On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points I

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27th November 2007

Abstract

It is known that the Julia set of the Newton's method of a non-constant polynomial is connected ([18]). This is, in fact, a consequence of a much more general result that establishes the relationship between simple connectivity of Fatou components of rational maps and fixed points which are repelling or parabolic with multiplier 1.

In this paper we study Fatou components of transcendental meromorphic functions, namely, we show the existence of such fixed points provided that immediate attractive basins or preperiodic components be multiply connected.

1 Introduction

The so-called *Newton's method* is, in all likelihood, the most common of the root-finding algorithms, mainly because of its simplicity, high efficiency index and quadratic order of convergence. Newton's method associated to a complex holomorphic function f is defined by the dynamical system

$$N_f(z) := z - \frac{f(z)}{f'(z)}\,.$$

As such, a natural question is what properties we might be interested in or, put more generally, what kind of study we want to make of it. From the dynamical point of view—and given the purpose of any root-finding

All authors were supported by MTM2005–02139. Also, the first and the second authors were supported by MTM2006–05849/Consolider (including a FEDER contribution), the second author by CIRIT no. 2005SGR–00550 and the first and the third authors by 2005SGR–01028.

algorithm—a fundamental issue is to understand the dynamics of N_f about its fixed points, as they correspond to the roots of the function f; in other words, we would like to understand the fixed *basins of attraction* of N_f, the sets of points that converge to a root of f under the iteration of N_f.

Basins of attraction are actually just one type of *stable* component or component of the *Fatou set* $\mathcal{F}(f)$, the set of points $z \in \widehat{\mathbb{C}}$ for which $\{f^n\}_{n\geq 1}$ is defined and normal in a neighbourhood of z (recall $\widehat{\mathbb{C}}$ stands for the *Riemann sphere*, the compact Riemann surface $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$). The *Julia set* or *set of chaos* is its complement, $J(f) := \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$.

At first, one could think that if the fixed points of N_f are exactly the roots of f, then Newton's method is a neat algorithm in the sense that it will always converge to one of the roots. But notice that not every stable component is a basin of attraction; even not every attracting behaviour is suitable for our purposes: Basic examples like Newton's method applied to cubic polynomials of the form $f_a(z) = z(z-1)(z-a)$, for certain values of $a \in \mathbb{C}$, lead to open sets of initial values converging to attracting periodic cycles. Actually, also the set of such parameters $a \in \mathbb{C}$, for this family of functions, is an open set of the corresponding parameter space. (See [6] or [8].)

A lot of literature concerning Newton's method's Julia and Fatou sets has been written, above all when applied to algebraic functions. Przytycki showed in [15] that every root of a polynomial P has a simply connected immediate basin of attraction for N_P. Meier [13] proved the connectivity of the Julia set of N_P when $\deg P = 3$, and later Tan Lei [20] generalised this result to higher degrees of P. In 1990, Shishikura [18] proved the result that actually sets the basis of our work: For any non-constant polynomial P, the Julia set of N_P is connected (or, equivalently, all its Fatou components are simply connected). In fact, he obtained this result as a corollary of a much more general theorem for rational functions, namely, the connectedness of the Julia set of rational functions with exactly one *weakly repelling fixed point*, i.e., a fixed point which is either repelling or parabolic of multiplier 1 (see Chapter 3).

The present work, however, deals with Newton's method applied to transcendental maps. In the same direction, in 2002 Mayer and Schleicher [12] extended Przytycki's theorem, showing that every root of a transcendental entire function f has a simply connected immediate basin of attraction for N_f, and this work has been recently continued by Rückert and Schleicher in [16], where they study Newton maps in the complement of such Fatou components. Our goal is to prove the natural transcendental versions of Shishikura's results—although this paper covers just part of it—, which can be conjectured as follows.

Conjecture 1.1. *If the Julia set of a transcendental meromorphic function f is disconnected, there exists at least one weakly repelling fixed point of f.*

We may assume that our transcendental meromorphic functions are defined on the plane $\mathbb{C}$, so infinity is an essential singularity.

REMARK. Notice that essential singularities are always in the Julia set of a transcendental meromorphic function f and therefore infinity can connect two unbounded connected components of $\mathcal{J}(f) \cap \mathbb{C}$ otherwise disconnected.

Now, transcendental meromorphic functions that come from applying Newton's method to transcendental entire functions happen to have no weakly repelling fixed points at all, so the next result is obtained forthwith.

Conjecture 1.2 (Corollary)**.** *The Julia set of the Newton's method of a transcendental entire function is connected.*

As it turns out, a possible proof of Conjecture 1.1 splits into several cases, according to different Fatou components, since the connectedness of the Julia set is equivalent to the simple connectedness of the connected components of its complement. In this paper we will see two of such cases, which, together, give raise to the following result.

Theorem 1.3. *Let f be a transcendental meromorphic function with either a multiply-connected attractive basin or a multiply-connected Fatou component with simply-connected image. Then, there exists at least one weakly repelling fixed point of f.*

Notice how this theorem actually connects with the result of Mayer and Schleicher mentioned above.

In order to prove this theorem, we use the method of *quasi-conformal surgery* and a theorem of Buff on *virtually repelling fixed points*. On the one hand, quasi-conformal surgery (see Section 2.1) is a powerful tool that allows to create holomorphic maps with some desired behaviour. One usually starts glueing together—or *cutting* and *sewing*, this is why this procedure is called 'surgery'—several functions having the required dynamics; in general, the map f obtained is not holomorphic. However, if we can create an appropriate almost complex structure on $\widehat{\mathbb{C}}$, the Measurable Riemann Mapping Theorem can be applied to find a holomorphic map g, plus some quasi-conformal homeomorphism that conjugates the functions f and g. On the other hand, the property of being virtually repelling is only slightly stronger than that of weakly repelling, and in some cases it might just be easier to prove the existence of a virtually repelling fixed point.

The paper is structured as follows: The next chapter gives some basic definitions and properties of complex dynamics and related topics; in particular, it puts stress upon quasi-conformal surgery and virtually repelling fixed points. Some of the cases that result from the proof of Theorem 1.3 use a surgery process quite similar to that of Shishikura's; thus, in Chapter 3 we recall his results and give part of his proof so as to show how surgery is used in our scenario. Later on, in our proof, we will focus on the differences between the two cases. Such proof, as well as the details on how Conjecture 1.1 splits, can be found in Chapter 4, dedicated to transcendental functions.

Acknowledgements. We wish to thank C. Henriksen, A. Douady, X. Buff, W. Bergweiler and A. Epstein for very valuable discussions and for their hospitality during several research stays.

2 Preliminaries and tools

This chapter provides some general background on holomorphic dynamics, to be used later on. After a few initial basic definitions and results, the settings on quasi-conformal surgery and virtually repelling fixed points are also presented.

We consider f to be a rational, transcendental entire or transcendental meromorphic function and use the term *complex function* to denote either case. We write f^n for the nth iteration of f, that is, $f^0(z) := z$ and $f^n(z) := f(f^{n-1}(z))$, when $n \geq 1$; as usual, f^{-n} represents $(f^n)^{-1}$, the set of all inverse branches of f^n.

We say that $z_0 \in \widehat{\mathbb{C}}$ is a *periodic point* of f of (minimal) period $n \in \mathbb{N}$ if $f^n(z_0) = z_0$ and $f^k(z_0) \neq z_0$, for all $0 < k < n$; the *multiplier* of a periodic point z_0 of period n is the value $\rho(z_0) := (f^n)'(z_0) \in \mathbb{C}$. A periodic point z_0 is called *attracting* if $|\rho(z_0)| < 1$, *repelling* if $|\rho(z_0)| > 1$ and *parabolic* if $\rho(z_0) = e^{2\pi i\theta}$, with $\theta \in \mathbb{Q}$. Also, z_0 is said to be *weakly repelling* if it is either repelling or parabolic of multiplier 1.

The following theorem of Fatou [10] will be a key tool in the cases where the surgery technique be used. Its proof can be found in [14].

Theorem 2.1 (Fatou)**.** *Any rational map of degree greater than one has, at least, one weakly repelling fixed point.*

The Fatou set is open by definition and its connected components are commonly referred to as *Fatou components*. The following is a first classification of such.

Definition. Let f be a complex function and U a (connected) component of $\mathcal{F}(f)$; U is said to be *preperiodic* if there exist integers $n > m \geq 0$ such that $f^n(U) = f^m(U)$. We say that U is *periodic* if $m = 0$, and *fixed* if $n = 1$. A Fatou component is called a *wandering domain* if it fails to be preperiodic.

The next classification of periodic Fatou components is essentially due to Cremer and Fatou, and was first stated in this form in [2].

Theorem 2.2 (Classification)**.** *Let U be a p-periodic Fatou component of a complex function f. Then U is one of the following:*

- immediate attractive basin: *U contains an attracting p-periodic point z_0 and $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$;*

- parabolic basin *or* Leau domain: *∂U contains a unique p-periodic point z_0 and $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$. Moreover $(f^p)'(z_0) = 1$;*

- Siegel disc: *there exists a holomorphic homeomorphism $\phi : U \to \mathbb{D}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$;*

- Herman ring: *there exist $r > 1$ and a holomorphic homeomorphism $\phi : U \to \{1 < |z| < r\}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$;*

- Baker domain: *∂U contains a point z_0 such that $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$, but $f(z_0)$ is not defined. In our context, z_0 is an essential singularity.*

Rational functions and transcendental entire functions of *finite type* (that is to say, with a finite number of singularities of the inverse function) have neither wandering domains nor Baker domains. The absence of wandering domains was proved by Sullivan [19] for rational functions and by Eremenko and Lyubich [9] and Goldberg and Keen [11] for such entire maps. As for Baker domains, while such Fatou components make no sense for rational functions because infinity is but a regular point, their absence for transcendental entire functions of finite type is, in fact, a consequence of a much stronger result of Eremenko and Lyubich [9], generalised to meromorphic maps by Bergweiler [3] using some of their ideas.

2.1 Quasi-conformal surgery

What is known today in holomorphic dynamics literature as *quasi-conformal surgery* is a technique to construct holomorphic maps with some prescribed dynamics. As mentioned, the term 'surgery' suggests that certain spaces and maps will be cut and sewed in order to construct the desired behaviour. This is usually the first step of the process and is known as *topological surgery*. On the other hand, the adjective 'quasi-conformal' indicates that the map one constructs in this first step is not holomorphic, but only quasi regular, and it needs to be made holomorphic by means of the Measurable Riemann Mapping Theorem. This second step is called *holomorphic smoothing*.

Quasi-conformal mappings were first introduced in complex dynamics in 1981 by Sullivan, in a seminar at the IHES, and applied to the study of polynomial-like mappings by Douady and Hubbard [8]. In 1985 Sullivan published his study in [19], and two years later Shishikura gave a great impulse to the technique in its application to rational functions (see [17]).

We now introduce some basic concepts in order to understand the main results.

Definition. Let $U \subset \mathbb{C}$ be an open set; a measurable function $\mu\colon U \to \mathbb{C}$ is called a *k-Beltrami coefficient* of U if $||\mu||_\infty = k < 1$.

Equivalently, one can associate to every k-Beltrami coefficient of U μ an *almost complex structure* σ, that is, a measurable field of (infinitesimal) ellipses in TU, defined up to multiplication by a positive real constant. More precisely, the argument of the minor axis of such ellipses at a point $z \in U$ is $\arg(\mu(z))/2$, and its ellipticity—i.e. the ratio between its axes—equals $(1 - |\mu(z)|)/(1 + |\mu(z)|)$. Notice that this value is bounded between $(1 - ||\mu||_\infty)/(1 + ||\mu||_\infty) > 0$ and 1 almost everywhere.

Definition. Let U and V be open sets in $\mathbb{C}$; a map $\phi\colon U \to V$ is said to be *k-quasi-regular* if it has locally square integrable weak derivatives and the function

$$\mu_\phi(z) := \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(z)$$

is a k-Beltrami coefficient. A *k-quasi-conformal map* is a quasi-regular homeomorphism.

It is easy to check that a quasi-regular map is locally the composition of a holomorphic function and a quasi-conformal map.

Definition. Let U and V be open sets in $\mathbb{C}$; a quasi-regular map $\phi\colon U \to V$ induces a contravariant functor $\phi^*\colon L^\infty(V) \to L^\infty(U)$ defined by

$$\phi^*\mu := \frac{\partial\phi/\partial\bar{z} + (\mu\circ\phi)\overline{(\partial\phi/\partial z)}}{\partial\phi/\partial z + (\mu\circ\phi)\overline{(\partial\phi/\partial\bar{z})}}\,.$$

Notice that if $\mu\colon V \to \mathbb{C}$ is a Beltrami coefficient, then so is its pull-back $\phi^*\mu\colon U \to \mathbb{C}$. Moreover, if ϕ is a holomorphic map, then $||\phi^*\mu||_\infty = ||\mu||_\infty$.

When the Beltrami coefficient μ is defined in terms of a quasi-regular map ψ as above ($\mu \equiv \mu_\psi$), one can check that $\phi^*\mu_\psi = \mu_{\psi\circ\phi}$.

Definition. We call *standard complex structure* the constant Beltrami coefficient $\mu_0 := 0$ or, equivalently, the associated field of circles σ_0.

By Weyl's Lemma, we have that a quasi-regular map ϕ is holomorphic if, and only if, $\phi^*\mu_0 = \mu_0$.

Now, it is clear that a quasi-conformal map ϕ defines a Beltrami coefficient μ_ϕ. Conversely, given a Beltrami coefficient μ and the so-called *Beltrami equation*

$$\frac{\partial\phi}{\partial\bar{z}} = \mu\cdot\frac{\partial\phi}{\partial z}\,,$$

can we find an actual quasi-conformal map ϕ such that $\mu_\phi \equiv \mu$? The celebrated measurable Riemann mapping theorem answers this question positively; the following is a $U = V = \mathbb{C}$ version of the statement (see also [1] or [7]).

Theorem 2.3 (Morrey, Bojarski, Ahlfors, Bers)**.** *Let μ be a Beltrami coefficient of $\mathbb{C}$; then, there exists a unique quasi-conformal map $\phi\colon \mathbb{C} \to \mathbb{C}$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\mu_\phi = \mu$.*

The application of this result to complex dynamics is the following. Suppose that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map whose dynamics we would like to see realised by a holomorphic map of $\widehat{\mathbb{C}}$. Then, Theorem 2.3 guarantees the existence of such a map as long as we can construct an appropriate f-invariant almost complex structure. The precise statement reads as follows.

Corollary 2.4. *Let μ be a Beltrami coefficient of $\mathbb{C}$ and $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a quasi-regular map such that $f^*\mu = \mu$; then, f is quasi-conformally conjugate to a holomorphic map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

Proof. Applying the measurable Riemann mapping theorem to μ, there exists a quasi-conformal map ϕ with $\mu = \phi^*\mu_0$. Now, let us define $g := \phi \circ f \circ \phi^{-1}$; we just need to see that g is indeed holomorphic:

$$g^*\mu_0 = (\phi f \phi^{-1})^*\mu_0 = (\phi^{-1})^* f^* \phi^* \mu_0 = (\phi^{-1})^* f^* \mu = (\phi^{-1})^*\mu = \mu_0.$$

□

Remark. Notice that the dynamical condition of f-invariancy is represented by the expression $f^*\mu = \mu$, that is, the function μ (and therefore the associated almost complex structure) is preserved under the dynamics of f.

2.2 On virtually repelling fixed points

We now introduce the concept of virtually repelling fixed point, which goes back to A. Epstein. It is slightly stronger than that of weakly repelling fixed point and its definition is based on the holomorphic index. (See also [5] or [14].)

Definition. The *holomorphic index* of a complex function f at a fixed point z is the residue

$$\iota(f, z) := \frac{1}{2\pi i} \oint_z \frac{dw}{w - f(w)}.$$

In the case of a simple fixed point ($\rho(z) \neq 1$), the index is also given by

$$\iota(f, z) = \frac{1}{1 - \rho(z)}.$$

If we have that $\mathrm{Re}(\iota(f,z)) < m/2$, where $m \geq 1$ denotes the multiplicity, the fixed point z is called *virtually repelling*.

REMARKS.

- Virtually repelling fixed points are in particular weakly repelling, as in the multiple case the multiplier is $\rho(z) = 1$, and in the simple case we have that
$$\mathrm{Re}\left(\frac{1}{1-\rho(z)}\right) < \frac{1}{2} \iff |\rho(z)| > 1\,.$$
- Virtual repellency, unlike weak repellency, is not preserved under topological conjugacy, since the residue index is only kept under analytic conjugacy (see [14]). See also [18] for a proof of this property in weakly repelling fixed points.

Theorem 2.5 (Buff)**.** *Let $U \subset \mathbb{D}$ be an open set and $f\colon U \to \mathbb{D}$ a proper holomorphic map of degree $d \geq 2$. If $|f(z) - z|$ is bounded away from zero as $z \in U$ tends to ∂U, then f has at least one virtually repelling fixed point.*

REMARK. Observe that if we require U to be compactly contained in $\mathbb{D}$, then f is a *polynomial-like mapping* (see [8]). By the Straightening Theorem, f is hybrid equivalent—in particular, quasi-conformally conjugate—to a polynomial P in U. It follows from Fatou's Theorem 2.1 applied to P that f must have a weakly repelling fixed point in U.

Of course, in our context we are not dealing with holomorphic maps, so we shall adapt Buff's result to our situation with the following version.

Corollary 2.6. *Let $f\colon V \to D$ be a proper transcendental holomorphic function with $V \subset D$ and $D \subset \widehat{\mathbb{C}}$ an open, simply connected set. If $|f(z)-z|$ is bounded away from zero as $z \in V$ tends to either ∂V or ∞, there exists at least one virtually repelling fixed point of f.*

Proof. Since the set D is open and simply connected, we have that there exists a conformal Riemann mapping $\varphi\colon D \to \mathbb{D}$. This map takes the subset V to some $\varphi(V) = U \subset \mathbb{D}$, as V is contained in D. (See Figure 1.)

Let us now define the map $g := \varphi \circ f \circ \varphi^{-1}$, which is clearly conjugate to f by the conformal conjugation φ. Observe that g is proper and $|g(z) - z|$ is bounded away from zero as $z \in U$ tends to ∂U, for so is $|f(z) - z|$ as $z \in V$ tends to either ∂V or the essential singularity. In this situation, g has at least one virtually repelling fixed point z_0 due to Theorem 2.5. Since conformal conjugacies preserve this property of fixed points, we have that there exists a virtually repelling fixed point $\varphi^{-1}(z_0)$ of f (in V). $\square$

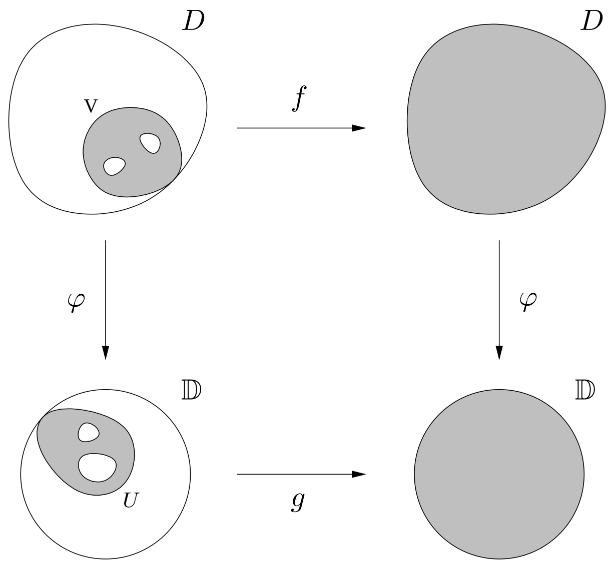

Figure 1: Sketch of the proof of Corollary 2.6. Observe that D or V could be unbounded.

Remark. In particular, Corollary 2.6 gives the existence of a weakly repelling fixed point of f, which is the property we shall use in our arguments.

3 Shishikura's rational case

Our work on connectivity of Julia sets of transcendental meromorphic functions is based on that of Shishikura's for rational maps. In this chapter we would like to show the main results in his paper, as well as part of their proofs, since they also cover some very specific situations of our transcendental result. The case chosen is that concerning immediate attractive basins and it has been rearranged so that the general structure matches the discourse on transcendental functions in Chapter 4.

The following theorem and corollary, along with all the other results and proofs in this chapter, are due to Shishikura and extracted from [18].

Theorem 3.1. *If the Julia set of a rational map f is disconnected, there exist two weakly repelling fixed points of f.*

Corollary 3.2. *The Julia set of a rational map with only one weakly repelling fixed point is connected; in other words, all its Fatou components are simply connected. In particular, the Julia set of the Newton's method of a non-constant polynomial is connected.*

Corollary 3.2 is an immediate consequence of the previous theorem, for the Newton's method of a non-constant polynomial has all its fixed points attracting except for the one fixed point at infinity, which is (weakly) repelling.

In order to prove Theorem 3.1, Shishikura uses a case-by-case approach, according to different types of Fatou component—for a general complex function, these are wandering domains, preperiodic components and periodic components, the latter ones described in the Classification Theorem 2.2. For the Julia set of a rational map to be disconnected, there must exist at least one multiply-connected Fatou component; namely, an immediate attractive basin, Leau domain, Herman ring or preperiodic component, since Siegel discs cannot be multiply connected and rational maps have neither wandering domains nor Baker domains. Furthermore, the preperiodic case may be treated in a slightly special way, since preperiodic components eventually landing on multiply-connected periodic components can clearly be omitted, so the image of a preperiodic Fatou component may be assumed simply connected.

The strategy that we have only just outlined can be shaped into the following theorem.

Theorem 3.3. *Let f be a rational map of degree greater than one. Then,*

- *if f has a multiply-connected immediate attractive or parabolic basin, there exist two weakly repelling fixed points;*
- *if f has a Herman ring, there exist two weakly repelling fixed points;*
- *if f has a multiply-connected Fatou component U such that $f(U)$ is simply connected, every component of $\widehat{\mathbb{C}} \setminus U$ contains a weakly repelling fixed point.*

The next sections contain a two-step version of part of Shishikura's proof for this result—namely, the attractive case. Thus, Section 3.1 deals but with fixed immediate attractive basins, while strictly periodic immediate attractive basins are left to Section 3.2. We refer to [18] for a complete proof of Theorem 3.3.

3.1 Fixed basin

Let us first sketch the process that forces the existence of at least two weakly repelling fixed points, provided that the rational map f has a multiply-connected fixed immediate attractive basin. Since the basin is multiply connected, there exist at least two components of its complement—we want to show that two of them contain a weakly repelling fixed point each. Using quasi-conformal surgery, we can construct a rational map g, conjugate to f where needed, with a weakly repelling fixed point in some suitable subset of the sphere so as for f to have such a point in one of the components of the complement of the basin.

Although this description applies to both fixed and periodic cases, in this section we just show the proof for the first one, that is to say: A rational map of degree greater than one with a multiply-connected fixed immediate attractive basin has, at least, two weakly repelling fixed points.

Let us call α the attracting fixed point of f contained in the multiply-connected fixed immediate attractive basin, $\mathcal{A}^*$. Take a small disc neighbourhood U_0 of α such that $\overline{f(U_0)} \subset U_0$. For each $n \geq 0$, let U_n be the connected component of $f^{-n}(U_0)$ that contains α.

From the choice of U_0, we have that

$$\mathcal{A}^* = \bigcup_{n \geq 0} U_n \, .$$

Therefore, there exists $n > 0$ such that U_n is multiply connected—otherwise, the union of the increasing simply-connected open sets U_n would be simply connected. More precisely, there exists $n_0 > 0$ such that U_{n_0} is multiply connected but $U_{n_0 - 1}$ is simply connected (see Figure 2). Rename $U := U_{n_0}$ for simplicity of the text.

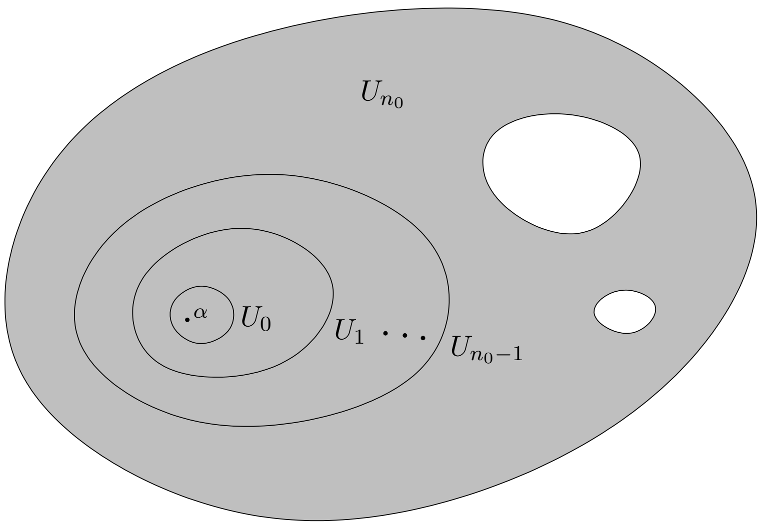

Figure 2: The increasing sequence of open neighbourhoods of α, where U_{n_0-1} is simply connected and U_{n_0} is multiply connected.

Since U is multiply connected, there exist at least two connected components of $\widehat{\mathbb{C}} \setminus U$; choose one of them and call it E. From the construction of U, notice that $f(U) = f(U_{n_0}) = U_{n_0-1} \subset U_{n_0} = U$ and, therefore, $f(U) \subset U \subset \mathcal{A}^*$.

Now that we have suitable sets to work with, the next step of this surgery process is the construction of some quasi-regular map—with certain desired dynamics—, to which the Measurable Riemann Mapping Theorem (see Section 2.1) can be applied. The following lemma produces exactly such a function.

Lemma 3.4 (Interpolation Lemma)**.** *Let V_0 and V_1 be simply-connected open sets in $\widehat{\mathbb{C}}$, with $\#(\widehat{\mathbb{C}} \setminus V_0) \geq 1$, and f a holomorphic map from a neighbourhood N of ∂V_0 to $\widehat{\mathbb{C}}$ such that $f(\partial V_0) = \partial V_1$ and $f(V_0 \cap N) \subset V_1$; choose a compact set K in V_0 and two points $a \in V_0$ and $b \in V_1$. Then, there exists a quasi-regular mapping $f_1 \colon V_0 \to V_1$ such that*

- *$f_1 = f$ in $V_0 \cap N_1$, where N_1 is a neighbourhood of ∂V_0 with $N_1 \subset N$;*
- *f_1 is holomorphic in a neighbourhood of K;*
- *$f_1(a) = b$.*

Shishikura's proof for the Interpolation Lemma is somewhat technical and can be found in [18], although Figure 3 offers a sketch of it.

In our situation (see Figure 4), we write $V_0 := \widehat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$, call $K := \overline{f(U)}$ and choose $a = b \in f(U)$ arbitrarily. This way, a quasi-regular mapping $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$ is obtained from Lemma 3.4.

Roughly speaking, the map f_1 simplifies f outside E, where its behaviour cannot be controlled, although it still agrees with f on the boundary of this set. We define yet another function $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by cutting and glueing f and f_1 where needed:

$$f_2 := \begin{cases} f & \text{on } E \\ f_1 & \text{on } \widehat{\mathbb{C}} \setminus E\,. \end{cases}$$

This function is quasi regular, since f is rational and so holomorphic, f_1 is quasi regular, and they coincide on an open annulus surrounding ∂E. Furthermore, we have—just from its definition—that f_2 is holomorphic in E and in a neighbourhood of $\overline{f(U)}$, and it has a fixed point at a, for $f_2(a) = f_1(a) = b = a$. Notice that $f_2(\widehat{\mathbb{C}} \setminus E) = f(U)$ and $\overline{f(U)} \subsetneq \widehat{\mathbb{C}} \setminus E$; hence $f(U)$ is invariant and the fixed point $a \in f(U)$ is a global attractor of f_2 in $\widehat{\mathbb{C}} \setminus E$. This concludes the topological step of the construction.

In order to apply the Measurable Riemann Mapping Theorem, it only remains to construct an appropriate f_2-invariant almost complex structure, so define

$$\sigma := \begin{cases} \sigma_0 & \text{on } f(U) \\ (f_2^n)^*\sigma_0 & \text{on } f_2^{-n}(f(U)), \text{ for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

(See Figure 5.)

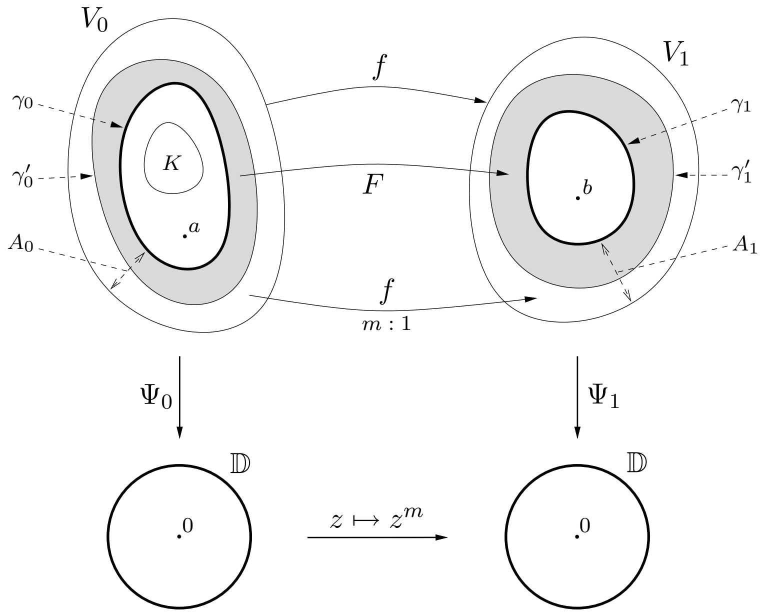

Figure 3: We first construct two annuli $A_0 \subset V_0 \cap N$ and $A_1 \subset V_1$, with $\partial A_i = \partial V_i \cup \gamma_i$ and $K \cap \overline{A_0} = \emptyset$, $a \notin \overline{A_0}$, $b \notin \overline{A_1}$, in such a way that the restriction $f_{|A_0}\colon A_0 \to A_1$ be a covering map of degree m and A_0 contain no critical points of f. Then we consider (conformal) Riemann mappings $\Psi_i\colon V_i \setminus \overline{A_i} \to \mathbb{D}$ such that $\Psi_0(a) = \Psi_1(b) = 0$, and define $\widetilde{f}$ on $V_0 \setminus \overline{A_0}$ as $\widetilde{f} := \Psi_1^{-1} \circ (z \mapsto z^m) \circ \Psi_0$. Thus both f and $\widetilde{f}$ are covering maps from γ_0 to γ_1 of the same degree without critical points, hence homotopic. Take $\gamma_1' \subset A_1$ and $\gamma_0' := f^{-1}(\gamma_1') \cap A_0$ as in the figure, and let F be the natural linear interpolation map defined between f on γ_0' and $\widetilde{f}$ on γ_0. Now the map $f_1\colon V_0 \to V_1$, defined as f between ∂V_0 and γ_0', F between γ_0' and γ_0, and $\widetilde{f}$ on $V_0 \setminus \overline{A_0}$, has the properties as required. The shaded regions indicate the dynamics of F.

By construction, $f_2^*\sigma = \sigma$ almost everywhere, since σ is defined based on the dynamics of f_2. Moreover, σ has bounded ellipticity: indeed, f_2 is holomorphic everywhere except in $X := \widehat{\mathbb{C}} \setminus (E \cup \overline{f(U)})$, where it is quasi regular. But orbits pass through X at most once, since $f_2(X) \subset f(U)$ and points never leave $f(U)$ under iteration of f_2.

These are precisely the hypothesis of Corollary 2.4, so there exists a map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, holomorphic on the whole sphere—and hence rational—, which is conjugate to f_2 by some quasi-conformal homeomorphism ϕ. Only for simplicity, let ψ be the inverse function of such homeomorphism, $\psi := \phi^{-1}$.

Now Theorem 2.1 ensures the existence of a weakly repelling fixed point z_0 of g, except when $\deg g = 1$ and g is an elliptic transformation. However, notice that

$$g(\psi(\widehat{\mathbb{C}} \setminus E)) = \psi(f_2(\widehat{\mathbb{C}} \setminus E)) = \psi(f(U)) \subsetneq \psi(U) \subsetneq \psi(\widehat{\mathbb{C}} \setminus E),$$

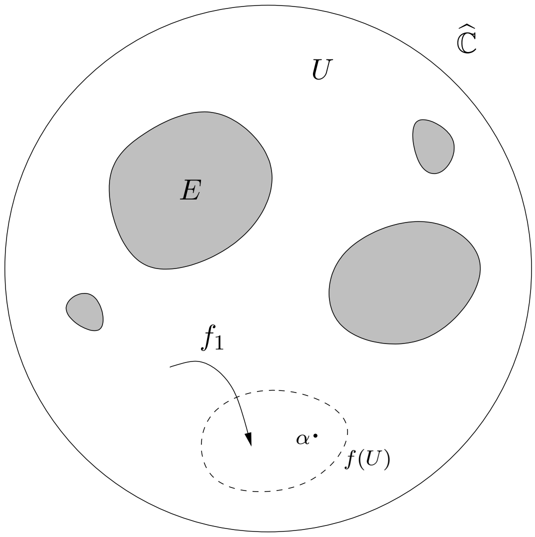

Figure 4: The sets U, $f(U)$ and E on the Riemann sphere. The shaded sets are connected components of $\widehat{\mathbb{C}} \setminus U$.

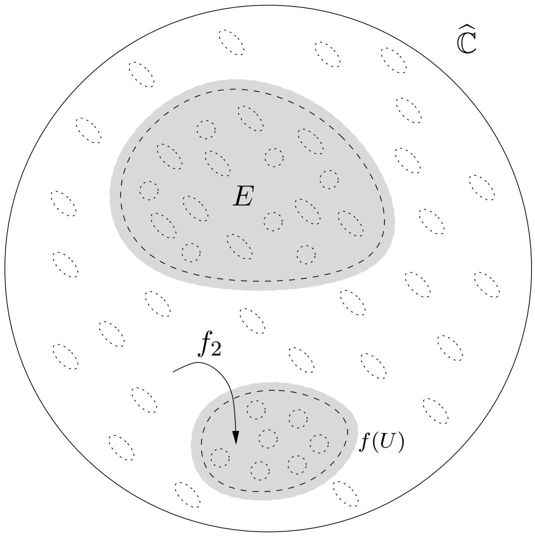

Figure 5: Construction of the almost complex structure σ. Recall that $U = \widehat{\mathbb{C}} \setminus E$. The grey area denotes the region where f_2 is holomorphic.

so g is a contraction and $\psi(a)$ is an attracting fixed point of g; in other words, g can never be an elliptic transformation. Also, observe that $\psi(\widehat{\mathbb{C}} \setminus E)$ is contained in the basin of $\psi(a)$.

Besides, the family $\mathcal{G} = \{g^n_{|\psi(\widehat{\mathbb{C}}\setminus E)}\}_{n\geq 1}$ omits the open set $\psi(X)$, therefore

$\mathcal{G}$ is normal in $\psi(\widehat{\mathbb{C}}\setminus E)$ by Montel's Theorem, that is, $\psi(\widehat{\mathbb{C}}\setminus E) \subset \mathcal{F}(g)$. But weakly repelling fixed points belong to the Julia set, so $z_0 \in \psi(E)$. Because such points are preserved under conjugacy, also f_2 has a weakly repelling fixed point $\phi(z_0)$, in E; and so does f, since both functions coincide precisely on this set (see Figure 6).

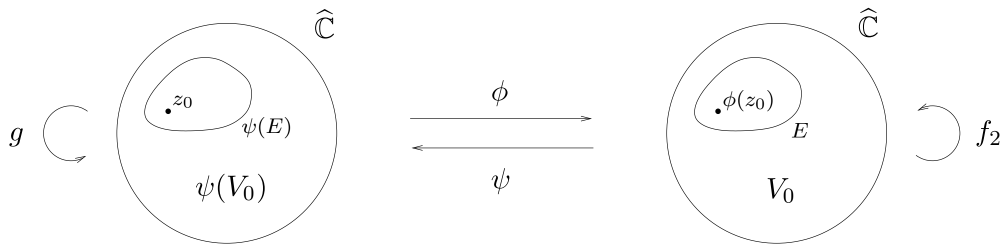

Figure 6: The properties of g (including the existence of a weakly repelling fixed point) are transferred to f_2 due to the conjugacy ϕ. Recall that $V_0 = \widehat{\mathbb{C}} \setminus E$.

The set E was arbitrarily chosen from at least two components of $\widehat{\mathbb{C}}\setminus U$, which means that f has at least two weakly repelling fixed points. This concludes the proof of Theorem 3.3 for fixed immediate attractive basins.

3.2 Periodic basin

In this section, we focus our attention on the case of periodic immediate attractive basins of period greater than one. The surgery process involved here is quite similar to that for fixed immediate attractive basins (see Section 3.1), so we will give the differences in detail and try to abridge the arguments when identical.

Analogously to the fixed case, let $\langle\alpha\rangle$ be the attracting cycle of f contained in the multiply-connected p-periodic immediate attractive basin, $\mathcal{A}^*$, and let $\mathcal{A}^*(\alpha)$ be the connected component of $\mathcal{A}^*$ containing α. Take a small disc neighbourhood U_0 of α such that $\overline{f^p(U_0)} \subset U_0$, and, for each $n \geq 0$, define U_n as the connected component of $f^{-n}(U_0)$ such that $U_n \cap \langle\alpha\rangle \neq \emptyset$.

As before, we can put $\mathcal{A}^*(\alpha)$ as

$$\mathcal{A}^*(\alpha) = \bigcup_{n\geq 0} U_{np}\,,$$

so, in the sequence $\{U_k\}_k$, there is a multiply-connected set U with simply-connected image. Shishikura formalises this statement with the following lemma.

Lemma 3.5. *Let f be a rational map of degree greater than one with a multiply-connected p-periodic immediate attractive basin. Then, there exists a connected open set U, contained in the basin, such that*

- *U is multiply connected and $f(U)$ is simply connected;*
- *U is a connected component of $f^{-1}(f(U))$;*
- *$\overline{f^p(U)} \subset U$.*

Next, let E be one of the connected components of the complement of U. Since $U \subset \mathcal{A}^*$ and $p > 1$, its image $f(U)$ must lie in either E or some other component of $\widehat{\mathbb{C}} \setminus U$. Then, let us assume that $k-1$ iterations of U under f belong to E and precisely the kth iteration lands outside it, with $k \in \mathbb{N}$; that is to say, $f^i(U) \subset E$, for all $0 < i < k$, and $f^k(U) \subset \widehat{\mathbb{C}} \setminus E$. (Notice that this assumption is not restrictive: Since $\overline{f^p(U)} \subset U$, necessarily k must range $0 < k \leq p$.) See Figure 7 for an overview of all possible cases.

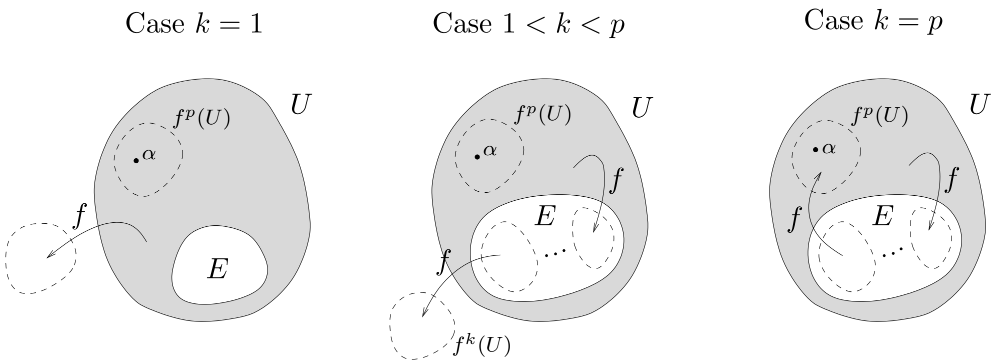

Figure 7: Three possible distributions—according to k—of the most relevant sets of this construction. U is shaded in grey.

In analogy to the fixed case, we will define a quasi-regular map $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that will map $\widehat{\mathbb{C}} \setminus E$ strictly inside itself, this time after k iterations. More precisely, set $V_0 := \widehat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$, which lies in either E (when $k > 1$) or $\widehat{\mathbb{C}} \setminus E$ (when $k = 1$). Set also $K := \overline{f^k(U)}$ and choose $b \in f(U)$ and $a = f^{k-1}(b) \in K$. By the Interpolation Lemma 3.4, there exists a quasi-regular map $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$ which agrees with f on ∂E, is holomorphic in a neighbourhood of K and satisfies $f_1(a) = b$.

Observe that if $k = 1$, then the situation is completely equal to the fixed case (see Figure 8).

From here on we proceed as in Section 3.1, setting $f_2 = f$ on E and $f_2 = f_1$ on $\widehat{\mathbb{C}} \setminus E$. This makes f_2 a quasi-regular map of $\widehat{\mathbb{C}}$, holomorphic in both E and a neighbourhood of $\overline{f^k(U)}$, with a k-periodic point $f_2^k(a) = f^{k-1}(f_1(a)) = f^{k-1}(b) = a$. Observe also that $f_2^k(\widehat{\mathbb{C}} \setminus E) = f^k(U)$ and $\overline{f^k(U)} \subsetneq \widehat{\mathbb{C}} \setminus E$; it follows that f_2^k is a contraction and a a global attractor in $\widehat{\mathbb{C}} \setminus E$.

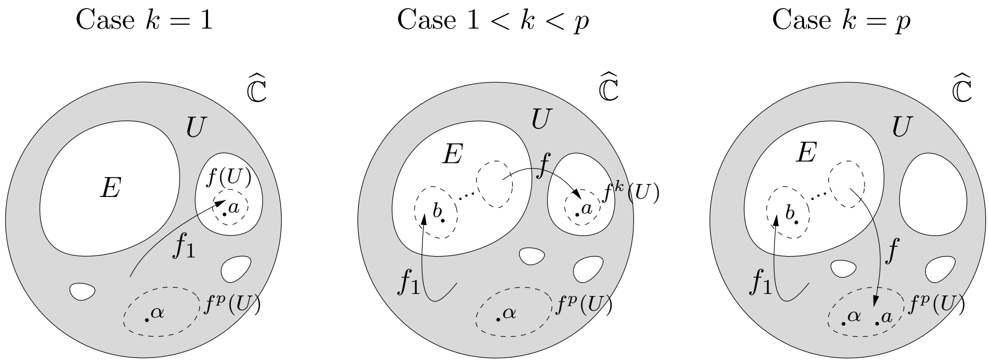

Figure 8: The topological surgery construction for the three possible cases, drawn on $\widehat{\mathbb{C}}$.

As before, we may define an almost complex structure σ by

$$\sigma := \begin{cases} \sigma_0 & \text{on } f(U) \\ (f_2^n)^*\sigma & \text{on } f_2^{-n}(f(U)), \text{ for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

Observe that $\sigma = \sigma_0$ on $\bigcup_{i=1}^k f^i(U)$ (see Figure 9).

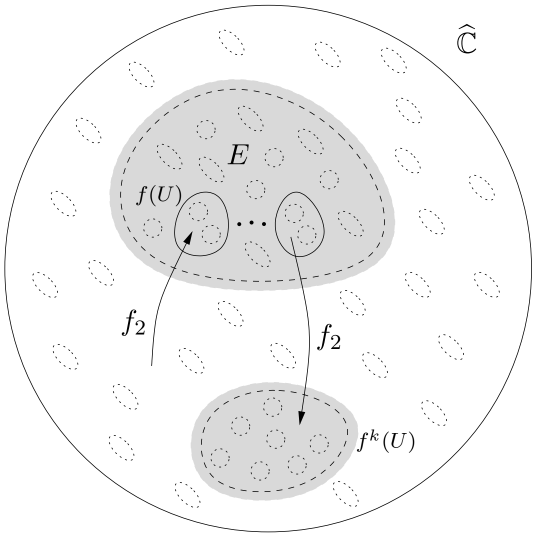

Figure 9: Construction of the almost complex structure σ. In grey we find the region where f_2 is holomorphic.

Furthermore, σ is f_2-invariant by construction and has bounded distortion, since orbits pass through $\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ (the set where f_2 is not holomorphic) at most once.

With this setting—and following the fixed case—, Corollary 2.4 and Theorem 2.1 guarantee the existence of a weakly repelling fixed point of f in E, which is exactly what we wanted to prove.

4 The transcendental case

Shishikura's Theorem 3.1 inspires the analogous result in the trascendental world, that is, our Conjecture 1.1 on connectedness of Julia sets of transcendental meromorphic functions and its relationship to the existence of weakly repelling fixed points.

Following Shishikura, we can use the Classification Theorem 2.2 to individualise the main statement according to Fatou components.

Conjecture 4.1. *Let f be a transcendental meromorphic function. Then,*

- *if f has a multiply-connected immediate attractive or parabolic basin, Baker domain or wandering domain, or*
- *if f has a Herman ring, or*
- *if f has a multiply-connected Fatou component U such that $f(U)$ is simply connected,*

there exists at least one weakly repelling fixed point of f.

Remark. The case of the multiply-connected wandering domain was already proved by Bergweiler and Terglane [4] in a different context, namely, in the search of solutions of certain differential equations with no wandering domains.

Now Theorem 1.3 clearly follows from the cases of the immediate attractive basin and the preperiodic Fatou component, which we shall prove in this chapter. The first two sections contain the proof of the first statement, rewritten as the following theorem, while the preperiodic case can be found in Section 4.3.

Theorem 4.2. *Let f be a transcendental meromorphic function with a multiply-connected p-periodic immediate attractive basin $\mathcal{A}^*$. Then, there exists at least one weakly repelling fixed point of f.*

We use two quite different strategies in order to prove this theorem. The first one is based on Shishikura's surgery construction and applies when either $\mathcal{A}^*$ is bounded, or preimages of a sufficiently small neighbourhood of the attractive point in $\mathcal{A}^*$ do not behave too wildly. The second technique, used in the rest of the cases, involves Buff's Theorem 2.5 on virtually repelling fixed points.

Let us first assume that $\mathcal{A}^*$ is bounded. In this very particular case we can also assume the existence of a connected open set $U \subset \mathcal{A}^*$ such as Lemma 3.5 gives—that is to say, multiply connected and such that $f(U)$ is simply connected, U is a connected component of $f^{-1}(f(U))$ and $f^p(\overline{U}) \subset U$—, since the basin has no accesses to infinity and therefore preimages of compact sets (in the construction of U) keep compact.

We have $U \subset \mathcal{A}^* \subset \mathcal{F}(f)$, so the essential singularity must be contained in the complement $\widehat{\mathbb{C}} \setminus U$. Moreover, since U is multiply connected, there exists at least one connected component E of $\widehat{\mathbb{C}} \setminus U$ which does not contain the singularity. As in the rational (periodic) case (see Section 3.2), we assume that the iterations of U under f do not jump outside E until the kth one, and proceed analogously to find a function f_2 that preserves f on E but has attracting dynamics (interpolation function f_1) on $\widehat{\mathbb{C}} \setminus E$.

Notice that f_2 is indeed quasi regular: On $\widehat{\mathbb{C}} \setminus E$, the map f_1 is quasi regular and infinity is no longer an essential singularity; on E, now f sends the poles to the (non-special) point at infinity—as f is meromorphic, f_2 is holomorphic on E as a map defined on the Riemann sphere—; by definition of f_1, the functions f and f_1 agree on the neighbourhood $V_0 \cap N_1$, so the glueing is continuous.

At this point, the topological step of the surgery process is done. The further holomorphic smoothing and end of the proof goes on exactly as in Section 3.2, therefore f has a weakly repelling fixed point in E.

As for the unbounded case, we cannot apply the previous surgery construction in general, since the existence of asymptotic values and Fatou components with the essential singularity on their boundary can lead to unbounded preimages of bounded sets, while trying to construct U. Instead, we will use this very property to force the situation described in Buff's Theorem 2.5 and, in particular, Corollary 2.6.

So let us assume from now on that $\mathcal{A}^*$ is unbounded. The cases of the fixed basin ($p = 1$) and the (strictly) periodic basin ($p > 1$) are next treated separately.

4.1 Fixed basin

In this case, the immediate attractive basin $\mathcal{A}^*$ consists of a single (fixed) Fatou component. Let $\alpha \in \mathcal{A}^*$ be its one attracting fixed point. We first construct a nested sequence of open sets containing α as follows: Let U_0 be a neighbourhood of α such that $\overline{f(U_0)} \subset U_0$, that is, put $U_0 := \varphi^{-1}(\Delta)$, where φ is the linearisation map of the fixed point α and Δ is a disc in its linearisation coordinates; and define U_n as the connected component of $f^{-n}(U_0)$ that contains α, for all $n \in \mathbb{N}$. Notice that $U_0 \subset U_1 \subset \dots$ because of the choice of the initial neighbourhood U_0.

Since $\mathcal{A}^*$ is multiply connected, there exists $n_0 \in \mathbb{N}$ such that $U_0, \dots, U_{n_0-1}$ are simply connected and U_{n_0} is multiply connected. This implies that the complement of U_{n_0} have at least one bounded connected component, since its fundamental group is $\pi_1(U_{n_0}) \neq \{0\}$. In view of this, let E be one of the bounded connected components of $\widehat{\mathbb{C}} \setminus U_{n_0}$ (see Figure 10).

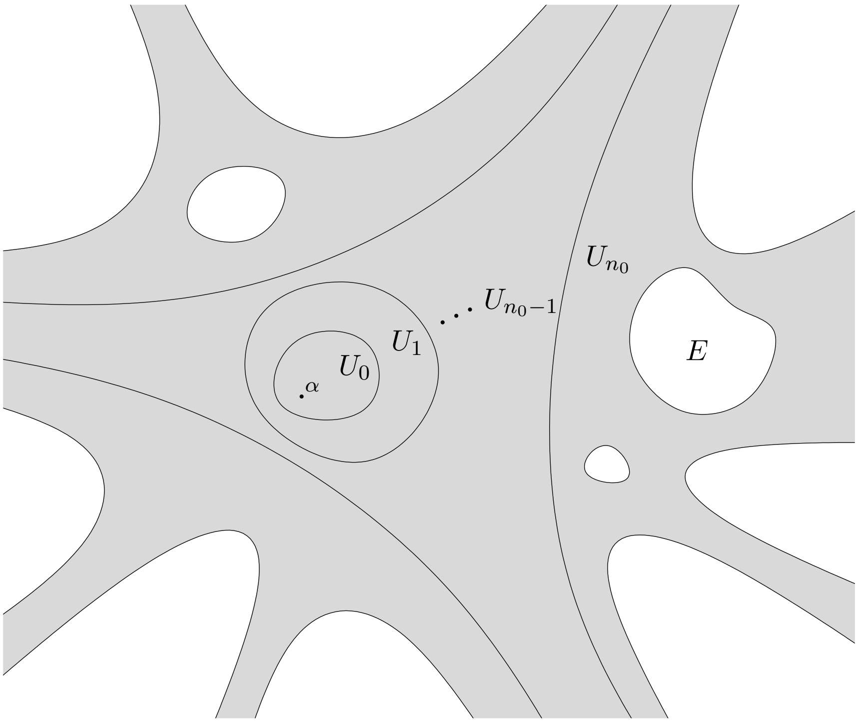

Figure 10: The sequence $\{U_k\}_k$ and the bounded set E. In grey, the multiply-connected set U_{n_0}.

As Figure 10 suggests, at some point the sets $\{U_k\}_k$ might become unbounded, so further preimages of such sets could have poles and prepoles on their boundaries. The actual condition for this fact to happen can be written in terms of the intersection set $\partial E \cap \mathcal{J}(f)$ and is specified in the following lemma.

Lemma 4.3. *Let f be a transcendental meromorphic function with an unbounded multiply-connected fixed immediate attractive basin $\mathcal{A}^*$, and let $\{U_k\}_{k=0}^{n_0}$ and E be as above. Then, the following are equivalent:*

(1). $U_0, \dots, U_{n_0-1}$ are all bounded;

(2). $\partial E \cap \mathcal{J}(f) = \emptyset$;

(3). ∂E contains no poles.

Proof. Let us first see how *(1)* implies *(2)*. The boundaries of $U_0, \ldots, U_{n_0-1}$ belong to the Fatou set and are bounded. Since ∂E is mapped onto ∂U_{n_0-1}, it follows that $\partial E \cap \mathcal{J}(f) = \emptyset$. Statement *(2)* trivially gives *(3)*. For *(3)* implies *(1)*, suppose there exists $k \in \mathbb{N}$, with $0 < k < n_0$, such that U_k is unbounded. Since this is an increasing sequence, $U_k, U_{k+1}, \ldots$ are all unbounded and in particular so is U_{n_0-1}. But $\partial U_{n_0-1} \subset f(\partial E)$, because U_{n_0-1} is simply connected, and the set E is bounded. Then ∂E must contain at least one pole, which contradicts *(3)*. $\square$

Therefore, in the case where ∂E never meets $\mathcal{J}(f)$, the set U_{n_0} can be renamed U and we have the following situation: U is multiply connected and $f(U) = f(U_{n_0}) = U_{n_0-1}$ is simply connected; U is a connected component of $f^{-1}(f(U)) = f^{-1}(U_{n_0-1})$, by definition; $f(\overline{U}) \subset \overline{U_{n_0-1}} \subset U_{n_0} = U$, since U_{n_0-1} is bounded and U open. Now this situation is but the setting we had in the case of $\mathcal{A}^*$ bounded, with $p = 1$ (see Figure 11). Surgery can thus be applied in the same fashion (see Section 3.1) to obtain a quasi-regular map that send $\widehat{\mathbb{C}} \setminus E$ to U_{n_0-1} and equal f on E. Observe that the essential singularity is no longer there and, therefore, the holomorphic map that we obtain from the surgery procedure is a rational map. This gives the desired weakly repelling fixed point in E.

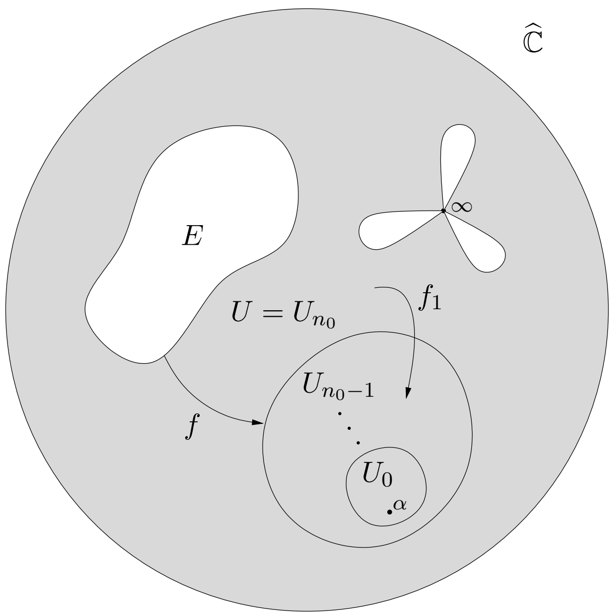

Figure 11: Sketch of the case where ∂E never meets the Julia set, on the Riemann sphere. The shaded set represents U. Surgery can be applied as in the case where $\mathcal{A}^*$ is bounded and $p = 1$; compare with Figure 4.

A different case is the situation where ∂E does intersect $\mathcal{J}(f)$. Lemma

4.3 asserts the existence of at least one pole P in ∂E. From now on, this is the situation we deal with.

As mentioned, in this case we no longer use quasi-conformal surgery, but Buff's Theorem 2.5—in other words, we want to find an open subset of $\widehat{\mathbb{C}}$ that contains a preimage of itself and whose boundary does not share fixed points with the boundary of such preimage. (We shall see it suffices that infinity not be on the preimage's boundary.)

Let us first construct a (shrinking) nested sequence of sets, in the complement of the open sets $\{U_k\}_k$, by defining V_n to be the connected component of $\widehat{\mathbb{C}} \setminus U_n$ that contains E, for all $0 \leq n \leq n_0$. Notice that the closed sets $V_0, \ldots, V_{n_0-1}$ are all unbounded, for U_{n_0} is the first multiply-connected set of its sequence, and $V_{n_0} = E$ is bounded by definition. Notice also that this component containing E is simply connected (since U_n is connected) and indeed unique, and that $V_0 \supset V_1 \supset \ldots \supset V_{n_0} = E$, since $U_0 \subset U_1 \subset \ldots$ and all the $\{V_k\}_k$ must contain E (see Figure 12).

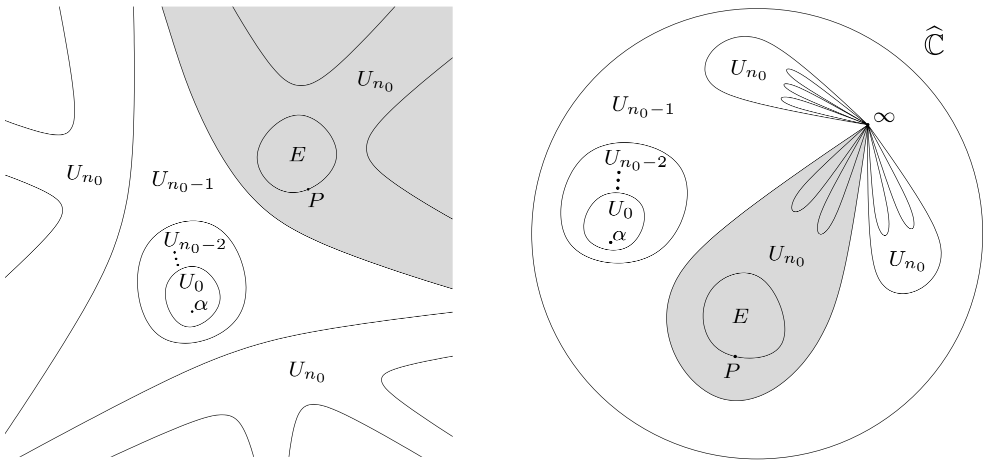

Figure 12: The increasing sequence of open sets $\{U_k\}_k$ and the decreasing one $\{V_k\}_k$. In this example, U_{n_0-1} is the first unbounded set in the sequence and, consequently, $V_n = \widehat{\mathbb{C}} \setminus U_n$ for all $n < n_0 - 1$. The shaded set corresponds to V_{n_0-1}, while $V_{n_0} = E$. The same situation has been drawn on the plane and on the Riemann sphere.

From Lemma 4.3 and from the fact that U_0 is bounded, there exists $n_1 \in \mathbb{N}$, with $0 < n_1 < n_0$, such that $U_0, \ldots, U_{n_1-1}$ are bounded and $U_{n_1}, U_{n_1+1}, \ldots$ are unbounded. Moreover, since the preimage of an unbounded set may contain poles on its boundary, we can assume there exists $n_2 \in \mathbb{N}$, with $0 < n_1 < n_2 \leq n_0$, such that $P \notin \partial V_0, \ldots, \partial V_{n_2-1}$ and $P \in \partial V_{n_2}$. The following lemma shows that, in this case, $P \in \partial V_n$ for all $n_2 \leq n \leq n_0$.

Lemma 4.4. *Suppose there exists $k < n_0$ such that $P \in \partial V_k$. Then, $P \in$*

∂V_j, *for all* $k \le j \le n_0$.

Proof. It is clear that $P \in \partial V_{n_0}$, given that $E = V_{n_0}$. Now, suppose there exists $k < j < n_0$ such that $P \notin \partial V_j$.

By definition, $E \subset V_j$ and therefore $P \in \mathring{V}_j$. However, on the other hand, since $V_j \subset \widehat{\mathbb{C}} \setminus U_j$, we have that $U_k \subset U_j \subset \widehat{\mathbb{C}} \setminus V_j$. It follows that $\overline{U_k} \subset \overline{U_j} \subset \overline{\widehat{\mathbb{C}} \setminus V_j}$ and hence $P \in \overline{\widehat{\mathbb{C}} \setminus V_j}$, given that $P \in \partial U_k$. But we assumed that $P \notin \partial V_j$, so we deduce that $P \in \mathrm{int}(\widehat{\mathbb{C}} \setminus V_j)$. This contradicts the fact that $P \in \mathring{V}_j$. □

If $n_2 = n_0$, the first set V_k which contains P on its boundary is E itself (see Figure 13). As V_{n_0-1} is unbounded, there exists some connected component X of $f^{-1}(V_{n_0-1})$ such that $P \in \partial X$. Furthermore, the preimage X must be contained in E, since points immediately outside E belong to U_{n_0} (whose image under f is U_{n_0-1}), and hence cannot be preimage of points in $V_{n_0-1} \subset \widehat{\mathbb{C}} \setminus U_{n_0-1}$. Of course the boundaries ∂V_{n_0-1} and ∂X do not have any common fixed point because $|f(z) - z|$ is bounded away from zero as $z \in X$ tends to ∂X, so the map $f\colon X \to V_{n_0-1}$ satisfies the hypothesis of Corollary 2.6 and therefore f has at least one weakly repelling fixed point.

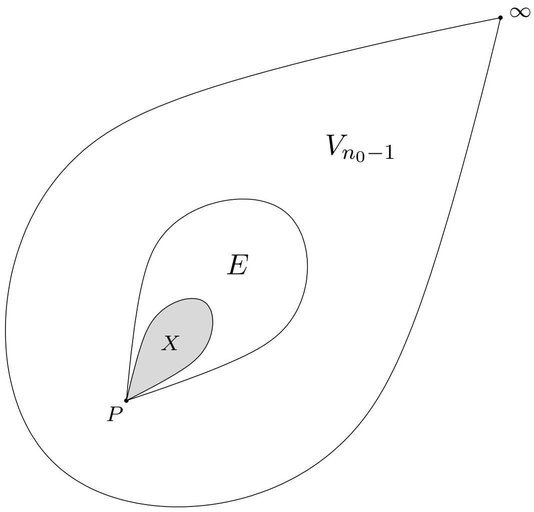

Figure 13: The situation where $n_2 = n_0$, i.e., the first set V_k that contains the pole P on its boundary is $V_{n_0} = E$ itself. Then, a preimage X of V_{n_0-1} must exist in E.

The most general case is that where $0 < n_1 < n_2 < n_0$. One example of this situation is given by Figure 14, namely when $n_2 = n_1+1$ and $n_0 = n_2+2$.

Observe that, in this case, the interior of the sets $\{V_k\}_k$ with $k \ge n_2$ might have more than one connected component (as shown in the example of Figure 14). In order to avoid this, in our setting we define yet another

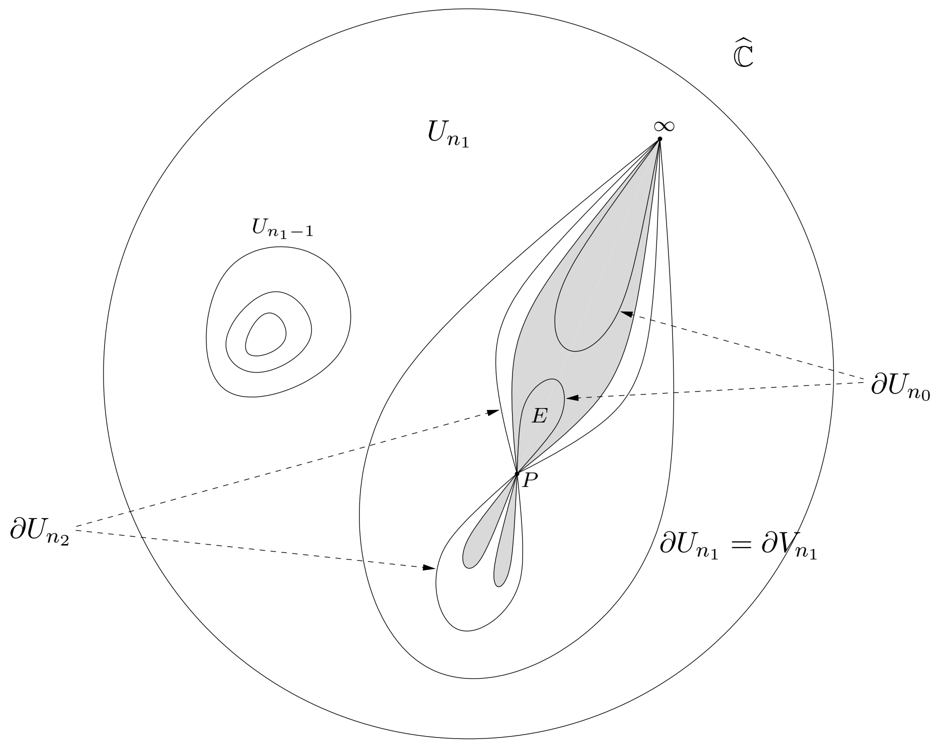

Figure 14: A possible distribution of the sets $U_1, \ldots, U_{n_0}$, with $0 < n_1 < n_2 < n_0$, and more precisely $n_2 = n_1 + 1$ and $n_0 = n_2 + 2$. To simplify, the sets $\{U_k\}_k$ have been drawn only with one access to infinity. Observe that $\mathring{V}_{n_2}$ and $\mathring{V}_{n_2+1}$ have two and three connected components, respectively. The shaded area represents V_{n_2+1}.

sequence $\{W_k\}_k$, where each W_n is the unbounded connected component of $\mathring{V}_n$, for all $n_2 \leq n < n_0$. Notice that such an unbounded component must be indeed unique, since the sets $\{V_k\}_k$ are all simply connected (see Figure 15).

With these tools, our proof will continue as follows: For every $n_2 \leq n < n_0$, we will first consider the preimage sets of W_n attached to P. If any connected component of $f^{-1}(W_n)$ happens to be bounded, then Buff's theorem can be applied and the proof will finish, as we will show in Lemma 4.5. But if all of them were unbounded, then it is clear both W_n and each of its preimages would have infinity as a fixed point (of the restricted map) on their boundaries, contradicting the hypotheses of Corollary 2.6. In this case we will jump to the next step and repeat the procedure with W_{n+1}. We will now make this argument precise.

As boundedness of preimages plays quite an important role, for clarity's sake we define for $n_2 \leq n < n_0$ the families of sets

$$\mathcal{X}_n := \{X \subset \widehat{\mathbb{C}} \text{ bounded connected component of } f^{-1}(W_n) : P \in \partial X\}.$$

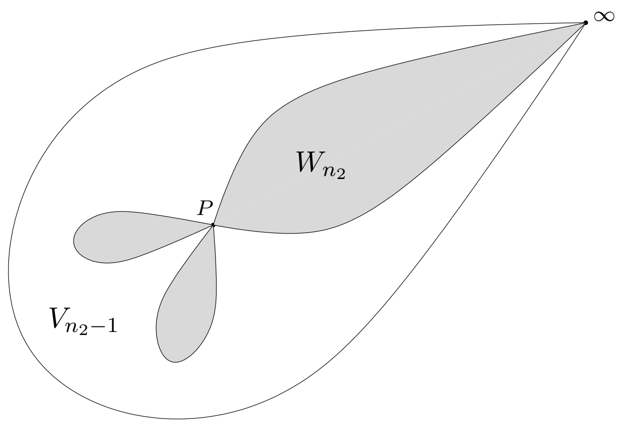

Figure 15: The open set W_{n_2} is the unbounded component of the interior of the (shaded) set V_{n_2}.

In other words, $\mathcal{X}_n$ is the set of bounded connected components of $f^{-1}(W_n)$ with P on their boundary. Now the following lemma proves the key point of our iterative process.

Lemma 4.5. *Fix $n^* \in \mathbb{N}$ such that $n_2 \leq n^* < n_0$ and suppose $\mathcal{X}_n = \emptyset$, for all $n_2 \leq n < n^*$, but $\mathcal{X}_{n^*} \neq \emptyset$. Then, there exists at least one weakly repelling fixed point of f.*

Proof. Let $X \in \mathcal{X}_{n^*}$. It is clear that $X \subset V_{n^*+1} \subset V_{n^*} \subset V_{n^*-1}$, where the first inclusion follows from the fact that $V_{n^*} \setminus V_{n^*+1} \subset U_{n^*+1}$ and its points never fall in W_{n^*} under iteration of f (see Figure 16). If $X \subset W_{n^*}$, then the map $f\colon X \to W_{n^*}$ satisfies the hypothesis of Corollary 2.6, which provides a weakly repelling fixed point of f. Otherwise, X is contained in one of the bounded components B of $\mathring{V}_{n^*}$ (see Figure 17). Consider preimages of W_{n^*-1}, that is to say, connected components of $f^{-1}(W_{n^*-1})$; since $W_{n^*} \subset W_{n^*-1}$, there exists a preimage Y of W_{n^*-1} such that $X \subset Y$. But also $Y \subset V_{n^*}$ (for the same reason that $X \subset V_{n^*+1}$), which means that $Y \subset B$ by continuity. This makes Y bounded, since so is B, therefore $Y \in \mathcal{X}_{n^*-1}$ and $\mathcal{X}_{n^*-1} \neq \emptyset$, contradicting our initial assumption. $\square$

Using this result, the end of the proof becomes straightforward: For every $n \in \mathbb{N}$ such that $n_2 \leq n < n_0$, check whether $\mathcal{X}_n \neq \emptyset$. As it turns out, the last family of sets of the sequence $\{\mathcal{X}_k\}_k$ always has this property, $\mathcal{X}_{n_0-1} \neq \emptyset$, since preimages of W_{n_0-1} with P on their boundary lie in $V_{n_0} = E$, which is bounded by definition. Therefore, take the smallest n for which $\mathcal{X}_n \neq \emptyset$ holds, and Lemma 4.5 gives a weakly repelling fixed point of f.

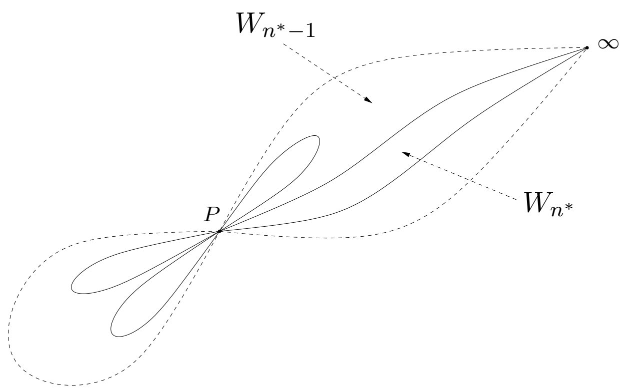

Figure 16: A bounded preimage X of W_{n^*-1} containing P on its boundary must be always in W_{n^*} and hence in W_{n^*-1}. Buff's theorem gives then a weakly repelling fixed point. Here the dashed lines represent V_{n^*-1}, while the continuous ones correspond to V_{n^*}.

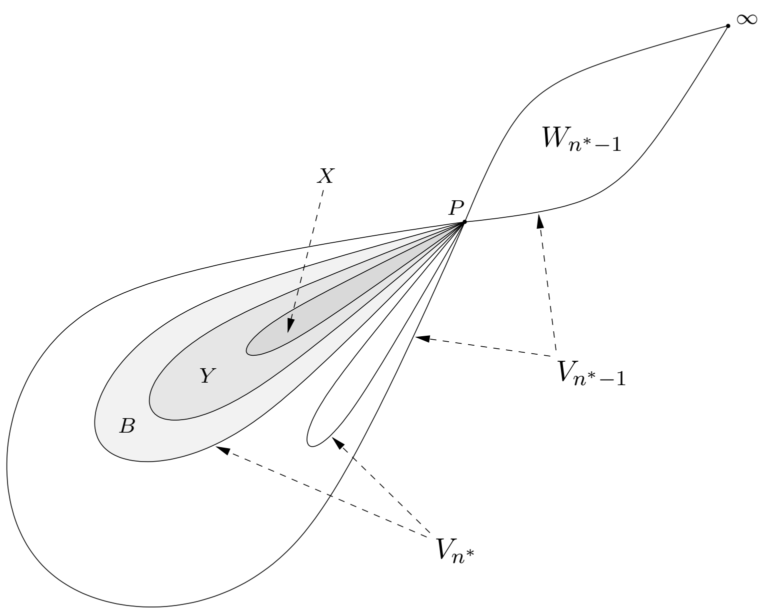

Figure 17: In the situation where X lies in one of the bounded components B of $\mathring{V}_{n^*}$, there exists a preimage Y of W_{n^*-1} such that $X \subset Y \subset B$.

4.2 Periodic basin

This case begins with the same setting as the fixed basin, although it soon becomes much simpler. Let $\mathcal{A}^*$ be the multiply-connected p-periodic immediate attractive basin of f and $\langle\alpha\rangle \subset \mathcal{A}^*$ be its attracting p-periodic cycle. As before, we define U_0 to be a suitable neighbourhood of α, so that $f^p(U_0) \subset U_0$, and U_n as the connected component of $f^{-n}(U_0)$ that intersects $\langle\alpha\rangle$, for all $n \in \mathbb{N}$. Analogously to the fixed case, we have that $U_l \subset U_{p+l} \subset U_{2p+l} \subset \ldots$, for all $0 \leq l < p$.

Again, there exists $n_0 \in \mathbb{N}$ such that $U_0, \ldots, U_{n_0-1}$ are simply connected and U_{n_0} is multiply connected, for so is $\mathcal{A}^*$. Call $U = U_{n_0}$ and let E be one of the bounded connected components of $\widehat{\mathbb{C}} \setminus U$ (see Figure 18).

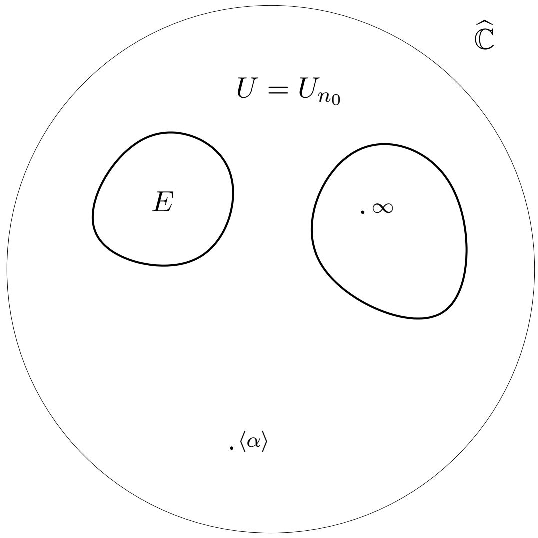

Figure 18: U is a multiply-connected subset of $\mathcal{A}^*$ such that $f(U)$ is simply connected. If U were unbounded, the point at infinity would be in $\partial(\widehat{\mathbb{C}} \setminus (U \cup E))$ (see Figure 19).

Remark. Notice the impossibility to use Lemma 4.3 to separate the different cases, as we did in the previous section. Indeed, in this periodic case the sequence $\{U_k\}_k$ is no longer nested so our proof cannot be extended beyond fixed basins.

When ∂E has no poles—analogously to the previous case—we will apply the periodic-case surgery described in Section 3.2 to find a weakly repelling fixed point of f. First notice the curve $f(\partial E)$ is bounded, since ∂E is bounded by definition and has no poles by hypothesis. It follows that $f(\partial E) = \partial U_{n_0-1}$, because $f(\partial E)$ is at least one of its connected components and U_{n_0-1} is simply connected. We conclude that U_{n_0-1} must be bounded, since so is $f(\partial E)$.

Now this means we can use the Interpolation Lemma 3.4 to obtain a quasi-regular map $f_1 \colon \widehat{\mathbb{C}} \setminus E \to U_{n_0-1} = f(U)$, as in the previous cases, and the surgery process goes on and finishes as it did in the rational periodic case.

When ∂E does contain a pole P, the image $f(U)$ must be unbounded and, therefore, contained in one of the unbounded connected components of $\widehat{\mathbb{C}} \setminus U$. Consider a simply-connected, unbounded, closed set $V \subset \widehat{\mathbb{C}}$, containing U but not its image $f(U)$ (see Figure 19)—this is always possible because we are in the case $p > 1$. Notice that also $E \subset V$ by construction of V (which is simply connected) and boundedness of E. Now there exists a preimage D of V, with $P \in \partial D$, and $D \subset E$ since points immediately outside E are in U and thus mapped to $f(U) \subset \widehat{\mathbb{C}} \setminus V$. Moreover, we have $D \subset E \subsetneq V$, so $\partial D \cap \partial V = \emptyset$ and Corollary 2.6 gives a weakly repelling fixed point of f.

This step concludes the periodic immediate attractive case and, with it, the proof of Theorem 4.2.

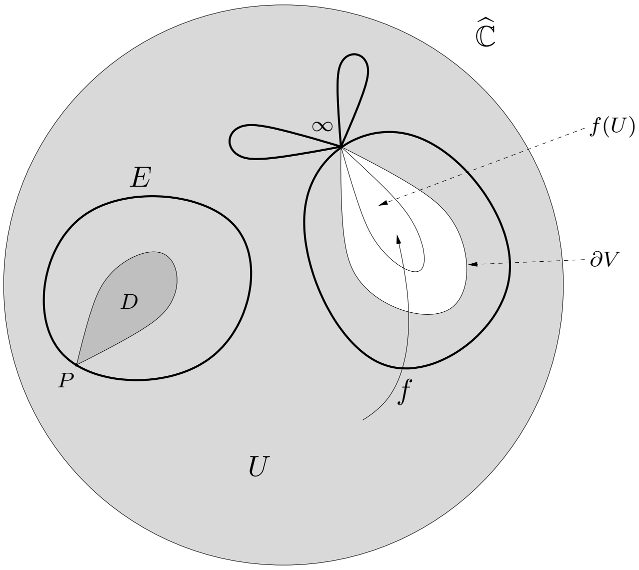

Figure 19: If there exists a pole P on ∂E, then there exists a set $D \subset E$ such that $f(D) = V$, where V is an unbounded simply-connected set that contains U but not $f(U)$. The thick lines correspond to ∂U, while the sets D and V appear dark- and light-shaded, respectively.

4.3 Preperiodic Fatou components

Recall that our main goal in this paper is to prove Theorem 1.3, as stated earlier in the introduction, and so far we have just closed one of its natural subcases, i.e., the immediate attractive basin. Notice, though, that our proof became specially laborious in those situations where we were unable to apply quasi-conformal surgery techniques, in other words, when we could not find a multiply-connected open set with simply-connected image.

However, the case we will deal with in this section starts exactly with and is actually defined by this very hypothesis, so it is no surprise that the preperiodic case shall be proved using only surgery—in fact, using surgery in a fashion very similar to that of Shishikura's for the rational (preperiodic) case. We want to prove the following.

Theorem 4.6. *Let f be a transcendental meromorphic function with a multiply-connected (strictly preperiodic) Fatou component U such that $f(U)$ is simply connected. Then, there exists at least one weakly repelling fixed point of f.*

It is clear that U is a connected component of $f^{-1}(f(U))$, since U is a Fatou component itself. Let E be one of the bounded components of $\widehat{\mathbb{C}} \setminus U$ (one such component always exists because U is multiply connected).

In analogy to the rational case, let us focus our attention on the sequence of iterations $\{f^k(U)\}_{k\in\mathbb{N}}$. Notice that, in the preperiodic case, such iterations will not necessarily eventually abandon E because they will never come back to U. This fact gives raise to two quite different situations, depicted in Figure 20.

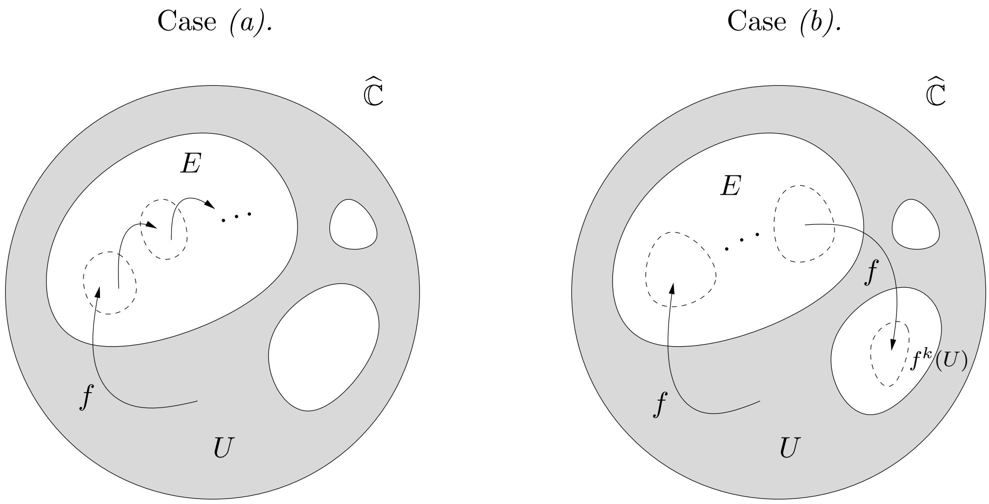

Figure 20: The two possible situations. In *(a)*, the iterations of U always stay in E, $f^k(U) \subset E$ for all $k \in \mathbb{N}$; whereas in *(b)*, there exists $k \in \mathbb{N}$ such that $f^i(U) \subset E$ for all $0 < i < k$ and $f^k(U) \subset \widehat{\mathbb{C}} \setminus E$.

Notice that Case *(b)* is exactly the situation we already treated in the attractive case, so an analogous procedure gives a global quasi-regular map f_2, with its conjugate rational function g, plus the subsequent weakly repelling fixed point of f in E.

For Case *(a)* we define a quasi-regular map $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in exactly the same way, i.e., via $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$. However, in this case we define our

f_2-invariant almost complex structure as

$$\sigma := \begin{cases} \sigma_0 & \text{on } f^n(U), \text{ for } n \in \mathbb{N} \\ f_2^*\sigma_0 & \text{on } \widehat{\mathbb{C}} \setminus E \\ (f_2^n)^*\sigma_0 & \text{on } f_2^{-n}(\widehat{\mathbb{C}} \setminus E), \text{ for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

(See Figure 21.)

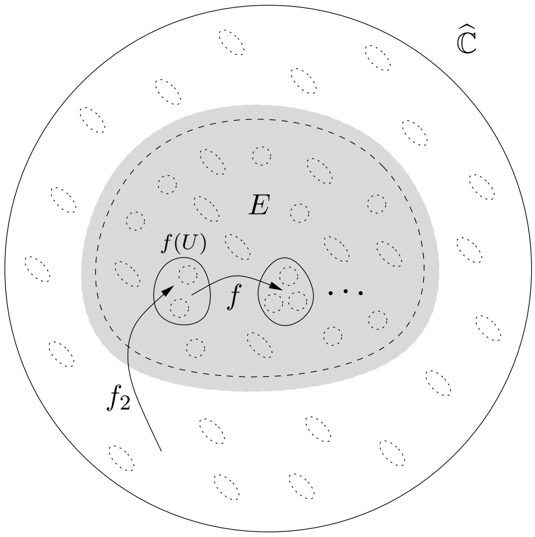

Figure 21: The new almost complex structure σ.

Therefore, we have that $f_2^*\sigma = \sigma$ almost everywhere, by construction, and that σ has bounded ellipticity, since f_2 is holomorphic everywhere except in $\widehat{\mathbb{C}}\setminus E$, where it is quasi regular but orbits clearly pass at most once through.

As usual, a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conjugate to f_2 is obtained from Corollary 2.4 and f inherits from it a weakly repelling fixed point in E.

References

[1] Lars Ahlfors and Lipman Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.

[2] I. Noel Baker, Janina Kotus, and Lü Yinian, *Iterates of meromorphic functions. III. Preperiodic domains*, Ergodic Theory Dynam. Systems **11** (1991), no. 4, 603–618.

[3] Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188.

[4] Walter BERGWEILER and Norbert TERGLANE, *Weakly repelling fixpoints and the connectivity of wandering domains*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 1–12.

[5] Xavier BUFF, *Virtually repelling fixed points*, Publ. Mat. **47** (2003), no. 1, 195–209.

[6] James H. CURRY, Lucy GARNETT, and Dennis SULLIVAN, *On the iteration of a rational function: computer experiments with Newton's method*, Comm. Math. Phys. **91** (1983), no. 2, 267–277.

[7] Adrien DOUADY and Xavier BUFF, *Le théorème d'intégrabilité des structures presque complexes*, The Mandelbrot set, theme and variations, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, Cambridge, 2000, pp. 307–324.

[8] Adrien DOUADY and John H. HUBBARD, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 2, 287–343.

[9] Alexandre È. ERËMENKO and Mikhail Yu. LYUBICH, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

[10] Pierre FATOU, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271, and **48** (1920), 33–94 and 208–314.

[11] Lisa R. GOLDBERG and Linda KEEN, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynam. Systems **6** (1986), no. 2, 183–192.

[12] Sebastian MAYER and Dierk SCHLEICHER, *Immediate and virtual basins of Newton's method for entire functions*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 2, 325–336.

[13] Hans-Günter MEIER, *On the connectedness of the Julia-set for rational functions*, preprint no. 4 RWTH Aachen, 1989.

[14] John MILNOR, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, Introductory lectures.

[15] Feliks PRZYTYCKI, *Remarks on the simple connectedness of basins of sinks for iterations of rational maps*, Dynamical systems and ergodic theory (Warsaw, 1986), Banach Center Publ., vol. 23, PWN, Warsaw, 1989, pp. 229–235.

[16] Johannes RÜCKERT and Dierk SCHLEICHER, *On Newton's method for entire functions*, preprint arXiv math.DS/0505652, 2005.

[17] Mitsuhiro Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 1–29.

[18] ———, *The connectivity of the Julia set and fixed points*, preprint Inst. Hautes Études Sci. M/90/37, 1990.

[19] Dennis Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), no. 3, 401–418.

[20] Tan Lei, *Branched coverings and cubic Newton maps*, Fund. Math. **154** (1997), no. 3, 207–260.