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The Cycle of Rents: a Model of Rational Bull-and-Bear Cycles in an Efficient Market

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Abstract: A widespread misbelief asserts that an efficient market would arbitrage out any cyclical or otherwise partially-predictable, non-random-walk pattern on the observed market prices time series. Hence, when such patterns are observed, they are often attributed to either irrational behavior or market inefficiency. Yet, strictly speaking, the efficient markets hypothesis only rules such patterns out of the expected (i.e. mean) path, whereas, if the probability diffusion process is asymmetric (as in most economic and financial stochastic models), the observed time series will approximate the median path, which is not subject to such constraint. This paper combines a general imperfect-competition production function specification (i.e. one generating economic rents) with the concept of time-to-build to develop a rational-expectations, efficient-markets model displaying a valuation cycle along its median path. This model may therefore help to explain the bull-and-bear cycles observed in asset markets generating economic rents e.g. real estate, commodities or, for that matter, most if not all of the assets quoted in the stock exchange.

JEL Codes: E32, E44, G1, G12, G14.

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1. Introduction

In 1997, Fred Foldvary, a somewhat heterodox economist, predicted, based on the 18-year real estate market cycle observed by Hoyt (1933), that “the next major bust, 18 years after the 1990 downturn, will be around 2008, if there is no major interruption such as a global war.” (Foldvary 1997). No global war ensued, and a major crash did indeed take place in 2008. Not that he would be the only author to make such an early, eerily accurate forecast: the same year, for example, another heterodox author, Fred Harrison (1997), also relied on this property price cycle to predict the same crash, having already anticipated the one in the early nineties in Harrison (1983).

Yet real estate is by no means the only market displaying such cycles. In commodity markets, for instance, several authors (e.g. Cuddington & Zellou 2012 or Erten & Ocampo 2013) identify cycles with average wavelengths of more than 30 years. Furthermore, in overall stock market returns, Fama & French (1988) and Poterba & Summers (1988) spot a 3–5 year cycle (aligned to Kitchin’s 1923 classical cycle – which, incidentally, Foldvary 1998 relied on to successfully predict the 1999-2000 dot-com crash), and Gracia (2012) finds a cycle with a wavelength of 31-33 years in both Tobin’s Q ratio and Shiller’s CAPE (Cyclically-Adjusted Price/Earnings) ratio – which, in turn, is generally recognized as a fair predictor of market upswings and downturns.

Such cycles seem difficult to explain from the viewpoint of rational expectations and the efficient markets hypothesis: indeed, if one could predict a price crash just by counting the number of years since the last one, then, why would smart investors not just place their bets on this opportunity until they arbitraged it out of existence? Faced with this apparent paradox, many authors either dismiss the phenomenon as spurious or attribute it to irrational behavior and/or to friction assumptions leading to some form of market inefficiency. Yet, in truth, these market cycles only constitute evidence against rational expectations under the light of an all-toocommonly held fallacy: that observable recurrent market price patterns departing from a random walk trajectory must reflect systematic arbitrage opportunities the market has not been able to preclude and, therefore, constitute prima facie evidence against the efficient markets hypothesis. Commonsensical as it may seem, this conclusion is fallacious because it assumes the observed path will over the long run approximate the mean (i.e. the "expected") path, where the efficient markets hypothesis does indeed require arbitrage opportunities to be instantly precluded, whereas in reality it will tend to approximate the median, which in asymmetric probability distributions may be very different from the mean, and where marker efficiency imposes no such constraints.

This is perhaps easiest to see with an example. Consider a gamble where we toss a fair coin: heads means the investor receives, say, twenty times the investment, whereas tails implies the loss of the entire investment. Assume now that we start by investing $1, and that the gamble must take place ten times in a row, each time reinvesting the proceeds of the previous toss. Obviously, the expected value at the end of the gamble is very high (specifically $\$10^{10}$ i.e. ten billion dollars) but so is the probability of ending empty-handed (namely $1 - 0.5^{10} = 99.9\%$). This implies that, if we observed this gamble as a time series, 99.9% of the times we would see investors putting down $1 and ending up with nothing (which is of course the median value), and we might therefore wonder why any rational investor would keep throwing away $1 bills.

That the median, not the mean, is the path that best predicts the observed time series of a stochastic variable is actually a fairly standard result in statistics (see Appendix 1 for more detail on this) and should therefore constitute no novelty at all. It is in fact implicit in every dynamic stochastic model where one or more representative agents are assumed to maximize their future expected utilities and then computer simulations are used to study how the model responds to random stochastic shocks – for, if the observed path approximated the mean, these computer simulations would tend to mimic the smoothness of the expected path. Unfortunately, the economic papers where the observed path's link to the median instead of the mean is explicitly acknowledged and used to draw some relevant conclusions are few and far between (although there are some e.g. Roll 1992, Gracia 2005 and Gracia 2012), which effectively, if not formally, leaves unchallenged the popular fallacy of treating observed data as if any arbitrage opportunities observed in them were also to be found in the expected path.

The purpose of this paper is to develop a model to explain long-term valuation cycles in perfectly rational and efficient yet imperfect-competition markets i.e. markets where some form of economic rent exists. It is thus meant to be applicable not just to real estate and commodity markets but also, to the extent every market in the real world generates some form of rent (be it under the form of returns on natural resources, or fixed capital invested, or agency rents, or efficiency wages, or government taxes, or any other), to any market within the economic system.

An economic rent is defined as the difference between the price of an item and its marginal cost: whatever its specific form, no rent can exist in a market with rational players unless there is some form of rigidity, barrier of entry or information asymmetry to make it possible – otherwise the price of any good would equal its marginal cost. Yet, no matter whether the rigidity is technical (as in mining and other fixed assets) or imposed as a means for someone to siphon resources from the system (as in taxes, agency rents and efficiency wages), process rigidity in a stochastic world must lead to some inefficiencies at least to the extent it limits the players' ability to adapt efficiently to any unexpected random shock. Hence, the heavier these rigidities (and thus also the larger the rents that result from them), the more they will act as amplifiers of any random market shock on the demand of productive resources – exactly in the same way as taking debt (whose interests constitute, of course, another form of rent) amplifies the risk exposure on a company's equity. This suffices to set the basis for a cycle. When things go well, companies borrow funds and invest fixed capital to make their processes more efficient, commodity prices soar, trade unions flex their muscle and welfare states become more generous (which must of course be financed through either taxes or debt). Rational players correctly discount the impact of these developments from their expectations, so the mean path forward remains cycle-free. Yet, as these additional rents (and their corresponding sources of rigidity) pile upon the system, it also becomes more fragile. for the rigidities amplify random shocks so that in the end it takes a very small one to trigger a major crisis. Eventually, once such a trigger shock has taken place, as the crisis unfolds the process gradually reverses: commodity prices drop, debt is defaulted, fixed capital is allowed to age, trade unions lose their negotiations and governments relearn the hard lessons of austerity until the burden of rents on the productive system is lightened enough for the process to restart. Since the observed trajectory will most approximate the median path, not the mean, then, as long as the probability diffusion process is asymmetric (as those in most of the models used today in economics and finance are), a median-path cycle of these characteristics is in principle perfectly compatible with both rational expectations and market efficiency.

The most direct precedents of the model put forward in this paper are Gracia (2005), (2011) and (2012). Specifically, Gracia (2005) puts forward an efficient markets' valuation model based on agency rents leading a predator-prey cycle along the median path of a company's solvency ratio and, complementarily to this, Gracia (2012) provides empirical evidence that a cycle of these characteristics could indeed explain the long-term valuation cycle observable in both Tobin's Q and Shiller's CAPE data series. Gracia (2011), on the other hand, develops the production function we use in this paper by deriving it from a set of strictly neoclassical axioms, and then performs a battery of empirical tests showing it to be a better fit to observed data than the Cobb-Douglas function (in fact rejecting the Cobb-Douglas specification at 99% confidence).

More broadly, and despite its many differences, Richard Goodwin's growth cycle model (Goodwin 1967) constitutes an obvious precedent in its portrayal of economic rents' fluctuations (specifically wages, which he modelled as a sort of scarcity rent) under the guise of a predator-prey cycle. Goodwin, who regarded himself as a Marxian economist, built his 'class struggle' model on the basis of assumptions that in his days were commonplace in mainstream literature (e.g. Phillips curve, constant-factor-intensity production function, differential saving propensities for capitalists and workers, etc.) but would not fit a rational-expectations, efficient-markets framework – not that, in fairness, would any of the business cycle models in vogue at the time (think e.g. of Samuelson's 1939 multiplier-accelerator model and its many offshoots). Goodwin's model remains, nevertheless, one of the most elegant analytical expressions of an old idea: that economic and social cycles are caused by people's efforts to capture and institutionalize economic rents for themselves, so that, over time, the burden of these rents gradually becomes heavier and, like a growing parasite, weakens the productive system so much that in the end even a small shock suffices to make it crash – at which point rents inevitably fall and, as they lighten their weight over the system, eventually growth resumes and the cycle restarts.

The model put forward in this paper, however, diverges from this tradition in its explicitly adopting the classical assumptions of rational behavior, market efficiency and frictionless price formation in a stochastic environment. This also distances it from the irrationality assumptions underlying Hyman Minsky's now-famous Financial Instability Hypothesis (e.g. Minsky 1992) as well as the most recent behavioral finance models but also the credit restriction and sticky prices assumptions behind rational expectations models such as Kiyotaki & Moore (1997), Bernanke, Gertler & Gilchrist (1999) or those following the very rich Rational Bubbles literature (see Martín & Ventura 2018 for a survey). Instead, it adopts the assumptions of existence of economic rents and time-to-build, the latter thus somehow linking to the tradition following Kydland &

Prescott (1982) or, going much further back, the old Austrian school (e.g. Böhm-Bawerk, Hayek or von Mises). It is actually the contention of this paper that, as long as a market does not operate under perfect competition, a set of relatively straightforward assumptions may suffice to explain the consistent presence of observable nonlinear behavior, including long-term cycles, in the price data series of a portfolio operating within a rational, efficient market.

Importantly, this paper's main purpose is to model the behavior of fixed-asset-heavy markets such as commodities or real estate within a larger overall market and, therefore, it does not aim to determine how the market-wide price of risk would be derived from investor's preferences but simply how, given a prevalent price of risk, arbitrage conditions suffice to determine the prices prevalent in any rent-generating sub-market. An obvious extension would of course be to perform the analysis for a closed market where the price of risk, instead of being exogenous, derived from the players' utility functions – but this would go beyond the scope of this paper.

From an analytical viewpoint it is also worth mentioning that this model operates within a continuous-time stochastic framework. Continuous time is not only conceptually more realistic but also analytically easier to manage, as it allows to make use of a set of very powerful analytical tools that simply do not exist for discrete time. The commonly held view that "discrete time is easier" springs from the fact that many models do not attempt to find explicit solutions beyond what discrete time allows so, for example, illustrate the time series generated by a given model via computer simulations instead of presenting an explicit time-dependent expression of the most likely observed path (which, as explained above, would be the median). Here, conversely, we assume continuous time and use it to identify closed-form paths where possible.

The structure of this paper is straightforward: Section 2 provides an intuitive rationale for the model's behavior, then Section 3 presents the analytical development of the model, and finally Section 4 summarizes the conclusions and potential future directions.

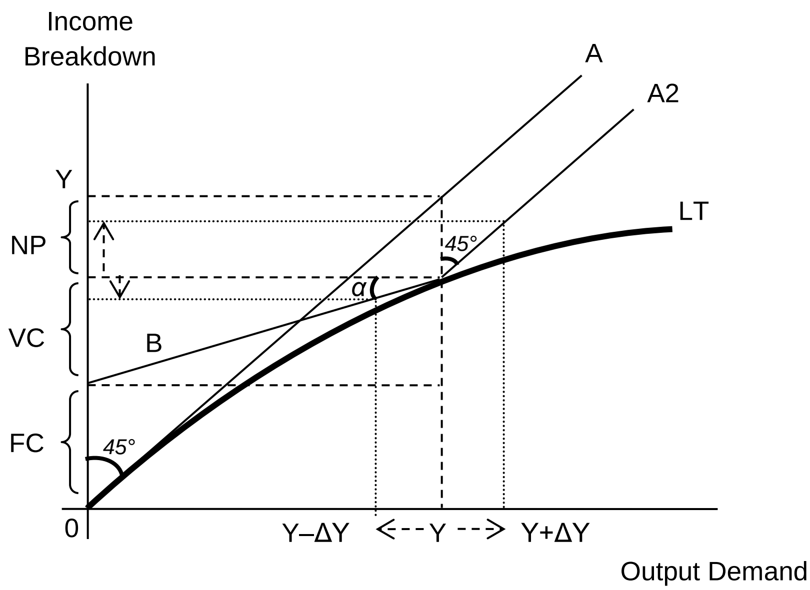
2. Model Rationale

2.1. The Production Function

Consider an economy composed of many production units, each one devoted to transforming inputs into a given set of outputs. For every given output volume, there are multiple productive processes or "techniques" available, each one requiring a given fixed investment in plant capacity in addition to the variable cost per unit produced. Each producer aims to select the output volume and productive process that maximizes real profit i.e. the difference between output value and input costs, measured in output units. We also assume the optimal technology curve that results from their selecting the most profitable technique at each production level displays economies of scale i.e. that, given an increase in production volume, there is always a technique that would allow to reduce the overall cost per unit (always expressed in terms of output units) and therefore increase the real profit. This is represented graphically by curve LT in Figure 1.

In this diagram, both the horizontal and vertical axes represent output, whereas every line in the quadrant represents the cost structure of a given productive process or technique. Hence, if we select an output demand level Y on the horizontal axis, its projection on the vertical axis according to the curve representing a given technique indicates how much of its output value corresponds to Fixed Cost (segment FC), Variable Cost (segment VC) or Net Profit (segment NP). For example, curve LT represents the optimal long-term production cost curve i.e. the lowest production cost possible for every given output level, regardless of how long it would take to deploy the associated production process: every point Y along this curve is tangential to a given optimal production technique (for example B) requiring a fixed upfront cost (i.e. FC) plus a certain variable cost per unit (represented by the slope of segment B), up to the plant capacity Y.

Figure 1: Long-term vs. Short-term Production Functions



Conversely, the bisector line A represents both income (i.e. if the output level is Y, the real income measured in output units is obviously also Y) and the cost profile of a production technique with constant variable costs per unit, no fixed costs and no barriers of entry, which would of course result in the unit price equating its marginal (i.e. variable) cost – hence it forms a 45° angle respective to the axes. Evidently, technique A has a steeper slope than B because its variable costs are higher, and it is more inefficient for a level of production Y because the overall cost per unit for technique B, including both fixed and variable costs, is lower than that of technique A – the difference being of course the net profit NP.

In economics we conventionally break down prices into marginal costs i.e. the incremental cost of producing the last unit of output (segment VC in Figure 1) and economic rents, that is, the difference between marginal cost and actual price (the sum of segments FC and NP). Rational, profit-maximizing producers will of course aim to produce up to the point where marginal cost equals marginal income but, unless they operate under perfect competition, this will not equal the price: if they enjoy any market power, it will be reflected as positive rents – and if, for instance, they happen to confront a monopsony, they might even have to live with negative rents.

Fixed investments represent sunk costs whose historical size is irrelevant for the maximization of future profit and whose return therefore constitutes an economic rent. To be sure, an investor considering whether to commit an upfront investment to a productive unit would require the present value of its expected return to equal or exceed the upfront cost so, in a deterministic world, it would make sense to model those future rents as equating marginal costs to marginal returns for that historical investment amount – yet, in an uncertain world, as soon as the conditions change so do the rents, after which no link may exist between them and the sunk costs.

The assumption behind the long-term curve LT is that, when planning for the long run, producers can jump from one technique to the next as their output volumes change, choosing for every level of production the technique with the lowest cost. Conversely, when unexpected shocks hit demand, it is not possible to do this in the short run, for the upfront investments to expand capacity and implement a more efficient production process cannot be deployed instantly, nor can installed capacity be instantly dismantled. Thus, if demand, for example, drops unexpectedly by a magnitude ΔY (i.e. down to $Y–\Delta Y$), the producer will use the same technique to produce at less than full capacity, whereas, if demand increases instead by ΔY (i.e. up to $Y+\Delta Y$), the producer will have to resort to the less efficient technique A to produce the supplementary units required above the available facilities' capacity. This means that, precisely because the long-term cost function is concave (i.e. has positive returns to scale), the short-term one must be convex (i.e. display diseconomies of scale) for, given a planned output level Y, actual production follows segment B when it falls below Y but segment A2 (parallel to A) when it raises above Y.

Note that, if we assume producers maximize their profit by producing up to the point where marginal cost equals marginal income, then segment A2 *must* necessarily be parallel to segment A (which represents income) and therefore cut the vertical axis at 45° for the level of production Y to be chosen as optimal. At the same time, if we assume segment B represents the long-term optimal technique, then, as capacity approaches that point from the left it *must* be tangential to the curve LT. The point where segments B and A2 meet is where the production capacity of the optimal facilities ends and thus less efficient methods must be employed. For diagrammatic convenience this is represented as a sharp 'peak', although it is perhaps more realistic (as well as more analytically convenient) to assume it to be smooth enough to be differentiable at this point: in the real world, after all, every plant and piece of machinery can usually be temporarily pushed just a bit beyond its natural capacity (e.g. by delaying maintenance, running at non-sustainable speeds, etc.) albeit obviously at a higher longer-term cost than under steady state conditions.

Imagine now that there are many industries in an economy, each producing a different set of goods and services but all subject to a similar cost function. If the structure of aggregate demand changes unexpectedly, so that demand for one product increases at the expense of another while the total consumer budget stays the same, the costs of those industries whose demand dropped will fall comparatively less than the costs of the industries with higher demand will raise. This will thus cause an aggregate productivity loss that will be more severe, other things being equal:

1. The larger the variability of demand (i.e. the increment ΔY in the diagram) and
2. The smaller the angle α between segment B and the horizontal line

In turn, since any increment in the share of economic rents (i.e. FC+NP) over the total revenue Y results in squeezing segment VC and thus flattening segment B and closing angle α, we may say that, under demand uncertainty, the higher the ratio $\frac{\mathrm{FC+NP}}{\mathrm{Y}}$, the lower the overall short-run productive efficiency. This ratio is far from novel in economics: it is actually Lerner's classical Market Power Index (Lerner 1934), which is defined as $L \equiv \frac{\mathrm{P-C'}}{\mathrm{P}}$ (where P represents the price and C' the marginal cost) and therefore, in terms of Figure 1, may be rewritten as $L \equiv \frac{\mathrm{FC+NP}}{\mathrm{Y}}$ or, in trigonometric terms, as $L \equiv 1 - \tan(\alpha)$. Note that for some of the analysis it will be more convenient to use instead the mark-up over marginal cost, whose definition is $\mu \equiv \frac{\mathrm{P}}{\mathrm{C'}}$ and thus, in terms of Figure 1, becomes $\mu \equiv \frac{\mathrm{Y}}{\mathrm{VC}}$ (or, in trigonometric terms, $\mu \equiv \cot(\alpha)$). Conversion between one and the other is trivial through the formula $L \equiv \frac{\mu-1}{\mu}$ or, in reverse, $\mu \equiv \frac{1}{1-L}$.

In sum, the model predicts that short-term production fluctuations will be a function of the variability of demand composition (which represents the average shock ΔY) amplified by the percentage of economic rents over total output (which determines the angle 'α' and therefore the degree of convexity), which provides a measure of the system's rigidity.

Under uncertainty, productivity in the short run behaves exactly opposite to the long run. In the long run what matters is curve LT and, since segment B represents the tangent to this curve, the larger Lerner's index, the flatter (i.e. the more efficient) the curve will be at that point. Conversely, in the short run, the larger this ratio (i.e. the flatter the LT curve), the smaller the angle α will be and, since the segment A2 must be the continuation of segment B at point Y (because otherwise Y would not represent the producer's optimum) and is bound to stand at a 45° of the axes, the higher the loss of efficiency due to any unplanned increment ΔY. In a deterministic world, to be sure, only the long-term function would matter, as rational agents would plan their entire future at one point in the beginning of time and never have to revise their expectations again. Yet, in a stochastic world, change takes place continuously and, as economic agents readjust to the new conditions, it is the short-term function that determines the initial response so, the larger the weight of rents over output, the lower the system's flexibility.

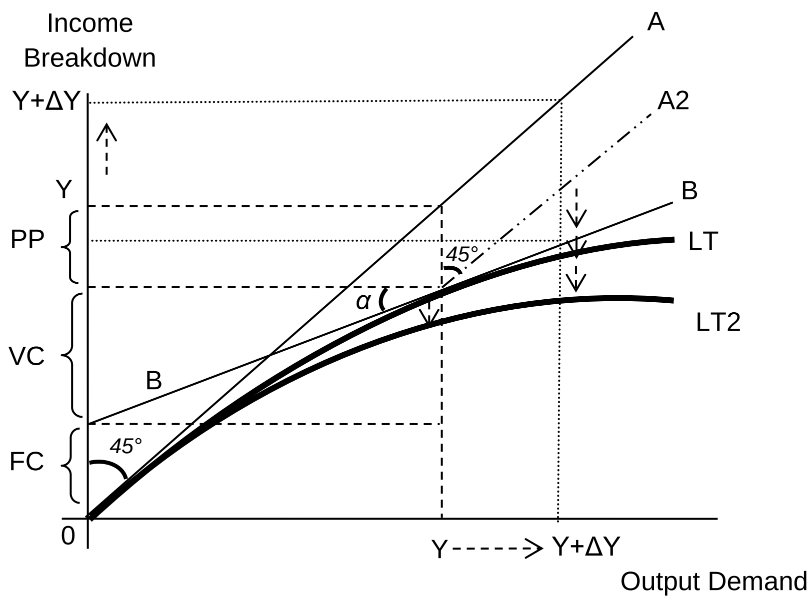
Note that Figure 1 is agnostic respective to the reasons why the long-term curve LT and therefore also segment B may flatten, as the angle α simply reflects a relative proportion between the overall output Y and the part of it that is distributed under the form of economic rents. Hence, although a higher Lerner index (or mark-up) necessarily implies lower flexibility and therefore higher inefficiency in the presence of uncertainty, a higher rent ratio may or may not be associated to higher absolute efficiency (i.e. higher production for the same input volume) in the long run. Fixed capital investment, for example, may indeed increase productivity through economies of scale and integration so, at the time of committing the resources, investors would logically expect their assets to generate a future rent to compensate for their present sacrifice. Yet rents can also be generated e.g. through coercive or lobbying power to enforce monopolies, or to charge tolls and taxes for governments, or to increase efficiency wages for trade unions, all of which are more likely to limit the system's productivity per input unit.

2.2. Productive and liquidation values

As we have seen, unexpected changes in demand (which we represented by ΔY) face a convex cost curve and therefore constitute a source of inefficiency in the short run whereas, if changes are expected, rational producers can prepare for them. In graphical terms (Figure 2), this means that a planned output increase ΔY would not meet segment A2 but a flatter segment (say, B2). If, for example, we assume output is increased at first by expanding the use of the same techniques represented by segment B, then segment B2 would just be a straight-line prolongation of segment B – which would leave angle 'α' unchanged. Then, of course, as further improvement takes place, segment B would gradually tilt downwards until becoming tangent to the long-term curve

LT at the point where output is Y+ΔY in the same way as it was before at the point Y (which would of course lead to angle α becoming smaller the flatter the curve LT is at the point Y+ΔY). Moreover, even if there is no output change, over time new opportunities to extract additional rents from the system would pop up and, to the extent they are economically attractive, would push the long-term productivity frontier LT further down (i.e. from LT to LT2 in Figure 2).

Figure 2: Transition from short to long term production functions



The key is that this planned production technique adjustment from its optimum at output volume Y to that of Y+ΔY and also from the technology level of long-term curve LT to that of LT2 gradually flattens segment B and, therefore, also reduces angle α, which, over time, amplifies the impact potential of any unexpected shocks – in other words, the production function becomes more fragile, more sensitive to negative random shocks.

Rational investors will of course discount the expected value of this inefficiency from the net present value of each production asset and decide whether to invest in more or less rent-intensive productive techniques on the basis of which option offers the highest expected value based on their degree of risk aversion. Hence, along the mean path (i.e. the trajectory where expectations are systematically fulfilled) there will by definition be no surprises, no bubbles or panics, and asset values will always grow at their market discount rate plus or minus the cash flows they absorb or generate. At the time rent-generating investments are made, rational players will expect them to yield the market return and, along the mean path, such expectations will always come true, so accumulated investment will indeed equal the net present value of capital.

Yet, as explained in the Introduction, to the extent the probability diffusion process is asymmetric (so that mean and median paths are different), along the observed path there will be continuous surprises that over a long-enough sample will approximate the median, not the mean, trajectory. As unexpected shocks take place, investors' original expectations may not be fulfilled, which means historical investments become sunk costs, so the return they yield under the form of rents may not bear any resemblance to the rate of return the market offers to fresh investments.

The market value of the production unit corresponds of course to the aggregate value of all its component assets i.e. of the assets tied up as necessary for the production process. Some of these assets, to be sure, would not have any value if detached from this process, but others (e.g. buildings, vehicles, cash holdings, etc.) are fungible enough to fetch some value if sold for redeployment into other activities. Of course, the aggregate liquidation value of these fungible assets is different from that of the production unit as a whole so, in a complete market, the owner of a productive unit always has two ways to exchange it for cash: either sell it as productive unit or dismantle it and sell out its fungible components. If the latter is higher, rational investors will obviously proceed to dismantle and liquidate the fungible assets whereas, if it is lower, additional fungible assets (e.g. new cash contributions) will be invested in expanding the productive unit, which will transform these contributions at least partially into non-fungible assets. In this context, however, we introduce three key assumptions:

1. Dismantling obviously means decommissioning the productive unit and therefore rendering it unable to generate any rents it might have been yielding before

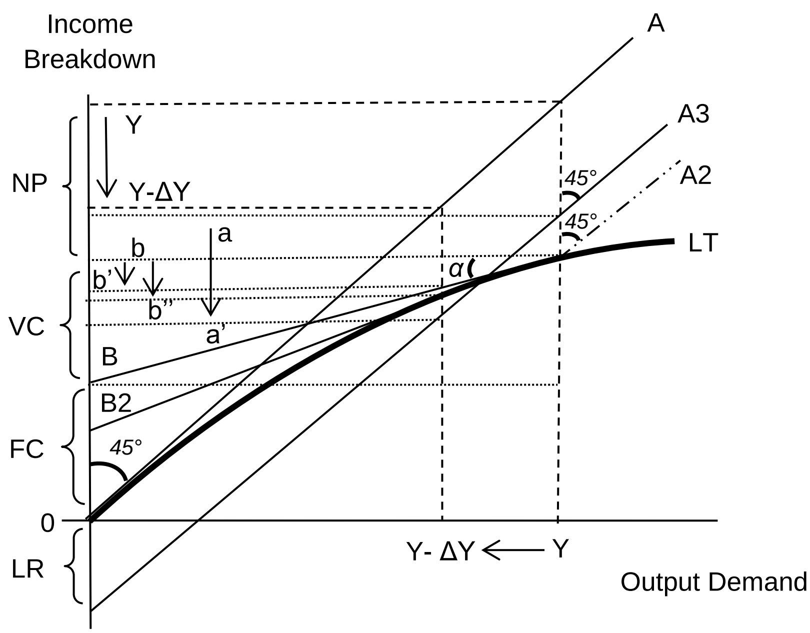
2. Building up productive assets or dismantling them for liquidation takes time. In the model we simplify this by introducing a positive constant $\theta \geq 0$ such that, the larger it is, the faster the processes of build-up and dismantling are assumed to be

3. Fungible assets are less risky than the other component assets of the productive unit, if nothing else because they can more easily be redeployed if the unit's profitability does not meet expectations. For the sake of simplicity, in the model we assume fungible assets yield the risk-free rate of return, whereas the unit as a whole is subject to a risk premium

As fungible assets are low risk (actually risk-free in the model), they can be pawned or mortgaged as credit collateral. Hence we will refer to the ratio of the unit's market value divided by its fungible assets' liquidation value as the "solvency ratio" (which we will represent as "*s*"), although it should be clear it represents the productive unit's ability to produce credit collateral, irrespective of whether it is actually used to borrow funds from investors outside the unit or not.

This solvency ratio therefore provides a yardstick against which rational producers can measure the relative economic appeal of each course of action. Figure 3 illustrates how this might happen in a very simple case: to the basic structure in Figures 1 and 2, Figure 3 adds an alternative, less capital-intensive technique associated to lower fixed and higher variable costs (segment B2), which could be achieved through partial fungible assets' liquidation, and also a positive rent (segment LR) that the producer could obtain by totally discontinuing production to free-up the fungible assets' returns. In the latter case, to be sure, the production unit would no longer produce, so output costs would equal market prices and thus the new supply line (segment A3) would run parallel to A so that profit always equal LR regardless of output volume.

Figure 3: Production vs. liquidation return



Faced with this choice, any rational producer would naturally take the most profitable option for the required output level but, as output demand changes randomly, the best option today may become the worst tomorrow. On Figure 3, for example, at an output level Y it is obviously better to stick to segment B than to either move to B2 or go all the way to A3. Yet, if demand experiences a negative shock bringing it down to Y–ΔY, then production costs following segment B go down only from b to b', thus squeezing the producer's net profit, whilst a partial liquidation to move to segment B2 would bring the producer's costs down to b'' and, even more drastically, proceeding to full liquidation would bring them down to a'. Assuming the net present value of these various options also follows the pattern in Figure 3, asset realignment would logically follow – and, as a result, would eventually lead to a gradual fall in the overall mark-up.

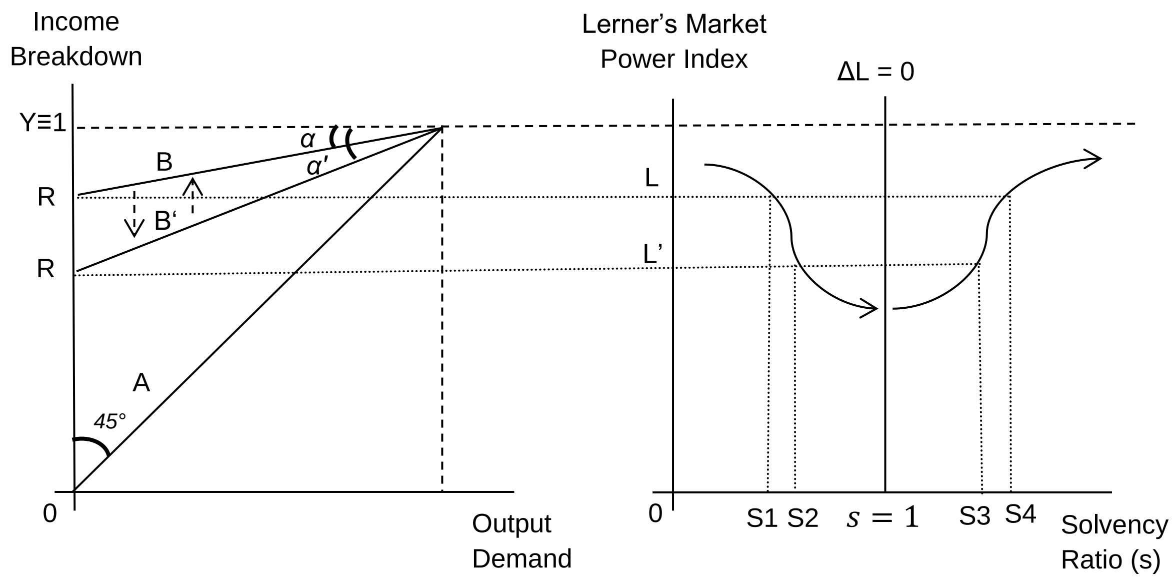
To put this in slightly more precise terms, if the solvency ratio is lower than unity (i.e. $s < 1$) it triggers a liquidation process, as it implies the cost of opportunity of keeping at least some of those assets locked within the production process is higher than the alternative. Dismantling, by widening angle α towards 45°, will of course tend to reduce the average mark-up over time; conversely, if the solvency ratio is above unity (i.e. $s > 1$) then more rent-generating asset build-up will take place and as a result the mark-up will tend to increase. There should therefore be a solvency value (in principle $s = 1$) where the two trends balance each other and thus the mark-up does not change.

2.3. Productivity and valuation cycle

If fungible assets carry a lower risk, in an efficient market with rational, risk-averse investors, they must also yield a lower market rate of return. To be sure, new cash injections and subtractions by investors could translate into changes in the mix of fungible and non-fungible assets. Yet if, for simplicity, we assume in the model that cash flows corresponding to fungible and non-fungible assets are proportional to their weight in the production unit's portfolio, then the higher risk premium of the solvency ratio's numerator respective to its denominator will, at least along the expected path, make solvency drift gradually towards higher values.

Figure 4 illustrates this model's behavior along the mean (i.e. expected) path, whilst Figure 5 does the same for the median path (which as we know would be the closest to the observed path). In each case the diagram on the left is just the one we saw in Figures 1, 2 and 3 – only, in this case we aggregate the rents FC and NP at the bottom of the vertical axis (and refer to the sum simply as $\mathrm{R} \equiv \mathrm{FC} + \mathrm{NP}$) so that the variable cost VC is represented by the segment immediately above (i.e. the vertical segment from R to Y) and then set total output to unity (i.e. $\mathrm{Y} \equiv 1$) to express rents as a percentage of total output and thus be able to set $\mathrm{R} \equiv L$ (where L represents Lerner's Market Power Index). Conversely, the diagram on the right-hand side plots the same vertical axis (only, now expressed in terms of Lerner's Index) against the solvency ratio on the horizontal axis. Perpendicular to this axis we trace a vertical line indicating the solvency ratio's value (in principle $s = 1$) that would lead to no change in Lerner's index (so that $\Delta L = 0$).

Figure 4: Output dynamics along the mean (i.e. expected) path



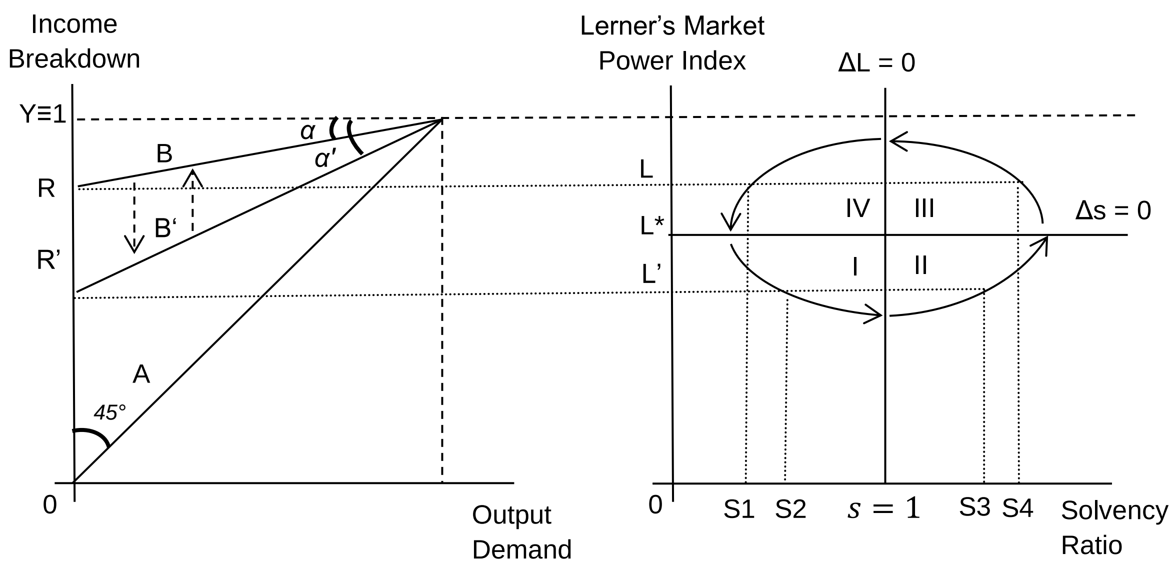
Under these conditions, starting from, say, the point where $s = \text{S1}$, the expected path will lead to a gradual increase in the solvency ratio due to the higher expected risk premium of its numerator (the net present value of future production rents) vs. its denominator (the liquidation value) whilst Lerner's index reduces as rent-generating assets are liquidated: therefore, as the solvency ratio grows from S1 to S2, Lerner's index drops from L to L' and the production function slope B slides to B', thus opening up angle α to become α'. As the process continues and the solvency ratio keeps growing, however, it eventually reaches and then exceeds the threshold value $s = 1$, at which point, since it no longer makes sense to liquidate productive assets, the dynamic reverses and rents grow again as a percentage of total output. Hence, as the solvency ratio proceeds from S3 to S4, Lerner's index climbs back from L' to L, the production slope flattens again from B' to B and its angle respective to the horizontal axis goes back from α' to α.

In short, there is no cycle along the mean path: regardless of the starting point, the expected trajectory under these assumptions leads to ever-higher values of the solvency ratio and, once it exceeds the threshold value $s = 1$, to ever-growing shares of rents over total output, which therefore will, after a long enough time, asymptotically approximate Lerner´s index to $L = 1$ or, what is the same, the mark-up to $\mu = \infty$.

The median path (which would be the closest to the observed path) is very different and, arguably, more interesting. As shown in Figure 5, the same factors as for the mean apply, but we need to add to them the impact of unexpected shocks on productivity (which by definition are excluded from the expected path) and take into account that, the higher the mark-up (i.e. the smaller angle α), the more sensitive the productive system becomes to these shocks. Therefore, although unexpected shocks come in many flavors and sizes, there must be a value of Lerner's index (let´s call it $L*$) such that the risk premium difference between the solvency ratio's numerator and denominator is offset by the differential impact of the median random shock, which is larger on an asset's productive value than on its liquidation value (because the former's longer financial duration makes it more exposed to such unexpected shocks – which is of course why it also has a higher expected risk premium) so that, on balance, when $L = L^*$ then $\Delta s = 0$. Note that this threshold point $L*$ does not exist on the expected path diagram (i.e. on Figure 4) simply because there the impact of random shocks is, *ex hypothesi*, fully discounted by rational investors out of the valuation and there are therefore no unexpected shocks on it – unlike, of course, on the median path.

These conditions can indeed lead to cyclical behavior along the median path (and therefore also on the observed path). Starting, for example, from a value such as L' in Quadrant I (i.e. where $L = L' < L^*$ and $s = S2 < 1$) the trajectory will for a while follow the same pattern as we described for the mean i.e. the solvency ratio will move from left to right, and the value of Lerner's index L will drop while $s < 1$ only to start climbing again as it enters Quadrant II, where $s > 1$. As Lerner's index eventually grows beyond $L*$, however, the probability of a negative shock that more than offsets the return differential between the asset's productive and liquidation values exceeds 50% and therefore the median path leads to lower solvency values[1] – which results in the value sliding towards the left through Quadrant III. Then, of course, when the value moves into Quadrant IV and thus $s < 1$ again, liquidations start to take place and the mark-up gradually reduces, until Lerner's index eventually falls again below $L*$ and the cycle restarts in Quadrant I.

Figure 5: Output dynamics along the median path (i.e. the one closest to the observed series)



Section 3, below, develops this same reasoning in a more analytical, rigorous manner.

[1] There are, however, a couple of theoretical cases in which this would not necessarily happen: if the market price of risk (i.e. the rate at which the market rewards asset variability) increases more than proportionally to the standard deviation just as Lerner's index goes up for this particular productive unit or industry, or if the non-diversifiable variability that this productive unit or industry is exposed to also drops more than proportionally to compensate for the increase of Lerner's index. For the sake of simplicity, in the model we have ruled out these possibilities by assuming that both the variances impacting the unit's non-diversifiable risk and the median path of the market price of risk associated to them are constant.

3. Analytical development

3.1. Definitions and identities

Consider an economic system within a probability space $(\Omega, I, \mathcal{P})$, where 'Ω' represents the universe of all possible states of nature $\omega \in \Omega$, 'I' the set of all possible subsets of Ω and '$\mathcal{P}$' the probability distribution function. In this space, for every point *t* in time there is a set $I_t \subset I$ representing the information available to market players at that time, which is assumed to be cumulative i.e. $\forall t \geq T$ we assume that $I_T \subset I_t$. Going forward we will represent with operator $E_t[\circ]$ the expected (i.e. mean) value of the variable within the brackets subject to the information available at time *t*, as $V_t[\circ]$ its variance and as $M_t[\circ]$ its median path.

The system encompasses $i = 1 \dots n$ commodities and $j = 1 \dots m$ production units, each one of which transforms (at any given point in a continuous time $t \in \Re$) a certain set of input quantities of goods and services $\boldsymbol{X} \equiv \{x_{1,j,t} \dots x_{n,j,t}\}$ into another set of *net* output quantities $\boldsymbol{Y} \equiv \{y_{1,j,t} \dots y_{n,j,t}\}$ i.e. output quantities from which the input stock variation experienced by each commodity during the production process has been subtracted. We refer to this transformation as a *technical production function* and represent it as a mapping $f_{j,t}: \mathbf{X} \to \mathbf{Y}$ or, what is the same, as a function $f_{j,t}(\cdots)$ such that $f_{j,t}(x_{1,j,t} \dots x_{n,j,t}) \equiv \{y_{1,j,t} \dots y_{n,j,t}\}$.

Following the usual convention, we count human labor as an input but not as an output so that, if we identify as "capital" the commodities numbered $i = 1 \dots \nu$ and as "labor" those numbered as $i = \nu + 1 \dots n$, then by definition $\forall i \in [\nu + 1, n] \; \rightarrow \; y_{i,j,t} \equiv 0$.

The commodities $i = 1 \dots n$ map to a set of market prices $\boldsymbol{P} \equiv \{p_{1,t} \dots p_{n,t}\}$ expressed in terms of a commodity (e.g. money) or a basket of commodities (e.g. average consumer's basket) selected as *numéraire* – which we will designate as ζ. Note that, without loss of generality, these commodities are defined granularly enough for their prices to be unique within the system – hence the price symbol $p_{i,t}$ includes a sub-index for the commodity (*i*) and another for time (*t*) but none for the production unit.

In this context we define a production unit's net total output value $\mathcal{Y}_{j,t}$ as the aggregate of the net commodities produced over a given time lapse Δt multiplied by their values plus the market value increment of the inputs' inventory over the same timeframe i.e.:

$$\mathcal{Y}_{j,t}\Delta t \equiv \sum_{i=1}^{\nu}(p_{i,t} + \Delta p_{i,t})y_{i,j,t}\,\Delta t + \sum_{i=1}^{\nu} x_{i,j,t}\Delta p_{i,t}$$

Going forward we will assume time to be continuous and thus every flow variable will be assumed to refer to a rate over an infinitesimal time lapse dt. Under this assumption the net output value $\mathcal{Y}_{j,t}$ may be rewritten as follows:

$$\mathcal{Y}_{j,t}dt \equiv \sum_{i=1}^{\nu} p_{i,t}y_{i,j,t}\,dt + \sum_{i=1}^{\nu} x_{i,j,t}dp_{i,t}$$

Note that, as this is a stochastic environment where new information can always pop up between any two points in time *t* and *t+dt*, the value of any flow variable such as $y_{i,j,t}$ or $\mathcal{Y}_{j,t}$ will not be known until time *t+dt*, whereas decisions to optimize the inputs $\{x_{1,j,t} \ldots x_{n,j,t}\}$ devoted to the production process will obviously have to rely on the information available at time *t*. To simplify the analysis, therefore, we define the variable $Y_{j,t} \equiv E_t[\mathcal{Y}_{j,t}]$ (i.e. the expected instantaneous net output value at the time the input allocation decision is made) and define the *economic* *production function* $F_{j,t}(x_{1,j,t} \dots x_{n,j,t})$ as a transformation $\Re^n \to \Re$ from the input quantities to the net output value flow i.e. $F_{j,t}(x_{1,j,t} \dots x_{n,j,t}) \equiv Y_{j,t}$.

We now define the net profit $\Pi_{j,t}$ of production unit *j* at time *t* as the difference between its net output value and its production cost expressed in terms of a basket of commodities we select as numéraire i.e. $\Pi_{j,t} \equiv \mathcal{Y}_{j,t} - \mathcal{C}_{j,t}$, where $\mathcal{C}_{j,t}$ represents the cost as a function of input quantities $\boldsymbol{X} \equiv \{x_{1,j,t} \dots x_{n,j,t}\}$. Importantly, since $\mathcal{Y}_{j,t}$ has already been defined as net of asset depreciation (or revaluation), the cost function $\mathcal{C}_{j,t}$ must exclude any depreciation constants and therefore, should all the inputs $\{x_{1,j,t} \dots x_{n,j,t}\}$ be zero, $\mathcal{C}_{j,t}$ would also be nil.

We also define the combined input $X_{j,t}$ as the aggregate of all production inputs weighted by their marginal costs i.e. $X_{j,t} \equiv \frac{1}{c_{\xi,j,t}} \boldsymbol{CX} \equiv \frac{1}{c_{\xi,j,t}} \sum_{i=1}^{n} c_{i,j,t} x_{i,j,t}$ where $\boldsymbol{C} \equiv \{c_{1,j,t} \dots c_{n,j,t}\}$ represent the marginal costs per input unit i.e. $c_{i,j,t} \equiv \frac{\partial \mathcal{C}_t}{\partial x_{i,t}}$ and $c_{\xi,j,t}$ represents the marginal cost of an input $\xi \in \{1 \dots n\}$ selected as aggregation unit – which could in principle be the same commodity selected as the aggregate output numéraire ζ (e.g. money) or a different one (e.g. labor).

We then define the mark-up ratio $\mu_{j,t}$ as the expected output value over aggregate marginal costs i.e. $\mu_{j,t} \equiv E_t \left[\frac{\mathcal{Y}_{j,t}}{c_{\xi,j,t} X_{j,t}}\right]$. Note that, as the purpose of the mark-up is to measure market power (for, as explained in Section 2, it translates into Lerner's index $L_{j,t}$ through the identity $L_{j,t} \equiv \frac{\mu_{j,t}-1}{\mu_{j,t}}$), it is more convenient to define it here as an expected value to weed out any unexpected output price or volume fluctuations that might not reflect any market power change.

Next, we define the financial variables. We designate by $K_{j,t}$ the market value (i.e. the price) at time *t* of the assets employed in production unit *j*, and the rate of return rate of return $\mathscr{r}_{j,t}$ as the ratio of the unit's net profit divided by its market price i.e. $\mathscr{r}_{j,t} \equiv \frac{\Pi_{j,t}}{K_{j,t}}$. For convenience, just as we did in the case of net output we also define a variable representing these rates' expected value at time *t* (i.e. at the start of the infinitesimal time lapse between time *t* and time *t*+*dt*), that is, $r_{j,t} \equiv E_t[\mathscr{r}_{j,t}]$. Furthermore, we define the net cash flow as the portion of net profit that is not reinvested to increment the asset value i.e. $C_{j,t} \equiv \Pi_{j,t} - \frac{dK_{j,t}}{dt}$, so that, rearranging terms, $\mathscr{r}_{j,t}K_{j,t}dt \equiv dK_{j,t} + C_{j,t}dt$. Analogously, we designate by $K^*_{j,t}$ the value of the unit's fungible assets, by $C^*_{j,t}$ the net cash flow associated to them, by $\mathscr{r}^*_{j,t}$ its rate of return (so that $\mathscr{r}^*_{j,t}K^*_{j,t}dt \equiv dK^*_{j,t} + C^*_{j,t}dt$) and by $s_{j,t} \equiv \frac{K_{j,t}}{K^*_{j,t}}$ the unit's solvency ratio. Last but not least, we represent as $\check{r}_t$ the market risk-free rate of return (so that, by definition, $\check{r}_t \equiv E_t[\check{r}_t]$), as $V_t[\mathscr{r}_{j,t}]$ the variance of the production unit's rate of return, as $\bar{\rho}_{j,t}$ the correlation coefficient between unit *j*'s return and that of the overall market portfolio, and as $\lambda_{j,t} \equiv \frac{E_t[\mathscr{r}_{j,t}]-\check{r}_t}{\bar{\rho}_{j,t}\sqrt{V_t[\mathscr{r}_{j,t}]}}$ the price of risk applicable to the unit.

3.2. Assumptions

<u>ASSUMPTION 1</u>: Complete, efficient markets

At any point *t* in time market transaction costs are negligible, there is a market price for every possible long or short asset, good or service in every possible future state of the world, and it is impossible to build a portfolio earning a risk-free return higher than the market risk-free rate (i.e. market prices continuously adjust so that no such arbitrage profit portfolio can be devised).

Comment: According to standard financial theory (e.g. the CAPM), in a complete, efficient market there is a market-wide price of risk λ_t such that $E_t[r_{j,t}] \equiv r_{j,t} = \lambda_t \bar{\rho}_{j,t} \sqrt{V_t[r_{j,t}]} + \check{r}_t$, where $\bar{\rho}_{j,t} \sqrt{V_t[r_{j,t}]}$ represents the undiversifiable part of the variability of unit *j*'s rate of return. Appendix 2 provides a standard proof of this assertion.

<u>ASSUMPTION 2</u>: *Profit maximization*

Producers select their input volumes to maximize their expected net profit $E_T[\Pi_{j,t}]$ for every point in time $t \geq T$ (where *T* represents any point in time selected as 'current' or 'reference').

<u>ASSUMPTION 3</u>: *Differentiability at optimal point*

The expected net profit function is differentiable respective to all its inputs at its maximum point.

<u>ASSUMPTION 4</u>: *Homogeneous economic production function*

The economic production function $Y_{j,t} \equiv F_{j,t}(x_{1,j,t} \dots x_{n,j,t})$ is continuous, twice-differentiable respective to every one of its inputs and also such that, for a scalar value $a \in \mathfrak{R}$, the expression $F_{j,t}(ax_{1,j,t} \dots . ax_{n,j,t}) = a^{h_{j,t}} F_{t,j}(x_{1,j,t} \dots . x_{n,j,t})$ always holds (where the degree of homogeneity $h_{j,t} \in \mathfrak{R}$ is independent of any of the inputs $\{x_{1,j,t} \dots x_{n,j,t}\}$).

Comment: This assumption is common to many production functions e.g. Cobb-Douglas, CES…

<u>ASSUMPTION 5</u>: *Inputs, costs and prices included in contemporary market information set*

The set I_t of information available at time *t* includes the contemporary information corresponding to input quantities, marginal costs, market prices and cash flows at that point in time i.e.:

$$\forall i \in \{1 \dots n\} \;\; ; \;\; \forall j \in \{1 \dots m\} \;\; ; \;\; \forall t \in \mathfrak{R} \qquad ; \qquad \{x_{\mathrm{i},j,t}\}\{c_{\mathrm{i},j,t}\}\{p_{j,t}\}\{K_{i,t}\}\{K_{i,t}^*\}\{C_{i,t}\}\{C_{i,t}^*\} \in I_t$$

Comment: Following Gracia (2011), if Assumptions 2 to 5 hold, then the economic production function of a production unit can be expressed as $Y_{j,t} = A_{j,t} X_{j,t}^{\frac{1}{\mu_{j,t}}}$ where the coefficient $A_{j,t}$ (which going forward we will refer to as "productivity coefficient") is a function of time independent of $X_{j,t}$ and $Y_{j,t}$ – a formal proof is provided in Appendix 3.

<u>ASSUMPTION 6</u>: Random walk perturbation on productivity coefficient

Productivity shocks follow an Itô stochastic diffusion process subject to a Wiener perturbation (that is, a linear, continuous, normally-distributed random-walk process otherwise known as "Brownian motion"). Specifically, we assume:

$$\frac{dA_{j,t}}{A_{j,t}} = E_t\left[\frac{dA_{j,t}}{A_{j,t}}\right] + \varsigma_{j.t} dW_{j,t}$$

Where $A_{j,t}$ represents the productivity coefficient, the operator $E_t[\circ]$ indicates the expected value according to the information available at instant t, $W_{j,t}$ is a Wiener process i.e. a stochastic process such that (given a reference time $t = 0$) $W_{j,0} \equiv 0$ and $dW_{j,t} \equiv \omega_{j,t}\sqrt{dt}$, where $\omega_{j,t}$ is a serially-uncorrelated, normally-distributed standardized white noise (i.e. $\omega_{j,t}$~N$[0,1]$), and the function $\varsigma_{j,t}$ represents the standard deviation of the increment $\frac{dA_{j,t}}{A_{j,t}}$.

<u>ASSUMPTION 7</u>: Random walk perturbation on expected output demand

The growth rate of expected output demand is also subject to a Wiener perturbation, i.e.:

$$\frac{dY_{j,t}}{Y_{j,t}} = E_t\left[\frac{dY_{j,t}}{Y_{j,t}}\right] + \varsigma_{j.t} dW_{j,t} + \sigma_{j,t} dZ_{j,t}$$

Where $Z_{j,t}$ represents another Wiener process i.e. a serially-uncorrelated, normally-distributed standardized white noise just as $W_{j,t}$ in Assumption 7, and $\sigma_{j,t}$ represents the associated standard deviation (i.e. $\sigma_{j,t} \equiv \frac{1}{Y_{j,t}} \frac{\partial Y_{j,t}}{\partial \check{Z}_{j,t}}$). Furthermore, we designate by $\rho_{j,t}$ the correlation coefficient between the two Wiener processes, so that $dW_{j,t} dZ_{j,t} \equiv \rho_{j,t} dt$ (where $-1 \leq \rho_{j,t} \leq 1$).

<u>ASSUMPTION 8</u>: Linear time to build and to dismantle

Expanding a productive unit (and thereby its ability to generate a mark-up over output) when its solvency ratio is above unity, as well as dismantling it to liquidate its fungible assets when the ratio is below unity, is a time-consuming process which we assume to follow the linear function:

$$\frac{d\mu_{j,t}}{\mu_{j,t}} = \theta_j (s_{j,t} - 1) dt \qquad\qquad where \quad 0 \leq \theta_j \leq \infty \quad is\ a\ constant$$

<u>ASSUMPTION 9</u>: Cost function homogeneous of degree one

The cost function $\mathcal{C}_{j,t} \equiv \mathcal{C}_{j,t}(x_{1,j,t} \ldots x_{n,j,t})$ is twice-differentiable respective to its inputs and, for a scalar value $a \in \Re$, the expression $F_{j,t}(ax_{1,j,t} \ldots . ax_{n,j,t}) = aF_{t,j}(x_{1,j,t} \ldots . x_{n,j,t})$ always holds.

Comment: Per Euler's formula this means that the cost (expressed in terms of the *numéraire* ζ) is:

$$\mathcal{C}_{j,t} = \sum_{i=1}^{n} \frac{\partial \mathcal{C}_{j,t}}{\partial x_{i,j,t}} x_{i,j,t} \equiv \sum_{i=1}^{n} c_{i,j,t} x_{i,j,t} \equiv c_{\xi,j,t} X_{j,t}$$

<u>ASSUMPTION 10</u>: Market returns' correlation to fundamentals

Changes in the expected rate of return for any given production unit $j \in \{1 \ldots m\}$ are perfectly correlated to the observed changes in the marginal costs in terms of the *numéraire* ζ i.e.:

$$\forall j,t \quad \rightarrow \quad \frac{dr_{j,t}}{r_{j,t}}\frac{dc_{\xi,j,t}}{c_{\xi,j,t}} = \left(\frac{dc_{\xi,j,t}}{c_{\xi,j,t}}\right)^2 \quad \text{where the reference item } \xi \text{ is set as } \xi = \zeta$$

Comment: We set the reference input ξ for the marginal cost $c_{\xi,j,t}$ as equal to the *numéraire* ζ ($i.e. \xi = \zeta$) simply because it is being compared to the return $r_{j,t} \equiv E_t\left[\frac{\Pi_{j,t}}{K_{j,t}}\right]$ where both $\Pi_{j,t}$ and $K_{j,t}$ are defined as expressed in terms of the *numéraire* ζ.

<u>*ASSUMPTION 11*</u>*: Risk-free fungible assets*

The fungible assets whose aggregate value is represented by $K^*_{j,t}$ are risk-free and therefore yield the risk-free rate of return i.e. $\forall j,t \quad \rightarrow \quad r^*_{j,t} = \check{r}_t$.

<u>*ASSUMPTION 12*</u>*: Neutral cash flows respective to asset composition*

Cash flows on average do not alter the overall asset mix into fungible assets backing up the liquidation value $K^*_{j,t}$ and other assets making up the balance to the total asset value $K_{j,t}$ so that:

$$\frac{C_{j,t}}{K_{j,t}} = \frac{C^*_{j,t}}{K^*_{j,t}}$$

The following two additional assumptions have been introduced for simplicity purposes:

<u>*ASSUMPTION 13: Constant market demand variance*</u>

The unit's non-productivity market demand variance's standard deviation $\sigma_{j,t}$ and its correlation coefficient $\bar{\rho}_{j,t}$ respective to the market portfolio's demand variability are both positive constants.

<u>*ASSUMPTION 14: Constant median market price of risk*</u>

The median market price of risk λ_t is a positive constant.

3.3. Propositions

Following are the main propositions put forward in this paper (proofs provided in Appendix 4).

LEMMA 1: *Under Assumptions 2 to 8, the combined input growth rate may be expressed as:*

$$\frac{dX_{j,t}}{X_{j,t}} = E_t\left[\frac{dX_{j,t}}{X_{j,t}}\right] + \mu_{j,t}X_{j,t}dZ_{j,t}$$

LEMMA 2: *Under Assumptions 1 to 10, a unit's capital value growth may be expressed as:*

$$\frac{dK_{j,t}}{K_{j,t}} = E_t\left[\frac{dK_{j,t}}{K_{j,t}}\right] + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

THEOREM: *Under Assumptions 1 to 12, a production unit's solvency ratio follows the path:*

$$\frac{ds_{j,t}}{s_{j,t}} = \lambda_t\bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

3.4. Dynamic median path

Adding assumptions 13 and 14 to the mix we obtain the median path (proof in Appendix 5):

$$Median\ Path = \left\{\begin{matrix} \dfrac{ds_{j,t}}{dt} = s_{j,t}\sigma_j\left(\lambda\bar{\rho}_j - \dfrac{\sigma_j}{2}\mu_{j,t}\right)\mu_{j,t} \\ ---------------- \\ \dfrac{d\mu_{j,t}}{dt} = \mu_{j,t}\theta_j\left(s_{j,t} - 1\right) \end{matrix}\right\}$$

This system describes an elliptic phase diagram (Figure 6 – which is really the equivalent of Figure 5 in Section 2) orbiting around a central fixed point $s_{j,t} = 1$, $\mu_{j,t} = \frac{2\lambda\bar{\rho}_j}{\sigma_j}$ with a frequency $\omega = \left|\lambda\bar{\rho}_j\sqrt{2\theta_j}\right|$ that is, in terms of time, a cycle (Figure 7).

Figure 6: Median path example (phase diagram)

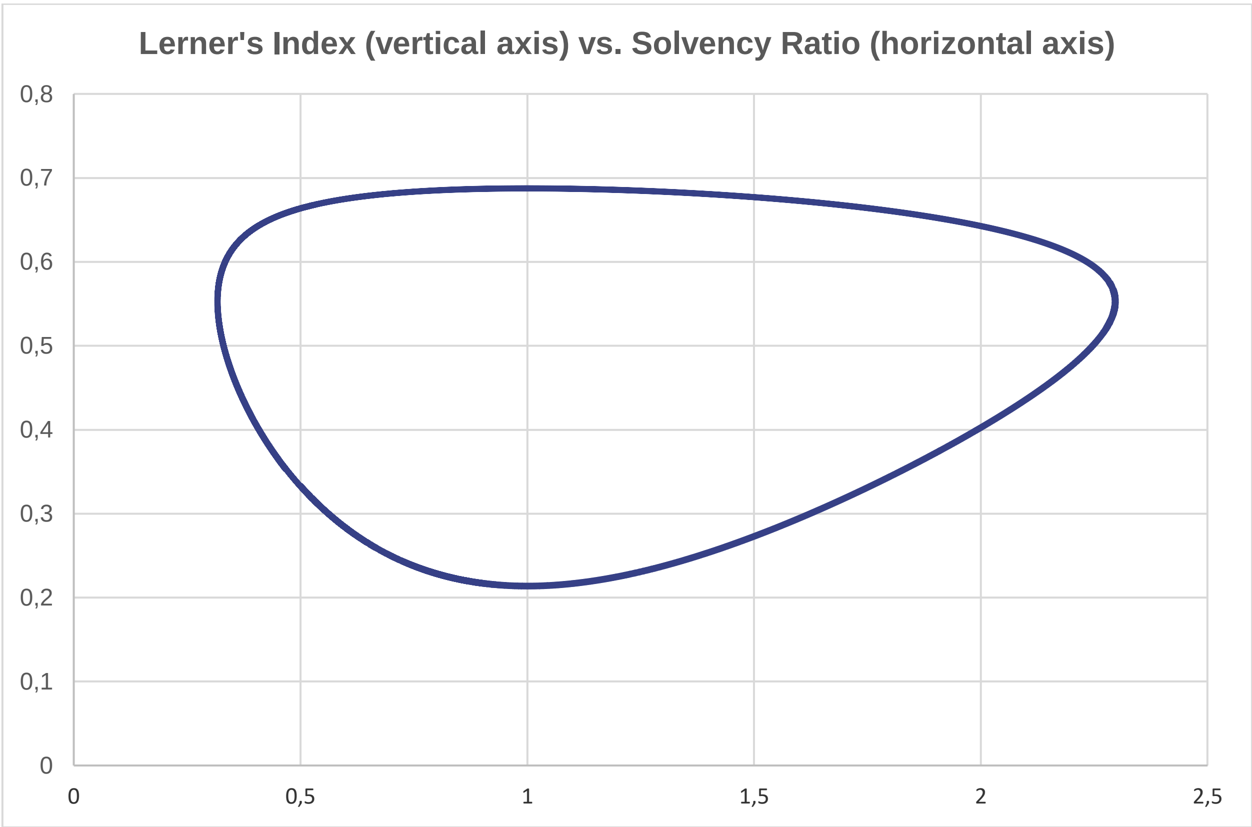
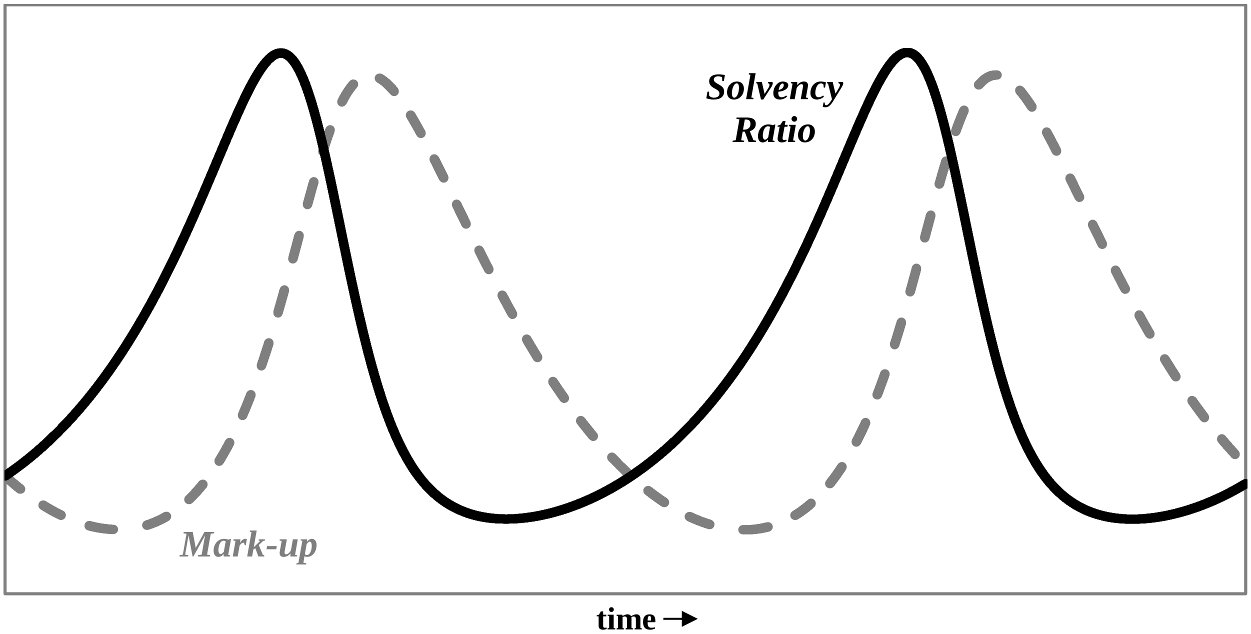


Figure 7: Median path example (time series)



4. Conclusions and future directions

The first objective of this paper was to illustrate how prices in an imperfect competition market (i.e. any market that generates some form of economic rents) could consistently display an observable bull-and-bear cycle even under full rationality and market efficiency. Since virtually all markets generate some form of economic rent, this model could help to explain the evidence of such patterns in real markets, and particularly in those (e.g. commodities and real estate) where fixed asset rents represent a large share of value. As a result, while investor irrationality and market inefficiency cannot be ruled out as real-world factors, their introduction as modeling devices to explain the observation of such patterns would not be necessary. In other words: ***irrational behavior and inefficient price setting constitute unnecessary assumptions to explain observed market patterns, such as recurrent bull-and-bear cycles with identifiable wavelengths, that substantially diverge from the expected path of a rational, efficient market – for the simple reason that the observed time series does not approximate the expected but the median path.***

The second objective of this paper was to rely on this fact to put forward a model of bull-and-bear cycles based on relatively generic assumptions. The specific model developed in this paper predicts that, over time, the observed solvency rate (i.e. the business-continuity value of productive assets expressed as a percentage of their liquidation value) as well as the profit margin (measured as mark-up, Lerner's index or similar metric) of any given industry will follow a cyclical path displaying a higher frequency (i.e. a shorter wavelength):

- The shorter the investment-to-return cycle of that industry (i.e. the higher θ_j) and
- The higher its correlation to the overall market variance (i.e. the higher $\bar{\rho}_j$)

Taken at face value, these predictions seem compatible with known stylized facts. For instance, the oil industry's notoriously long investment time horizons could explain why its price "super cycle" lasts around 30 years on average whereas that of real estate, which has a somewhat shorter cash-to-cash turnaround, has been measured as lasting 18 from peak to peak. Similarly, assets invested in industries whose demand is closely correlated to the market's overall fluctuations are more exposed to its random shocks and would therefore follow a median path cycle with more frequent ups and downs synchronized with those overall market shocks.

Note that, since the economic system is composed of many different industries, this prediction naturally fits Joseph Schumpeter's old view that the cycle waves with different wavelengths previous authors (Kitchin, Juglar, Kondratiev…) had identified in GDP fluctuations corresponded to the cyclical behavior of the different industries underlying them, with longer-term investments leading to longer-term waves (Schumpeter 1939). Furthermore, the resulting valuation cycle, as reflected on the behavior of the solvency ratio, is characterized by periodic hikes and crashes, where the crashes are visibly steeper than the hikes, which is also consistent with the historical experience of bull and bear markets, where the drop is generally faster than the rise.

An obvious extension would be to expand this model into a general-equilibrium framework to explore its implications from a macroeconomic perspective – a direction already outlined to some extent in Gracia (2012), albeit under very restrictive assumptions regarding the potential sources of the rents' cycle (which in that specific paper was assumed to result from agency rents' behavior). Another logical next step would of course be to test the model's predictions against empirical data (along the lines of the tests already conducted in Gracia 2011 and 2012).

The model as it stands, however, already offers a potential explanation for why rent-generating asset markets and industries such as commodities, real estate (or, for that matter, most if not all of the assets quoted in the stock exchange) experience recurrent bull-and-bear cycles with an observable average wavelength punctuated by more-or-less periodical hikes and crashes.

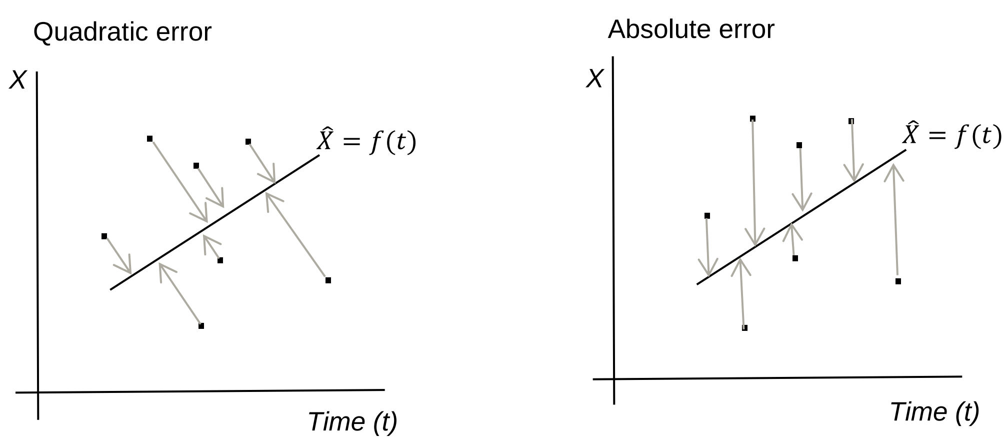
5. Appendices

5.1. Appendix 1: Observed path estimation via the median path

This appendix aims to clarify why it is the median instead of the mean path the one that best approximates the observed trajectory of a stochastic variable's time series.

Given a data cloud generated by a stochastic process, the mean path is defined as the path that minimizes the average quadratic error between the data points and the path itself. The best estimator of a data series, however, would be the path that best predicts the observed value *at every given point in time* i.e. for every arbitrarily chosen point in time, minimizes the error between the predicted value and the subset of data points that refer to that same point in time – that is, as Figure 5.1.A illustrates, the absolute (as opposed to quadratic) error:

Figure 5.1.A: Quadratic vs. absolute error



In other words, if we aim to minimize the modelled path's error respective to the observed data series (i.e. to minimize the difference between the observations made at every new point in time and their modelled forecast), then it is the absolute error we need to focus on – and, as we prove below, the probability path that minimizes the absolute error is the median, not the mean.

There is a fairly standard proof that the median is the path minimizing the absolute error (here we follow a nicely straightforward proof provided in his blog by David E. Giles, retired Professor of Econometrics at the University of Victoria, Canada http://web.uvic.ca/~dgiles/blog/median2.pdf). Given a stochastic variable X subject to a distribution function $F(X)$, we define a value M such that it minimizes the expected absolute error respective to the observed values of X (i.e. $E[X - M]$ where the operator $E[\circ]$ represents the expected value). Evidently the value of this error respective to the observations where $X > M$ is positive, whereas the one for those where $X < M$ is negative, so the overall sum of errors is:

$$E[X - M] = \int_{-\infty}^{M} (M - X) dF + \int_{M}^{\infty} (X - M)\, dF$$

To minimize this function, we therefore need to differentiate it respective to M so that, applying Leibniz's integral differentiation rule, we obtain:

$$\frac{dE[X - M]}{dM} = (M - M) + \int_{-\infty}^{M} dF - (M - M) - \int_{M}^{\infty} dF = 0$$

Which leads to:

$$\int_{-\infty}^{M} dF = \int_{M}^{\infty} dF$$

By definition, the distribution function is such that $\int_{-\infty}^{\infty} dF = 1$ so, for $\int_{-\infty}^{M} dF = \int_{M}^{\infty} dF$ to hold, M must be chosen so that $\int_{-\infty}^{M} dF = \int_{M}^{\infty} dF = \frac{1}{2}$, that is, it must be the median.

Just to double-check this is not a maximum, if we differentiate again respective to M we obtain:

$$\frac{d^2 E[X-M]}{dM^2} = \frac{d}{dM}\int_{-\infty}^{M} dF - \frac{d}{dM}\int_{M}^{\infty} dF = dF(M) + dF(M) = 2dF(M) \geq 0$$

Which, again based on the definition of distribution function, must be positive or zero, as no individual probability $dF(X)$ in the density function can be negative – so M corresponds to a minimum (or, at the extreme, to a saddle point).

To illustrate what this means for the most common asymmetric probability diffusion processes we resort (following Gracia 2005 and Gracia 2012) to a widely used one: the geometric Brownian motion with drift $dX_t = X_t(\mu dt + \sigma dW_t)$, where X_t represents a stochastic function, t designates time, $\mu, \sigma \in \Re$ are real constants and W_t is a Wiener process such that $W_0 \equiv 0$ (for the point in time $t = 0$ chosen as the reference) and $dW_t = \omega\sqrt{dt}$, where ω is a standardized normally-distributed (i.e. symmetrical) function such that $\omega \sim N(0,1)$. Assuming $X_t, W_t \in I_t$ (where I_t represents the set of information available at time t) the expected value of this function is:

$$E_0\left[\frac{dX_t}{X_t}\right] = \mu dt + \sigma \underbrace{E_0[dW_t]}_{=0} = \mu dt$$

And therefore, integrating this expression deterministically:

$$E_0[X_t] = X_0 e^{\mu t}$$

Conversely, to find the complete distribution path we must integrate stochastically (using Itô's lemma). To do this we first calculate the differential of the variable's natural logarithm i.e.:

$$d \ln X_t = \frac{dX_t}{X_t} - \frac{1}{2}\left(\frac{dX_t}{X_t}\right)^2 = \mu dt + \sigma dW_t - \frac{\sigma^2}{2} dt$$

Which after integration becomes:

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Since we know the Brownian motion W_t is symmetrically-distributed around the reference value $W_0 \equiv 0$, the median will be the path where $W_t = 0$ and therefore (using the operator $M[\circ]$ to represent the median value):

$$M_0[X_t] = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t} \qquad \Leftrightarrow \qquad M_0\left[\frac{dX_t}{X_t}\right] = \left(\mu - \frac{\sigma^2}{2}\right)dt$$

Now consider an external observer intending to estimate the parameters that drive this stochastic variable e.g. through an ordinary least-squares regression on the basis of an observed data time series. To do so, the observer will have to linearize the function to fit the form:

$$\Delta \ln X_t = \alpha + \beta \Delta t + u_t$$

Where the operator Δ represents discrete increments (in the same way d represents infinitesimal ones), $\alpha, \beta \in \Re$ are the regression parameters and u_t is a normally-distributed residual. At the same time, the underlying function to be estimated is:

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \qquad \Leftrightarrow \qquad \Delta \ln X_t = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma(W_{t+\Delta t} - W_t)$$

Since the perturbation $\sigma(W_{t+\Delta t} - W_t)$ is normally-distributed, assuming the observed data series is long and representative enough, any such econometric analysis will result in the estimated regression parameters approximating $\alpha = 0$ and $\beta = \left(\mu - \frac{\sigma^2}{2}\right)$ i.e. not the values corresponding to the mean but to the *median path*.

5.2. Appendix 2: Complete, efficient markets and the price of risk

This appendix follows a standard approach to derive the price of risk in a complete, efficient market (see examples in e.g. Malliaris & Brock 1982 pp. 236-238 or Björk 1998 pp. 245-247).

LEMMA 5.2.A: *In a complete, efficient market there is a single price of risk λ_t such that the expected rate of return of the overall market portfolio will take the form $E_t[r_t] = \lambda_t\sigma_t + \check{r}_t$.*

PROOF: Consider two contracts, whose market values at time *t* we indicate as $P_{A,t}$ and $P_{B,t}$, representing different derivatives on the market portfolio (e.g. forward contracts, or options or just the portfolio itself), so that their expected returns (which we will represent as $r_{A,t}$ and $r_{B,t}$) and their standard deviations (respectively $\sigma_{A,t}$ and $\sigma_{B,t}$) are different but, as a result of their underlying asset being the same, the random shocks they are exposed to (which we will represent as the white noise W_t) are perfectly correlated to each other. Therefore, as long as none of the contracts has reached a termination date, their returns will of course be $r_{A,t} + \sigma_{A,t}W_t$ and $r_{B,t} + \sigma_{B,t}W_t$. In a complete market it would always be possible to build a riskless portfolio (whose value we represent as V_t) by buying one unit of asset *a* and simultaneously short-selling a number *b* of units of asset *B*: the volume *b* of units of *B* sold short would be fine-tuned to ensure the resulting portfolio is risk free. Such a portfolio would therefore have a value equal to:

$$V_t = P_{A,t} - bP_{B,t}$$

In an efficient market, however, prices instantly adjust to preclude any potential arbitrage opportunity so, if this portfolio is risk-free, then to rule out any arbitrage its return must be the market risk-free rate, which we will represent as $\check{r}_t$ i.e.:

$$\check{r}_t V_t = (r_{A,t} + \sigma_{A,t}W_t)P_{A,t} - b(r_{B,t} + \sigma_{B,t}W_t)P_{B,t}$$

Of course, for the portfolio to be risk free when each one of the two contracts carries some risk separately, the variabilities of *A* and *B* must cancel out so that $\sigma_{A,t}P_{A,t} = b\sigma_{B,t}P_{B,t}$, which can be achieved by selecting *b* so that:

$$b = \frac{\sigma_{A,t}P_{A,t}}{\sigma_{B,t}P_{B,t}}$$

Replacing into the portfolio value equation this leads to:

$$V_t = P_{A,t} - \frac{\sigma_{A,t}P_{A,t}}{\sigma_{B,t}P_{B,t}}P_{B,t} = \frac{\sigma_{B,t} - \sigma_{A,t}}{\sigma_{B,t}}P_{A,t}$$

Which, replacing now into the return equation implies that:

$$\breve{r}_t \frac{\sigma_{B,t} - \sigma_{A,t}}{\sigma_{B,t}} P_{A,t} = r_{A,t}P_{A,t} - r_{B,t}\frac{\sigma_{A,t}P_{A,t}}{\sigma_{B,t}P_{B,t}}P_{B,t}$$

$$\breve{r}_t\left(\sigma_{B,t} - \sigma_{A,t}\right) = r_{A,t}\sigma_{B,t} - r_{B,t}\sigma_{A,t}$$

$$\frac{r_{A,t} - \breve{r}_t}{\sigma_{A,t}} = \frac{r_{B,t} - \breve{r}_t}{\sigma_{B,t}}$$

Now, as in complete, efficient markets this expression must be valid for every conceivable asset within any timeframe, we conclude that there is a single market price of risk (which we represent by λ_t) that applies to the overall market portfolio and all its derivatives i.e.:

$$\lambda_t = \frac{r_t - \breve{r}_t}{\sigma_t}$$

Q.E.D.

LEMMA 5.2.B: *In a complete, efficient market the expected return of any asset A will take the form* $E_t[r_{A,t}] = \lambda_t \bar{\rho}_{A,t} \sigma_{A,t} + \check{r}_t$.

PROOF: Consider a portfolio *V* combining a specific asset *A* with a weight of a over the total portfolio value and the market portfolio representing the rest (i.e. $1 - a$). Its total return will be:

$$r_{V,t} = a r_{A,t} + (1-a) r_t = a\left(E_t[r_{A,t}] + \sigma_{A,t} W_{A,t}\right) + (1-a)\left(E_t[r_t] + \bar{\sigma}_t W_t\right)$$

Where $W_{A,t}, W_t$ are the white noises of the returns of asset *A* and the market portfolio (whose correlation coefficient is, as defined above, $\bar{\rho}_{A,t}$). Hence the expected portfolio return will be:

$$E_t[r_{V,t}] = a E_t[r_{A,t}] + (1-a) E_t[r_t]$$

Whereas the portfolio standard deviation will be:

$$\sigma_{V,t} = \sqrt{a^2 \sigma_{A,t}^2 + (1-a)^2 \bar{\sigma}_t^2 + 2 \bar{\rho}_{A,t} \bar{\sigma}_t \sigma_{A,t} a (1-a)}$$

In Lemma 5.2.A we established that the expected market portfolio return is $E_t[r_t] = \lambda_t \bar{\sigma}_t + \check{r}_t$ and therefore its differential respective to the standard deviation is $\frac{dE_t[r_t]}{d\bar{\sigma}_t} = \lambda_t$. Hence, of course, in the particular case of $a = 0$ the differential of the expected return of portfolio *V* respective to its standard deviation will of course also be:

$$a = 0 \quad \Rightarrow \quad \frac{dE_t[r_{V,t}]}{d\sigma_{V,t}} = \lambda_t = \frac{E_t[r_t] - \check{r}_t}{\bar{\sigma}_t}$$

At the same time, the calculus composition rule allows to decompose expression $\frac{dE_t[r_{V,t}]}{d\sigma_{V,t}}$ as follows:

$$\frac{dE_t[r_{V,t}]}{d\sigma_{V,t}} = \frac{dE_t[r_{V,t}]}{d\sigma_{V,t}} \frac{da}{da} = \frac{dE_t[r_{V,t}]}{da} \left(\frac{d\sigma_{V,t}}{da} \right)^{-1}$$

Therefore, if we differentiate $E_t[r_{V,t}]$ and $\sigma_{V,t}$ by a and then equate $a = 0$ we obtain that:

$$\frac{dE_t[r_{V,t}]}{d\sigma_{V,t}} = \frac{dE_t[r_{V,t}]}{da} \left(\frac{d\sigma_{V,t}}{da} \right)^{-1} = \frac{E_t[r_{A,t}] - E_t[r_t]}{\bar{\rho}_{A,t}\sigma_{A,t} - \bar{\sigma}_t}$$

Thus, equating both expressions we obtain:

$$\frac{dE_t[r_{V,t}]}{d\sigma_{V,t}} = \frac{E_t[r_{A,t}] - E_t[r_t]}{\bar{\rho}_{A,t}\sigma_{A,t} - \bar{\sigma}_t} = \frac{E_t[r_t] - \check{r}_t}{\bar{\sigma}_t}$$

$$E_t[r_{A,t}] = \bar{\rho}_{A,t} \frac{\sigma_{A,t}}{\bar{\sigma}_t} E_t[r_t] - \left(\bar{\rho}_{A,t} \frac{\sigma_{A,t}}{\bar{\sigma}_t} - 1 \right) \check{r}_t = \bar{\rho}_{A,t} \frac{\sigma_{A,t}}{\bar{\sigma}_t} (E_t[r_t] - \check{r}_t) + \check{r}_t$$

$$E_t[r_{A,t}] = \lambda_t \bar{\rho}_{A,t} \sigma_{A,t} + \check{r}_t$$

Q.E.D.

5.3. Appendix 3: The production function

This appendix follows Gracia (2011) to derive the production function from Assumptions 2 to 5.

LEMMA 5.3.A: *Under Assumptions 2 to 5, the economic production function may be expressed as* $Y_{j,t} = A_{j,t} X_{j,t}^{\frac{1}{\mu_{j,t}}}$, *where the integration coefficient* $A_{j,t}$ *is a function independent of* $X_{j,t}$ *and* $Y_{j,t}$.

PROOF: Since producers fine-tune their demand for inputs in order to maximize their expected net profit (Assumption 2) and a finite maximum profit point exists and at that point the profit function is twice-differentiable respective to the input variables (Assumption 3) then at the maximum point the differential must equal zero. Furthermore, as both the input quantities $x_{i,j,t}$ and the marginal costs $c_{i,j,t}$ are known at the time *t* when they are selected (Assumption 5), then the net profit maximum is at a point where, for every input $x_{i,j,t}$:

$$E_t\left[\frac{\partial \Pi_{j,t}}{\partial x_{i,j,t}}\right] = E_t\left[\frac{\partial \mathcal{Y}_{j,t}}{\partial x_{i,j,t}} - \frac{\partial \mathcal{C}_{j,t}}{\partial x_{i,j,t}}\right] = \frac{\partial Y_{j,t}}{\partial x_{i,j,t}} - c_{i,j,t} = 0 \qquad \Leftrightarrow \qquad \frac{\partial Y_{j,t}}{\partial x_{i,j,t}} = c_{i,j,t}$$

Which, replacing into the definition of mark-up, leads to:

$$\frac{1}{\mu_{j,t}} \equiv c_{l,j,t}\frac{X_{j,t}}{Y_{j,t}} \equiv \frac{1}{Y_{j,t}}\sum_{i=1}^{n} c_{i,j,t}x_{i,j,t} = \frac{1}{Y_{j,t}}\sum_{i=1}^{n}\frac{\partial Y_{j,t}}{\partial x_{i,j,t}}x_{i,j,t}$$

On the other hand, we know that, per Euler's homogeneous function theorem, Assumption 4 implies that, for every production unit $j = 1 \ldots m$:

$$b_{j,t}Y_{j,t} = \sum_{i=1}^{n}\frac{\partial Y_{j,t}}{\partial x_{i,j,t}}x_{i,j,t}$$

And therefore, combining the two expressions we find that:

$$b_{j,t} = \frac{1}{\mu_{j,t}}$$

Which implies that, since $b_{j,t}$ is by definition independent of the input, so must be $\mu_{j,t}$.

Selecting commodity *a* as aggregation unit of the combined input $X_{j,t} \equiv \frac{1}{c_{l,j,t}}\sum_{i=1}^{n} c_{i,j,t}x_{i,j,t}$ means that an increment of one unit of $x_{l,j,t}$, other things being equal, leads to an increment of the same magnitude in $X_{j,t}$ (i.e. $\frac{\partial X_{j,t}}{\partial x_{l,j,t}} \equiv 1$) and therefore:

$$\frac{1}{\mu_{j,t}} \equiv c_{l,j,t}\frac{X_{j,t}}{Y_{j,t}} = \frac{\partial Y_{j,t}}{\partial x_{1,j,t}}\frac{X_{j,t}}{Y_{j,t}} = \frac{\partial Y_{j,t}}{\partial X_{j,t}}\frac{X_{j,t}}{Y_{j,t}}$$

$$\frac{\partial Y_{j,t}}{\partial X_{j,t}} = \frac{1}{\mu_{j,t}}\frac{Y_{j,t}}{X_{j,t}}$$

Thus, integrating this partial differential equation does indeed lead to the following homogeneous function of degree $\frac{1}{\mu_{j,t}}$:

$$Y_{j,t} = A_{j,t}X_{j,t}^{\frac{1}{\mu_{j,t}}}$$

Where the integration coefficient $A_{j,t}$ (which we will refer to as "productivity coefficient") is necessarily a function independent of both $X_{j,t}$ and $Y_{j,t}$ (but not necessarily of the mark-up $\mu_{j,t}$).

Q.E.D.

5.4. Appendix 4: Production unit's solvency path

The purpose of this appendix is to provide a formal proof of the propositions in subsection 3.3.

LEMMA 1: *Under Assumptions 2 to 8, the combined input growth rate may be expressed as:*

$$\frac{dX_{j,t}}{X_{j,t}} = E_t\left[\frac{dX_{j,t}}{X_{j,t}}\right] + \mu_{j,t}X_{j,t}dZ_{j,t}$$

PROOF: If we take the logarithm of Lemma 5.3.A (Appendix 3) for any given unit we obtain:

$$Y_{j,t} = A_{j,t}X_{j,t}^{\frac{1}{\mu_{j,t}}} \quad \Leftrightarrow \quad \ln Y_{j,t} = \ln \mathrm{A}_{j,t} + \frac{\ln X_{j,t}}{\mu_{j,t}}$$

Differentiating then stochastically (through Itô's lemma) both sides of this expression:

$$d\ln Y_{j,t} = d\ln \mathrm{A}_{j,t} + \frac{d\ln X_{j,t}}{\mu_{j,t}} - \frac{d\mu_{j,t}}{\mu_{j,t}^2}\ln X_{j,t} + \frac{1}{2}\frac{\left(d\mu_{j,t}\right)^2}{\mu_{j,t}^3}\ln X_{j,t} - \frac{d\mu_{j,t}}{\mu_{j,t}^2}d\ln X_{j,t}$$

Which, resorting again to Itô's lemma to differentiate the natural logarithms, becomes:

$$\frac{dY_{j,t}}{Y_{j,t}} - \frac{1}{2}\left(\frac{dY_{j,t}}{Y_{j,t}}\right)^2 = \frac{dA_{j,t}}{A_{j,t}} - \frac{1}{2}\left(\frac{dA_{j,t}}{A_{j,t}}\right)^2 + \frac{1}{\mu_{j,t}}\left[\frac{dX_{j,t}}{X_{j,t}} - \frac{1}{2}\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2\right] - \frac{d\mu_{j,t}}{\mu_{j,t}^2}\ln X_{j,t} + \frac{1}{2}\frac{\left(d\mu_{j,t}\right)^2}{\mu_{j,t}^3}\ln X_{j,t} - \frac{d\mu_{j,t}}{\mu_{j,t}^2}\left[\frac{dX_{j,t}}{X_{j,t}} - \frac{1}{2}\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2\right]$$

Per Assumption 5, the solvency rate $s_{j,t} \equiv \frac{K_{j,t}}{K_{j,t}^*}$ is deterministic at time t (i.e. $s_{j,t} \subset I_t$). Hence, squaring Assumption 8 on both sides, $\left(\mu_{j,t}\lambda_j\left[s_{j,t} - 1\right]\right)^2 = \left(d\mu_{j,t}\right)^2 = d\mu_{j,t}dX_{j,t} = d\mu_{j,t}dt = 0$, which here implies that:

$$\frac{dY_{j,t}}{Y_{j,t}} - \frac{1}{2}\left(\frac{dY_{j,t}}{Y_{j,t}}\right)^2 = \frac{dA_{j,t}}{A_{j,t}} - \frac{1}{2}\left(\frac{dA_{j,t}}{A_{j,t}}\right)^2 + \frac{1}{\mu_{j,t}}\left[\frac{dX_{j,t}}{X_{j,t}} - \frac{1}{2}\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2\right] - \frac{d\mu_{j,t}}{\mu_{j,t}^2}\ln X_{j,t}$$

Now, if we replace $\frac{dY_{j,t}}{Y_{j,t}}$ and $\frac{dA_{j,t}}{A_{j,t}}$ according to Assumptions 7 and 8, this translates into:

$$E_t\left[\frac{dY_{j,t}}{Y_{j,t}}\right] + \varsigma_{j,t}dW_{j,t} + \sigma_{j,t}dZ_{j,t} - \frac{1}{2}\left(\frac{dY_{j,t}}{Y_{j,t}}\right)^2 = E_t\left[\frac{dA_{j,t}}{A_{j,t}}\right] + \varsigma_{j,t}dW_{j,t} - \frac{1}{2}\left(\frac{dA_{j,t}}{A_{j,t}}\right)^2 + \frac{1}{\mu_{j,t}}\left[\frac{dX_{j,t}}{X_{j,t}} - \frac{1}{2}\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2\right] - \frac{d\mu_{j,t}}{\mu_{j,t}^2}\ln X_{j,t}$$

Which, rearranging terms, becomes:

$$\sigma_{j,t}dZ_{j,t} = E_t\left[\frac{dA_{j,t}}{A_{j,t}}\right] - E_t\left[\frac{dY_{j,t}}{Y_{j,t}}\right] + \frac{1}{2}\left(\frac{dY_{j,t}}{Y_{j,t}}\right)^2 - \frac{1}{2}\left(\frac{dA_{j,t}}{A_{j,t}}\right)^2 + \frac{1}{\mu_{j,t}}\left[\frac{dX_{j,t}}{X_{j,t}} - \frac{1}{2}\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2\right] - \frac{d\mu_{j,t}}{\mu_{j,t}^2}\ln X_{j,t}$$

As the only stochastic component of the right hand side of this expression is $\frac{1}{\mu_{j,t}}\frac{dX_{j,t}}{X_{j,t}}$, if we square both sides we obtain that $\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2 = \left(\mu_{j,t}\sigma_{j,t}\right)^2 dt$ and therefore:

$$\frac{dX_{j,t}}{X_{j,t}} = E_t\left[\frac{dX_{j,t}}{X_{j,t}}\right] + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

Q.E.D.

LEMMA 2: *Under Assumptions 2 to 10, a unit's capital value growth may be expressed as:*

$$\frac{dK_{j,t}}{K_{j,t}} = E_t\left[\frac{dK_{j,t}}{K_{j,t}}\right] + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

PROOF: If we obtain the expected value of capital and then apply Assumptions 5 and 9 then:

$$E_t[\mathcal{r}_{j,t}K_{j,t}] = \underbrace{E_t[\mathcal{r}_{j,t}]}_{\equiv r_{j,t}}K_{j,t} = E_t[\Pi_{j,t}] = E_t[\mathcal{Y}_{j,t} - c_{\xi,t}X_{j,t}] = Y_{j,t} - c_{\xi,j,t}X_{j,t}$$

$$r_{j,t}K_{j,t} = (\mu_{j,t} - 1)\frac{\partial Y_{j,t}}{\partial X_{j,t}}X_{j,t} = (\mu_{j,t} - 1)c_{\xi,j,t}X_{j,t}$$

So, if we now compute the natural logarithm of this expression:

$$d\ln K_{j,t} + d\ln r_{j,t} = d\ln(\mu_{j,t} - 1) + d\ln c_{\xi,j,t} + d\ln X_{j,t}$$

And then differentiate per Itô's lemma:

$$\frac{dK_{j,t}}{K_{j,t}} - \frac{1}{2}\left(\frac{dK_{j,t}}{K_{j,t}}\right)^2 = \frac{d\mu_{j,t}}{\mu_{j,t} - 1} - \frac{1}{2}\left(\frac{d\mu_{j,t}}{\mu_{j,t} - 1}\right)^2 + \frac{dX_{j,t}}{X_{j,t}} - \frac{1}{2}\left(\frac{dX_{j,t}}{X_{j,t}}\right)^2 + \frac{dc_{\xi,j,t}}{c_{\xi,j,t}} - \frac{1}{2}\left(\frac{dc_{\xi,j,t}}{c_{\xi,j,t}}\right)^2 - \frac{dr_{j,t}}{r_{j,t}} + \frac{1}{2}\left(\frac{dr_{j,t}}{r_{j,t}}\right)^2$$

Assumption 8 implies that $(d\mu_{j,t})^2 = 0$ and Assumption 10 that the variabilities of $c_{\xi,j,t}$ and $r_{j,t}$ are perfectly correlated so they cancel each other. Squaring both sides we obtain $\left(\frac{dK_{j,t}}{K_{j,t}}\right)^2 = \left(\frac{dX_{j,t}}{X_{j,t}}\right)^2$ so, resorting now to Lemma 1, the rate of increase of the unit's asset value becomes:

$$\frac{dK_{j,t}}{K_{j,t}} = \frac{d\mu_{j,t}}{\mu_{j,t}-1} + \frac{dc_{\xi,j,t}}{c_{\xi,j,t}} + \frac{dX_{j,t}}{X_{j,t}} - \frac{dr_{j,t}}{r_{j,t}} = E_t\left[\frac{dK_{j,t}}{K_{j,t}}\right] + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

Q.E.D.

THEOREM: *Under Assumptions 1 to 12, a production unit's solvency ratio follows the path:*

$$\frac{ds_{j,t}}{s_{j,t}} = \lambda_t \bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

PROOF: If we take a production unit's solvency ratio, calculate its natural logarithm and then differentiate again per Itô's lemma we obtain:

$$s_{j,t} \equiv \frac{K_{j,t}}{K^*_{j,t}} \qquad \Leftrightarrow \qquad \ln s_{j,t} \equiv \ln K_{j,t} - \ln K^*_{j,t}$$

$$\frac{ds_{j,t}}{s_{j,t}} - \frac{1}{2}\left(\frac{ds_{j,t}}{s_{j,t}}\right)^2 = \frac{dK_{j,t}}{K_{j,t}} - \frac{1}{2}\left(\frac{dK_{j,t}}{K_{j,t}}\right)^2 - \frac{dK^*_{j,t}}{K^*_{j,t}} + \frac{1}{2}\left(\frac{dK^*_{j,t}}{K^*_{j,t}}\right)^2$$

If we apply Assumption 11 (which implies $K^*_{j,t}$ is risk-free, so that $\left(\frac{dK^*_{j,t}}{K^*_{j,t}}\right)^2 = 0$) and square both sides of this expression we find that $\left(\frac{ds_{j,t}}{s_{j,t}}\right)^2 = \left(\frac{dK_{j,t}}{K_{j,t}}\right)^2$ so, replacing above, we obtain:

$$\frac{ds_{j,t}}{s_{j,t}} = \frac{dK_{j,t}}{K_{j,t}} - \frac{dK^*_{j,t}}{K^*_{j,t}}$$

If we now replace $\frac{dK_{j,t}}{K_{j,t}}$ and $\frac{dK^*_{j,t}}{K^*_{j,t}}$ with their definitions this expression translates into:

$$\frac{ds_{j,t}}{s_{j,t}} = \left(r_{j,t} - \frac{C_{j,t}}{K_{j,t}}\right)dt - \left(r^*_{j,t} - \frac{C^*_{j,t}}{K^*_{j,t}}\right)dt$$

Where, squaring both sides of the expression, we obtain that $\left(\frac{ds_{j,t}}{s_{j,t}}\right)^2 = \left(\frac{dK_{j,t}}{K_{j,t}}\right)^2 = \left(\mathscr{r}_{j,t}dt\right)^2$ which, combined with Lemma 2, yields:

$$\mathscr{r}_{j,t}dt = \underbrace{E_t\left[\mathscr{r}_{j,t}\right]}_{\equiv r_{j,t}} + \mu_{j,t}\sigma_{j,t}dZ_{j,t} = r_{j,t} + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

Since we know from Assumption 11 that the return of the liquidation value is the market risk-free interest rate i.e. $r^*_{j,t} = \check{r}_t$. Hence, if we apply Assumption 1 i.e. express the risk premium $r_{j,t} - \check{r}_t$ as the market price of risk λ_t multiplied by the non-diversifiable portion of its risk then:

$$\frac{ds_{j,t}}{s_{j,t}} = \lambda_t\bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt + \mu_{j,t}\sigma_{j,t}dZ_{j,t} + \left(\frac{C^*_{j,t}}{K^*_{j,t}} - \frac{C_{j,t}}{K_{j,t}}\right)dt$$

Which, applying now Assumption 12, simplifies into:

$$\frac{ds_{j,t}}{s_{j,t}} = \lambda_t\bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt + \mu_{j,t}\sigma_{j,t}dZ_{j,t}$$

Q.E.D.

5.5. Appendix 5: Median cycle path

The purpose of this appendix is to obtain the analytical expression of the median path cycle in subsection 3.4. To obtain the median we develop the increments of the logarithm of $s_{j,t}$ i.e.:

$$d\ln s_{j,t} = \frac{ds_{j,t}}{s_{j,t}} - \frac{1}{2}\left(\frac{ds_{j,t}}{s_{j,t}}\right)^2 = \underbrace{\lambda_t\bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt + \mu_{j,t}\sigma_{j,t}dZ_{j,t}}_{=\frac{ds_{j,t}}{s_{j,t}}} - \frac{1}{2}\underbrace{\mu^2_{j,t}\sigma^2_{j,t}dt}_{=\left(\frac{ds_{j,t}}{s_{j,t}}\right)^2}$$

Therefore, by integrating this expression we obtain:

$$s_{j,t} = s_{j,0}Exp\left[\int \lambda_t\bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt + \int \mu_{j,t}\sigma_{j,t}dZ_{j,t} - \frac{1}{2}\int \mu_{j,t}^2\sigma_{j,t}^2dt\right]$$

Since $dZ_{j,t}$ is defined as a symmetric, normally-distributed stochastic process, so that its median path is zero, whereas, per Assumptions 8 and 13, the increments of both $\mu_{j,t}$ and $\sigma_{j,t}$ in the lapse between *t* and *t*+*dt* are known at time *t*, then $\int \mu_{j,t}\sigma_{j,t}dZ_{j,t} = 0$, so the solvency median path is:

$$M_t\left[\frac{ds_{j,t}}{s_{j,t}}\right] = \lambda_t\bar{\rho}_{j,t}\mu_{j,t}\sigma_{j,t}dt - \frac{1}{2}\mu_{j,t}^2\sigma_{j,t}^2dt$$

If we now combine this median path with Assumptions 13 and 14 as well as with the mark-up growth path defined in Assumption 8 then we obtain the following path expression:

$$Median\ Path = \left\{\begin{matrix} \dot{s}_{j,t} \equiv \dfrac{ds_{j,t}}{dt} = s_{j,t}\sigma_j\left(\lambda\bar{\rho}_j - \dfrac{\sigma_j}{2}\mu_{j,t}\right)\mu_{j,t} \\ ----------- \\ \dot{\mu}_{j,t} \equiv \dfrac{d\mu_{j,t}}{dt} = \mu_{j,t}\theta_j\left(s_{j,t} - 1\right) \end{matrix}\right\}$$

Where $\sigma_j, \bar{\rho}_j, \theta_j$ and λ are positive constants, $s_{j,t}$ and $\mu_{j,t}$ are variables and $\dot{s}_{j,t}$ and $\dot{\mu}_{j,t}$ their differentials respective to time. To study the system's dynamics, we build its Jacobian matrix i.e.:

$$J\left(s_{j,t}, \mu_{j,t}\right) = \begin{pmatrix} \dfrac{\partial\dot{s}_{j,t}}{\partial s_{j,t}} = \sigma_j\left(\lambda\bar{\rho}_j - \dfrac{\sigma_j}{2}\mu_{j,t}\right)\mu_{j,t} & ; & \dfrac{\partial\dot{s}_{j,t}}{\partial\mu_{j,t}} = s_{j,t}\sigma_j\left(\lambda\bar{\rho}_j - \sigma_j\mu_{j,t}\right) \\ ------- & -- & ------- \\ \dfrac{\partial\dot{\mu}_t}{\partial s_{j,t}} = \mu_{j,t}\theta_j & ; & \dfrac{\partial\dot{\mu}_t}{\partial\mu_{j,t}} = \theta_j\left(s_{j,t} - 1\right) \end{pmatrix}$$

Resorting to the Linearization or Hartman–Grobman theorem we can characterize the dynamic behavior of this function around any given point $s_{j,t} = S,\ \mu_{j,t} = M$ by analyzing its eigenvalues Λ_1 and Λ_1, which would be the solutions of the determinant:

$$Det \begin{vmatrix} \Lambda - \sigma_j \left(\lambda\bar{\rho}_j - \frac{\sigma_j}{2} M\right) M & ; & -S\sigma_j\left(\lambda\bar{\rho}_j - \sigma_j M\right) \\ -------- & -- & -------- \\ -M\theta_j & ; & \Lambda - \theta_j(S-1) \end{vmatrix} = 0$$

Or, what is the same, the solutions of the characteristic function:

$$\left(\Lambda - \sigma_j \left(\lambda\bar{\rho}_j - \frac{\sigma_j}{2} M\right) M\right)\left(\Lambda - \theta_j(S-1)\right) - M\theta_j S\sigma_j\left(\lambda\bar{\rho}_j - \sigma_j M\right) = 0$$

In this system there is an obvious fixed point at $S = 1,\ M = \frac{2\lambda\bar{\rho}_j}{\sigma_j}$ where both the solvency rate and the mark-up remain constant i.e. $\dot{s}_{j,t} = \dot{\mu}_{j,t} = 0$. Hence its Jacobian matrix is:

$$J(\,S,\mathrm{M}) = \begin{pmatrix} 0 & ; & -\sigma_j\lambda\bar{\rho}_j \\ -------- & -- & -------- \\ \theta_j \frac{2\lambda\bar{\rho}_j}{\sigma_j} & ; & 0 \end{pmatrix}$$

So the eigenvalues for this fixed point will be:

$$\Lambda^2 + 2\lambda^2\bar{\rho}_j^2\theta_j = 0 \qquad \Leftrightarrow \qquad \Lambda = \pm i\lambda\bar{\rho}_j\sqrt{2\theta_j} \qquad where\ i \equiv \sqrt{-1}$$

Per Hartman–Grobman’s theorem, a fixed point is stable (i.e. small deviations tend to revert back to it) if the real part of both roots is negative, whereas if at least one eigenvalue is positive then the fixed point is unstable and, when both real parts are zero (as is the case here), then the system is dynamically stable and describes an elliptic orbit around the fixed point $s_{j,t} = 1,\ \mu_{j,t} = \frac{2\lambda\bar{\rho}_j}{\sigma_j}$ with a frequency of $\omega = \left|\sqrt{\Lambda_1\Lambda_2}\right|$, which in this case means $\omega = \left|\lambda\bar{\rho}_j\sqrt{2\theta_j}\right|$, and therefore also a wavelength from peak to peak (or trough to trough) of $\mathcal{T} = \frac{2\pi}{\left|\lambda\bar{\rho}_j\sqrt{2\theta_j}\right|}$ (where $\pi \equiv 3.14159$ ...).

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