. The single-valued solution ρ, as defined to show the independence of the properties in Theorem 1, meets all properties but *weak self consistency*, and the marginal contribution solution meets all properties but *weak fairness*.

Observe that *weak self consistency*, *weak fairness* and the *null player property* do not characterize the Shapley value on the domain of all games. Clearly, the *null player property* and *weak self consistency* do not imply *efficiency*, take for instance the zero solution. If we additionally impose *efficiency*, the Shapley value meets all these properties but uniqueness still remains open.

A natural question is to ask for the applicability of Theorem 4 to some domains of games. Because of the definition of self reduced game, it only makes sense to consider classes of games such that all subgames belong to the class, like the well-established domain of convex games. Although the domain of convex games is not closed under the self reduction operation for the Shapley value, Hokari and van Gellekom (2002) show that it satisfies *weak self consistency*. This observation allows us to follow the same arguments as in the proof of Theorem 4 to characterize the Shapley value on the domain of convex games.

Theorem 5. *The Shapley value is the unique single-valued solution on the domain of convex games that satisfies weak self consistency, weak fairness and the dummy player property.*

To finish, note that all the games used in Theorem 3 are convex. It might be the case that, for some $\alpha \in \mathcal{A}$, the class of convex games is not closed under the α-max reduction operation. Given a class of games Γ', a way to overcome such a problem is replacing α-*weak consistency* by the weaker property of **conditional α-weak consistency** imposing that the original agreement must be reconfirmed in the reduced game only when it belongs to the class Γ'. The incompatibility stated in Theorem 3 still hold on the class of convex games imposing *conditional α-weak consistency*. Moreover, since one-player games are convex and the *dummy player property* together with α-*weak consistency*, $\alpha \in \mathcal{A}$, imply *efficiency*, Corollary 1 also holds on the domain of convex games imposing *conditional α-weak consistency*.

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