

ABSTRACT

We consider holomorphic maps $f : U \rightarrow U$ for a hyperbolic domain U in the complex plane, such that the iterates of f converge to a boundary point ζ of U . By a previous result of the authors, for such maps there exist nice absorbing domains $W \subset U$. In this paper we show that W can be chosen to be simply connected, if f has doubly parabolic type in the sense of the Baker–Pommerenke–Cowen classification of its lift by a universal covering (and ζ is not an isolated boundary point of U). We also provide counterexamples for other types of the map f and give an exact characterization of doubly parabolic type in terms of the dynamical behaviour of f .

Absorbing sets and Baker domains for holomorphic maps

Krzysztof Barański, Núria Fagella, Xavier Jarque and Bogusława Karpińska

1. Introduction

In this paper we study iterates $f^n = \underbrace{f \circ \dots \circ f}_{n \text{ times}}$ of a holomorphic map $f : U \rightarrow U$,

where U is a hyperbolic domain in the complex plane \mathbb{C} (i.e. a domain whose complement in \mathbb{C} contains at least two points) and f has no fixed points, that is $f(z) \neq z$ for $z \in U$. In the special case when U is the unit disc \mathbb{D} (or, equivalently, the right half-plane \mathbb{H}), the dynamical behaviour of f has been extensively studied, starting from the works of Denjoy, Wolff and Valiron in the 1920's and 1930's (see [?, ?, ?, ?] and a more detailed explanation in Section ??). In particular, the celebrated Denjoy–Wolff Theorem asserts that under this assumption, the iterates of f converge almost uniformly (i.e. uniformly on compact subsets of U) as $n \rightarrow \infty$ to a point ζ in the boundary of U . Changing the coordinates by a Möbius map, we can conveniently assume in this case $U = \mathbb{H}$, $\zeta = \infty$. Baker and Pommerenke [?, ?] and Cowen [?] proved that f on \mathbb{H} is semi-conjugate to a Möbius map $T : \Omega \rightarrow \Omega$ by a holomorphic map $\varphi : \mathbb{H} \rightarrow \Omega$, where the following three cases can occur:

- (i) $\Omega = \mathbb{H}$, $T(\omega) = a\omega$ for some $a > 1$ (*hyperbolic type*)
- (ii) $\Omega = \mathbb{H}$, $T(\omega) = \omega \pm i$ (*simply parabolic type*),
- (iii) $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$ (*doubly parabolic type*),

(see Section ?? for a precise formulation). The terms ‘simply’ and ‘doubly’ are used due to the following fact: if f has, respectively, simply or doubly parabolic type and extends holomorphically to a neighbourhood of infinity in the Riemann sphere, then ∞ becomes a parabolic fixed point with one or two petals, respectively (see e.g. [?, ?]). An alternative terminology for simply and doubly parabolic types, used in [?], is ‘parabolic type II’ and ‘parabolic type I’, respectively.

For an arbitrary hyperbolic domain $U \subset \mathbb{C}$, the problem of describing the dynamics of a holomorphic map $f : U \rightarrow U$ without fixed points is more complicated. To this aim, one can consider a lift $g : \mathbb{H} \rightarrow \mathbb{H}$ of f by a universal covering $\pi : \mathbb{H} \rightarrow U$. Some results on the dynamics of f were obtained by Marden and Pommerenke [?] and Bonfert [?], who proved that if f has no *isolated boundary fixed points* (i.e. points ζ in the boundary of U in $\bar{\mathbb{C}}$ such that f extends holomorphically to $U \cup \{\zeta\}$ with $f(\zeta) = \zeta$, see Definition ??), then it is semi-conjugate to a Möbius map on \mathbb{C} or on a hyperbolic domain in \mathbb{C} . In 1999, König [?] extended the Baker–Pommerenke–Cowen result on the semi-conjugacy of f to a Möbius map for the case when $f^n \rightarrow \infty$ as $n \rightarrow \infty$ and every closed loop in U is eventually contractible in U under iteration of f (see Theorem ??).

One can extend the classification of f into the three types (hyperbolic, simply parabolic and doubly parabolic), defining its type by the type of its lift g (see Section ??). In [?], König

2000 *Mathematics Subject Classification* Primary 30D05, 37F10, 30D30. Secondary 30F20, 30F45.

Supported by Polish NCN grant decision DEC-2012/06/M/ST1/00168. The second and third authors were partially supported by the Catalan grant 2009SGR-792, and by the Spanish grant MTM2011-26995-C02-02.

characterised the three types of f (under the restriction on eventual contractibility of loops in U) in terms of the behaviour of the sequence $|f^{n+1}(z) - f^n(z)|/\text{dist}(f^n(z), \partial U)$ for $z \in U$, where ∂U denotes the boundary of U in \mathbb{C} and

$$\text{dist}(f^n(z), \partial U) = \inf_{u \in \partial U} |f^n(z) - u|$$

(see Theorem ??).

In this paper we present a characterization of maps f of doubly parabolic type in terms of their dynamical properties in the general case, where f is an arbitrary holomorphic map without fixed points on a hyperbolic domain $U \subset \mathbb{C}$. More precisely, we prove the following.

THEOREM A. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then the following statements are equivalent:*

- (a) f has doubly parabolic type,
- (b) $\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0$ as $n \rightarrow \infty$ for some $z \in U$,
- (c) $\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0$ as $n \rightarrow \infty$ almost uniformly on U ,
- (d) $|f^{n+1}(z) - f^n(z)|/\text{dist}(f^n(z), \partial U) \rightarrow 0$ as $n \rightarrow \infty$ for some $z \in U$,
- (e) $|f^{n+1}(z) - f^n(z)|/\text{dist}(f^n(z), \partial U) \rightarrow 0$ as $n \rightarrow \infty$ almost uniformly on U ,

where ϱ_U denotes the hyperbolic distance in U .

For the other two types of f we prove that if $\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) > 0$, then f has hyperbolic type (see Proposition ?? and Remark ??).

Another question we consider in this paper is the existence and properties of *absorbing domains* in U for f .

DEFINITION 1.1 Absorbing domain. Let U be a domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map. A domain $W \subset U$ is called *absorbing* in U for f , if $f(W) \subset W$ and for every compact set $K \subset U$ there exists $n \geq 0$, such that $f^n(K) \subset W$.

The problem of the existence of suitable absorbing domains for holomorphic maps has a long history and is related to the study of the local behaviour of a holomorphic map near a fixed point and properties of the Fatou components in the theory of the dynamics of rational, entire and meromorphic maps. (For basic information about the dynamics of holomorphic maps we refer to [?, ?].) For instance, if U is a neighbourhood of an attracting fixed point ζ of f (e.g. if U is the immediate basin of an attracting periodic point ζ of period p of a meromorphic map \tilde{f} , where $f = \tilde{f}^p|_U$), then f is conformally conjugate (by a map ϕ) to the map $w \mapsto f'(\zeta)w$ (if $0 < |f'(\zeta)| < 1$) or $w \mapsto w^k$ for some integer $k > 1$ (if $f'(\zeta) = 0$) near $w = 0$, and $W = \phi^{-1}(\mathbb{D}(0, \varepsilon))$ for a small $\varepsilon > 0$ is a simply connected absorbing domain in U for f , such that $f(\overline{W}) \subset W$ and $\bigcap_{n \geq 0} f^n(\overline{W}) = \{\zeta\}$ (see e.g. [?]).

From now on, assume that

$$f : U \rightarrow U$$

is a holomorphic map on a hyperbolic domain $U \subset \mathbb{C}$ and the iterates of f converge to a boundary point ζ of U . Changing the coordinates by a Möbius map, we can assume $\zeta = \infty$, so

$$f^n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

almost uniformly on U . Since the above definition of an absorbing domain is quite wide, (observe for instance that the whole domain U is always absorbing for f), we introduce a notion of a *nice absorbing domain*.

DEFINITION 1.2 Nice absorbing domain. An absorbing domain W in a domain $U \subset \mathbb{C}$ for a holomorphic map $f : U \rightarrow U$ with $f^n \rightarrow \infty$ is called *nice*, if

- (a) $\overline{W} \subset U$,
- (b) $f^n(\overline{W}) = \overline{f^n(W)} \subset f^{n-1}(W)$ for every $n \geq 1$,
- (c) $\bigcap_{n=1}^{\infty} f^n(\overline{W}) = \emptyset$.

An example of a nice absorbing domain is an attracting petal W in a basin U of a parabolic p -periodic point $\zeta = \infty$ for a rational map \tilde{f} , where $f = \tilde{f}^p|_U$ (see e.g. [?]).

The question of the existence of absorbing regions in hyperbolic domains U is particularly interesting in studying the dynamics of entire and meromorphic maps with Baker domains. Recall that a p -periodic Baker domain for a transcendental meromorphic map $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}$ is a Fatou component $U \subset \mathbb{C}$, such that $\tilde{f}^p(U) \subset U$ and $\tilde{f}^{pn} \rightarrow \infty$ as $n \rightarrow \infty$. A 1-periodic domain is called invariant.

Note that periodic Baker domains for entire maps are always simply connected (see [?]), while in the transcendental meromorphic case they can be multiply connected. The dynamical properties of Baker domains have been studied in many papers, see e.g. [?, ?, ?, ?, ?, ?, ?, ?] and a survey [?].

The Baker–Pommerenke–Cowen results [?, ?, ?] imply that for a holomorphic map $f : \mathbb{H} \rightarrow \mathbb{H}$ with $f^n \rightarrow \infty$ as $n \rightarrow \infty$, there exists a nice simply connected absorbing domain W in U for f , such that the map φ , which semi-conjugates f to a Möbius map $T : \Omega \rightarrow \Omega$, is univalent on W . Hence, by the use of a Riemann map, one can construct nice simply connected absorbing domains for $f : U \rightarrow U$ with $f^n \rightarrow \infty$, if $U \subset \mathbb{C}$ is simply connected.

The existence of such absorbing regions in non-simply connected hyperbolic domains U , in particular Baker domains for transcendental meromorphic maps, was an open question addressed e.g. in [?, ?, ?], related to the question of the existence of so-called virtual immediate basins for Newton’s root-finding algorithm for entire functions.

In [?], König showed that if U is an arbitrary hyperbolic domain in \mathbb{C} and every closed loop in U is eventually contractible in U under iteration of f , then there exists a nice simply connected absorbing domain in U for f . In particular, this holds if U is a p -periodic Baker domain for a transcendental meromorphic map \tilde{f} with finitely many poles, where $f = \tilde{f}^p|_U$ (see Theorem ??).

In a recent paper [?], the authors constructed nice absorbing domains for $f : U \rightarrow U$ with $f^n \rightarrow \infty$ for an arbitrary hyperbolic domain $U \subset \mathbb{C}$ (see Theorem ??). In particular, the construction was used to prove that the Baker domains of Newton’s method for entire functions are always simply connected.

In this paper we consider the question of the existence of simply connected absorbing domains W in U for f . In fact, this is equivalent to the condition that every closed loop in U is eventually contractible in U under iteration of f (see Proposition ??). We prove the following.

THEOREM B. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map, such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$ and ∞ is not an isolated point of the boundary of U in the Riemann sphere $\overline{\mathbb{C}}$. If f has doubly parabolic type, then there exists a nice simply connected absorbing domain W in U for f .*

Note that the assumption on the point at infinity is necessary. In fact, if ∞ is an isolated point of the boundary of U in $\overline{\mathbb{C}}$, then a simply connected absorbing domain cannot exist for any type of the map f (see Proposition ??).

We also provide counterexamples for maps which are not of doubly parabolic type.

THEOREM C. *There exist transcendental meromorphic maps $f : \mathbb{C} \rightarrow \overline{\mathbb{C}}$ with an invariant Baker domain $U \subset \mathbb{C}$, such that $f|_U$ is not of doubly parabolic type and there is no simply connected absorbing domain W in U for f . The examples are constructed in two cases:*

- (i) $\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) > 0$ ($f|_U$ has hyperbolic type),
- (ii) $\varrho_U(f^{n+1}(z), f^n(z)) \not\rightarrow 0$ as $n \rightarrow \infty$ for $z \in U$ and $\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) = 0$.

We also provide examples of simply connected absorbing domains W in U for f of doubly parabolic type. In all three types of examples, the map f has the form

$$f(z) = z + 1 + \sum_{p \in \mathcal{P}} \frac{a_p}{(z - p)^2}, \quad a_p \in \mathbb{C} \setminus \{0\},$$

where \mathcal{P} is the set of poles of f . Moreover, $\{\infty\}$ is a singleton component of $\overline{\mathbb{C}} \setminus U$, in particular it is a singleton component of the Julia set of f . To our knowledge, these are the first examples of Baker domains of this kind. A detailed description of the examples is contained in Theorem ??.

The plan of the paper is the following. In Section ?? we present notation, definitions and a more detailed description of the classical results mentioned in the introduction, together with some other facts used in the proofs of Theorems A, B, C. In Section ?? we characterize doubly parabolic type (Theorem A) and in Section ?? we prove Theorem B. The examples described in Theorem C are constructed in Section ??.

ACKNOWLEDGEMENTS. We wish to thank the Institut de Matemàtica de la Universitat de Barcelona (IMUB) for the hospitality during the stay of two of the authors at the Universitat de Barcelona. We are grateful to the referee for useful comments and remarks.

2. Background

For $z \in \mathbb{C}$ and $A, B \subset \mathbb{C}$ we write

$$\text{dist}(z, A) = \inf_{a \in A} |z - a|, \quad \text{dist}(A, B) = \inf_{a \in A, b \in B} |a - b|.$$

The symbols \overline{A} , ∂A denote, respectively, the closure and boundary of A in \mathbb{C} . The Euclidean disc of radius r centred at $z \in \mathbb{C}$ is denoted by $\mathbb{D}(z, r)$ and the unit disc $\mathbb{D}(0, 1)$ is simply written as \mathbb{D} .

Let $U \subset \mathbb{C}$ be a hyperbolic domain, i.e. a domain whose complement in \mathbb{C} contains at least two points. By the Uniformization Theorem, there exists a universal holomorphic covering π from the right half-plane \mathbb{H} onto U . Every holomorphic map $f : U \rightarrow U$ can be lifted by π to a holomorphic map $g : \mathbb{H} \rightarrow \mathbb{H}$, such that the diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\ \downarrow \pi & & \downarrow \pi \\ U & \xrightarrow{f} & U \end{array}$$

commutes. By $\varrho_U(\cdot)$ and $\varrho_U(\cdot, \cdot)$ we denote, respectively, the density of the hyperbolic metric and the hyperbolic distance in U , defined by the use of the hyperbolic metric in \mathbb{H} . The disc of radius r centred at z with respect to the hyperbolic metric in U is denoted by $\mathcal{D}_U(z, r)$.

Recall the classical Schwarz–Pick Lemma and Denjoy–Wolff Theorem.

LEMMA 2.1 Schwarz–Pick’s Lemma [?, Theorem 4.1]. *Let U, V be hyperbolic domains in \mathbb{C} and let $f : U \rightarrow V$ be a holomorphic map. Then*

$$\varrho_V(f(z), f(z')) \leq \varrho_U(z, z')$$

for every $z, z' \in U$. In particular, if $U \subset V$, then

$$\varrho_V(z, z') \leq \varrho_U(z, z'),$$

with strict inequality unless $z = z'$ or f lifts to a Möbius automorphism of \mathbb{H} .

THEOREM 2.2 Denjoy–Wolff Theorem [?, Theorem 3.1]. *Let $g : \mathbb{H} \rightarrow \mathbb{H}$ be a non-constant holomorphic map, which is not a Möbius automorphism of \mathbb{H} . Then there exists a point $\zeta \in \overline{\mathbb{H}} \cup \{\infty\}$, such that g^n tends to ζ as $n \rightarrow \infty$ almost uniformly on \mathbb{H} .*

The following estimate relates the hyperbolic density ϱ_U to the quasi-hyperbolic density $1/\text{dist}(z, \partial U)$.

LEMMA 2.3 [?, Theorem 4.3]. *Let $U \subset \mathbb{C}$ be a hyperbolic domain. Then*

$$\varrho_U(z) \leq \frac{2}{\text{dist}(z, \partial U)} \quad \text{for } z \in U \quad (2.1)$$

and

$$\varrho_U(z) \geq \frac{1 + o(1)}{\text{dist}(z, \partial U) \log(1/\text{dist}(z, \partial U))} \quad \text{as } z \rightarrow \partial U. \quad (2.2)$$

Moreover, if U is simply connected, then

$$\varrho_U(z) \geq \frac{1}{2 \text{dist}(z, \partial U)} \quad \text{for } z \in U.$$

The above lemma implies the following standard estimate of the hyperbolic distance. We include the proof for completeness.

LEMMA 2.4. *Let U be a hyperbolic domain in \mathbb{C} and let $z, z' \in U$. Then*

$$\frac{|z - z'|}{\text{dist}(z, \partial U)} \geq 1 - e^{-\varrho_U(z, z')/2}.$$

Proof. Suppose that there exist $z, z' \in U$ such that

$$\frac{|z - z'|}{\text{dist}(z, \partial U)} < 1 - e^{-\varrho_U(z, z')/2} \quad (2.3)$$

and let γ be the straight line segment connecting z and z' . In particular, (2.3) implies that $|z - z'| < \text{dist}(z, \partial U)$, so $\gamma \subset U$ and $|u - z| < \text{dist}(z, \partial U)$ for $u \in \gamma$. Thus, by (2.1),

$$\begin{aligned} \varrho_U(z, z') &\leq \int_{\gamma} \varrho_U(u) |du| \leq 2 \int_{\gamma} \frac{|du|}{\text{dist}(u, \partial U)} \leq 2 \int_{\gamma} \frac{|du|}{\text{dist}(z, \partial U) - |u - z|} \\ &= 2 \int_0^{|z - z'|} \frac{ds}{\text{dist}(z, \partial U) - s} = 2 \ln \frac{\text{dist}(z, \partial U)}{\text{dist}(z, \partial U) - |z - z'|} = -2 \ln \left(1 - \frac{|z - z'|}{\text{dist}(z, \partial U)} \right), \end{aligned}$$

which contradicts (2.3). \square

The lower bounds from Lemma 2.4 can be improved in the presence of dynamics. The following result was proved by Rippon in [?] (actually, it was formulated under an additional assumption $f^n \rightarrow \infty$ as $n \rightarrow \infty$, but the proof does not use this).

THEOREM 2.5 [?, Theorem 1]. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then for every compact set $K \subset U$ there exists a constant $C > 0$ such that*

$$\frac{|f^n(z) - f^n(z')|}{\text{dist}(f^n(z), \partial U)} \leq C \varrho_U(f^n(z), f^n(z'))$$

for every $z, z' \in K$ and every $n \geq 0$.

The following result proved by Bonfert in [?] describes a relationship between the dynamical behaviour of f and its lift g .

THEOREM 2.6 [?, Theorem 1.1]. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points. Let $g : \mathbb{H} \rightarrow \mathbb{H}$ be a lift of f by a universal covering map $\pi : \mathbb{H} \rightarrow U$. Then*

$$\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0 \iff \varrho_{\mathbb{H}}(g^{n+1}(1), g^n(1)) \rightarrow 0$$

as $n \rightarrow \infty$ for any $z \in U$.

An obvious consequence of this theorem is that the left hand side of the equivalence is either satisfied for every $z \in U$ or for none.

The next theorem summarizes the results of Baker–Pommerenke–Cowen [?, ?, ?] on the dynamics of holomorphic maps in \mathbb{H} . We use the notation from [?]. (The equivalence of the Baker–Pommerenke and Cowen approaches were shown by König in [?].)

THEOREM 2.7 Cowen’s Theorem [?, Theorem 3.2], see also [?, Lemma 1]. *Let $g : \mathbb{H} \rightarrow \mathbb{H}$ be a holomorphic map such that $g^n \rightarrow \infty$ as $n \rightarrow \infty$. Then there exist a simply connected domain $V \subset \mathbb{H}$, a domain Ω equal to \mathbb{H} or \mathbb{C} , a holomorphic map $\varphi : \mathbb{H} \rightarrow \Omega$, and a Möbius transformation T mapping Ω onto itself, such that:*

- (a) V is absorbing in \mathbb{H} for g ,
- (b) $\varphi(V)$ is absorbing in Ω for T ,
- (c) $\varphi \circ g = T \circ \varphi$ on \mathbb{H} ,
- (d) φ is univalent on V .

Moreover, φ and T depend only on g . In fact (up to a conjugation of T by a Möbius transformation preserving Ω), one of the following cases holds:

- (i) $\Omega = \mathbb{H}$, $T(\omega) = a\omega$ for some $a > 1$ (hyperbolic type),
- (ii) $\Omega = \mathbb{H}$, $T(\omega) = \omega \pm i$ (simply parabolic type),
- (iii) $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$ (doubly parabolic type).

REMARK 2.8. An equivalent description of the three cases can be given by taking

- (i) $\Omega = \{z \in \mathbb{C} : 0 < \text{Im}(z) < b\}$ for some $b > 0$ (hyperbolic type),
- (ii) $\Omega = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ (simply parabolic type),
- (iii) $\Omega = \mathbb{C}$ (doubly parabolic type).

and $T(\omega) = \omega + 1$ in all three cases.

The following theorem gathers König’s results from [?].

THEOREM 2.9 [?]. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$. Suppose that for every closed curve $\gamma \subset U$ there exists $n > 0$*

such that $f^n(\gamma)$ is contractible in U . Then there exist a simply connected domain $W \subset U$, a domain Ω and a transformation T as in Cowen's Theorem ??, and a holomorphic map $\psi : U \rightarrow \Omega$, such that:

- (a) W is absorbing in U for f ,
- (b) $\psi(W)$ is absorbing in Ω for T ,
- (c) $\psi \circ f = T \circ \psi$ on U ,
- (d) ψ is univalent on W .

Moreover,

- (i) T has hyperbolic type if and only if

$$\inf_{z \in U} \inf_{n \geq 0} \frac{|f^{n+1}(z) - f^n(z)|}{\text{dist}(f^n(z), \partial U)} > 0,$$

- (ii) T has simply parabolic type if and only if

$$\liminf_{n \rightarrow \infty} \frac{|f^{n+1}(z) - f^n(z)|}{\text{dist}(f^n(z), \partial U)} > 0 \text{ for every } z \in U \quad \text{and} \quad \inf_{z \in U} \overline{\lim}_{n \rightarrow \infty} \frac{|f^{n+1}(z) - f^n(z)|}{\text{dist}(f^n(z), \partial U)} = 0,$$

- (iii) T has doubly parabolic type if and only if

$$\lim_{n \rightarrow \infty} \frac{|f^{n+1}(z) - f^n(z)|}{\text{dist}(f^n(z), \partial U)} = 0 \quad \text{for every } z \in U.$$

Furthermore, if $\tilde{f} : \mathbb{C} \rightarrow \overline{\mathbb{C}}$ is a meromorphic map with finitely many poles, and U is a periodic Baker domain of period p , then the above assumptions are satisfied for $f = \tilde{f}^p|_U$, and consequently, there exists a simply connected domain W in U with the properties (a)–(d) for $f = \tilde{f}^p$.

In fact, if under the assumptions of Theorem ??, we take V and φ from Cowen's Theorem ?? for a lift g of f by a universal covering $\pi : \mathbb{H} \rightarrow U$, then π is univalent in V and we can set $W = \pi(V)$ and $\psi = \varphi \circ \pi^{-1}$, which is well defined in U .

REMARK 2.10. It follows from results proved in [?] that under the conditions of Cowen's Theorem ?? or Theorem ??, one can choose the absorbing domain W to be nice.

DEFINITION 2.11 **Isolated boundary fixed point.** Let U be a hyperbolic domain in \mathbb{C} and ζ an isolated point of the boundary of U in $\overline{\mathbb{C}}$. Then there exists a neighbourhood $V \subset \overline{\mathbb{C}}$ of ζ such that $V \setminus \{\zeta\} \subset U$. Let $f : U \rightarrow U$ be a holomorphic map. Since $f(U) \subset U$ and U is hyperbolic, f on $V \setminus \{\zeta\}$ omits at least three values in $\overline{\mathbb{C}}$, so by Picard's Theorem, f extends holomorphically to V . If $f(\zeta) = \zeta$, we say that ζ is an *isolated boundary fixed point* of f .

The following theorem was proved by Bonfert in [?].

THEOREM 2.12 [?, Theorem 1.4]. Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then there exist a domain $\Upsilon \subset \mathbb{C}$, a non-constant holomorphic map $\Psi : U \rightarrow \Upsilon$ and a Möbius transformation S mapping Υ onto itself, such that

$$\Psi \circ f = S \circ \Psi.$$

Moreover, if $\rho_U(f^{n+1}(z), f^n(z)) \rightarrow 0$ for $z \in U$, then $\Upsilon = \mathbb{C}$, $S(\omega) = \omega + 1$. Otherwise, the domain Υ is hyperbolic.

The existence of nice absorbing regions in arbitrary hyperbolic domains was proved by the authors in [?].

THEOREM 2.13 [?, Theorem A]. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map, such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists a nice absorbing domain W in U for f , such that f is locally univalent on W . Moreover, for every $z \in U$ and every sequence of positive numbers r_n , $n \geq 0$ with $\lim_{n \rightarrow \infty} r_n = \infty$, the domain W can be chosen such that*

$$W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(f^n(z), r_n).$$

REMARK 2.14. If f has parabolic type, then W can be chosen such that $W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(f^n(z), b_n)$ for a sequence b_n with $\lim_{n \rightarrow \infty} b_n = 0$ and $b_n < b$ for an arbitrary given $b > 0$, see [?, Proposition 3.1].

3. Characterization of doubly parabolic type: Proof of Theorem A

Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points. Consider a universal covering $\pi : \mathbb{H} \rightarrow U$ and a lift $g : \mathbb{H} \rightarrow \mathbb{H}$ of the map f by π . Then g has no fixed points, so by the Denjoy–Wolff Theorem ??, $g^n \rightarrow \zeta$ for a point ζ in the boundary of \mathbb{H} in $\overline{\mathbb{C}}$. Conjugating g by a Möbius map, we can assume $\zeta = \infty$. Consider the map $T : \Omega \rightarrow \Omega$ from Cowen’s Theorem ?? for the map g . By properties of a universal covering, for different choices of π and g , the suitable maps T are conformally conjugate, so in fact the type of T does not depend on the choice of π and g . Hence, we can state the following definition.

DEFINITION 3.1 Type of f . Let U be a hyperbolic domain in \mathbb{C} , $f : U \rightarrow U$ a holomorphic map without fixed points and $g : \mathbb{H} \rightarrow \mathbb{H}$ a lift of f by a universal covering of U . The map f has hyperbolic, simply parabolic or doubly parabolic type if the same holds for its lift g .

THEOREM A. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then the following statements are equivalent:*

- (a) f has doubly parabolic type,
- (b) $\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0$ as $n \rightarrow \infty$ for some $z \in U$,
- (c) $\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0$ as $n \rightarrow \infty$ almost uniformly on U ,
- (d) $|f^{n+1}(z) - f^n(z)| / \text{dist}(f^n(z), \partial U) \rightarrow 0$ as $n \rightarrow \infty$ for some $z \in U$,
- (e) $|f^{n+1}(z) - f^n(z)| / \text{dist}(f^n(z), \partial U) \rightarrow 0$ as $n \rightarrow \infty$ almost uniformly on U .

Proof. First we prove (a) \Rightarrow (b). Since f has doubly parabolic type, we have $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$ in Cowen’s Theorem ?? for a lifted map $g : \mathbb{H} \rightarrow \mathbb{H}$. We claim that for every $\omega \in \mathbb{C}$ there exists $m \in \mathbb{N}$ and a sequence $d_n > 0$ with $d_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$\mathbb{D}(T^n(\omega), d_n) \subset \varphi(V) \quad \text{for every } n \geq m, \tag{3.1}$$

for φ and V from Cowen’s Theorem ?? for g . Indeed, if (??) does not hold, then $\mathbb{D}(T^n(\omega), d) \not\subset \varphi(V)$ for some $d > 0$ and infinitely many n , which is impossible, since by the assertion (b) of Cowen’s Theorem ?? for $K = \overline{\mathbb{D}(\omega, d)}$, we have $\mathbb{D}(T^n(\omega), d) = T^n(K) \subset \varphi(V)$ for sufficiently large n .

Since $|T^{n+1}(\omega) - T^n(\omega)| = |\omega + n + 1 - (\omega + n)| = 1$ and $d_n \rightarrow \infty$, Lemma ?? implies

$$\varrho_{\mathbb{D}(T^n(\omega), d_n)}(T^{n+1}(\omega), T^n(\omega)) \rightarrow 0,$$

as $n \rightarrow \infty$, so by (??) and the Schwarz–Pick Lemma ??,

$$\varrho_{\varphi(V)}(T^{n+1}(\omega), T^n(\omega)) \leq \varrho_{\mathbb{D}(T^n(\omega), d_n)}(T^{n+1}(\omega), T^n(\omega)) \rightarrow 0.$$

By Cowen’s Theorem ??, φ is univalent on V , so for $\omega \in \varphi(V)$ and $z = \pi((\varphi|_V)^{-1}(\omega))$, by the Schwarz–Pick Lemma ?? applied to the map $\pi \circ (\varphi|_V)^{-1}$, we have

$$\varrho_U(f^{n+1}(z), f^n(z)) \leq \varrho_{\varphi(V)}(T^{n+1}(\omega), T^n(\omega)) \rightarrow 0,$$

which shows (b).

Now we show (b) \Rightarrow (a). By Theorem ??, we have $\varrho_{\mathbb{H}}(g^{n+1}(w), g^n(w)) \rightarrow 0$ for some $w \in \mathbb{H}$. Suppose that g is not of doubly parabolic type. Then $\Omega = \mathbb{H}$ in Cowen’s Theorem ?? for the map g , so by the Schwarz–Pick Lemma ?? applied to the map φ , we have

$$\varrho_{\mathbb{H}}(T^{n+1}(\varphi(w)), T^n(\varphi(w))) = \varrho_{\mathbb{H}}(\varphi(g^{n+1}(w)), \varphi(g^n(w))) \leq \varrho_{\mathbb{H}}(g^{n+1}(w), g^n(w)) \rightarrow 0,$$

which is not possible, since for $T(\omega) = a\omega$ or $T(\omega) = \omega \pm i$ we have

$$\varrho_{\mathbb{H}}(T^{n+1}(\varphi(w)), T^n(\varphi(w))) = \varrho_{\mathbb{H}}(T(\varphi(w)), \varphi(w)) > 0.$$

Hence, $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$ and g has doubly parabolic type.

The implication (c) \Rightarrow (b) is trivial. To show (b) \Rightarrow (c), note first that by Theorem ??, the pointwise convergence holds for every $z \in U$. Take a compact set $K \subset U$ and suppose that the convergence is not uniform on K . This means that there exist sequences $z_j \in K$, $n_j \rightarrow \infty$ as $j \rightarrow \infty$, and a constant $c > 0$, such that

$$\varrho_U(f^{n_j+1}(z_j), f^{n_j}(z_j)) > c.$$

Passing to a subsequence, we can assume $z_j \rightarrow z$ for some $z \in K$. Then $\varrho_U(z_j, z) \rightarrow 0$, so by the Schwarz–Pick Lemma ??, for every $n \geq 0$

$$\varrho_U(f^n(z_j), f^n(z)) \leq \varrho_U(z_j, z) \rightarrow 0$$

as $j \rightarrow \infty$. Hence, since $\varrho_U(f^{n_j+1}(z), f^{n_j}(z)) \rightarrow 0$ by the pointwise convergence, we have

$$\begin{aligned} 0 < c &< \varrho_U(f^{n_j+1}(z_j), f^{n_j}(z_j)) \\ &\leq \varrho_U(f^{n_j+1}(z_j), f^{n_j+1}(z)) + \varrho_U(f^{n_j+1}(z), f^{n_j}(z)) + \varrho_U(f^{n_j}(z), f^{n_j}(z_j)) \\ &\leq 2\varrho_U(z_j, z) + \varrho_U(f^{n_j+1}(z), f^{n_j}(z)) \rightarrow 0, \end{aligned}$$

which is a contradiction. This ends the proof of (b) \Leftrightarrow (c).

The implication (c) \Rightarrow (e) follows from Theorem ?? for $z' = f(z)$ and the implication (e) \Rightarrow (d) is trivial. To show (d) \Rightarrow (b), we use Lemma ?? for the points $f^n(z)$, $f^{n+1}(z)$. In this way we have proved the equivalences (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e). \square

By Theorem A and Theorem ??, we immediately obtain the following.

COROLLARY 3.2. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then f has doubly parabolic type if and only if we have $\Upsilon = \mathbb{C}$, $S(\omega) = \omega + 1$ in Theorem ??.*

The following proposition gives a sufficient condition for a map f to be of hyperbolic type.

PROPOSITION 3.3. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map without fixed points and without isolated boundary fixed points. If*

$$\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) > 0,$$

then f has hyperbolic type.

REMARK 3.4. By the Schwarz–Pick Lemma ??, the sequence $\varrho_U(f^{n+1}(z), f^n(z))$ for $z \in U$ is decreasing, so $\lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z))$ always exists and

$$\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) > 0 \iff \inf_{z \in U} \varrho_U(f(z), z) > 0.$$

Proof of Proposition ??. In view of Theorem A, to prove the proposition it is sufficient to show that if f has simply parabolic type, then

$$\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) = 0.$$

We proceed as in the proof of the implication (a) \Rightarrow (b) in Theorem A. Let $g : \mathbb{H} \rightarrow \mathbb{H}$ be a lift of f by a universal covering $\pi : \mathbb{H} \rightarrow U$. Since f has simply parabolic type, we have $\Omega = \mathbb{H}$, $T(\omega) = \omega \pm i$ in Cowen’s Theorem ?? for the map g .

Take a small $\varepsilon > 0$ and $\omega \in \mathbb{H}$ with $\operatorname{Re}(\omega) = 1/\varepsilon$. Let

$$D_n = \mathbb{D} \left(T^n(\omega), \frac{1}{2\varepsilon} \right)$$

for $n \geq 0$. Then $\overline{D_n} = T^n(\overline{D_0}) \subset \mathbb{H}$, so $D_n \subset \varphi(V)$ for large n (depending on ε, ω), by the assertion (b) of Cowen’s Theorem ?. We have $|T^{n+1}(\omega) - T^n(\omega)| = 1$ and $\operatorname{dist}(T^n(\omega), \partial D_n) = 1/(2\varepsilon)$, so Lemma ?? implies

$$\varrho_{D_n}(T^{n+1}(\omega), T^n(\omega)) \leq 2 \ln \frac{1}{1 - 2\varepsilon} \leq \frac{4\varepsilon}{1 - 2\varepsilon} < 5\varepsilon$$

for sufficiently small ε . Since φ is univalent on V , by the Schwarz–Pick Lemma ?? for the map $\pi \circ (\varphi|_V)^{-1}$, we have

$$\varrho_U(f^{n+1}(z), f^n(z)) \leq \varrho_{\varphi(V)}(T^{n+1}(\omega), T^n(\omega)) \leq \varrho_{D_n}(T^{n+1}(\omega), T^n(\omega)) < 5\varepsilon$$

for large n , where $z = \pi((\varphi|_V)^{-1}(\omega))$. Hence, for any arbitrarily small $\varepsilon > 0$, there exists $z \in U$ such that $\varrho_U(f^{n+1}(z), f^n(z)) < 5\varepsilon$ for n large enough. It follows that

$$\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) = 0.$$

□

REMARK 3.5. If the images under f^n of any closed curve in U are eventually contractible in U (e.g. when U is a Baker domain of a meromorphic map with finitely many poles), then by Theorem ??, one obtains a characterization of all three types of f in terms of its dynamical behaviour. In the general case, apart from the characterization of doubly parabolic type in Theorem A, Proposition ?? gives a sufficient condition for f to be of hyperbolic type. It would be interesting to determine whether the condition is necessary, and to obtain a characterization of all three types of f in terms of its dynamical behaviour in the general case.

4. Simply connected absorbing domains: Proof of Theorem B

With the goal of proving Theorem B, we present a condition equivalent to the existence of a simply connected absorbing domain W in U for f .

PROPOSITION 4.1. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map, such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$. Then the following statements are equivalent:*

- (a) *There exists a simply connected absorbing domain W in U for f .*
- (b) *There exists a nice simply connected absorbing domain W in U for f .*
- (c) *For every closed curve $\gamma \subset U$ there exists $n > 0$ such that $f^n(\gamma)$ is contractible in U .*

Proof. The implication (a) \Rightarrow (c) follows by the absorbing property of W , the implication (c) \Rightarrow (b) is due to Theorem ?? and Remark ??, and the implication (b) \Rightarrow (a) is trivial. \square

DEFINITION 4.2. For a compact set $X \subset \mathbb{C}$ we denote by $\text{ext}(X)$ the connected component of $\overline{\mathbb{C}} \setminus X$ containing infinity. We set $K(X) = \overline{\mathbb{C}} \setminus \text{ext}(X)$.

Before proving Theorem B, we show that if ∞ is an isolated point of the boundary of U , then a simply connected absorbing domain W does not exist.

PROPOSITION 4.3. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map, such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$. If ∞ is an isolated point of the boundary of U in $\overline{\mathbb{C}}$, then there is no simply connected absorbing domain W in U for f .*

Proof. By assumption, U is a punctured neighbourhood of ∞ in $\overline{\mathbb{C}}$. Since U is hyperbolic and $f(U) \subset U$, the set $f(U)$ omits at least three points in $\overline{\mathbb{C}}$, so by the Picard Theorem, the map f extends holomorphically to $U \cup \{\infty\}$. Let $V = \{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ for a large $R > 0$. Since $f^n \rightarrow \infty$ uniformly on the boundary of V in $\overline{\mathbb{C}}$, by the openness of f^n , the closure of $f^n(V)$ in $\overline{\mathbb{C}}$ is contained in V for every sufficiently large n . This easily implies that ∞ is an attracting fixed point of the extended map f . Hence, f in a neighbourhood of ∞ is conformally conjugate (by a map ψ) to the map $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$ or $z \mapsto z^k$ for some integer $k \geq 2$ in a neighbourhood of $z = 0$. Let $\gamma = \psi^{-1}(\partial\mathbb{D}(0, r))$ for a small $r > 0$. Then for every $n > 0$ we have $f^n(\gamma) \subset \text{ext}(\gamma)$ and $K(f^n(\gamma)) \supset K(\gamma)$, so $f^n(\gamma)$ is not contractible in U and we can use Proposition ?? to end the proof. \square

Now we prove the main result of this section.

THEOREM B. *Let U be a hyperbolic domain in \mathbb{C} and let $f : U \rightarrow U$ be a holomorphic map, such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$ and ∞ is not an isolated point of the boundary of U in $\overline{\mathbb{C}}$. If f has doubly parabolic type, then there exists a nice simply connected absorbing domain W in U for f .*

Proof. Note first that f has no fixed points in U . Moreover, we will show that f has no isolated boundary fixed points. Indeed, suppose that ζ_0 is an isolated point of the boundary of U in $\overline{\mathbb{C}}$ and f extends holomorphically to $U \cup \{\zeta_0\}$ with $f(\zeta_0) = \zeta_0$. By assumption, $\zeta_0 \neq \infty$. Take a Jordan curve $\gamma_0 \subset U$ surrounding ζ_0 in a small neighbourhood of ζ_0 , such that $D \setminus \{\zeta_0\} \subset U$, where D is the component of $\mathbb{C} \setminus \gamma_0$ containing ζ_0 . Since $f^n \rightarrow \infty$ uniformly on γ_0 and $\zeta_0 = f^n(\zeta_0) \in f^n(D)$, by the Maximum Principle we obtain

$$\mathbb{C} \setminus \{\zeta_0\} = \bigcup_{n=0}^{\infty} f^n(D) \setminus \{\zeta_0\} \subset \bigcup_{n=0}^{\infty} f^n(D \setminus \{\zeta_0\}) \subset U,$$

so in fact $U = \mathbb{C} \setminus \{\zeta_0\}$, which is impossible since U is hyperbolic. Hence, f has no isolated boundary fixed points.

Take a closed curve $\gamma \subset U$. We will show that there exists $n > 0$ such that $f^n(\gamma)$ is contractible in U . (By Proposition ??, this will prove the existence of a nice simply connected absorbing domain W in U for f .) Suppose this is not true. By Theorem ?? for $K = \gamma \cup f(\gamma)$, there exists a constant $C > 0$ such that for every $z \in \gamma$ and every $n \geq 0$,

$$\frac{|f^{n+1}(z) - f^n(z)|}{\text{dist}(f^n(z), \partial U)} \leq C \varrho_U(f^{n+1}(z), f^n(z)),$$

so by the assertion (c) of Theorem A, there exists $n_0 \geq 0$ such that for every $z \in \gamma$ and every $n \geq n_0$,

$$|f^{n+1}(z) - f^n(z)| < \frac{1}{2} \text{dist}(f^n(z), \partial U). \quad (4.1)$$

Take an arbitrary point $v \in \mathbb{C} \setminus U$. By (??), for every $z \in \gamma$ and every $n \geq n_0$ we have

$$|f^{n+1}(z) - f^n(z)| < \frac{1}{2} |f^n(z) - v|.$$

This implies that for $n \geq n_0$, the point $f^{n+1}(z) - v$ lies in a disc D of center $f^n(z) - v$ and radius $\frac{1}{2}|f^n(z) - v|$. Clearly $0 \notin D$ and a simple calculation shows that D is included in a sector of vertex 0 an angle of measure $\pi/3$, symmetric with respect to the straight line containing 0 and $f^n(z) - v$. Hence, there exists a branch Arg of the argument function in D such that in a neighbourhood of z we have

$$|\text{Arg}(f^{n+1}(z) - v) - \text{Arg}(f^n(z) - v)| < \frac{\pi}{6}.$$

Taking the analytic continuation of this branch while z goes around γ , we see that the winding number of $f^n(\gamma)$ around v is the same as the winding number of $f^{n+1}(\gamma)$ around v . In particular, v is in a bounded component of $\mathbb{C} \setminus f^n(\gamma)$ if and only if v is in a bounded component of $\mathbb{C} \setminus f^{n+1}(\gamma)$. Using this inductively, we show that for every $v \in \mathbb{C} \setminus U$ and every $m \geq n \geq n_0$,

$$v \in K(f^n(\gamma)) \quad \text{if and only if} \quad v \in K(f^m(\gamma)). \quad (4.2)$$

Take $n \geq n_0$. By assumption, $f^n(\gamma)$ is not contractible in U , so there exists a point $v_0 \in K(f^n(\gamma)) \setminus U$. Since $f^k \rightarrow \infty$ as $k \rightarrow \infty$ uniformly on γ , there exists $m > n$ such that $f^m(\gamma) \cap K(f^n(\gamma)) = \emptyset$. We cannot have $K(f^m(\gamma)) \subset \text{ext}(f^n(\gamma))$, because then we would have $v_0 \notin K(f^m(\gamma))$, which contradicts (??) for $v = v_0$. Hence, $K(f^n(\gamma)) \subset K(f^m(\gamma))$, so

$$K(f^n(\gamma)) \setminus U \subset K(f^m(\gamma)) \setminus U.$$

On the other hand,

$$K(f^n(\gamma)) \setminus U \supset K(f^m(\gamma)) \setminus U,$$

because otherwise there would exist a point $v_1 \in \mathbb{C} \setminus U$, such that $v_1 \in K(f^m(\gamma)) \setminus K(f^n(\gamma))$, which contradicts (??) for $v = v_1$.

We conclude that for every $n \geq n_0$ there exists $m > n$, such that

$$K(f^n(\gamma)) \subset K(f^m(\gamma)) \quad \text{and} \quad K(f^n(\gamma)) \setminus U = K(f^m(\gamma)) \setminus U.$$

Using this inductively, we construct a strictly increasing sequence $n_j, j \geq 0$, such that

$$K(f^{n_j}(\gamma)) \subset K(f^{n_{j+1}}(\gamma)) \quad \text{and} \quad K(f^{n_j}(\gamma)) \setminus U = K(f^{n_0}(\gamma)) \setminus U \quad (4.3)$$

for every j . Since $f^{n_j} \rightarrow \infty$ as $j \rightarrow \infty$ uniformly on γ , (??) implies that

$$\bigcup_{j=0}^{\infty} K(f^{n_j}(\gamma)) = \mathbb{C}$$

and the set

$$\mathbb{C} \setminus U = \bigcup_{j=0}^{\infty} K(f^{n_j}(\gamma)) \setminus U = K(f^{n_0}(\gamma)) \setminus U$$

is a compact subset of \mathbb{C} . Hence, U contains a punctured neighbourhood of ∞ in $\overline{\mathbb{C}}$, so ∞ is an isolated point of the boundary of U in $\overline{\mathbb{C}}$, which is a contradiction. \square

5. Examples

Throughout this section, let

$$f(z) = z + 1 + \sum_{p \in \mathcal{P}} \frac{a_p}{(z - p)^2}, \quad a_p \in \mathbb{C} \setminus \{0\}, \quad (5.1)$$

where $\mathcal{P} \subset \mathbb{C}$ has one of the following three forms:

- (i) $\mathcal{P} = i\mathbb{Z} = \{im : m \in \mathbb{Z}\}$,
- (ii) $\mathcal{P} = \mathbb{Z}$ or \mathbb{Z}_+ ,
- (iii) $\mathcal{P} = \mathbb{Z} + i\mathbb{Z} = \{j + im : j, m \in \mathbb{Z}\}$.

It is obvious that for sufficiently small $|a_p|$, the map (??) is transcendental meromorphic, with the set of poles equal to \mathcal{P} .

Let

$$\tilde{\mathcal{P}} = \bigcup_{j=0}^{\infty} (\mathcal{P} - j) = \{p - j : p \in \mathcal{P}, j = 0, 1, \dots\}.$$

(In the case $\mathcal{P} = \mathbb{Z}$ or $\mathcal{P} = \mathbb{Z} + i\mathbb{Z}$ we have $\tilde{\mathcal{P}} = \mathcal{P}$.) The assertions of Theorem C and other results mentioned in Section ?? are gathered in the following theorem.

THEOREM 5.1. *For every sufficiently small $\delta > 0$, there exists a map f of the form (??) with an invariant Baker domain U , such that $U \supset \mathbb{C} \setminus \bigcup_{p \in \tilde{\mathcal{P}}} \mathbb{D}(p, \delta)$. The cases (i)–(iii) regarding the form of the set \mathcal{P} can be characterised as follows.*

- In case (i) we have

$$\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for } z \in U$$

and $f|_U$ has doubly parabolic type, so there exists a simply connected absorbing domain W in U for f .

- In case (ii) we have

$$\varrho_U(f^{n+1}(z), f^n(z)) \not\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for } z \in U,$$

$$\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) = 0,$$

and there does not exist a simply connected absorbing domain W in U for f .

- In case (iii) we have

$$\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) > 0,$$

$f|_U$ has hyperbolic type, and there does not exist a simply connected absorbing domain W in U for f .

Moreover, in all three cases, $\{\infty\}$ is a singleton component of $\overline{\mathbb{C}} \setminus U$, in particular it is a singleton component of the Julia set of f .

Let

$$e(z) = \sum_{p \in \mathcal{P}} \frac{a_p}{(z - p)^2} = f(z) - (z + 1)$$

and notice for further purposes that for all $n \geq 1$,

$$f^n(z) - (z + n) = \sum_{k=0}^{n-1} e(f^k(z)). \quad (5.2)$$

To prove Theorem ??, we will use the following two lemmas.

LEMMA 5.2. *For every sufficiently small $\varepsilon > 0$, there exist non-zero complex numbers a_p , $p \in \mathcal{P}$, such that for every $n \geq 0$ and $z_0, \dots, z_n \in \mathbb{C}$, if $\text{dist}(z_0, \tilde{\mathcal{P}}) \geq \varepsilon$ and $|z_k - z_0 - k| < \varepsilon/2$ for $k = 0, \dots, n$, then*

$$\sum_{k=0}^n |e(z_k)| < \frac{\varepsilon}{2}.$$

Proof. Take a small $\varepsilon > 0$ and points z_0, \dots, z_n fulfilling the condition stated in the lemma. Then $z_k = z_0 + k + \zeta_k$, where $\zeta_k \in \mathbb{C}$, $|\zeta_k| < \varepsilon/2 < 1$.

First, consider case (iii). Then, writing $\mathcal{P} \ni p = j + im$ for $j, m \in \mathbb{Z}$, we have

$$\sum_{k=0}^n |e(z_k)| \leq \sum_{k=0}^n \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{|a_{j+im}|}{|z_k - j - im|^2}. \quad (5.3)$$

Note that

$$|z_k - j - im| = |z_0 + k + \zeta_k - j - im| \geq |z_0 + k - j - im| - |\zeta_k| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}, \quad (5.4)$$

since $|z_0 - j + k - im| \geq \text{dist}(z_0, \tilde{\mathcal{P}}) \geq \varepsilon$. Let

$$k_0 = [\text{Re}(z_0)], \quad m_0 = [\text{Im}(z_0)].$$

We will show that

$$|z_k - j - im| \geq \frac{\varepsilon}{8} (|k + k_0 - j| + |m - m_0|). \quad (5.5)$$

To prove (??), we note that if $|k + k_0 - j| + |m - m_0| \leq 4$, then by (??),

$$|z_k - j - im| \geq \frac{\varepsilon}{2} \geq \frac{\varepsilon}{8} (|k + k_0 - j| + |m - m_0|),$$

which gives (??). Otherwise, $|k + k_0 - j| + |m - m_0| \geq 5$, so

$$\begin{aligned} |z_k - j - im| &= |z_0 + k + \zeta_k - j - im| \\ &\geq \frac{\sqrt{2}}{2} (|\text{Re}(z_0 + k + \zeta_k - j - im)| + |\text{Im}(z_0 + k + \zeta_k - j - im)|) \\ &= \frac{\sqrt{2}}{2} (|\text{Re}(z_0) + k + \text{Re}(\zeta_k) - j| + |\text{Im}(z_0) + \text{Im}(\zeta_k) - m|) \\ &\geq \frac{\sqrt{2}}{2} (|k + k_0 - j| + |m - m_0| - |\text{Re}(z_0) - k_0| \\ &\quad - |\text{Im}(z_0) - m_0| - |\text{Re}(\zeta_k)| - |\text{Im}(\zeta_k)|) \\ &\geq \frac{\sqrt{2}}{2} (|k + k_0 - j| + |m - m_0| - 4) \\ &\geq \frac{\sqrt{2}}{10} (|k + k_0 - j| + |m - m_0|), \end{aligned}$$

which shows (??) for sufficiently small ε . Using (??) together with (??) for $k = j - k_0$ and (??) for $k \neq j - k_0$, we obtain

$$\begin{aligned} \sum_{k=0}^n |e(z_k)| &< \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |a_{j+im}| \left(\frac{4}{\varepsilon^2} + \frac{64}{\varepsilon^2} \sum_{k \in \mathbb{Z} \setminus \{j-k_0\}} \frac{1}{(|k+k_0-j| + |m-m_0|)^2} \right) \\ &\leq \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |a_{j+im}| \left(\frac{4}{\varepsilon^2} + \frac{64}{\varepsilon^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k^2} \right) < \frac{\varepsilon}{2} \end{aligned}$$

if $|a_{j+im}|$ are chosen to be sufficiently small.

The remaining cases (i) and (ii) are proved in the same way as case (iii) by setting, respectively, $j = 0$ and $m = 0$. \square

For a small $\varepsilon > 0$ let

$$V = \mathbb{C} \setminus \bigcup_{p \in \tilde{\mathcal{P}}} \mathbb{D}(p, \varepsilon), \quad \tilde{V} = \mathbb{C} \setminus \bigcup_{p \in \tilde{\mathcal{P}}} \mathbb{D}(p, 2\varepsilon).$$

LEMMA 5.3. *For every $\varepsilon > 0$, there exist non-zero complex numbers a_p , $p \in \mathcal{P}$, such that for every $z \in \tilde{V}$ and every $n \geq 0$,*

$$|f^n(z) - z - n| < \frac{\varepsilon}{2}.$$

In particular, this implies $f^n(z) \in V$ for every $n \geq 0$.

Proof. Take $\varepsilon > 0$. Obviously, we can assume that it is small enough to apply Lemma ?? . Hence, we can take numbers a_p , $p \in \mathcal{P}$ satisfying the conditions of Lemma ?? for this value of ε .

Let $z \in \tilde{V}$. To prove the lemma we show, by induction on n , the following claim:

$$\text{dist}(f^n(z), \tilde{V}) \leq \sum_{k=0}^{n-1} |e(f^k(z))| \quad \text{and} \quad |f^n(z) - z - n| < \frac{\varepsilon}{2} \quad (5.6)$$

(with the convention $\sum_{k=0}^{-1} = 0$ for $n = 0$). For $n = 0$ the claim (??) is trivial. Suppose it holds for $0, \dots, n$. By the definition of $e(z)$,

$$f^{n+1}(z) = f^n(z) + 1 + e(f^n(z)),$$

so by induction,

$$\begin{aligned} \text{dist}(f^{n+1}(z), \tilde{V}) &\leq \text{dist}(f^n(z) + e(f^n(z)), \tilde{V}) \\ &\leq \text{dist}(f^n(z), \tilde{V}) + |e(f^n(z))| \\ &\leq \sum_{k=0}^n |e(f^k(z))| \end{aligned}$$

and $|f^k(z) - z - k| < \varepsilon/2$ for $k = 0, \dots, n$. Hence, the points $z_k = f^k(z)$, $k = 0, \dots, n$ fulfil the assumptions of Lemma ?? and hence

$$\sum_{k=0}^n |e(f^k(z))| < \frac{\varepsilon}{2}.$$

This together with (??) implies

$$|f^{n+1}(z) - z - n - 1| = \left| \sum_{k=0}^n e(f^k(z)) \right| \leq \sum_{k=0}^n |e(f^k(z))| < \frac{\varepsilon}{2},$$

which proves the claim (??) for $n + 1$, completing the induction.

Finally, note that the second part of (??) implies $f^n(z) \in V$, since $z + n \in \tilde{V}$ and $\text{dist}(\tilde{V}, \mathbb{C} \setminus V) = \varepsilon$, by the definitions of V and \tilde{V} . This ends the proof of the lemma. \square

Proof of Theorem ??. Take a small $0 < \delta < 1/4$. Set $\varepsilon = \delta/2$ and consider a map of the form (??) for the numbers $a_p, p \in \mathcal{P}$ satisfying the conditions of Lemma ?? for this value of ε . In particular, Lemma ?? implies that $f^n \rightarrow \infty$ as $n \rightarrow \infty$ almost uniformly on \tilde{V} . Since \tilde{V} is connected, it follows that $\mathbb{C} \setminus \bigcup_{p \in \mathcal{P}} \mathbb{D}(p, \delta) = \tilde{V}$ is contained in an invariant Baker domain U of f . Note that $U \cap \mathcal{P} = \emptyset$ since the poles are contained in the Julia set of f .

Now we characterise cases (i)–(iii). In case (i), we have $U \supset \tilde{V} \supset \{z \in \mathbb{C} : \text{Re}(z) \geq 2\varepsilon\}$, in particular $1 \in \tilde{V} \subset U$. By Lemma ??,

$$|f^{n+1}(1) - f^n(1)| < 1 + \varepsilon < 2$$

and

$$\text{dist}(f^n(1), \partial U) \geq \text{dist}(f^n(1), \partial \tilde{V}) > n + 1 - \frac{5}{2}\varepsilon > n$$

for $n \geq 0$. Hence, by Lemma ??,

$$\varrho_U(f^{n+1}(1), f^n(1)) \leq 2 \ln \frac{1}{1 - |f^{n+1}(1) - f^n(1)| / \text{dist}(f^n(1), \partial U)} < 2 \ln \frac{1}{1 - 2/n} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore, Theorem A implies that $\varrho_U(f^{n+1}(z), f^n(z)) \rightarrow 0$ as $n \rightarrow \infty$ for $z \in U$ and $f|_U$ has doubly parabolic type. It is easy to check that $W = \{z \in \mathbb{C} : \text{Re}(z) > 1\}$ is a nice simply connected absorbing domain in U for f .

Now consider case (ii). Then $U \supset \tilde{V} \supset \{z \in \mathbb{C} : |\text{Im}(z)| \geq 2\varepsilon\}$, in particular $ik \in \tilde{V} \subset U$ for every positive integer k . Hence, by Lemma ??,

$$\frac{1}{2} < 1 - \varepsilon < |f^{n+1}(ik) - f^n(ik)| < 1 + \varepsilon < 2$$

and

$$\frac{k}{2} < k - \frac{5}{2}\varepsilon < \text{dist}(f^n(ik), \partial \tilde{V}) \leq \text{dist}(f^n(ik), \partial U) < k + \frac{\varepsilon}{2} < 2k$$

for $n \geq 0$. Hence, by Theorem ?? for $K = \{ik, f(ik)\}$, we have

$$\varrho_U(f^{n+1}(ik), f^n(ik)) \geq \frac{1}{C} \frac{|f^{n+1}(ik) - f^n(ik)|}{\text{dist}(f^n(ik), \partial U)} > \frac{1}{4Ck} > 0,$$

so Theorem A implies $\varrho_U(f^{n+1}(z), f^n(z)) \not\rightarrow 0$ as $n \rightarrow \infty$ for $z \in U$. Moreover, by Lemma ??,

$$\varrho_U(f^{n+1}(ik), f^n(ik)) \leq 2 \ln \frac{1}{1 - |f^{n+1}(ik) - f^n(ik)| / \text{dist}(f^n(ik), \partial U)} < 2 \ln \frac{1}{1 - 4/k} \rightarrow 0$$

as $k \rightarrow \infty$, which shows $\inf_{z \in U} \lim_{n \rightarrow \infty} \varrho_U(f^{n+1}(z), f^n(z)) = 0$ (cf. Remark ??).

Now consider case (iii). We will show that in this case we have $\inf_{z \in U} \varrho_U(f(z), z) > 0$. First, take $z \in \tilde{V}$. We can choose $p \in \mathcal{P}$ such that

$$|z - p| \leq \frac{\sqrt{2}}{2}. \tag{5.7}$$

Lemma ?? implies that

$$|f(z) - z| > 1 - \frac{\varepsilon}{2} > \frac{1}{2}. \tag{5.8}$$

Note that $p, p+1 \notin U$, so setting $w = z - p$, $w' = f(z) - p$ and using the Schwarz–Pick Lemma ??, we obtain

$$\varrho_U(f(z), z) \geq \varrho_{\mathbb{C} \setminus \{p, p+1\}}(f(z), z) = \varrho_{\mathbb{C} \setminus \{0, 1\}}(w, w').$$

Let γ be the hyperbolic geodesic connecting w and w' in $\mathbb{C} \setminus \{0, 1\}$. By (??) and (??), we have $w \in \mathbb{D}(0, \sqrt{2}/2)$ and $|w - w'| > 1/2$, so there exists a curve $\tilde{\gamma} \subset \gamma$ of Euclidean length $1/2$, such that $\tilde{\gamma} \subset \mathbb{D}(0, (1 + \sqrt{2})/2)$. By (??), there exists $c > 0$ such that $\varrho_{\mathbb{C} \setminus \{0, 1\}}(u) > c$ for every $u \in \mathbb{D}(0, (1 + \sqrt{2})/2) \setminus \{0, 1\}$. Hence,

$$\varrho_{\mathbb{C} \setminus \{0, 1\}}(w, w') = \int_{\gamma} \varrho_{\mathbb{C} \setminus \{0, 1\}}(u) du \geq \int_{\tilde{\gamma}} \varrho_{\mathbb{C} \setminus \{0, 1\}}(u) du > \frac{c}{2},$$

which shows

$$\inf_{z \in \tilde{V}} \varrho_U(f(z), z) \geq \frac{c}{2}. \quad (5.9)$$

Now take $z \in U \setminus \tilde{V}$. If there exists $n > 0$ such that $f^n(z) \in \tilde{V}$, then by the Schwarz–Pick Lemma ?? and (??), we have $\varrho_U(f(z), z) \geq \varrho_U(f^{n+1}(z), f^n(z)) \geq c/2$. Hence, we can assume that $f^n(z) \in U \setminus \tilde{V} \subset \bigcup_{p \in \mathbb{Z} + i\mathbb{Z}} \mathbb{D}(p, 2\varepsilon)$ for every $n \geq 0$. Since $f^n(z) \rightarrow \infty$ as $n \rightarrow \infty$, there exists $k \geq 0$ and $p, p' \in \mathbb{Z} + i\mathbb{Z}$, $p \neq p'$, such that $|f^k(z) - p| < 2\varepsilon$, $|f^{k+1}(z) - p'| < 2\varepsilon$, which implies $|f^{k+1}(z) - f^k(z)| > 1 - 4\varepsilon > 1/2$. Hence, (??) and (??) are satisfied for $f^k(z)$ instead of z . Repeating the previous arguments, we show $\varrho_U(f^{k+1}(z), f^k(z)) \geq c/2$, so by the Schwarz–Pick Lemma ??,

$$\varrho_U(f(z), z) \geq \varrho_U(f^{k+1}(z), f^k(z)) \geq c/2,$$

which implies

$$\inf_{z \in U} \varrho_U(f(z), z) \geq \frac{c}{2}.$$

By Remark ?? and Proposition ??, $f|_U$ has hyperbolic type.

Finally, note that in all three cases (i)–(iii), the boundary of the square

$$Q_k = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq k + 1/2, |\operatorname{Im}(z)| \leq k + 1/2\}$$

is contained in $\tilde{V} \subset U$ for every integer $k \geq 0$, which shows that $\{\infty\}$ is a singleton component of $\mathbb{C} \setminus U$. Moreover, in cases (ii) and (iii), if $\gamma = \partial Q_0$, then $f^n(\gamma)$ is not contractible in U . To see this, notice that by Lemma ??, the points of $f^n(\gamma)$ are $\varepsilon/2$ -close to the suitable points of the boundary of the square $Q_0 + n$, which winds once around the pole n of f . Similarly as in the proof of Theorem B, this implies that $f^n(\gamma)$ winds once around the pole n , so it is not contractible in U . By Proposition ??, we conclude that in cases (ii) and (iii) there is no simply connected absorbing domain W in U for f . □

Krzysztof Barański
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
Poland
baranski@mimuw.edu.pl

Núria Fagella
Departament de Matemàtica Aplicada i
Anàlisi
Universitat de Barcelona
08007 Barcelona
Spain
fagella@maia.ub.es

Xavier Jarque
Departament de Matemàtica Aplicada i
Anàlisi
Universitat de Barcelona
08007 Barcelona
Spain
xavier.jarque@ub.edu

Bogusława Karpińska
Faculty of Mathematics and Information
Science
Warsaw University of Technology
ul. Koszykowa 75
00-662 Warszawa
Poland
bkarpin@mini.pw.edu.pl