INCOME DISTRIBUTIONS IN INPUT- OUTPUT MODELS

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Abstract

This paper addresses a gap in the existing input-output literature regarding income distribution issues; little is available to see what actually happens if the distribution of an economy's national income is changing.

We propose that income distributions can excellently be studied by restructuring the basic input-output relations in terms of a so-called augmented input coefficients matrix. This matrix is the sum of the intermediate input coefficients matrix and newly constructed matrices of sector-specific input coefficients that represent the existing distribution of income. We show that shifts in the distribution can be modelled by attributing weights to these matrices and vary these according to system-specific rules. The basic framework is embedded in the Leontief tradition, but we incorporate elements of the Sraffian tradition as well, such as its attention for specific standardizations and numeraires. Numerical illustrations based on existing literature are given throughout the text.

Keywords:

Income distribution; Factor remuneration; Augmented input coefficients matrix; System numeraire

1. Introduction

The study of income distribution (ID) is generally understood as the study of which part of an economy's national income (NI) goes to which party, and why. Time and again, insight into the distribution and its dynamics has proven to be of prime importance for economic analysis and policy.

Leontief-based input-output (IO) analysis offers an excellent perspective for detailed study here. A first reason is the degree of detail that IO *tables* normally maintain. A second one is the double entry nature of the tables which means that the information on surplus and distribution is accounted for in two ways, i.e. in the value added rows and the final demand columns. Both registrations are inter-connected in that total earnings are equal to total spending, while also links between individual rows and columns can exist. IO *models* nowadays straightforwardly can be used to calculate the output and employment effects of, say, an increase in one or more final demand categories. Taking the outcomes for all factors together, we can observe in which way absolute quantities will be affected, in which way the distribution will change, etc., following shifts in exogenous final demand.

Typical ID modeling, nonetheless, came rather late to the fore.[1] Here the contributions in Miyazawa and Masegi (1963) and Miyazawa (1976) should be mentioned, tracing back their origins to Keynes-Kaldor types of multiplier analysis. Miyazawa proposed a way forward via a special extension of the real output model, closing it for household income groups represented by group-characteristic consumption bundles. In this way several types of multipliers could be presented, showing the effects of shifts in demand on the selected groups. Kurz (1985) demonstrated how changes in income distribution will affect income-expenditure multipliers in a multi-commodity setting. Later models explored effects of further disaggregation of the payments sectors into income classes (Rose, Nakayama and Stevens, 1982). Schefold (1976) analysed the effects on ID of forms of technical progress, while Leontief (1985, 1986) studied the distributional effects of the arrival of new technologies in the US.

However, by and large, much of ID analysis is relegated to extensions and subfields built around special tools and instruments such as Social Accounting Matrices or SAMs.[2] This line to a large extent goes back to pioneering work such as Pyatt and Roe (1977) in studies focusing on poverty and income inequality. Many variants have been proposed. One line focuses on a fuller coverage of household characteristics, see Batey and Weeks (1989), or Duchin (1998); see also Pyatt (2001) for a recent perspective in terms of various types of multiplier matrices.

Contrarily to Leontief's basic model, ID determining mechanisms are very much part of the Sraffian basic model (Sraffa, 1960). The Sraffian model, essentially, consists of a price equation and a so-called "actual system" (basically equal to Leontief's open, static model) for providing a physical measure of output. It focuses on the distribution of income between two

1 See Batey and Rose (1990).
2 To illustrate, in Miller and Blair (2009) the term "income distribution" appears only thrice, i.e. on pp. 7, 499 and 681, and then only in connection to Social Accounting Matrices.

parties, labour and capital. The model, most importantly, recognizes that price changes will accompany a shift in the distribution of income, the necessary links being administered (*put in place*, *taken care of*) by expressing the wage in terms of NI. The theoretical framework contains several normalizations and numeraires unique for the Sraffian approach such as the so-called "standard commodity". Differences with Leontief analysis are substantial, which may explain the lack of cross-fertilization between both schools of thought. [3] [4]

In fact, the standard IO model as it is presented nowadays is not well-endowed to answer questions regarding ID issues. We may be interested, for example, in the consequences of decreasing access to primary inputs, or the effects of redistributive policies. In such cases, given the interdependence in the system, a shift in one or more parties' positions will have implications for the other ones, which, in effect, may be "crowded out". However, there is no way to trace this, there being no transmission mechanism that links, say, shifts in factor remuneration to shifts in net output, or vice-versa. This means that it is not possible, in the present context, to systematically explore ID-determining mechanisms in a variety of relevant cases. This leads us directly to the *central issue* of this paper, i.e. to show in which way the necessary tools for ID analysis can be built in.

Below we shall start with the well-known observation that in standard Leontief models factor prices are given exogenously, say in dollars or euro's per unit of factor input. Changes in these prices have no influence on the real sphere, commodity prices will reflect changes in factor prices, but this has no further consequences.

However, adopting exogenous factor prices ignores the fact that direct links between both spheres do exist. In fact, such links are readily observed if (factor) remunerations are expressed in real terms, i.e. in terms of the commodity bundles constituting the various categories of the relevant national product. We shall explore these links further below.

We start by expressing the factor remunerations in real terms. For labour e.g. this straightforwardly translates into a bundle of consumed commodities per unit of (labour) input. Each sector using labour, these bundles can be assembled in a (square) matrix the columns of which stand for the real wage per unit of output of each sector.

Accounting for the wage in terms of these matrices makes it possible to consider the factor remunerations as *inputs*, which means that they can be treated as bundles of commodities analogous to the intermediate inputs into the sectoral production processes. We show that in this way an 'initial situation' can be defined which can serve as a benchmark for introducing *shifts* in ID. We further show that shifts in factor remuneration directly can be translated into shifts in ID, and vice versa, by attaching weights to the newly introduced 'factor remuneration matrices'. A separate problem then is to find a rule (or set of rules) that allows us to consistently vary the distribution of income. We shall show that such a rule is provided by the

[3] To illustrate, and again referring to Miller and Blair (2009), only two references to Sraffa are provided, both in the final chapter.

[4] See Steedman (2000) on differences and on further references to the intellectual history.

mathematical rules *(constraints, properties attributed to)* that guard the IO model's internal consistency. As we shall see, in this context specific 'numeraires' will have an important role.

Central will be a special matrix, to be called the system's augmented input coefficients matrix. This matrix is the sum of the intermediate input coefficients matrix and newly constructed matrices of sector-specific input coefficients which represent existing *claims* on NI, and which can be changed according to *specific* rules to reflect shifts in NI. We start, in section 2, with the standard open IO model with one primary factor. In section 3 we discuss the Sraffian approach and return in section 4 to a general class of IO models, where we shall pay attention to the fact that there is no a priori built-in one-to-one relation between the value added and final demand categories. In section 5 we conclude with a number of remarks.

2. Augmented Input Coefficients Matrices

We start our proposition with the standard IO model with one primary factor (to be called labour), which consumes the entire net product. Three equations are sufficient to provide the essentials. We have,

(1) $x = Ax + f$

where A, the (n,n) matrix of intermediate input coefficients, is square, nonnegative, indecomposable, and of full rank with Perron-Frobenius (PF) eigenvalue smaller than 1; f and x stand for, respectively, the (n,1) vectors of net product, here consumed in its entirety by the single primary factor, and total output. We also have

(2) $L = l'x$

where $l' > 0$ stands for the (1,n) row vector of direct labour input coefficients and L for the size of the labour force employed. [5] The corresponding price equation is

(3) $p' = p'A + wl'$,

p′ standing for the (1,n) row vector of commodity prices and w (a scalar) for the money wage per worker.

We observe that a shift in w will not have any influence on the real sphere as given by equations (1) and (2). This is a fundamental property of the model; w is exogenously determined, and the model is constructed without any direct connection between w and the real sphere as expressed by (1) and (2). The model clearly is transparent, but there is a price to pay; the signalled transparency deprives us from studying interactions between shifts in the wage and the real sphere.

However, the above three equations can be reformulated to provide the basis for an extension that allows us to study income distributions in detail. In fact, we shall show that by "expressing the wage in terms of NI", we obtain a direct link between the price and output

[5] The notation $l' > 0$ means that each element of the vector l′ is positive.

equations that can be further exploited. This implies, as we shall see, adopting specific ”standards” or “numeraires” which reflect the real sphere.

The link can be established by expressing the wage in real terms, here an (n,1) commodity bundle. If we do this for all sectors combined, we can write them as an *input*. We have,

(4) $pf = p'(f/L)L = p'(f/L)lx = [p'(f/L)lx = ([p'(f/L)l)x$

If we now adopt the rule

(5) $w = p'(f/L)$ [6]

we have expressed the wage in terms of NI (per head). We subsequently need a standardization to fix the value of w, e.g. by fixing the value of the bundle f/L at unity, in which case it becomes a numeraire. If all workers receive the same real wage, we now can write equation (3) as

(6) $$\begin{aligned} p' &= p'A + p'(f/L)l' \\ &= p'(A + (f/L)l') \end{aligned}$$

from which we observe that the elements of the (standardized) final deliveries vector now appear as part of the economy’s cost structure. [7] Correspondingly, we have for the real output system

(7) $x = (A + (f/L)l')x$

So we have rewritten the model given by equations (1-3) in terms of two eigenequations where p′ and x stand for, respectively, the left and right hand PF eigenvectors of matrix $(A + (f/L)l')$. It is useful to have a compact notation for the wage part of this matrix. With

(8) $B \equiv (f/L)l'$

we have

(9) $p' = p'(A + B)$ [8]

With $f > 0$, we then have $B > 0$, and $r(B) = 1$, where r(B) stands for the rank of matrix B. Correspondingly, we find:

(10) $x = (A + B)x$

with $Bx = [(f/L)l']x = f$, using $L = l'x$. We shall call, following Seton (1977), matrix (A + B) an “augmented input coefficients matrix”. We continue by rewriting (10) as

(11) $[I - (A + B)]x = 0.$

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6 Each element of the column vector f/L is obtained by dividing the corresponding element of f by L, a scalar.
7 Further below we shall introduce shifts in ID by attaching weights to matrix B and similar input coefficients matrices.
8 So matrix B is the outer product of the column vector f/L and the row vector l′.

From (11) we have that the columns of matrix I – (A + B) are interdependent. This implies that its determinant must vanish, i.e.:

(12) $$|I - (A + B)| = 0$$

At this point we should observe that the matrix (A + B) contains what we may call ‘complete’ or ‘full’ information on the various segments of the economy. We do have only one final demand category (labour) but it is fully represented in terms of its consumption bundle and value added row. It is this ‘completeness’ that allows us to reformulate the available information in another form, i.e. equation (12). This equation and its variants provide all information that we need to study IDs. Later on, e.g., we shall say that matrix B has a weight “1” in the interpretation of equation (12) that we shall further explore.

Equation (12) provides invaluable insight in the system’s structure in that it provides an exact, quantitative statement on the interconnection between the coefficients of matrices A and B. For example, if one or more coefficients of matrix (A + B) would change, this implies that also one or more of the remaining coefficients must change. We shall explore this connection in more detail further below. We also can observe that equation (12), viewed as a statement on the system (1) – (3), is quite different from well-known statements on the system’s productivity like $|I - A| > 0$. [9] The latter statement is of a qualitative nature, and provides a type of information quite different from what equation (12) does. The difference between both statements is a consequence of the fact that equation (12) includes all segments of the economy, while a statement like $|I - A| > 0$ only concerns the intermediate part.

We may observe that equations (9) and (10) allow us to distinguish between ‘structure’ and ‘scale’. We shall say that ‘structure’ is represented by the properties of matrix A + B, while ‘scale’ is represented by vectors p′ and x. These vectors, being left and right-hand eigenvectors of matrix (A + B), can be increased or decreased in size by simple scalar multiplication without affecting the structure of the system; specific numeraires will be discussed in the next sections. [10]

Below we shall see that we can study NI and the distribution of it by varying the coefficients of matrices A and B. Suppose e.g. than an innovation reduces the use of, say, good 1 in all production processes. [11] This will have consequences for the economy’s net output. If the proportions of the final demand bundle do not change, and if L does not change, we can look at

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[9] This is one of a series of equivalent statements on an economy’s productivity, see Miller and Blair (2009, apps. 2.1 and 2.2).

[10] We should observe that the model of equations (8) and (9) is fundamentally different from the so-called closed Leontief model (see e.g. Pasinetti, 1977, ch. IV, or Miller and Blair, 2009, section 2.5). In that model the production of a commodity called labour is interpreted just like the production of any other commodity, i.e. in terms of fixed input coefficients. In our model the final demand vector f stands for the households consumption bundle which changes if preferences change.

[11] This is one of the basic concepts of the so-called RAS-method; for origins, see Leontief (1951) or Stone and Brown (1962).

(13) $\quad x = (\underline{A} + \beta B)x$

where $\underline{A}$ is the post-innovation matrix of intermediate input coefficients, β a proportionality factor (where we would expect $\beta > 1$), and x the total output vector. We now can analyze the relation between *shifts* in the original intermediate input coefficients matrix and the scalar β by considering

(14) $\quad \left| I - (\underline{A} + \beta B) \right| = 0,$

a relation which must be satisfied. This, in broad lines, is the approach we shall pursue further below.

Equation (12) has additional properties that we can use. One such property concerns the case where $r(B) = 1$, as above. We have:

Lemma 1.

Let all symbols be as before, and let $A_{\#}$ denote the adjoint of matrix A. Then

(15) $\quad \det(A + (f/L)l') = \det(A) + l'A_{\#}(f/L)$ [12]

In the next two sections we shall explore the properties of equation (12) and its variants in two ways. In section 3 we shall explore the relation between NI and matrices A and B in a Sraffian context. In section 4 we shall introduce further claims on net output in a Leontief-type of context, the claims on NI being modelled by introducing additional coefficient matrices, similar to matrix B in the single primary factor case. In both sections, ID determining mechanisms will be studied by attaching weights to the individual matrices, and by varying these weights. We thereby shall propose two additional lemma's, both based on lemma 1.

Before continuing, we should mention that adding the 'material' and labour costs is quite well-known in certain subfields. For example, Seton (1977, p. 17), in discussing Marxian production systems, points out that this sum sometimes is referred to as the economy's "augmented technology", representing the total cost of each output unit to the capitalist producer. Pasinetti (1977, appendix to ch. V) used a similar construct in discussing the structure of particular Marxian inspired models. However, the idea of combining final demand and value added categories was only put forward for the wage-labour input connection, without further discussion of systems where links between all categories are explored. As far as we know, such constructs have never been employed to study trade-offs in ID schemes of the type we shall be dealing with further below.

3. Capital and Labour as Competing Parties

[12] For proofs, see e.g. Rao and Bhimasankaram (2000, section 6.7), or Trenkler (2000).

In this section we shall see in which way augmented input coefficient matrices can be applied in Sraffian analysis. As mentioned already in section 1, the Sraffian approach is quite different from Leontief's. Differences include a focus on distribution issues, and the role of specific standardizations and numeraires therein in providing the necessary structure. The original model, as presented by Sraffa (1960) in his "Production of Commodities by Means of Commodities", has initiated many comments, further propositions, and extensions, but interaction with the Leontief "school of thought" has been minimal. [13]

Many aspects of Sraffa's work, especially those concerning the validity of neo-classical theorizing and the revival of classical thought, are important in any contribution discussing differences and similarities with Leontief's work. However, these are outside the scope of the present contribution. For a recent set of essays on Sraffa's life and works, we should refer to Bharadwaj and Schefold (1990), and to Kurz and Salvadori (2003) for a collection of original contributions.

Central in the Sraffian approach is the price model, traditionally written as

(16) $p' = (1+r)p'A + wl'$

where r and w stand for, respectively, the rate of profit and the wage rate. [14] The problem is to determine the ID when w and r shift, in relation to changing prices. Solving for p′ results in

(17) $p' = wl'[I - (1 + r)A]^{-1}$

Thus, price *proportions* (which do not depend on w) can be obtained straightforwardly. If w would be in money terms (say in \$ or £ per unit of labour), prices also are in \$ or £. In that case, there is no direct relation between r and w in the sense that knowledge of one enables us to calculate the other. Adopting a particular commodity as numeraire will establish a fixed relation, though. Following Kurz and Salvadori (1995, ch. 4, section 1), we may adopt a particular commodity bundle, say d, and assign the value 1 to that bundle, i.e. $p'd = 1$. For given r, this fixes the value of p′ and, hence, w. We thus can introduce a relation between r and w based on the bundle d, provided the inverse of matrix $[I - (1+r)A]$ exists. In this way a range of IDs can be obtained, each based on the particular commodity bundle chosen as the numeraire. If we decide to select a net output bundle as given by equation (1), we obtain a distribution based on that particular output bundle. We may observe here that this method is

[13] We should remark that Sraffa makes no assumption regarding the presence of constant returns to scale. For the argument in this section, no such assumption will be required, though. Sraffa, however, adds that there also is no harm in adopting such an assumption as a temporary working hypothesis if such an assumption is found to be helpful (Sraffa, 1960, p. v; see on this also Kurz and Salvadori, 1995, pp. 423-424 in a comparison of the analyses of von Neumann (1945) and Sraffa.

[14] See e.g. Pasinetti (1977, section V,3). We should add that the row vector l′ is obtained via the aggregation of different labour types via wage rates. Sraffa's assumption of labour to be of uniform quality is equivalent to assuming that any differences in quality have been reduced to equivalent differences in quantity such that each unit of labour receives the same wage (Sraffa, 1960, p. 10; see also Kurz and Salvadori, 1995, pp. 324-325). We are indebted to one referee for pointing this out.

not based on the notion that the economy's final demand vector has an interpretation in terms of the real wage bundle, as put forward in our sections 1 and 2. Also, as we shall see in section 4, this method cannot readily serve as a basis for analyzing more complex cases where more than two parties are involved.

When the wage is expressed in terms of NI, w î $[0,1]$ with $w = 1$ when $r = 0$. Similarly, r î $[0,R]$, where R is the maximum rate of profit, obtained when $w = 0$. A special role is played by the standard commodity. This is the net output bundle of an IO system (the so-called standard system) with the special property that its net output vector has the same proportions as the vector of the aggregated means of production in the system as a whole (or, equivalently, its total output vector). Adopting the standard commodity as a numeraire for prices is Sraffa' answer to the problem of finding a standard that is 'invariable' with respect to changes in ID. In fact, this composite commodity, used as a numeraire, enables us to formulate the distribution determining relations independent of any price movements, because (as a unit of measurement for prices) it eliminates the influence of the characteristics of the individual production processes from the relation between r and w. If this standard commodity is used as a numeraire, we will have, by construction, a case where the w-r relation is linear. [15]

In the Sraffian literature –as far as we know- up to now no *general* approach to characterizing the structural form of the ID over the entire range of r and w(as proposed in the previous section), has been presented. Below, therefore, we shall show how adopting the method described in our earlier sections, the (r,w)-relation can be derived for the general case.

We shall proceed following the approach proposed in the previous section. With given net output bundle f, the economy's NI is $p'f$. The Sraffian approach contains two important standardizations, i.e. $p'f = 1$ and $L = 1$. This means that also the income per head, $p'(f/L) = 1$. We now may rewrite the price equation as

(18) $$p' = p'[(1+r)A + w(f/L)l'],$$

or

(19) $$p'([I - [(1+r)A + w(f/L)l']) = 0$$

again with $p'(f/L) = 1$ and w î $[0,1]$. The sought after wage-profit relation then is

(20) $$\varphi(r,w) = \left| I - [(1 + r)A + w(f/L)l'] \right| = 0$$

Equation (19) is the Sraffian equivalent of equation (12). We see that matrices A and B ($= (f/L)l'$) both have received weights, $(1 + r)$ and w, respectively. A closer look at equation (20) gives us the relation between the weight factors, and thus the sought after ID. Starting from (20) and defining

(21) $$M_{r,w} \equiv (1+r)A + w(f/L)l'$$

15 See further Pasinetti (1977, ch. 5), Abraham-Frois and Berrebi (1979, ch. 2), Kurz and Salvadori (1995, 2007) or Steenge (1995) on this. Similarly, also in the case where l' in (3) is proportional to p', we have a linear relation, see Pasinetti (1977, ch. 5). See also Kurz and Salvadori (2007) for a recent contribution on the origin of this particular line of research.

we obtain the simplified form

(22) $\varphi(r,w) = |I - M_{r,w}| = 0$

which gives us the sought-after relation between r and w, a polynomial in both variables. We should note that analysis of the price equation does *not* require a simultaneous specification of the physical output side. In fact, we only need to possess knowledge of the final demand bundle f that is produced in the “actual system”. We have two additional lemma’s here which will enable us to obtain analytical expressions for the r-w relation:

Lemma 2.

Let $M_{\alpha,\beta} = [\alpha A + \beta(f/L)l']$, all symbols as before and α and β scalars. Then

(23) $|I - M_{\alpha,\beta}| = |I - \alpha A| - \beta l'(I - \alpha A)_{\#}(f/L)$

Proof: $|I - M_{\alpha,\beta}| = |I - [(\alpha A + \beta(f/L)l']| = |(I - \alpha A) - \beta(f/L)l']| =$

$|I - \alpha A| - \beta l'(I - \alpha A)_{\#}(f/L)$.

In our case, we then have:

Lemma 3.

Let $M_{\alpha,\beta}$ be as in lemma 2 and let $|I - M_{\alpha,\beta}| = 0$. Then:

(24) $\beta = |I - \alpha A| / l'(I - \alpha A)_{\#}(f/L)$

That is, lemma 3 informs us that β is a function of α.

a) Frontiers and prices

Below we shall present an example using Sraffa’s data on his 'actual system′ (see also Pasinetti, 1977, chs. 2 and 5 on this system). We have, rounding in three decimals,

$$A = \begin{bmatrix} 0.413 & 2.571 & 0.500 \\ 0.027 & 0.286 & 0.050 \\ 0.020 & 0.286 & 0.250 \end{bmatrix}$$

and

$$l' = \begin{bmatrix} 0.040 & 0.571 & 0.500 \end{bmatrix}$$

Final demand equals

$$f = \begin{bmatrix} 180 \\ 0 \\ 30 \end{bmatrix}$$

Straightforward calculation gives, after rounding,

$$x = \begin{bmatrix} 450 \\ 21 \\ 60 \end{bmatrix}$$

and

$$L = 60$$

Substituting the values of f, l and L in equation (20), we obtain the expression

$$\varphi(r,w) = 0.241-0.579r - 0.241w + 0.170r^2 + 0.119rw - 0.010r^3 - 0.010r^2w,$$

a polynomial of 3rd degree in r and first degree in w. Putting $\varphi(r,w) = 0$ and solving for w, we find

$$w(r) = \frac{0.241 - 0.579r + 0.170r^2 - 0.010r^3}{0.241 - 0.119r + 0.010r^2}$$

Clearly, w(r) is non-linear. To illustrate, the Taylor expansion of w(r) around the point r = 0 by a polynomial of degree 2 gives

$$w(r) = 1.000 – 1.909r – 0.279r^2 + O(r^3)$$

Figure 1 gives the graph of w(r). The economically relevant is the part of the left-most curve in the first quadrant, r ranging from 0 to 0.482, its maximum value. Prices are obtained by substituting an admissible point (r,w) in (20), and solving for p′, the left hand PF eigenvector of matrix $M_{r,w}$; standardization then gives absolute prices. For example, for r = 0.2 we find w = 0.606. From (20) we then have,

$$M_{0.2,0.606} = \begin{bmatrix} 0.569 & 4.125 & 1.509 \\ 0.032 & 0.343 & 0.060 \\ 0.036 & 0.516 & 0.452 \end{bmatrix}$$

with standardized left-hand PF eigenvector, [16]

$$p'_{0.2,0.606} = \begin{bmatrix} 0.205 & 1.891 & 0.771 \end{bmatrix}$$

Figure 1 about here

[16] That is, $(p'_{0.2,0.606})f = 60$.

b) Adopting the standard commodity as numeraire

The standard commodity is defined as the net output vector (indicated by the symbol f*) of an economic system (the so-called standard system, see above) that produces its net output in the same proportions as its aggregated means of production Ax*, or, equivalently, its total output vector x*. It can be shown that these proportions are those of the right-hand PF eigenvector of the matrix of intermediate input coefficients, i.e. matrix A. [17] In this case w(r) is linear, as mentioned. Keeping the size of the labour force unchanged (i.e. 60 units), we find that

$$f^* = \begin{bmatrix} 141.77 \\ 11.58 \\ 14.47 \end{bmatrix}$$

and

$$x^* = \begin{bmatrix} 435.57 \\ 35.59 \\ 44.48 \end{bmatrix}$$

If we adopt f*/L as our numeraire, we have, again using Sraffa's data, the case of a linear trade-off between w and r. Straightforward calculation gives

$$w(r) = \frac{0.241 - 0.579r + 0.170r^2 - 0.010r^3}{0.241 - 0.080r + 0.005r^2}$$

or, in terms of its Taylor expansion, [18]

$$w(r) = 1.000 - 2.073r + 0.000r^2 + O(r^3)$$

Figure 2 captures the changed situation in the same wider picture as the general case of the previous sub-section. Again the economically relevant part is given by the values of r in the closed interval [0,R], with R = 0.482 the maximum rate of profit.

Figure 2 about here

Price proportions are given by the left hand PF eigenvector of matrix

$$M_{0.2,0.585} = \begin{bmatrix} 0.551 & 3.876 & 1.292 \\ 0.037 & 0.407 & 0.116 \\ 0.030 & 0.424 & 0.371 \end{bmatrix}$$

17 See e.g. Pasinetti (1977, section V,9).

18 The curve w(r) is not completely linear due to rounding.

The standardized price vector is

$$p'_{0.2,0.585} = [0.198 \quad 1.827 \quad 0.744],$$

with standardization to $(p'_{0.2,0.585})f^* = 60$. We observe that price proportions have changed, in correspondence to the new numeraire (i.e. f*). In this way, we find, over the entire range of r and w, the corresponding price adaptations. [19]

4. Income Distribution in the Leontief Model

In section 2 we constructed matrix B as a matrix of what we termed "imputed input coefficients". By construction, r(B) = 1, B being the outer product of the column vector f/L and the row vector l. The columns of matrix B thus stood for the real wage per unit of output of the relevant sector, and its rows for the imputed quantities of the relevant commodity.

In the above case, the imputed input coefficients were obtained by combining information from two sources, i.e. from the net output and the primary input categories. However, this need not standard be the case; often the necessary information –to determine IDs- is available in another form. An example is matrix rA in the previous section. There the imputed input coefficients were readily available as the elements of the intermediate input coefficients matrix A, multiplied by the rate of profit r. This procedure, i.e. obtaining the required coefficients from other sources, also can be used in Leontief-based distribution problems. We provide an illustration in section 4-a. In section 4-b the case of two primary factors is discussed, while in section 4-c the general case is discussed.

Again the proportions of the relevant output and price vectors are obtained as, respectively, left and right hand PF eigenvectors of the system's augmented input coefficients matrix, after which standardizing NI results in absolute levels. Subsequent varying the weights of the sector-specific input coefficients matrices (see also the previous section) allows us to trace the ID patterns.

4-a) New Claims on the National Income

Suppose that a country is confronted, in a relatively short time, with challenges of an entirely new nature. Suppose also that these challenges are of a scale that is significant in terms of its NI. We may think here of ambitious climate change oriented initiatives, issues related to shifting patterns in international trade, or problems related to population dynamics. [20] In such a situation many issues –often up to then dormant- may become critical. One of these is the

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[19] Leontief (1985, 1986), in exploring the effects of technological change on ID, employed a price equation quite similar to Sraffa's. However, also here we may observe that it is difficult to get insight in the general shape of the wage rate – rate of profit relation, and that this method is difficult to generalize.

[20] An interesting example is provided by present-day water policy in the Netherlands. Investment expenditures in water management by central and local government, water boards, and industry amount to about 5 billion Euros annually (Brouwer and van der Veeren, 2009). New findings, possibly related to climate change policies, may easily ask for a rapid increase of this figure; see further Barro (2006) on the sheer scale of the potential welfare loss in advanced economies due to infrequently occurring disasters.

supply of primary resources. If this supply is not sufficient to meet the new demands, their entry will be at the cost of the more traditional claims on NI. The "older" claims can be "crowded out", and the question then becomes what precisely will happen in terms of which parties will benefit and which will not, at what rate, which prices will realize, etc. [21] Stated otherwise, are there also here "laws" of the type presented in the previous section? And if so, what do they tell us?

Below we shall, via an analogy with the model of the previous section, show that problems of the above sketched nature can be addressed employing a framework which is able to "weigh" all economic activities within the structure of an augmented input coefficients matrix. Suppose we have a matrix of intermediate input coefficients,

$$A = \begin{bmatrix} 0.25 & 0.05 & 0.15 \\ 0.09 & 0.19 & 0.06 \\ 0.07 & 0.09 & 0.39 \end{bmatrix}$$

and a final demand vector,

$$f = \begin{bmatrix} 100 \\ 250 \\ 150 \end{bmatrix}$$

with corresponding labour input coefficients vector,

$$l' = \begin{bmatrix} 0.20 & 0.30 & 0.15 \end{bmatrix}$$

We shall suppose furthermore that the "new challenges" manifest themselves in the form of extra production being required to fulfil the various needs, old and new, and that the costs of producing the additional goods and services will be borne by the industrial sectors according to some earlier agreed on distribution scheme, say on the basis of sectoral output. If the additionally required production is known in its totality, we may proceed in a way analogous to the previous section. Suppose that the estimated extra production over time is given by the vector

$$h = z\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

where z is a scale parameter, and that the program specifies that in each specific period a certain part of h must be produced and put in place. An arrangement like that means that each commodity price now must reflect the new commitment in the form of a separate cost category to be dealt with. If costs are allocated strictly proportional according to the following matrix D where

[21] Regarding this, we should recall that quite a number of studies have appeared the last decade based on forecasting, scenario's, or other types of analysis. Hirsch et al. (2005), e.g., predict that price volatility will increase dramatically, with associated enormous economic, social and political costs. In addition, the owners of the relevant resources may be expected to claim a much greater share of total income. Friedrichs (2010) predicts three possible scenarios, i.e. predatory militarism, totalitarian retrenchment or socio-economic adaptation, also against a background of a redistribution of power and wealth from resource importers to exporters.

$$D = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.10 & 0.10 & 0.10 \\ 0.15 & 0.15 & 0.15 \end{bmatrix}$$

only a proportionality constant has to be determined to obtain the consequences per period. Concentrating on a specific period, say one year, and with x standing for the total output vector for that year, Dx then is the vector of the additionally required production. Let further $\delta \geq 0$ be the proportionality constant we referred to. The vector δDx then indicates which part of Dx must be produced during the year under consideration. Assuming that labour is still the only primary factor and that the real wage part again is given by a matrix B of the type defined in section 2, the new price equation reads [22]

(25) $\quad p' = p'(A + \delta D + \beta B)$

From (25) we see that we have here a variant of the Sraffian price equation, now with matrix D performing the role of matrix A in equation (18). For the real outputs we have correspondingly

(26) $\quad x = (A + \delta D + \beta B)x$

Both equations directly lead to a determinantal equation of the type we discussed before, i.e.,

$$\varphi(\beta,\delta) = 0.354 - 0.354\beta + 0.017\beta\delta - 0.205\delta = 0$$

which gives, solving for δ, again a hyperbola:

$$\delta = \frac{0.354 - 0.354\beta}{0.205 - 0.017\beta}$$

It is useful to consider the Taylor expansion around the point $\beta = 0$. This gives

$$\delta = 1.727 - 1.584\beta - 0.131\beta^2 + O(\beta^3)$$

which shows that δ is nearly linear in β in the economically relevant part of the curve, see Figure 3.

Figure 3 about here

We observe that price and output proportions can be obtained by substituting admissible values for β and δ (see Figure 3) in equation (25) or (26), and subsequent calibrating by

22 That is, industry 1 faces additional costs per unit of production as given by the first column of matrix D multiplied by δ, etc. These additional costs can consist of specific taxes to be paid, claims by capital owners, costs associated with R & D expenses, etc. We may wish to compare this with the role of profit in the price equation of section 4 where profits are an integral part of each commodity price.

selecting an appropriate standard, say by keeping total employment constant. For example, for $\beta = 1$ and $\delta = 0$, we find that total employment, L, equals 200.202, and

$$x_{1,0} = \begin{bmatrix} 221.973 \\ 357.312 \\ 324.092 \end{bmatrix}$$

with corresponding standardized price vector,

$$p'_{1,0} = [0.354 \quad 0.434 \quad 0.376],$$

We may straightforwardly vary β and δ over their entire admissible range. For example, for $\beta = 0.2$ and $\delta = 0.902$, we find, calibrating at $L = 200.202$,

$$x_{0.5,0.902} = \begin{bmatrix} 225.445 \\ 314.608 \\ 404.871 \end{bmatrix}$$

with standardized price vector,

$$p'_{0.5,0.902} = [0.369 \quad 0.393 \quad 0.434],$$

The above example shows that it is not necessary to introduce a new and clearly identifiable primary factor in the context of the new program. The introduction of the “new challenges” is at the expense of labour’s real wage at a rate dictated by the above equations. In this sense, the model of this sub-section has a lot in common with the Sraffian model of the previous section.

4-b) Several Factors and Final Demand Categories

In this sub-section we shall discuss the case where two factors and two final demand categories are distinguished, a situation different from the previous one. We shall assume that for both factors a direct relation exists between their demand for final products and the amount of services they provide in return. We then can construct coefficient matrices analogous to matrix B in section 2. For convenience the two factors shall be called labour and capital again; regarding capital, we shall assume that it consists of a well-specified bundle of capital goods that will be distributed over the sectors according to their needs.

We shall use the same intermediate input coefficients matrix as in section 4-a. The final demand bundle now consists of two parts, one part (f_1) going to households in the form of real wage for the labour contributed, while a second part (f_2) goes to capital owners for investment, depreciation allowances, replacement and other capital related purposes. Correspondingly we have, next to the (1,n) row vector of direct labour input coefficients l' (as in section 4-a, a second (1,n) row vector, k′, standing for the required capital related inputs into production.[23] Let us assume

$$f_1 = \begin{bmatrix} 100 \\ 70 \\ 80 \end{bmatrix}$$

and

$$f_2 = \begin{bmatrix} 0 \\ 180 \\ 70 \end{bmatrix}$$

(so $f_1 + f_2 = f$ with f as in section 4-a). For the labour and capital-related vectors of input coefficients we have, respectively,

$$l' = \begin{bmatrix} 0.20 & 0.30 & 0.15 \end{bmatrix}$$

(as before), and

$$k' = \begin{bmatrix} 0.05 & 0.02 & 0.03 \end{bmatrix}$$

Calculation gives

$$x = \begin{bmatrix} 221.973 \\ 357.312 \\ 324.092 \end{bmatrix}$$

Total employment, L, is 200.202 (with $L = l'x$). With K denoting the total supply of capital related input contributions, we find K = 27.968 (via $K = k'x$). The input coefficients matrices now can be simply obtained. With

(27) $\quad \underline{M} = A + B + H$

we have

(28) $\quad B = (f_1/L)l'$

and

(29) $\quad H = (f_2/K)k'$

Substitution gives,

$$\underline{M} = \begin{bmatrix} 0.350 & 0.200 & 0.225 \\ 0.482 & 0.424 & 0.306 \\ 0.275 & 0.260 & 0.525 \end{bmatrix}$$

The corresponding price proportions are given by the left hand PF eigenvector of $\underline{M}$.
Calibrating, as in section 4-a, at $p'f = 200.202$, we find

[23] Questions concerning the way in which the bundle f_2 will be allocated between the sectors for replacement and other purposes fall outside this paper. If necessary, appropriate mechanisms can be built-in without undue effort.

$p'_{1,1} = [0.450 \;\; 0.355 \;\; 0.442]$ [24]

The shares in NI are, respectively, 105.280 and 94.922.

Let us now again introduce distribution parameters, β for the wage part and γ for the capital related part. Employing the symbol $\underline{M}_{\beta,\gamma}$ for the corresponding augmented input coefficients matrix, we then have,

(30) $\underline{M}_{\beta,\gamma} = A + \beta B + \gamma H$ [25]

First, let us now consider the relation between β and γ for the case at hand. We have

(31) $\left| I - \underline{M}_{\beta,\gamma} \right| = 0$

Written out, we have

$$\varphi(\beta,\gamma) = 0.354 - 0.169\beta - 0.149\gamma - 0.035\beta\gamma = 0$$

Figure 4 gives, for the interval $-1 \leq \beta \leq 3$, the relation between β and γ.

Figure 4 about here.

It is interesting to note that φ(β,γ) is convex in its economically relevant part. In this respect we should realize that its coefficients are determined by the characteristics of the technological and the behavioural parameters of the economy in case. Further analysis for the general case here may be asked for.

To illustrate, let us assume $\beta = 1.25$. Calculation gives $\gamma = 0.735$. This results in the following augmented input coefficients matrix

$$\underline{M}_{(1.25,0.735)} = \begin{bmatrix} 0.375 & 0.237 & 0.244 \\ 0.414 & 0.416 & 0.268 \\ 0.262 & 0.277 & 0.520 \end{bmatrix}$$

Price and output proportions again are given by the left and right hand PF eigenvectors of $\underline{M}_{(1.25,0.735)}$. Calibrating at NI = 200.202, we find,

$p'_{1.25,0.735} = [0.419 \;\; 0.369 \;\; 0.418]$

and, assuming that the supply of capital will be sufficient,

24 We can check that x satisfies $p'f = p'x - p'Ax$.

25 So matrix $\underline{M}$ reflects the case where both β and γ have unit value, with $L = l'x = 200.202$ and $K = k'x = 27.968$.

$$x_{(1.25,0.735)} = \begin{bmatrix} 254.964 \\ 332.049 \\ 330.629 \end{bmatrix}$$

while shares in NI are 126.516 and 73.688, respectively.

4-c) The General Case

The general case is now straightforward. Again, we may include the additional factor-related terms in a new and extended system matrix. Assembling the new terms, we then obtain:

(45) $x = (A + B + H_1 + \ldots + H_k)x,$

where k reflects the number of additional final demand/value added categories we wish to distinguish. This gives, correspondingly, the following equation which contains the basic information we need for identifying IDs via the process of attaching weights to the individual constituting (sub)matrices:

(46) $\left| I - (A + \beta B + \gamma_1 H_1 + .. + \gamma_k H_k) \right| = 0$

Hereafter we may proceed by selecting values for the distribution parameters, thereby –as before- taking care of the role of possible critical values for one or more of the primary factors.

At this point we should be precise about the results we wish to obtain. In the Leontief models a choice is offered: we may start from the classification as given by the final demand categories, or from the classification of the vale added categories. Our ultimate choices should reflect the research issues we are interested in.

A different point is that finding a satisfactory “match” between final demand and value added categories may not be straightforward. We may, e.g., wish to start with relatively less complex matchings such as those between forms of personal consumption and wages and salaries. Investment (gross or net) may be coupled to capital returns, depreciation, and other capital-related categories, taxes to government based expenditure, and so on. In carrying out such exercises, we should recall that the total of all final demand categories is equal, by construction, to the total of all value added categories. Starting with the less complex categories, the remaining categories may be determined successively.

A separate problem is the fact that the IO model’s primary resources are not the standard ones as distinguished traditionally in other areas of economics. For example, there is no clear classification in terms of the rents of primary factors such as land, labour and capital, including their sub-divisions. The lack of harmonization with the standard approaches in other fields of economics may mean that additional decisions regarding classification and possible sub-classification will have to be made.

5. Final Remarks

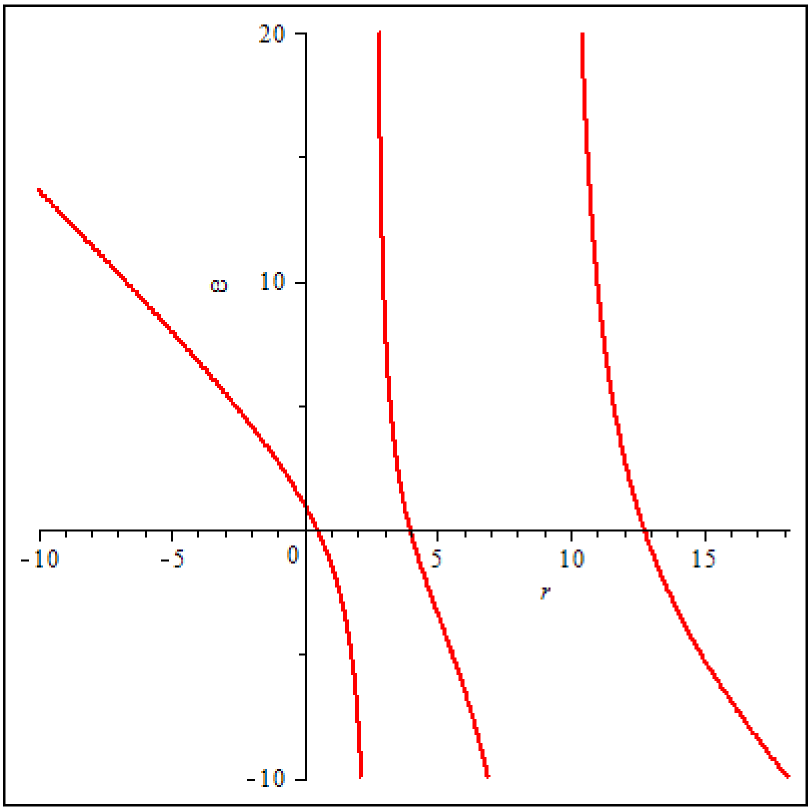
In this paper we have shown that mechanisms determining income distribution can fruitfully be introduced in a standard IO framework. Central here is the so-called augmented input coefficients matrix, a matrix that consists of an economy's intermediate input coefficients matrix supplemented by a number of specially constructed coefficients matrices that translate shares of NI into cost categories for the individual sectors.

Basically these newly constructed coefficients are extensions of the concepts of an intermediate input coefficient. Such extensions have been proposed before (see section 2), but only confined to labour input. We have shown that the principle can be extended to include *all* parties that claim parts of NI. The resulting augmented coefficients matrix gives a complete picture of all such claims on NI, formulated in terms of sectoral cost categories. Income distributions are obtained by attributing *weights* to these newly constructed coefficient matrices. We have proposed three lemmas that allow us to trace the relations between these weights, a sufficient condition for obtaining the relevant distributions of income. We also have shown that prices and quantities (which express the size or scale of the system), are straightforwardly obtained as standardized left and right hand PF eigenvectors of the augmented input coefficients matrix.

The above definitely will not be the last word. Very likely, additional connections will be found and explored as well. In this respect, the methodology can be refined in a number of ways. For example, we may try to establish a link between the weights of the sub-matrices we have introduced and other concepts such as substitution or other elasticities, possibly in correspondence to behavioural shifts motivated by changed circumstances. Also the role of government directed tax-subsidy programs can be investigated, including the distributive effects of specific policy initiatives. Finally, we also may wish to start the other way around, i.e. with changes in the prevailing prices, and see which shifts in distribution these may trigger.

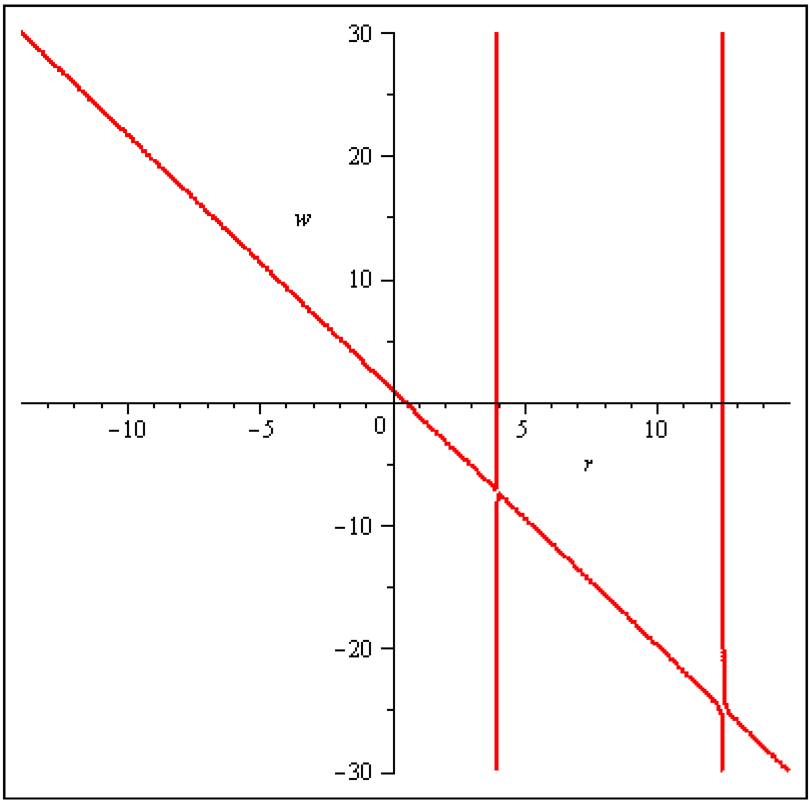
We should remark that, depending on the way the final demand and value added categories have been defined, this may involve a certain amount of re-arranging of the data in the value added rows. For example, household consumption may be paid for out of wages and salaries, and capital income. If the value added classification is not conform, the researcher will have to go back to the original underlying data and construct data according the appropriate classification. If the original data are not available or of low quality, additional research may be required.

Figure 1



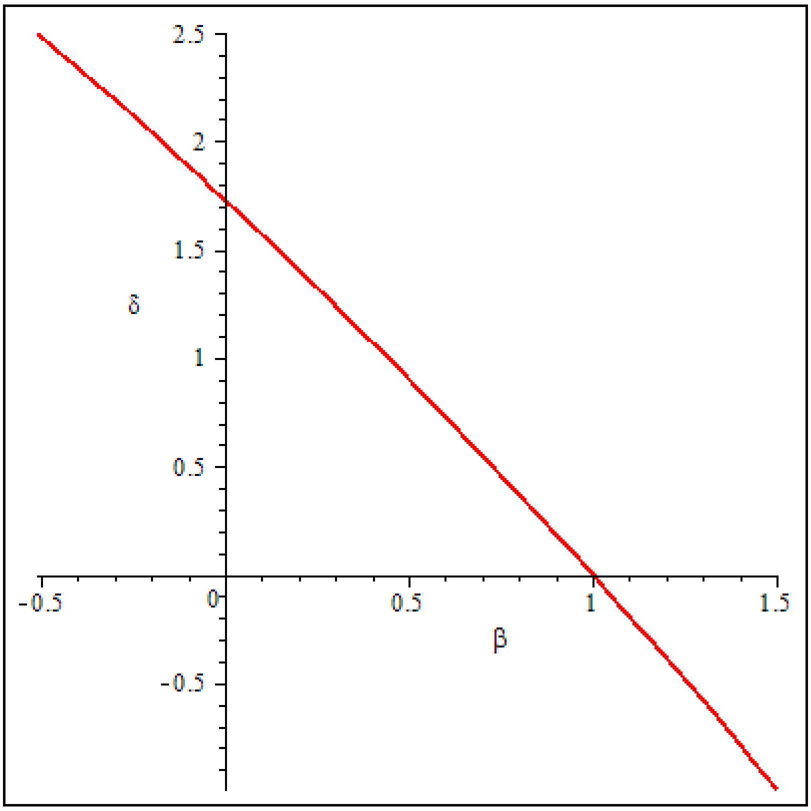
$$\text{graph of } w(r) = \frac{0.240 - 0.580r + 0.170r^2 - 0.010r^3}{0.240 - 0.118r + 0.010r^2}$$

Figure 2



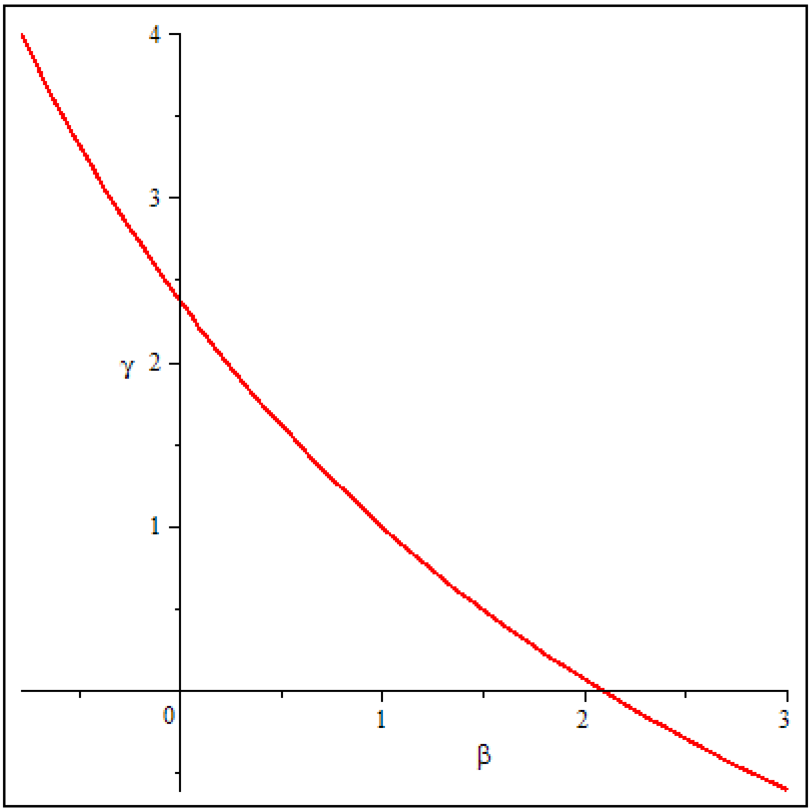
$$\text{graph of } \mathrm{w(r)} = \frac{0.240 - 0.580r + 0.170r^2 - 0.010r^3}{0.240 - 0.079r + 0.001r^2}$$

Figure 3



Graph of $\delta = \dfrac{0.354 - 0.354\beta}{0.205 - 0.071\beta}$

Figure 4



Graph of $\varphi(\beta,\gamma) = 0.354 - 0.169\beta - 0.149\gamma - 0.035\beta\gamma = 0, -1 \leq \beta \leq 3$

References

Abraham-Frois, G. and E. Berrebi, 1979, Theory of Value, Prices and Accumulation: a Mathematical Integration of Marx, von Neumann and Sraffa, Cambridge University Press, Cambridge.

Barro, R., 2006, Rare Disasters and Asset Markets in the Twentieth Century, Quarterly Journal of Economics, vol. 121, 823-866.

Batey and Rose, 1990, Extended Input-Output Models: Progress and Potential, International Regional Science Review, vol. 13(1-2), 27-49.

Batey, P.W.J. and M.J. Weeks, 1989, The effects of household disaggregation in extended input-output models, in: Frontiers of Input-Output Analysis, R.E. Miller, K.R. Polenske and A.Z. Rose, eds., Oxford University Press, New York, 119-133.

Bharadwaj, K. and B. Schefold, eds., 1990, Essays on Piero Sraffa; Critical Perspectives on the Revival of Classical Theory, Unwin Hyman, London, UK.

Brouwer, R. and R. van der Veeren, 2009, Costs and Benefits of Water Policy, in: Water Policy in the Netherlands, S. Reinhard and H. Folmer, eds., Resources for the Future, Washington DC, USA, 2009, 91-107.

Duchin, F., 1998, Structural Economics, Measuring Change in Technology, Lifestyles, and the Environment, Island Press, Washington , D.C.

Friedrichs, J., 2010, Global energy crunch: how different parts of the world would react to a peak oil scenario, Energy Policy, 38 (8), 4562-4569.

Hirsch, R.L., R.H. Bezdek, and R.M. Wendling (2005). Peaking of World Oil Production: Effects, Mitigation, and Risk Management, U.S. Department of Energy NETL, www.hilltoplancers.org/stories/hirsch0502.pdf

Kurz, H.D., 1985, Effective Demand in a “classical” model of value and distribution: the multiplier in a Sraffian framework, The Manchester School, vol. 53, issue 2, 121-137.

Kurz, H.D. and N. Salvadori, 1995, Theory of Production: A long-period Analysis, Cambridge University Press, Cambridge, New York and Melbourne.

Kurz, H.D., and N. Salvadori, eds., 2003, The Legacy of Piero Sraffa, two vols., Edward Elgar, Cheltenham, UK.

Kurz, H.D. and N. Salvadori, 2007, On the collaboration between Sraffa and Besicovitch: the 'Proof of Gradient', in: G. Chiodi and L. Ditta, eds., Sraffa or An Alternative Economics, Palgrave Macmillan, Houndmills, U.K., etc., 260-274.

Leontief, W.W., 1951, The Structure of American Economy, 1919-1939, 2d ed., Oxford University Press, New York.

Leontief, W.W., 1985, The Choice of Technology, Scientific American, vol. 252, 37-45.

Leontief, W.W., 1986, Technological Change, Prices, Wages, and Rates of Return on Capital in the U.S. Economy, Ch. 19 in: Input-Output Economics (2nd Edition), Oxford University Press, New York.

Miller, R.E. and P.D. Blair, Input-Output Analysis; Foundations and Extensions, 2009, Cambridge University Press, Cambridge.

Miyazawa, K., 1976, Input-Output Analysis and the Structure of Income Distribution, Springer-Verlag, New York.

Miyazawa, K. and S. Masegi, 1963, Interindustry analysis and the structure of Income-Distribution, Metroeconomica, vol. 15, II-III, 89-103.

Neumann, J. von, 1945, A Model of General Economic Equilibrium, Review of Economic Studies, vol. 13, pp. 1-9.

Pasinetti, L.L., 1977, Lectures on the Theory of Production, Columbia University Press, New York.

Pyatt, G. and A. Roe, 1977, Social accounting for development planning with special reference to Sri Lanka, Cambridge University Press, Cambridge.

Pyatt, G., 2001, Some Early Multiplier Models of the Relationship between Income Distribution and Production Structure, Economic Systems Research, vol. 13(2), 139-163.

Rao, A.R. and P. Bhimasankaram, 2000, Linear Algebra (2nd Edition), Hindustan Book Agency, New Delhi.

Rose, A., B. Nakayama and B. Stevens, 1982, Modern Energy Region Development and Income Distribution: An Input-Output Analysis, Journal of Environmental Economics and Management, vol. 9, 149-164.

Schefold, B., 1976, Different forms of technical progress, The Economic Journal, vol. 86, no. 344, 806-819.

Seton, F., 1977, The Question of Ideological Obstacles to Rational Price Setting in Communist Countries, Ch. 2 in The Socialist Price Mechanism, Alan Abouchar (ed), Duke University Press, Durham, NC.

Sraffa, P., 1960, Production of Commodities by Means of Commodities, Cambridge University Press, Cambridge.

Steedman, I., 2000, Income Distribution, Foreign Trade and the Value-Added Vector, Economic Systems Research, vol. 12, no. 2, 221-230.

Steenge, A.E., 1995, Sraffa and Goodwin: a unifying framework for standards of value in the income distribution problem, Journal of Economics, vol. 62, no 1, 55-75.

Stone, R. and A. Brown, 1962, A Computable Model of Economic Growth (A Programme for Growth, Vol. 1), Chapman and Hall, London.

Trenkler, G., 2000, On a Generalisation of the Covariance Matrix of the Multinomial Distribution, in: Innovations in Multivariate Statistical Analysis, R.D.H. Heijmans, D.S.G. Pollock and A. Satorra, eds., Kluwer Academic Publishers, Dordrecht. 67-73.