HERMAN RINGS AND ARNOLD DISKS.

XAVIER BUFF, NÚRIA FAGELLA, LUKAS GEYER, AND CHRISTIAN HENRIKSEN

Abstract. For $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$, let $f_{\lambda,a}$ be the rational map defined by

$$f_{\lambda,a}(z) = \lambda z^2 \frac{az+1}{z+a}.$$

If $\alpha \in \mathbb{R}/\mathbb{Z}$ is a Bruno number, we let $\mathcal{D}_\alpha$ be the set of parameters (λ, a) such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α (we consider that $(\mathrm{e}^{2i\pi\alpha}, 0) \in \mathcal{D}_\alpha$). The results obtained in [McS] imply that for any $g \in \mathcal{D}_\alpha$ the connected component of $\mathcal{D}_\alpha \cap (\mathbb{C}^* \times (\mathbb{C}\setminus\{0,1\}))$ which contains g is isomorphic to a punctured disk.

In this article, we show that there is an isomorphism $\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha$ such that

$$\mathcal{F}_\alpha(0) = (\mathrm{e}^{2i\pi\alpha}, 0) \quad \text{and} \quad \mathcal{F}'_\alpha(0) = (0, r_\alpha),$$

where r_α is the conformal radius at 0 of the Siegel disk of the quadratic polynomial $z \mapsto \mathrm{e}^{2i\pi\alpha} z(1+z)$. In particular, $\mathcal{D}_\alpha$ is a Riemann surface isomorphic to the unit disk.

As a consequence, we show that for $a \in (0, 1/3)$, if $f_{\lambda,a}$ has a fixed Herman ring with rotation number α and if m_a is the modulus of the Herman ring, then, as $a \to 0$, we have

$$\mathrm{e}^{\pi m_a} = \frac{r_\alpha}{a} + \mathcal{O}(a).$$

We finally explain how to adapt the results to the complex standard family $z \mapsto \lambda z \mathrm{e}^{\frac{a}{2}(z-1/z)}$.

1. Introduction.

In this article, we are mainly concerned with the dynamics of rational maps of the form

$$f_{\lambda,a}(z) = \lambda z^2 \frac{az+1}{z+a}, \quad \lambda \in \mathbb{C}^*, \ a \in \mathbb{C}.$$

Note that $f_{\lambda,a}$ is conjugate to $f_{\lambda,-a}$ via the conjugacy $z \mapsto -z$. If $\lambda \in S^1$ and $a = 0$, the map $f_{\lambda,a}$ is the rotation $z \mapsto \lambda z$. Observe that when a is real and $|\lambda| = 1$, the map $f_{\lambda,a}$ is a Blaschke fraction $z \mapsto \lambda z^2 \frac{z+b}{1+\bar{b}z}$ with $b = 1/a$. But as opposed to families of Blaschke fractions which only depend $\mathbb{R}$-analytically on parameters, our family depends $\mathbb{C}$-analytically on the parameters λ and a and is, in some sense, the simplest one that exhibits families of Herman rings.

For all $\alpha \in \mathbb{R}/\mathbb{Z}$ we denote by $\mathcal{R}_\alpha$ the rigid rotation of the complex plane: $\mathcal{R}_\alpha(z) = \mathrm{e}^{2i\pi\alpha} z$. When $\lambda \in S^1$ and $a \in (-1/3, 1/3)$, the map $f_{\lambda,a}$ restricts to a diffeomorphism of S^1 which has a rotation number $\rho(\lambda, a) \in \mathbb{R}/\mathbb{Z}$. Given $a \in (-1/3, 1/3)$, the function $t \mapsto \rho(\mathrm{e}^{2i\pi t}, a)$ is continuous and weakly increasing [P]. Moreover, for each fixed $a \in (-1/3, 1/3)$ and for each irrational number α, there is a unique angle $t \in \mathbb{R}/\mathbb{Z}$ such that $\rho(\mathrm{e}^{2i\pi t}, a) = \alpha$ (see e.g. [dMvS]). By a theorem of Denjoy [D], when $\alpha = \rho(\lambda, a)$ is irrational, $f_{\lambda,a} : S^1 \to S^1$ is topologically conjugate to the rotation $\mathcal{R}_\alpha : S^1 \to S^1$.

Figure 1 shows some of the level sets:

$$\mathcal{T}_\alpha = \{(t, a) \in \mathbb{R}/\mathbb{Z} \times [0, 1/3] \mid \rho(\mathrm{e}^{2i\pi t}, a) = \alpha\}.$$

Those sets are called *Arnold tongues* and they intersect the line $\{a = 0\}$ at the point $t = \alpha$. If α is a

Date: February 27, 2004.
1991 *Mathematics Subject Classification.* 37F10,30D20 (how to make 2000???)
GRANTS HERE.

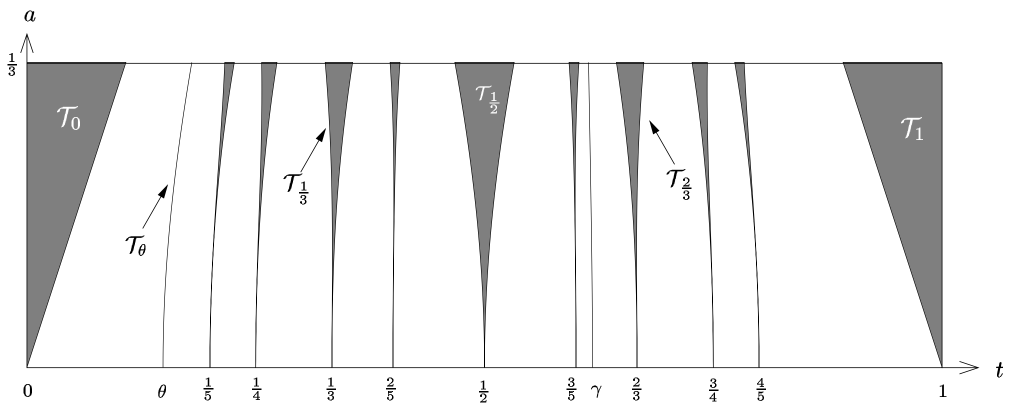

FIGURE 1. Rational Arnold tongues in the parameter space of the family $f_{\lambda,a}$ for $\lambda = e^{2\pi i t}$, $t \in \mathbb{R}/\mathbb{Z}$, up to denominator 5. Irrational tongues for $\gamma = \frac{\sqrt{5}-1}{2}$ and $\theta = \sqrt[5]{2} - 1$.

rational number, the Arnold tongue $\mathcal{T}_\alpha$ has interior and if α is irrational, the Arnold tongue $\mathcal{T}_\alpha$ is a Lipschitz curve [A].

If $\alpha \in \mathbb{R}/\mathbb{Z}$ is a Bruno number and if (t, a) belongs to $\mathcal{T}_\alpha$ with a sufficiently close to 0, the restriction of $f_{e^{2i\pi t},a}$ to S^1 is $\mathbb{R}$-analytically conjugate to the rotation of angle α [Br]. This conjugacy extends to a conjugacy in a neighborhood of S^1 and so, $f_{e^{2i\pi t},a}$ has a fixed Herman ring. Figure 2 shows the example of such a Herman ring for $a = 1/4$ and $t = 0.61517321588\ldots$.

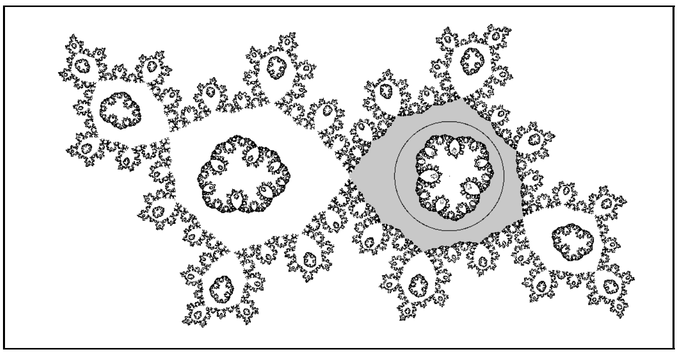

FIGURE 2. For $t = 0.61517321588\ldots$, the rational map $f_{e^{2i\pi t},1/4}$ leaves the circle $\mathbb{S}^1$ invariant and has a Herman ring.

Even in the case where $\lambda \notin S^1$ and $a \notin (0, 1/3)$, the rational map $f_{\lambda,a}$ may have a fixed Herman ring. However, the unit circle is no longer invariant, and it is more difficult to locate parameters for which one can find a Herman ring.

It is known that in this particular family $f_{\lambda,a}$, there is a Herman ring if and only if the rotation number is a Bruno number (see e.g. [Sh1] or [H]). This result is proved using a surgery construction due to Shishikura, and the optimality of the Bruno condition for the existence of Siegel disks in the family of quadratic polynomials proved by Bruno [Br] and Yoccoz [Y]. More precisely, it uses the fact that the quadratic polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha} z(1 + z)$ is linearizable at 0 if and only if α is a Bruno number. Figure 3 shows the Siegel disk of the quadratic polynomial P_α for $\alpha = (\sqrt{5} - 1)/2$.

The goal of this article is to study the set of complex parameters (λ, a) for which $f_{\lambda,a}$ has a fixed Herman ring with a given rotation number α. (If $\lambda = e^{2i\pi\alpha}$ and $a = 0$, we consider that there is a Herman ring which is equal to $\mathbb{C}^*$, and so, has infinite modulus.)

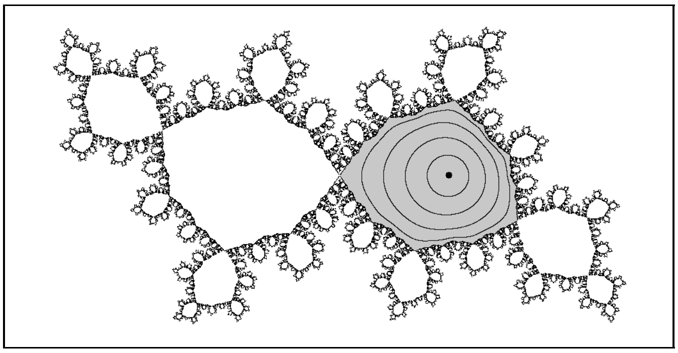

FIGURE 3. The quadratic polynomial P_α with $\alpha = (\sqrt{5}-1)/2$ has a Siegel disk. We have drawn the orbits of some points in the Siegel disk. Each orbit accumulates on an $\mathbb{R}$-analytic circle.

Definition 1.1. Given a Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, we let Δ_α be the Siegel disk of the quadratic polynomial P_α and we let $\mathcal{D}_\alpha$ be the set of parameters $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$ such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α. We shall call $\mathcal{D}_\alpha$ the *Arnold disk of rotation number* α.

In some sense, this set is the complexification of the Arnold tongue $\mathcal{T}_\alpha$ and it was studied for general families from a local point of view in [R]. The name Arnold *disk* is justified by the following theorem which will be proved in section 4. We first recall the definitions of conformal radius and modulus.

Definition 1.2. If $U \subsetneq \mathbb{C}$ is a simply connected open subset containing 0, the *conformal radius* of U is $\mathrm{rad}(U) = |\phi'(0)|$ where $\phi : (\mathbb{D}, 0) \to (U, 0)$ is any isomorphism. If α is a Bruno number, we define $r_\alpha = \mathrm{rad}(\Delta_\alpha)$, where Δ_α is the Siegel disk of P_α.
If A is a round annulus $A = \{z \in \mathbb{C} \mid r < |z| < R\}$, the *modulus* of A is $\mathrm{mod}(A) = \frac{1}{2\pi}\log\frac{R}{r}$. If H is any annulus conformally equivalent to A we define its modulus to be equal to $\mathrm{mod}(A)$.

Theorem A. *For any Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, the set $\mathcal{D}_\alpha$ is a Riemann surface isomorphic to the unit disk and there is an isomorphism $\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha$ such that*

$$\mathcal{F}_\alpha(0) = (\mathrm{e}^{2i\pi\alpha}, 0) \quad \text{and} \quad \mathcal{F}'_\alpha(0) = (0, r_\alpha).$$

Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $f_{\mathcal{F}_\alpha(\delta)}$ is equal to $\dfrac{1}{\pi}\log\dfrac{1}{|\delta|}$.

Let us give an intuitive idea of how the parametrization $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$ is chosen. For any $\delta \in \mathbb{D}$, the map $f_{\mathcal{F}_\alpha(\delta)}$ possesses a Herman ring whose modulus determines by $|\delta|$; the argument of δ corresponds to the *twist parameter*, which roughly indicates how much one boundary of the ring is rotated with respect to the other (see Sect. 2 and Fig. 7). Maps in the Arnold Tongue $\mathcal{T}_\alpha$ correspond to $\delta \in (0,1)$. (This parametrization was given in [H] for the rational family and in [FG] for the complex standard family.)

Theorem B. *Assume α is a Bruno number. Then,*

(a) *the Arnold disk can be locally parameterized by a in a neighborhood of* $(\mathrm{e}^{2i\pi\alpha}, 0)$ *(i.e., it is locally the graph of a holomorphic map $a \mapsto \lambda(a)$)*
(b) *as $|a| \to 0$, the modulus m_a of the Herman ring of $f_{\lambda(a),a}$ satisfies*

$$\mathrm{e}^{\pi m_a} = \frac{r_\alpha}{|a|} + \mathcal{O}(a).$$

Corollary 1.3. *Assume α is a Bruno number. For $a \in (0,1/3)$, let $t_a \in \mathbb{R}/\mathbb{Z}$ be the unique parameter such that $(t_a, a) \in \mathcal{T}_\alpha$ and let m_a be the modulus of the Herman ring of $f_{e^{2i\pi t_a},a}$ ($m_a = 0$ if there is no Herman ring). Then, as $a \to 0$, we have*

$$e^{\pi m_a} = \frac{r_\alpha}{a} + \mathcal{O}(a).$$

This improves the estimate one would get by using the techniques developped by Fagella, Seara and Villanueva [FSV] in the case of the complex standard family. Indeed, we would obtain $\mathcal{O}(\log a)$ instead of $\mathcal{O}(a)$.

In the last section, we explain how to adapt the arguments to the case of the complex standard family $f_{\lambda,a}(z) = \lambda z e^{\frac{a}{2}(z-1/z)}$.

The following is work in progress. Theorem A describes the topology of any given Arnold disk and hence a natural question is to ask how these disks coexist in $\mathbb{C}^2$ when we move the rotation number. We think that there is a nice lamination: if $(\alpha_n)_{n\geq 1}$ is a sequence of Bruno numbers converging to a Bruno number α_0 such that $\lim_{n\to\infty} r_{\alpha_n} = r$, then, the sequence of maps $\mathcal{F}_{\alpha_n}$ converges uniformly on every compact subset of $\mathbb{D}$ to the map

$$\delta \mapsto \mathcal{F}_{\alpha_0}\left(\frac{r}{r_{\alpha_0}}\delta\right)$$

(the map $\delta \mapsto \mathcal{F}_{\alpha_0}(r\delta/r_{\alpha_0})$ is well defined because, the conformal radius r_α depends upper semi-continuously on α and so, $r \leq r_{\alpha_0}$).

We think we can use this result to prove the existence of Bruno numbers $\alpha \in \mathbb{R}/\mathbb{Z}$ such that the boundary of $\mathcal{D}_\alpha$ is a C^∞ Jordan curve and such that for all $(\lambda, a) \in \partial\mathcal{D}_\alpha$, the map $f_{\lambda,a}$ restricts to a diffeomorphism of a C^∞ Jordan curve and is C^∞ (but not $\mathbb{R}$-analytically) conjugate to the rotation of angle α on this curve. These results will appear in a forthcoming paper.

2. Preliminaries.

In the whole section, we assume that α is a Bruno number and that (λ, a) is in $\mathcal{D}_\alpha$. For simplicity, we set $f = f_{\lambda,a}$. By definition of $\mathcal{D}_\alpha$, the map f has a fixed Herman ring H on which it is conjugate to the rotation $\mathcal{R}_\alpha$. More precisely, let m be the modulus of the Herman ring H and set $r = e^{-2\pi m}$. Moreover, denote by A_r the round annulus

$$A_r = \{z \in \mathbb{C}^* \mid r < |z| < 1\}.$$

Then, there exists an isomorphism $\phi : A_r \to H$ which conjugates $\mathcal{R}_\alpha : A_r \to A_r$ to $f : H \to H$ and preserves the orientation in $\mathbb{C}$ of the invariant curves (i.e., morally sends S^1 to the outer boundary of H). This isomorphism is unique up to precomposition with a rotation centered at 0.

2.1. Basic Properties.

The rational map $f_{\lambda,a}$ has four critical points. One is fixed at 0, one is fixed at ∞. Therefore, there are only two *free* critical points. If $f_{\lambda,a}$ has a Herman ring, then the orbits of those critical point are *trapped*: the closure of those orbits must contain the boundary components of the Herman ring. In particular, the immediate basins of the superattracting fixed points at 0 and ∞ are simply connected. We claim that the Herman ring separates those two basins.

Proposition 2.1. *Suppose $f = f_{\lambda,a}$ has a fixed Herman ring H. Then, H separates 0, one of the critical point in $\mathbb{C}^*$ and the pole $-a$ on the one hand, from ∞, the other critical point in $\mathbb{C}^*$ and the zero $-1/a$ on the other hand.*

Proof. Let γ be an invariant curve in H, let V_0 be the bounded connected component of $\mathbb{P}^1 \setminus \gamma$ and let V_∞ be the unbounded one. Set $U_\infty = f^{-1}(V_\infty) \cap V_0$ and $U_0 = f^{-1}(V_0) \cap V_0$. Since f is proper and since $f(\gamma) = \gamma$, we see that $f : U_\infty \to V_\infty$ and $f : U_0 \to V_0$ are proper mappings. Let d_∞ and d_0 be their degrees.

We first claim that $d_\infty = 1$ and $d_0 = 2$. Indeed, the image of γ (which is γ itself) turns exactly once around every point in V_0. So, by the argument principle, $d_0 - d_\infty = 1$. Since ∞ has at most one preimage in V_0, we see that the only possibilities for (d_∞, d_0) are $(0,1)$ or $(1,2)$. The case $(0,1)$ is not possible since otherwise, $f : V_0 \to V_0$ would be an isomorphism, V_0 would be contained in the Fatou set of f and this would contradict the fact that it contains a boundary component of the Herman ring H.

It follows that U_∞ is a topological disk compactly contained in V_0 and U_0 is a topological annulus. By the Riemann-Hurwitz formula, U_0 contains two critical points of f. The two remaining critical points of f are contained in an annulus $W_\infty \subset V_\infty$ mapped with degree 2 to V_∞. Finally, there is a topological disk $W_0 \subset V_\infty$ mapped to V_0 with degree 1 (see Fig. 4).

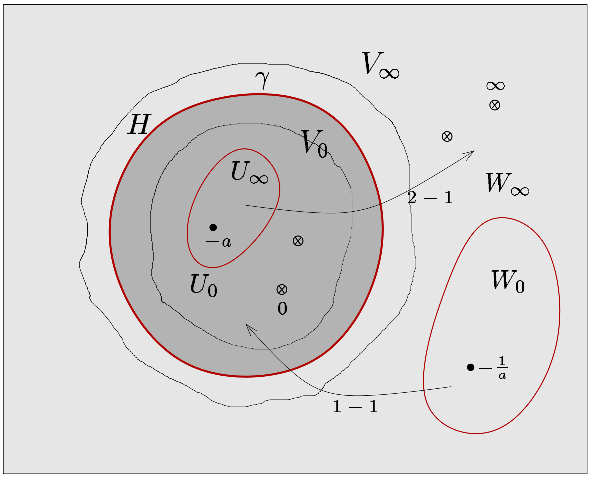

FIGURE 4. Sketch illustrating the proof of proposition 2.1.

We must finally show that $0 \in U_0$, which will complete the proof. This last result is more subtle since we must use a dynamical characterization of 0: it is a superattracting fixed point. There are several possible arguments. It is possible to show that each component of $\mathbb{P}^1 \setminus \gamma$ must contain a critical point whose orbit accumulates a boundary component of H. Or we can argue that every connected component of $\mathbb{P}^1 \setminus \gamma$ must contain two fixed points, one of which is repelling or has multiplier 1 (see for example [Sh2] or [Bu]). ■

As we mentioned previously, it is quite difficult to locate parameters in $\mathcal{D}_\alpha$ when a is not real. However, the following proposition asserts that all the sets $\mathcal{D}_\alpha$ are contained in some common compact subset of $\mathbb{C}^* \times \mathbb{C}$.

Proposition 2.2. *Assume α is a Bruno number and $(\lambda, a) \in \mathcal{D}_\alpha$. Then*

$$\frac{1}{4} \leq |\lambda| \leq 4 \quad \text{and} \quad |a| \leq 1.$$

Proof. As mentioned previously, the immediate basins of the superattracting fixed points at 0 and ∞ are simply connected and the Herman ring separates those two basins. Let W_0 (respectively W_∞) be the immediate basin of 0 (respectively ∞). Let $\phi_0 : W_0 \to B(0, r_0)$ (respectively $\phi_\infty : W_\infty \to \mathbb{P}^1 \setminus \overline{B(0, r_\infty)}$) be the conformal representation which is tangent to the identity at 0 (respectively ∞).

Then, ϕ_0 (respectively ϕ_∞) conjugates $f_{\lambda,a}$ to a proper mapping of $B(0, r_0)$ (respectively $\mathbb{P}^1 \setminus \overline{B(0, r_\infty)}$) of degree 2 and which has a superattracting fixed point at 0 (respectively ∞). This proper mapping is $z \mapsto z^2/c_0$ (respectively $z \mapsto z^2/c_\infty$). Since this map must have a fixed point on the boundary of $B(0,r_0)$ (resp. $B(0,r_\infty)$), it follows that $|c_0| = r_0$ (resp. $|c_\infty| = r_\infty$). Now, note that

$$f_{\lambda,a}(z) \underset{z\to 0}{\sim} \frac{\lambda}{a}z^2 \quad \text{and} \quad f_{\lambda,a}(z) \underset{z\to\infty}{\sim} \lambda a z^2.$$

Therefore, we deduce that

$$r_0 = \frac{|a|}{|\lambda|} \quad \text{and} \quad r_\infty = \frac{1}{|\lambda a|}.$$

Now, W_0 does not contain $-a$ which is a preimage of ∞ and W_∞ does not contain $-1/a$ which is a preimage of 0. So, by the Koebe one-quarter theorem, we have

$$r_0 \le 4|a| \quad \text{and} \quad r_\infty \ge \frac{1}{4|a|}.$$

This, in turn, yields

$$\frac{1}{4} \le |\lambda| \le 4.$$

Finally, the basins W_0 and W_∞ are disjoint. It follows that $r_0 \le r_\infty$ (see for example [BE] section 3). As a consequence, we have

$$\frac{|a|}{|\lambda|} \le \frac{1}{|\lambda a|}$$

which gives $|a| \le 1$. ■

2.2. **Shishikura's surgery.** We explain here a surgery construction originally due to Shishikura [Sh1] that will be used in several instances along this paper. The general idea of the construction is, starting from a rational map with a Herman ring, to obtain a polynomial with a Siegel disk, by means of gluing a rigid rotation to "fill in the hole" of the Herman ring. In the particular case of our family of rational maps $f_{\lambda,a}$, the polynomial that we obtain is precisely the quadratic polynomial P_α. The result is summarized in the following proposition.

Proposition 2.3. *Suppose $f = f_{\lambda,a}$ has a fixed Herman ring H with rotation number α, and let $\gamma \subset H$ denote an invariant curve. Let U_0 be the bounded part of $\mathbb{P}^1 \setminus \gamma$ and U_∞ be the unbounded one. Then, the polynomial P_α has a Siegel disk $\Delta = \Delta_\alpha$ and there exists a quasiconformal homeomorphism $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ and a P_α invariant curve Γ in Δ such that:*

1. *ψ maps γ to Γ and U_∞ to the unbounded component, say V, of $\mathbb{P}^1 \setminus \Gamma$;*
2. *ψ conjugates $f : \overline{U}_\infty \to \mathbb{P}^1$ to $P_\alpha : \overline{V} \to \mathbb{P}^1$;*
3. *$\partial\psi/\partial\bar{z} = 0$ a.e. on $\mathbb{P}^1 \setminus \bigcup_{n\ge 0} f^{-n}(U_0)$ (in particular ψ is conformal in the interior of this set).*

Proof. Shishikura proves this using a surgery construction that is by now classical. We will give an outline of the procedure. Let $\phi : A_r \to H$ be a conformal map that conjugates the rigid rotation $\mathcal{R}_\alpha : A_r \to A_r$ to $f : H \to H$.

Notice that $\phi^{-1}(\gamma)$ is a circle centered at 0 with radius ρ, $r < \rho < 1$. Denote by $\hat{\phi} : \mathbb{D} \to H \cup U_0$ a quasiconformal mapping that agrees with ϕ on A_ρ, maps $\mathbb{D}_\rho$ onto U_0 and fixes 0. Define a map $\hat{f} : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$\hat{f} = \begin{cases} f & \text{on } \overline{U}_\infty \\ \hat{\phi} \circ \mathcal{R}_\alpha \circ \hat{\phi}^{-1} & \text{on } \overline{U}_0. \end{cases}$$

The map $\hat{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is a proper and quasiregular mapping which maps infinity to itself with local degree 2. By proposition 2.1 f has a pole in U_0, so by counting the preimages of infinity we see that $\hat{f}$ has degree 2. It fixes 0 and is conjugate to the rotation $\mathcal{R}_\alpha$ in a neighborhood of 0. Finally, since it is of degree two it has a unique critical point ω in $\mathbb{C}$.

The map $\hat{f}$ is not holomorphic on U_0, but there it preserves the complex structure defined by the Beltrami form

$$\mu = \frac{\bar{\partial}\hat{\phi}^{-1}}{\partial\hat{\phi}^{-1}}.$$

Pulling back this Beltrami form via $\hat{f}$, we see that there exists a Beltrami form $\hat{\mu}$ which coincides with μ on U_0, which vanishes on $\mathbb{P}^1 \setminus \bigcup_{n\geq 0} \hat{f}^{-n}(U_0)$ and which is invariant by $\hat{f}$:

$$\hat{f}^*\hat{\mu} = \hat{\mu}.$$

By the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ which fixes 0 and ∞, sends ω to $-1/2$ and such that

$$\hat{\mu} = \frac{\bar{\partial}\psi}{\partial\psi}.$$

Then, the map $\psi \circ \hat{f} \circ \psi^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ is proper and holomorphic of degree 2, it fixes 0 and is conjugate to the rotation $\mathcal{R}_\alpha$ in a neighborhood of 0, it has a superattracting fixed point at infinity and a critical point at $-1/2$. The only such map is the quadratic polynomial P_α and thus

$$P_\alpha = \psi \circ \hat{f} \circ \psi^{-1}.$$

The map ψ is the required conjugacy. ∎

Remark 2.4. The modulus of the annulus bounded by γ and the outer boundary of H is equal to the modulus of the annulus bounded by Γ and the boundary of the Siegel disk Δ_α (since ψ gives a conformal isomorphism between those two annuli).

2.3. **Case where a is real.**

Proposition 2.5. *Suppose $(\lambda, a) \in \mathcal{D}_\alpha$ for some Bruno number α. If $a \in (0, 1/3)$ then $|\lambda| = 1$. If also $(\lambda', a) \in \mathcal{D}_\alpha$ then $\lambda' = \lambda$.*

Proof. Suppose that $a \in (0, 1/3)$ and $|\lambda| < 1$. Then f maps the unit circle onto the circle of radius $|\lambda| < 1$ as a smooth diffeomorphism. We distinguish between two cases:

(a) $\mathbb{D} \subset \mathbb{P}^1 \setminus H$,
(b) $\mathbb{D}$ intersects H.

(a) We can cut out the dynamics in the unit disk and replace it by the attracting dynamics $z \mapsto \lambda z$. Indeed, define $\phi : S^1 \to S^1$ by $\phi(z) = f(z)/\lambda$. Extend ϕ to $\overline{\mathbb{D}}$ quasiconformally such that ϕ is the identity on $\lambda\overline{\mathbb{D}}$. Define a new map $\hat{f} : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$\hat{f} = \begin{cases} f & \text{on } \mathbb{P}^1 \setminus \mathbb{D} \quad \text{and} \\ \hat{\phi}^{-1} \circ \Lambda \circ \hat{\phi} & \text{on } \overline{\mathbb{D}}, \end{cases}$$

where Λ denotes multiplication by $\lambda : \Lambda(z) = \lambda z$. Note that we have constructed ϕ so that the two definitions agree on S^1. Now $\hat{f}$ is a proper quasiregular mapping of degree 2 (we removed the pole in $\mathbb{D}$). It is not holomorphic, but there exists an invariant Beltrami form μ that vanishes on $\mathbb{P}^1 \setminus \bigcup\limits_{n\geq 0} f^{-n}(\mathbb{D})$ and satisfies

$$\mu = \frac{\bar{\partial}\hat{\phi}}{\partial\hat{\phi}}$$

on $\mathbb{D}$. It follows that $\hat{f}$ is holomorphic with respect to the complex structure induced by μ, and one can verify that $\hat{f}$ is quasiconformally conjugate to a quadratic polynomial. However, H is a Herman ring for $\hat{f}$, which is impossible.

(b) Since $\mathbb{D}$ intersects H, we can choose an invariant curve $\gamma \in H$ so that $\mathbb{D}$ intersects the unbounded component of $\mathbb{P}^1 \setminus \gamma$. Let ψ and U_0 be defined by Proposition 2.3. Set $W = \psi(\mathbb{D} \cup U_0)$ and $W' = \psi(\lambda\mathbb{D} \cup U_0)$. Then, $P_\alpha(W) \subset W'$ and W' is strictly contained in W. So $P_\alpha : W \to W'$ is a strong contraction which contradicts that zero is an indifferent fixed point.

We have shown that we cannot have $|\lambda| < 1$. The proof that we cannot have $|\lambda| > 1$ is analogous.

Now suppose that $(\lambda', a) \in \mathcal{D}_\alpha$. By what we have just shown, $|\lambda'| = 1$, so $f_{\lambda',\alpha}$ is equal to the composition $\mathcal{R}_\beta \circ f_{\lambda,\alpha}$ for some rigid rotation $\mathcal{R}_\beta$. For an arbitrary circle homeomorphism h, the rotation number of $\mathcal{R}_\beta \circ h$ is a weakly increasing function of β, and there is only one value of β (modulo 1) for which the rotation number is equal to some given irrational number (see e.g. [dMvS]). Since $\mathcal{R}_\beta \circ f_{\lambda,\alpha}|_{S^1}$, and $f_{\lambda,\alpha}|_{S^1}$ have the same irrational rotation number we have $\beta = 0$ (modulo 1) and $\lambda = \lambda'$. ■

3. The twist coordinate

As above let f have a fixed Herman ring H of rotation number α and let $\phi : A_r \to H$ be a conformal isomorphism that conjugates $\mathcal{R}_\alpha : A_r \to A_r$ to $f : H \to H$.

Our goal in this section is to define a *twist coordinate* for the map $f_{\lambda,a}$. To fix ideas, we will first define it for the easiest case, that is the case when the boundaries of the Herman ring H are Jordan curves and contain the two critical points of the map, say ω_1 on the outer boundary and ω_2 on the inner boundary.

In such a situation, the linearizing map ϕ extends as a homeomorphism $\phi : \partial A_r \to \partial H$. Denote by $C(c, r)$ the circle centered at $c \in \mathbb{C}$ of radius $r \in [0, \infty)$. Let $\tilde{\omega}_1 \in C(0, 1)$ and $\tilde{\omega}_2 \in C(0, r)$ be the preimages of ω_1 and ω_2 under ϕ and let $\Theta_1 = \frac{1}{2\pi}\arg(\tilde{\omega}_1)$ and $\Theta_2 = \frac{1}{2\pi}\arg(\tilde{\omega}_2)$, taken in $\mathbb{R}/\mathbb{Z}$.

We define the *twist coordinate* of $f_{\lambda,a}$ as the difference between these two arguments (see Figure 5), i.e.,

$$\Theta = \Theta_2 - \Theta_1.$$

Note that even though ϕ is uniquely defined only up to rotation (and hence so are $\tilde{\omega}_1$ and $\tilde{\omega}_2$), the

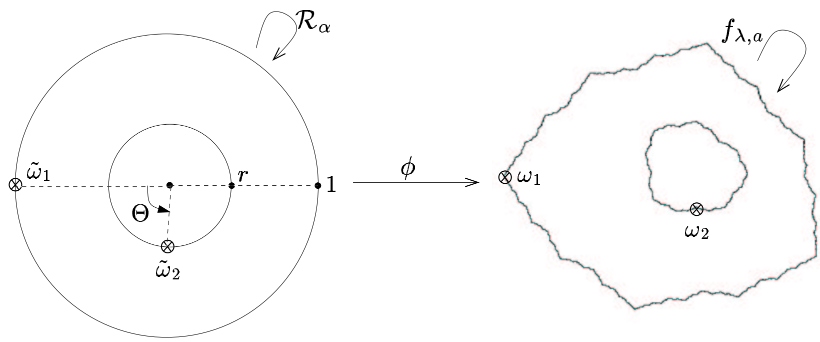

FIGURE 5. The definition of the twist coordinate in the simplest case, when both critical points lie on the boundary of H.

twist coordinate is independent of this choice.

Intuitively, the twist parameter measures the rotation of the boundaries with respect to each other.

We now turn to the general case, that is, when the critical points are not on the boundary of H or ϕ cannot be extended as a homeomorphism $\partial A_r \to \partial H$. For this reason the definition of Θ in this situation will be different. We then show that the two definitions coincide.

We first recall the definition of the *equator* of an annulus.

Definition 3.1. Let H be an annulus conformally equivalent to A_r, by a conformal map $\phi : A_r \to H$. We define the *equator* of H (or *core geodesic*) as the simple closed curve

$$\gamma = \phi(C(0, \sqrt{r})) \subset H.$$

Remark 3.2. The curve γ subdivides the Herman ring H into two annuli of equal moduli (half the original modulus of H).

Let γ be the equator of H and $\psi : \mathbb{P}^1 \to \mathbb{P}^1$, U_0, U_∞, Γ and V be as in Proposition 2.3. Recall that ψ conjugates $f_{\lambda,a}$ on U_∞ (the unbounded component of $\mathbb{P}^1 \setminus \gamma$), to the polynomial P_α on V (the unbounded component of $\mathbb{P}^1 \setminus \Gamma$ with $\Gamma = \psi(\gamma)$). Moreover, ψ fixes 0 and ∞ and sends the critical point in U_∞ to $-1/2$.

We now proceed to make a parallel construction with the map $\hat{f}(u) = \frac{1}{\overline{f(1/\bar{u})}}$, i.e., the conjugate of f under the change of coordinates $u = \tau(z) = \frac{1}{\bar{z}}$. Observe that $\hat{f}$ has a Herman ring $\widehat{H} = \tau(H)$ of rotation number α. More precisely, let $\hat{\gamma} = \tau(\gamma)$ be the equator of $\widehat{H}$ and let $\hat{\psi}$, $\widehat{U}_0, \widehat{U}_\infty$, $\widehat{\Gamma}$ and $\widehat{V}$ be given by Proposition 2.3: $\hat{\psi}$ conjugates $\hat{f}$ in $\widehat{U}_\infty$ to P_α on $\widehat{V}$.

We claim that $\widehat{\Gamma} = \Gamma$ and $\widehat{V} = V$. Indeed, the modulus of H is the same as the modulus of $\widehat{H}$. Since γ and $\hat{\gamma}$ are equators of those annuli, the moduli of the annuli bounded by γ and $\hat{\gamma}$ on the one hand and by the outer boundaries of H and $\widehat{H}$ on the other hand, are equal. Therefore, the moduli of the annuli bounded by the curves Γ and $\widehat{\Gamma}$ on the one hand, and by the boundary of the Siegel disk on the other hand, are equal. So, $\Gamma = \widehat{\Gamma}$ and $V = \widehat{V}$.

We now denote by $\varphi : \mathbb{D} \to \Delta_\alpha$ a linearizing map of the Siegel disk. The map φ conjugates the rigid rotation $\mathcal{R}_\alpha : \mathbb{D} \to \mathbb{D}$ to $P_\alpha : \Delta_\alpha \to \Delta_\alpha$ and is unique up to pre-composition with other rigid rotations. See Figure 6.

We define a map $\mathcal{C} : \varphi^{-1}(\Gamma) \to \varphi^{-1}(\Gamma)$ by

$$\mathcal{C} = \varphi^{-1} \circ \psi \circ \tau \circ \hat{\psi}^{-1} \circ \varphi.$$

Observe that, by construction, $\mathcal{C}$ is a conjugacy between the rotation $\mathcal{R}_\alpha$ and itself on the Euclidean circle $\varphi^{-1}(\Gamma)$. But any such map can only be a rigid rotation itself, that is, $\mathcal{C}(z) = \mathrm{e}^{2\pi i \Theta} z$ for some $\Theta \in \mathbb{R}/\mathbb{Z}$. We then define Θ to be the twist coordinate of $f_{\lambda,a}$.

It only remains to prove the following.

Proposition 3.3. *When the two boundary components of the Herman ring H are Jordan curves each containing a critical point, the two preceding definitions coincide.*

Proof. We first observe that if $\phi : A_r \to H$ conjugates $\mathcal{R}_\alpha$ to f, then the map $\hat{\phi} : A_r \to \widehat{H}$ defined by $\hat{\phi}(u) := \tau \circ \phi(r/\bar{u})$ conjugates $\mathcal{R}_\alpha$ to $\hat{f}$. Hence, observing the commutative diagram in Figure 6, we see that the maps

$$R = \varphi^{-1} \circ \psi \circ \phi$$

and

$$\widehat{R} = \varphi^{-1} \circ \hat{\psi} \circ \hat{\phi}$$

are conformal maps from the annulus $A_{\sqrt{r}}$ to the Euclidean annulus bounded by $\varphi^{-1}(\Gamma)$ and S^1, and they conjugate the rigid rotation $\mathcal{R}_\alpha$ to itself (this in fact implies that $\varphi^{-1}(\Gamma)$ is the Euclidean circle of radius $\sqrt{r}$). Such maps can only be rigid rotations themselves, whose angles we will determine now.

Let $\tilde{\omega}_1 \in C(0,1)$ and $\tilde{\omega}_2 \in C(0,r)$ be, as before, the preimages of the two critical points ω_1 and ω_2 under ϕ. If we follow the outer one, $\tilde{\omega}_1$, along the diagram, we see that it is first mapped to ω_1 by ϕ, then to $\omega = -1/2$ by ψ and finally to some unit vector by φ^{-1}, say $\mathrm{e}^{2\pi i \beta}$. Hence

$$R = \mathcal{R}_{\beta - \Theta_1}$$

where we recall that $\Theta_1 = \frac{1}{2\pi} \arg(\tilde{\omega}_1)$.

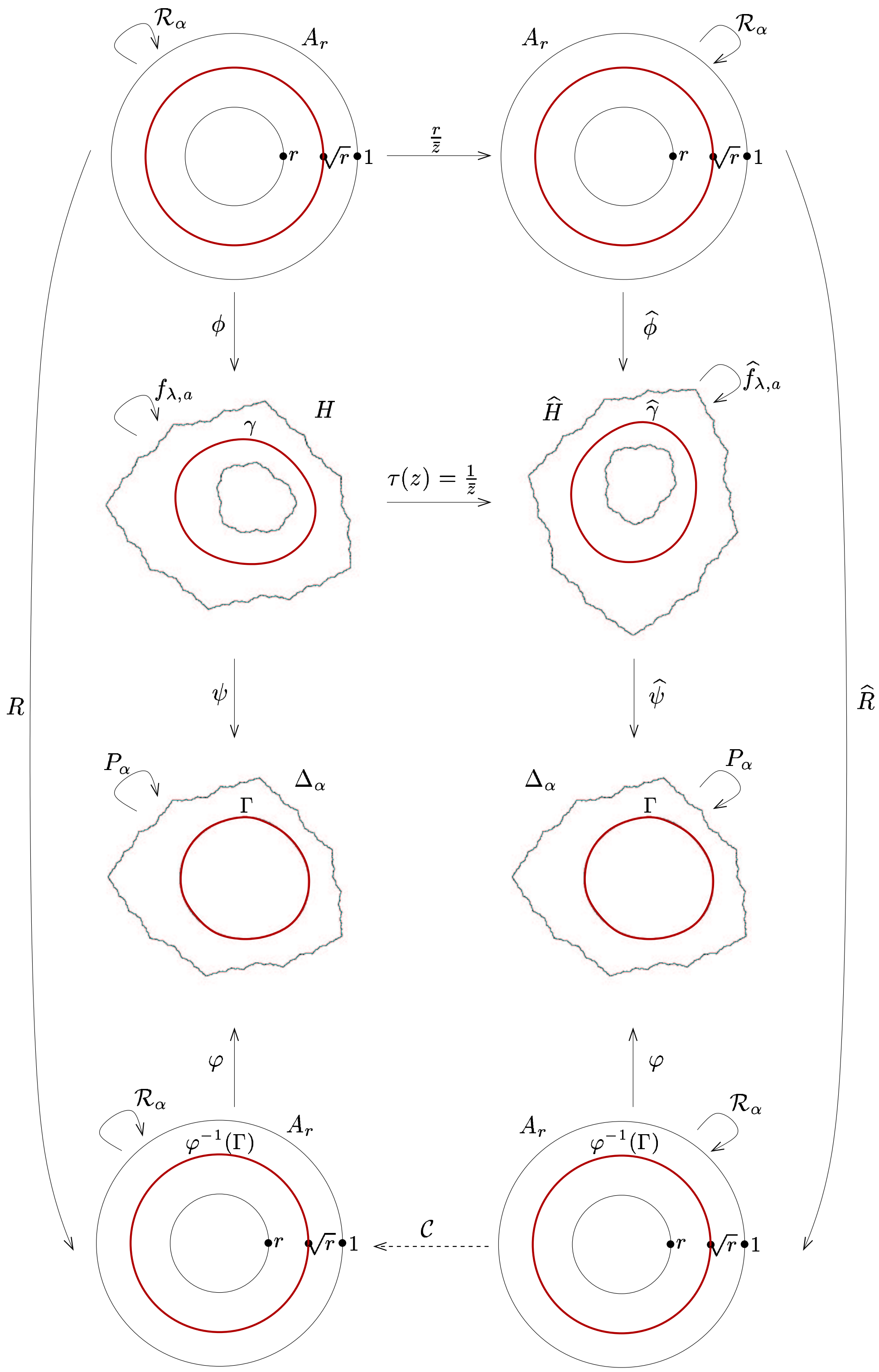

FIGURE 6. Commutative diagram illustrating the general definition of the twist coordinate of $f_{\lambda,a}$. The conclusion is that $\mathcal{C}(z) = e^{2\pi i\Theta}z$ for some $\Theta \in \mathbb{R}/\mathbb{Z}$ which we define as the twist coordinate.

We now do the same for $\widehat{R}$. First observe that the map $r\tau$ sends all points of modulus r to points of modulus 1 and viceversa, but preserves their arguments. Therefore the critical points are interchanged, i.e., the point $\tilde{\omega}_2$ is sent to a point of modulus 1 and argument Θ_2. This point is in turn mapped to the critical point on the outer boundary of $\widehat{H}$ which is later sent to $\omega = -1/2$ by $\hat{\psi}$. Finally, its image is, as before, $e^{2\pi i\beta}$. We deduce then that

$$\widehat{R} = \mathcal{R}_{\beta-\Theta_2}.$$

Now, the outmost diagram says that

$$\mathcal{C} = R \circ r\tau \circ \widehat{R}^{-1},$$

or equivalently that

$$\Theta = (\beta - \Theta_1) - (\beta - \Theta_2) = \Theta_2 - \Theta_1.$$

■

We now want to illustrate the twist coordinate with a picture. If $\lambda \in S^1$ and $a \in (0, 1/3)$, the twist coordinate is always equal to 0, because of the symmetry $z \mapsto 1/\bar{z}$ with respect to S^1. Now, if $\lambda \in S^1$ and $a \in i\mathbb{R}$ there is a symmetry $z \mapsto -1/\bar{z}$, and if $f_{\lambda,a}$ has a Herman ring, the twist parameter is equal to $1/2$. Even if S^1 is not invariant, it is still possible to locate parameters (λ, a) such that $f_{\lambda,a}$ has a Herman ring. For example, for $a = i/4$ and $t = 0.624098187\ldots$, the rational map $f_{e^{2i\pi t},a}$ has a fixed Herman ring with rotation number $(\sqrt{5}-1)/2$. Now, the twist coordinate is better seen by working in the universal covering of $\mathbb{C}^*$ given by $Z \mapsto z = e^{2i\pi Z}$. Figure 7 shows the Julia set of two rational maps having Herman rings with rotation numbers $(\sqrt{5}-1)/2$ and with twist coordinates 0 on the top and $1/2$ on the bottom. In both cases, we marked the critical points with arrows. The real part of Z ranges from -1 to 1.

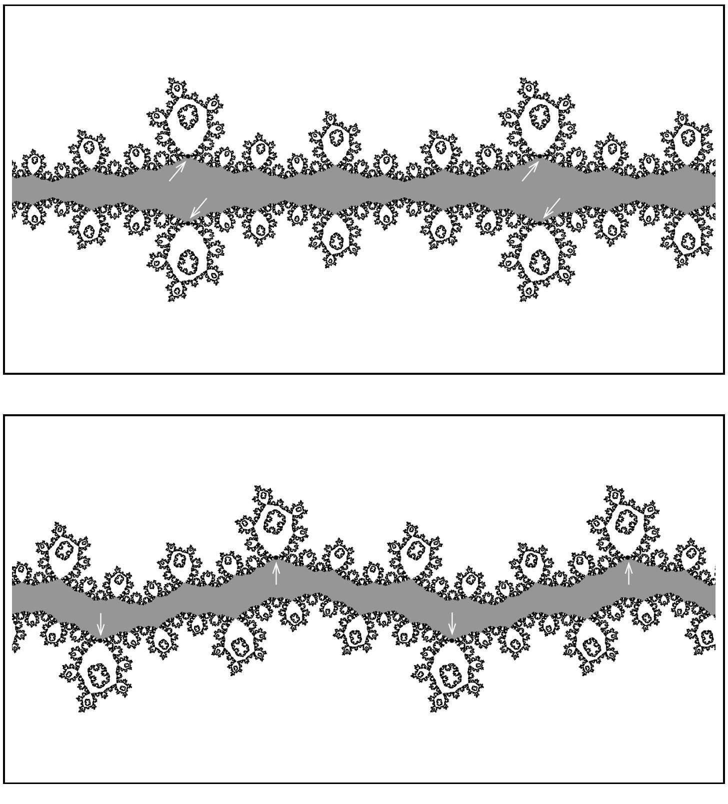

FIGURE 7. The universal covering of two Herman rings with rotation number $(\sqrt{5}-1)/2$ and with twist coordinate 0 on the top and $1/2$ on the bottom.

4. Parametrization of Arnold disks. Proof of Theorem A.

We will work with a slightly different family of rational maps. The purpose of introducing a new family of rational map is to get rid of the conjugacy between $f_{\lambda,a}$ and $f_{\lambda,-a}$.

Definition 4.1. For $\lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$, we define

$$g_{\lambda,b} : w \mapsto \lambda w^2 \frac{1+w}{b+w}.$$

The map $f_{\lambda,a}$ is conjugate to the map g_{λ,a^2} via $z \mapsto w = az$; i.e.,

$$f_{\lambda,a}(z) = \frac{1}{a} g_{\lambda,a^2}(az).$$

Now, if g_{λ_1,b_1} is conjugate to g_{λ_2,b_2} by a scaling map, then $(\lambda_1, b_1) = (\lambda_2, b_2)$. Moreover, the maps $g_{\lambda,b}$ are exactly the cubic rational maps which have superattracting fixed points at 0 and ∞ and which map -1 to 0. Observe that $g_{\lambda,0}(z) = \lambda z(1+z)$, a quadratic polynomial, and finally note that if $\lambda \in S^1$ and $b > 0$, the invariant circle now has radius $\sqrt{b}$.

Definition 4.2. Given a Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, let $\mathcal{D}'_\alpha$ be the set of parameters (λ, b) such that $g_{\lambda,b}$ has a fixed Herman ring with rotation number α.

By convention, we consider that for $\lambda = \mathrm{e}^{2i\pi\alpha}$ and $b = 0$, the quadratic polynomial $g_{\lambda,b} = P_\alpha$ has a Herman ring $\Delta_\alpha \setminus \{0\}$ of infinite modulus. We will now state a weak version of Theorem A for the family $g_{\lambda,b}$.

Proposition 4.3. *For any Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, there is an isomorphism $\mathcal{G}_\alpha : \mathbb{D} \to \mathcal{D}'_\alpha$ with $\mathcal{G}_\alpha(0) = (\mathrm{e}^{2i\pi\alpha}, 0)$. Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $g_{\mathcal{G}_\alpha(\delta)}$ is equal to $\frac{1}{2\pi}\log\frac{1}{|\delta|}$.*

4.1. **Proof of Proposition 4.3.** We will prove this result in several steps and later see how it implies Theorem A.

In the previous section, we defined a twist coordinate from $\mathcal{D}_\alpha$ to $\mathbb{R}/\mathbb{Z}$. Similarly, we can define a twist coordinate $\Theta : \mathcal{D}'_\alpha \to \mathbb{R}/\mathbb{Z}$. Let us also consider the modulus coordinate $m : \mathcal{D}'_\alpha \to (0, +\infty]$ which maps $(\lambda, b) \in \mathcal{D}'_\alpha$ to the modulus of the Herman ring of $g_{\lambda,b}$. We can define a map $\Pi : \mathcal{D}'_\alpha \to \mathbb{D}$ by

$$\Pi(\lambda, b) = \exp\big(-2\pi m(\lambda, b) + 2i\pi\Theta(\lambda, b)\big).$$

We will show that this map provides an isomorphism between $\mathcal{D}'(\alpha)$ and $\mathbb{D}$. See Figure 8.

For this purpose, given $(\lambda_1, b_1) \in \mathcal{D}'_\alpha$, we will construct an analytic map $\mathcal{G} : \mathbb{D} \to \mathbb{C}^* \times \mathbb{C}$ whose image is contained in $\mathcal{D}'_\alpha$ and such that

$$\mathcal{G} \circ \Pi(\lambda_1, b_1) = (\lambda_1, b_1), \quad \mathcal{G}(0) = (\mathrm{e}^{2i\pi\alpha}, 0) \quad \text{and} \quad \Pi \circ \mathcal{G} = \mathrm{Id},$$

which shows that Π is injective. We will then show that the map $\mathcal{G}$ does not depend on the choice of (λ_1, b_1), which proves that $\mathcal{G}$ is an isomorphism between $\mathbb{D}$ and $\mathcal{D}'_\alpha$, and Π is its inverse.

Step 1. Model homeomorphisms. Given a complex number η in the right half-plane $\mathbb{H}_+ = \{Z \in \mathbb{C} \mid \mathrm{Re}(Z) > 0\}$, let us first consider the $\mathbb{R}$-linear map $L_\eta : \mathbb{C} \to \mathbb{C}$ which is the identity on $i\mathbb{R}$ and maps 1 to η:

$$L_\eta(Z) = \frac{1}{2}\left((\eta+1)Z + (\eta-1)\bar{Z}\right),$$

or, equivalently,

$$L_\eta(x + iy) = x\eta + iy = x(\mathrm{Re}(\eta)) + i(y + x\mathrm{Im}(\eta)).$$

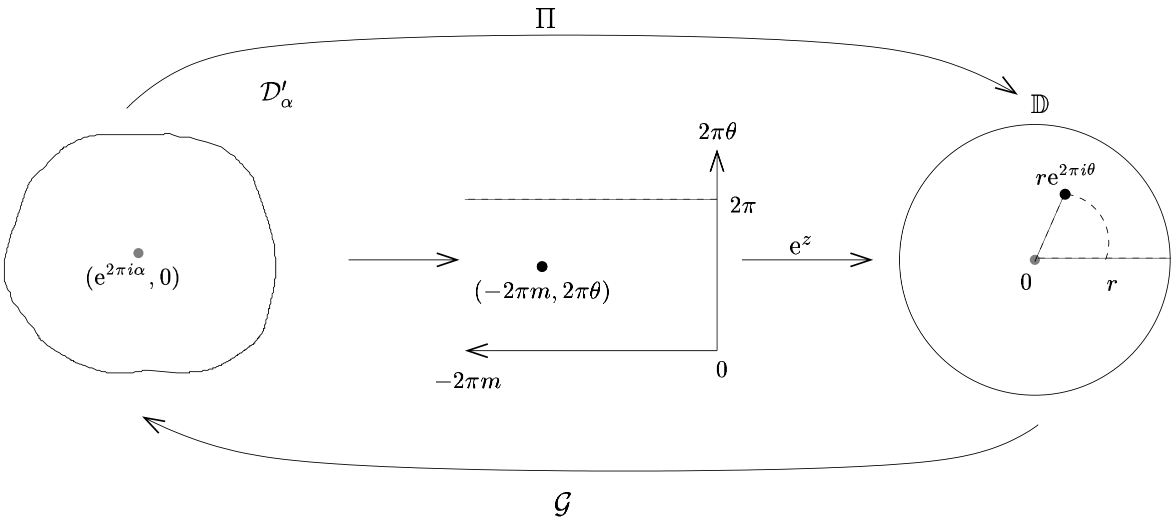

FIGURE 8. The map Π and its inverse $\mathcal{G}$.

This map is a K-quasiconformal homeomorphism of $\mathbb{C}$ with

$$K = \left|\frac{\eta - 1}{\eta + 1}\right|.$$

Now, let τ_1 be a complex number in the left half-plane $\mathbb{H}_- = \{Z \in \mathbb{C} \mid \mathrm{Re}(Z) < 0\}$ and set

$$r_1 = |e^{\tau_1}|, \quad \tau_\eta = L_\eta(\tau_1), \quad \text{and} \quad r_\eta = |e^{\tau_\eta}|.$$

Note that L_η is a homeomorphism between the vertical strips

$$\{Z \in \mathbb{C} \mid \mathrm{Re}(\tau_1) < \mathrm{Re}(Z) < 0\} \quad \text{and} \quad \{Z \in \mathbb{C} \mid \mathrm{Re}(\tau_\eta) < \mathrm{Re}(Z) < 0\},$$

which commutes with the vertical translation by $2\pi i$. See Figure 9. Thus, it projects to a K-quasiconformal homeomorphism $h_\eta : A_{r_1} \to A_{r_\eta}$. Moreover,

$$h_\eta(e^{\tau_1}) = e^{\tau_\eta} \quad \text{and} \quad h_\eta = \mathrm{Id} \text{ on } S^1.$$

Finally, for any angle $\alpha \in \mathbb{R}/\mathbb{Z}$, the homeomorphism $h_\eta : A_{r_1} \to A_{r_\eta}$ conjugates the rotation $\mathcal{R}_\alpha : A_{r_1} \to A_{r_1}$ to the rotation $\mathcal{R}_\alpha : A_{r_\eta} \to A_{r_\eta}$. As a consequence, the rotation $\mathcal{R}_\alpha : A_{r_1} \to A_{r_1}$ preserves the complex structure defined by the Beltrami form $\bar{\partial} h_\eta / \partial h_\eta$.

Step 2. A map $\mathbf{S_{g_1}} : \mathbb{H}_+ \to \mathcal{D}'_\alpha$. For simplicity, we will identify $(\lambda, b) \in \mathcal{D}'_\alpha$ with the rational map $g_{\lambda,b}$. Let us consider a rational map $g_1 := g_{\lambda_1,b_1} \in \mathcal{D}'_\alpha$. Let H_1 be the Herman ring of g_1 and (m_1, Θ_1) be the modulus and twist coordinate of H_1. Set $\tau_1 = 2\pi(-m_1 + i\Theta_1)$ and $r_1 = e^{-2\pi m_1}$ and let $\phi_1 : A_{r_1} \to H_1$ be an isomorphism which conjugates $\mathcal{R}_\alpha : A_{r_1} \to A_{r_1}$ to $g_1 : H_1 \to H_1$.

Now, let $\eta \in \mathbb{H}_+$ be an arbitrary complex number and let r_η and $h_\eta : A_{r_1} \to A_{r_\eta}$ be as in Step 1. Since $\mathcal{R}_\alpha : A_{r_1} \to A_{r_1}$ preserves the complex structure defined by the Beltrami form $\bar{\partial} h_\eta / \partial h_\eta$ and since $\phi_1 : A_{r_1} \to H_1$ conjugates $\mathcal{R}_\alpha : A_{r_1} \to A_{r_1}$ to $g_1 : H_1 \to H_1$, we see that $g_1 : H_1 \to H_1$ preserves the complex structure defined by the Beltrami from

$$\mu_\eta = \frac{\bar{\partial}\left(h_\eta \circ \phi_1^{-1}\right)}{\partial\left(h_\eta \circ \phi_1^{-1}\right)}.$$

Remark 4.4. Note that μ_η does not depend on the choice of isomorphism $\phi_1 : A_{r_1} \to H_1$.

There exists a unique extension of μ_η to $\mathbb{C}$ which is g_1–invariant and is 0 outside the preimages of H_1. This Beltrami form μ_η depends holomorphically on $\eta \in \mathbb{H}_+$.

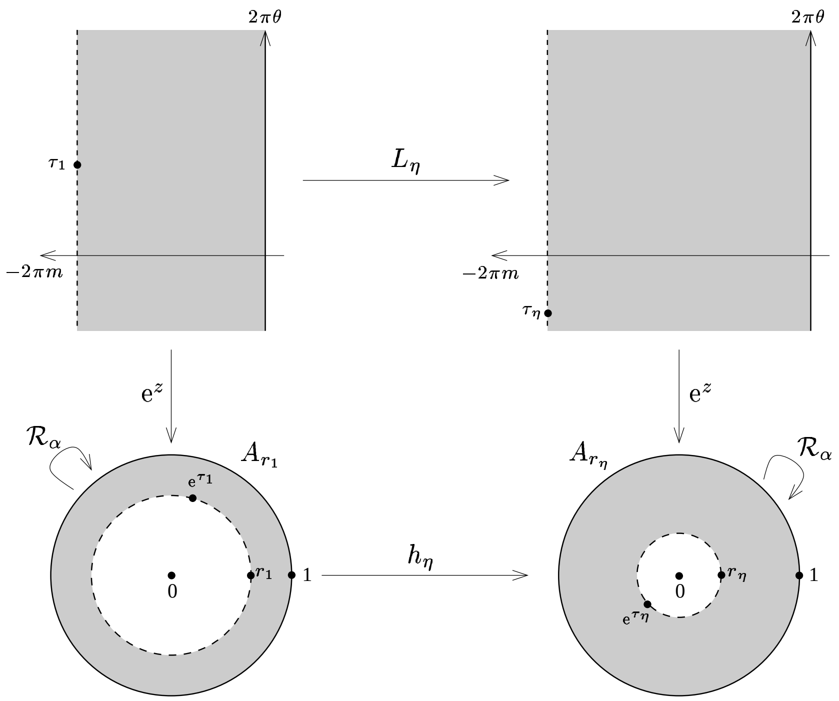

FIGURE 9. The quasiconformal homeomorphism L_η.

Next, let $\chi_\eta : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal homeomorphism which fixes 0 and -1 and integrates μ_η:

$$\mu_\eta = \frac{\bar\partial \chi_\eta}{\partial \chi_\eta}.$$

Since g_1 preserves the complex structure defined by μ_η, the map

$$g_\eta = \chi_\eta \circ g_1 \circ \chi_\eta^{-1}$$

is holomorphic. Observe that g_η must be a rational map of degree 3 which has superattracting fixed points at 0 and ∞ and maps -1 to 0. Moreover, by construction, $H_\eta := \chi_\eta(H_1)$ is a Herman ring for g_η with rotation number α. Thus, $g_\eta \in \mathcal{D}'_\alpha$ and the above construction defines a map $S_{g_1} : \mathbb{H}_+ \to \mathcal{D}'_\alpha$ which maps η to g_η.

The construction above is summarized in the following commutative diagram.

$$\begin{tikzcd}
(A_{r_\eta}, \mu_0) \arrow[r, "\mathcal{R}_\alpha"] & (A_{r_\eta}, \mu_0) \\
(A_{r_1}, \frac{\bar\partial h_\eta}{\partial h_\eta}) \arrow[u, "h_\eta"] \arrow[r, "\mathcal{R}_\alpha"] \arrow[d, "\phi_1"'] & (A_{r_1}, \frac{\bar\partial h_\eta}{\partial h_\eta}) \arrow[u, "h_\eta"'] \arrow[d, "\phi_1"] \\
(H_1, \mu_\eta) \arrow[r, "g_1"] \arrow[d, "\chi_\eta"'] & (H_1, \mu_\eta) \arrow[d, "\chi_\eta"] \\
(H_\eta, \mu_0) \arrow[r, "g_\eta"] & (H_\eta, \mu_0)
\end{tikzcd}$$

Finally, μ_η depends holomorphically on η: we have

$$\left.\frac{\partial g_\eta}{\partial \bar\eta}\right|_{\chi_\eta(z)} + \left.\frac{\partial g_\eta}{\partial z}\right|_{\chi_\eta(z)} \cdot \left.\frac{\partial \chi_\eta}{\partial \bar\eta}\right|_z + \left.\frac{\partial g_\eta}{\partial \bar z}\right|_{\chi_\eta(z)} \cdot \left.\overline{\frac{\partial \chi_\eta}{\partial \eta}}\right|_z = \left.\frac{\partial \chi_\eta}{\partial \bar\eta}\right|_{g_1(z)},$$

and since $\partial\chi_\eta/\partial\bar\eta \equiv 0$, $\partial g_\eta/\partial \bar z \equiv 0$, we get

$$\left.\frac{\partial g_\eta}{\partial \bar\eta}\right|_{\chi_\eta(z)} \equiv 0.$$

As a consequence, the map $S_{g_1} : \mathbb{H}_+ \to \mathcal{D}'_\alpha$ is analytic.

Step 3. Modulus and twist of $\mathbf{H}_\eta$. Our goal is to compute the modulus and the twist coordinate of the Herman ring H_η of the new map $g_\eta = S_{g_1}(\eta)$, in terms of η, m_1 and Θ_1. We will prove the following.

Proposition 4.5. *Let H_1 be the Herman ring of g_1 and let m_1 and Θ_1 be its modulus and twist coordinate. Given $\eta \in \mathbb{H}_+$ let H_η be the fixed Herman ring of $g_\eta = S_{g_1}(\eta)$. Denote its modulus by m_η and its twist coordinate by Θ_η. Then,*

(1) $m_\eta = m_1 \mathrm{Re}(\eta)$, *and*
(2) $\Theta_\eta = \Theta_1 - m_1 \mathrm{Im}(\eta)$.

Proof. For the proof we shall use the notation in Step 2.

We start with the modulus. Observe that the map $\phi_\eta := \chi_\eta \circ \phi_1 \circ h_\eta^{-1} : A_{r_\eta} \to H_\eta$ is holomorphic by construction and it conjugates $\mathcal{R}_\alpha : A_{r_\eta} \to A_{r_\eta}$ to $g_\eta : H_\eta \to H_\eta$. Hence it is a linearizing map for the Herman ring, and the modulus of H_η will be that of A_{r_η}. That is,

$$m_\eta = \frac{1}{2\pi} \log \frac{1}{r_\eta}.$$

Now, recall that

$$\tau_\eta = L_\eta(-2\pi m_1 + i\Theta_1) = -2\pi m_1 \mathrm{Re}(\eta) + 2\pi i(\Theta_1 - m_1 \mathrm{Im}(\eta)).$$

Hence $r_\eta = \mathrm{e}^{-2\pi m_1 \mathrm{Re}(\eta)}$ and we conclude $m_\eta = m_1 \mathrm{Re}(\eta)$.

We proceed now to compute Θ_η. To this end, we need to make the general construction of Section 3 with the map g_η and its Herman ring H_η. Using the notation in that section, observe that we have the following diagram (compare to Figure 6).

$$\begin{tikzcd}
A_{r_\eta} \arrow[d, "\hat\phi_\eta"'] & A_{r_\eta} \arrow[l, "r_\eta/\bar z"'] \arrow[d, "\phi_\eta"'] & A_1 \arrow[l, "h_\eta"'] \arrow[r, "r_1/\bar z"] \arrow[d, "\phi_1"] \arrow[lll, bend right, "\hat h_\eta"'] & A_1 \arrow[d, "\hat\phi_1"] \\
\widehat H_\eta \arrow[d, "\hat\psi_\eta"'] & H_\eta \arrow[l, "1/\bar z"'] \arrow[d, "\psi_\eta"'] & H_1 \arrow[l, "\chi_\eta"'] \arrow[r, "1/\bar z"] \arrow[d, "\psi_1"] & \widehat H_1 \arrow[d, "\hat\psi_1"] \\
\Delta_\alpha & \Delta_\alpha & \Delta_\alpha \arrow[l, "\chi"] & \Delta_\alpha \\
\mathbb{D} \arrow[u, "\varphi"'] \arrow[r, "\mathcal{C}_\eta"] & \mathbb{D} \arrow[u, "\varphi"'] & \mathbb{D} \arrow[u, "\varphi"] \arrow[l, "\mathcal{C}"'] & \mathbb{D} \arrow[u, "\varphi"] \arrow[l, "\mathcal{C}_1"'] \arrow[lll, bend left, "\widehat{\mathcal{C}}"]
\end{tikzcd}$$

In this diagram we introduce three new composition maps: $\psi, \mathcal{C}$ and $\widehat{\mathcal{C}}$, which are defined as to make the diagram commute.

We first comment on the bottom row which is the most important for our argument. Observe that $\mathcal{C}_1$ and $\mathcal{C}_\eta$ are the maps that define the twist coordinates for g_1 and g_η. Indeed, let $\gamma_1, \gamma_\eta, \Gamma_1, \Gamma_\eta$ be as in Section 3. Then $\mathcal{C}_1$ (resp. $\mathcal{C}_\eta$) restricted to the circle $\varphi^{-1}(\Gamma_1)$ (resp. $\varphi^{-1}(\Gamma_\eta)$) is a rigid rotation of angle Θ_1 (resp. Θ_η).

Next observe that both maps $\mathcal{C}, \widehat{\mathcal{C}} : \varphi^{-1}(\Gamma_1) \to \varphi^{-1}(\Gamma_\eta)$ are also analytic maps that conjugate the rigid rotation $\mathcal{R}_\alpha$ to itself. Hence, they must be rigid rotations composed with scaling maps. That is

$$\mathcal{C}|_{\varphi^{-1}(\Gamma_1)}(z) = s\mathrm{e}^{2\pi i\Theta} z$$

and

$$\widehat{\mathcal{C}}|_{\varphi^{-1}(\Gamma_1)}(z) = s\mathrm{e}^{2\pi i\widehat{\Theta}} z,$$

where $s \in \mathbb{R}$ and $\Theta, \widehat{\Theta} \in \mathbb{R}/\mathbb{Z}$. Our goal is then to compute the angles Θ and $\widehat{\Theta}$ since we have

$$\Theta_\eta = -\widehat{\Theta} + \Theta_1 + \Theta. \tag{1}$$

A key observation that we shall use later is the following.

Claim 4.6. The map $\mathcal{C}$ and $\widehat{\mathcal{C}}$ extend as the identity to the unit circle.

Proof. Observe that the map $\chi := \psi_\eta \circ \chi_\eta \circ \psi_1^{-1}$ is a globally defined quasiconformal map which conjugates the polynomial P_α to itself. It follows that χ can be extended as the identity to the boundary of the Siegel disk (for example, one can argue that χ must fix the orbit of the critical point which accumulates on the boundary of the Siegel disk).

Moreover, the restriction of χ to the annulus bounded by Γ and the boundary of the Siegel disk is quasiconformal. It follows that χ can be extended as the identity to the ideal boundary of the Siegel disk and hence $\mathcal{C}$ extends as the identity to the unit circle. The same argument applied to $\hat{\chi} := \hat{\psi}_\eta \circ \tau \circ \chi_\eta \circ \tau \circ \hat{\psi}_1^{-1}$ shows the same for $\widehat{\mathcal{C}}$. ∎

We proceed by computing some rotation angles of the maps in the topmost row. It is easy to check from the expressions of L_η and h_η that

$$h_\eta(r\mathrm{e}^{2\pi i\theta}) = \begin{cases} \mathrm{e}^{2\pi i\theta} & \text{if } r = 1 \\ \sqrt{r_\eta}\,\mathrm{e}^{2\pi i\theta - \pi i m_1 \mathrm{Im}(\eta)} & \text{if } r = \sqrt{r_1} \\ r_\eta\,\mathrm{e}^{2\pi i\theta - 2\pi i m_1 \mathrm{Im}(\eta)} & \text{if } r = r_1 \end{cases}$$

and

$$\hat{h}_\eta(r\mathrm{e}^{2\pi i\theta}) = \begin{cases} \mathrm{e}^{2\pi i\theta - 2\pi i m_1 \mathrm{Im}(\eta)} & \text{if } r = 1 \\ \sqrt{r_\eta}\,\mathrm{e}^{2\pi i\theta - \pi i m_1 \mathrm{Im}(\eta)} & \text{if } r = \sqrt{r_1} \\ r_\eta\,\mathrm{e}^{2\pi i\theta} & \text{if } r = r_1. \end{cases}$$

To transfer this information to the bottom maps we need to consider the vertical compositions. Let us then consider all the maps of the diagram restricted to the annuli bounded by the equator curves and the outer boundaries. Then, all the "vertical" compositions are conformal. It follows that the circle $\varphi^{-1}(\Gamma_1)$ (resp. $\varphi^{-1}(\Gamma_\eta)$) has radius $\sqrt{r_1}$ (resp. $\sqrt{r_\eta}$) and hence we have

$$\begin{tikzcd}
A_{\sqrt{r_\eta}} \arrow[d, "\widehat{R}_\eta"'] & A_{\sqrt{r_\eta}} \arrow[l, "r_\eta/\bar{z}"'] \arrow[d, "R_\eta"'] & A_{\sqrt{r_1}} \arrow[l, "h_\eta"'] \arrow[d, "R_1"] \arrow[r, "r_1/\bar{z}"] & A_{\sqrt{r_1}} \arrow[d, "\widehat{R}_1"] \arrow[lll, bend right, "\hat{h}_\eta"'] \\
A_{\sqrt{r_\eta}} \arrow[r, "\mathcal{C}_\eta"] & A_{\sqrt{r_\eta}} & A_{\sqrt{r_1}} \arrow[l, "\mathcal{C}"'] & A_{\sqrt{r_1}} \arrow[l, "\mathcal{C}_1"'] \arrow[lll, bend left, "\widehat{\mathcal{C}}"]
\end{tikzcd}$$

where $R_1, R_\eta, \widehat{R}_1$ and $\widehat{R}_\eta$ are rigid rotations

We first analize the innermost diagram to observe that, given that h_η commutes with rigid rotations, the map $\mathcal{C}$ is equal to h_η up to rotation. But then, we observed in Claim 4.6 that $\mathcal{C}$ extends as the identity to the unit circle as h_η does. It follows that $\mathcal{C} = h_\eta$ on $A_{\sqrt{r_1}}$ and hence,

$$\Theta = -\frac{1}{2} m_1 \mathrm{Im}(\eta).$$

Studying the outermost diagram and arguing as above, we have that $\widehat{\mathcal{C}}$ is equal to $\hat{h}_\eta$ up to composition with a rigid rotation. But again, $\widehat{\mathcal{C}}$ is the identity on S^1. It then follows that on $A_{\sqrt{r_1}}$

$$\widehat{\mathcal{C}} = \mathrm{e}^{2\pi i m_1 \mathrm{Im}(\eta)} \cdot \hat{h}_\eta$$

and hence, by knowing $\hat{h}_\eta$ on the equator, we conclude

$$\widehat{\Theta} = \frac{1}{2} m_1 \mathrm{Im}(\eta).$$

Finally, from equation (1) we obtain

$$\Theta_\eta = -\frac{1}{2} m_1 \mathrm{Im}(\eta) + \Theta_1 - \frac{1}{2} m_1 \mathrm{Im}(\eta) = \Theta_1 - m_1 \mathrm{Im}(\eta).$$

■

Step 4. A group property. Let $S : \mathcal{D}'_\alpha \times \mathbb{H}_+ \to \mathcal{D}'_\alpha$ be the map $S : (g, \eta) \mapsto S_g(\eta)$. Assume η_1 and η_2 are two complex numbers in $\mathbb{H}_+$, and $g_1 \in \mathcal{D}'_\alpha$. Our goal is to prove the following.

Lemma 4.7.

$$S(S(g_1, \eta_1), \eta_2) = S(g_1, L_{\eta_2}(\eta_1)).$$

Proof. Set

$$g_2 = S(g_1, \eta_1) \quad \text{and} \quad g_3 = S(g_2, \eta_2).$$

Then we want to show that

$$g_3 = S(g_1, \eta_3) \quad \text{with} \quad \eta_3 = L_{\eta_2} \circ L_{\eta_1}(1).$$

Indeed, for $i = 1$, 2 or 3, let H_i be the Herman ring of g_i and m_i be the modulus of H_i and set $r_i = \mathrm{e}^{-2\pi m_i}$. Observe that $L_{\eta_3} = L_{\eta_2} \circ L_{\eta_1}$. Those homeomorphisms project to homeomorphisms $h_1 : A_{r_1} \to A_{r_2}$, $h_2 : A_{r_2} \to A_{r_3}$ and $h_3 : A_{r_1} \to A_{r_3}$ such that $h_3 = h_2 \circ h_1$.

Now, let μ_1 be the Beltrami form which coincides with 0 outside $\bigcup_{n \geq 0} g_1^{-n}(H_1)$, with

$$\frac{\bar{\partial}(h_1 \circ \phi_1^{-1})}{\partial(h_1 \circ \phi_1^{-1})}$$

on H_1 and satisfies $g_1^*(\mu_1) = \mu_1$ everywhere. Let $\chi_1 : \mathbb{C} \to \mathbb{C}$ be the integrating map that fixes 0 and -1. Observe that the map

$$\phi_2 = \chi_1 \circ \phi_1 \circ h_1^{-1} : A_{r_2} \to H_2$$

is a homeomorphism that preserves the standard complex structure, thus, an isomorphism (and a linearizing map). So, we can define μ_2 on H_2 and $\chi_2 : \mathbb{C} \to \mathbb{C}$ and as above,

$$\phi_3 = \chi_2 \circ \phi_2 \circ h_2^{-1} : A_{r_3} \to H_3$$

is a linearizing map. Now, we have the following commutative diagram:

$$\begin{array}{ccccc} A_{r_1} & \xrightarrow{h_1} & A_{r_2} & \xrightarrow{h_2} & A_{r_3} \\ {\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow{\scriptstyle \phi_3} \\ H_1 & \xrightarrow{\chi_1} & H_2 & \xrightarrow{\chi_2} & H_3 \end{array}$$

Let us define $\chi_3 = \chi_2 \circ \chi_1$. On the other hand, let $\tilde{\chi}_3$, $\tilde{\mu}_3$ be the map and complex structure used in the construction of $\tilde{g}_3 := S(g_1, \eta_3)$. We will show that the complex structure μ_3 induced by ϕ_3 equals $\tilde{\mu}_3$. Since χ_3 and $\tilde{\chi}_3$ are normalized in the same way, these two integrating maps must be the same. Hence we can conclude that $g_3 = \tilde{g}_3$.

To this end, observe that the maps ϕ_i are holomorphic and $h_3 = h_2 \circ h_1$. Therefore on H_1, we have

$$\mu_3 = \frac{\bar{\partial}\chi_3}{\partial\chi_3} = \frac{\bar{\partial}(h_3 \circ \phi_1^{-1})}{\partial(h_3 \circ \phi_1^{-1})} = \tilde{\mu}_3.$$

Moreover, χ_3 conjugates g_1 to the holomorphic map g_3 and so, the Beltrami form μ_3 is g_1-invariant. So it follows that μ_3 and $\tilde{\mu}_3$ coincide on the set $\bigcup\limits_{n\geq 0} g_1^{-n}(H_1)$. Finally, since χ_1 is holomorphic outside $\bigcup\limits_{n\geq 0} g_1^{-n}(H_1)$ and χ_2 is holomorphic outside $\bigcup\limits_{n\geq 0} g_2^{-n}(H_2) = \chi_1\left(\bigcup\limits_{n\geq 0} g_1^{-n}(H_1)\right)$, we see that χ_3 is holomorphic outside $\bigcup\limits_{n\geq 0} g_1^{-n}(H_1)$. Therefore, $\mu_3 = \tilde{\mu}_3 = 0$ outside $\bigcup\limits_{n\geq 0} g_1^{-n}(H_1)$ and we conclude that $\mu_3 = \tilde{\mu}_3$ everywhere. ■

Step 5. $\mathbf{S_{g_1}(\eta + i/m) = S_{g_1}(\eta)}$**.** Again, assume $g_1 \in \mathcal{D}'_\alpha$, let H_1 be the Herman ring, m_1 be the modulus of H_1 and set $r_1 = \mathrm{e}^{-2\pi m_1}$. Let $\phi_1 : A_{r_1} \to H_1$ be an isomorphism, and let the map $S_{g_1} : \mathbb{H}_+ \to \mathcal{D}'_\alpha$ be as in Step 2. We will show that

$$(\forall\ \eta \in \mathbb{H}_+) \quad S_{g_1}(\eta + i/m) = S_{g_1}(\eta).$$

Thanks to the previous step, we can assume, without loss of generality, that $\eta = 1$. Indeed, if we know that $S_h(1 + \frac{i}{m}) = S_h(1)$ for all $h \in \mathcal{D}'_\alpha$ then, taking $h = S_{g_1}(\eta)$ the general equality follows.

We will construct a quasiconformal homeomorphism $\tilde{\chi}$ that satisfies

$$g_1 = \tilde{\chi} \circ g_1 \circ \tilde{\chi}^{-1}.$$

We will then show that $\tilde{\chi}$ integrates $\mu_{1+i/m}$ and fixes $0, \infty$ and -1. Therefore $\chi = \tilde{\chi}$ and

$$S_{g_1}(1) = g_1 = \tilde{\chi} \circ g_1 \circ \tilde{\chi}^{-1} = \chi \circ g_1 \circ \chi^{-1} = S_{g_1}(1 + i/m).$$

The quasiconformal homeomorphism $L_{1+i/m}$ preserves vertical strips and, in particular, it preserves

$$B = \{Z \in \mathbb{C} \mid \log r_1 < \mathrm{Re}(Z) < 0\}$$

It coincides with the identity on $i\mathbb{R}$ and with the translation by $-2i\pi$ on $\log r_1 + i\mathbb{R}$. Therefore, the quasiconformal homeomorphism $h_{1+i/m} : A_{r_1} \to A_{r_1}$ extends by the identity to the two boundary components of A_{r_1} (it is a Dehn twist).

Let us define $\tilde{\chi} = \phi_1 \circ h_{1+i/m} \circ \phi_1^{-1}$ on the Herman ring. Note that $\bar{\partial}\tilde{\chi}/\partial\tilde{\chi} = \mu_{1+i/m}$ on H_1 (since it preserves the standard complex structure) and $\tilde{\chi}$ is equal to the identity on the two ideal boundary components of H_1. In particular, $\tilde{\chi}$ extends by the identity to the two boundary components of H_1 in $\mathbb{C}$ (see [EMc]). Next, given a connected component U of $\bigcup_{n\geq 0} g_1^{-n}(H_1)$, let $n \geq 0$ be the smallest integer such that $g_1^n(U) = H_1$. The map $g_1^n : U \to H_1$ is an isomorphism and we set

$$\tilde{\chi}|_U = (g_1^{\circ n}|_U)^{-1} \circ \tilde{\chi} \circ g_1^n.$$

Note that $\tilde{\chi}|_U$ extends by the identity to ∂U. We can extend $\tilde{\chi}$ to a homeomorphism $\tilde{\chi} : \mathbb{C} \to \mathbb{C}$ such that the restriction of $\tilde{\chi}$ to $\mathbb{C} \setminus \bigcup_{n\geq 0} g_1^{-n}(H_1)$ is the identity. By a lemma of Rickman, $\tilde{\chi} : \mathbb{C} \to \mathbb{C}$ is quasiconformal. By construction,

$$\tilde{\chi} \circ g_1 = g_1 \circ \tilde{\chi} \quad \text{and} \quad \frac{\bar{\partial}\tilde{\chi}}{\partial\tilde{\chi}} = \mu_{1+i/m}.$$

Finally observe that -1, 0 and ∞ are not eventually mapped to H_1. Hence $\tilde{\chi}$ fixes these 3 points.

Step 6. Definition of the map $\mathbf{G_{g_1}} : \mathbb{D} \to \mathcal{D}'_\alpha$.

Let us still denote by m_1 the modulus of the Herman ring H_1 and let us now denote by θ_1 the twist coordinate of g_1. As above set $r_1 = \mathrm{e}^{-2\pi m_1}$. We can define $p : \mathbb{H}_+ \to \mathbb{D}^*$ as the universal covering

$$p : \eta \mapsto \delta = \exp(-2\pi m_1 \eta + 2i\pi\theta_1) = \exp(L_\eta(\tau_1)).$$

Since $S_{g_1}(\eta + i/m) = S_{g_1}(\eta)$, there exists a holomorphic map $G_{g_1} : \mathbb{D}^* \to \mathcal{D}'_\alpha$ such that $S_{g_1} = G_{g_1} \circ p$. That is, the following diagram commutes:

$$\begin{tikzcd} \mathbb{H}_+ \arrow[r, "S_{g_1}"] \arrow[d, "p"'] & \mathcal{D}'_\alpha \\ \mathbb{D}^* \arrow[ur, "G_{g_1}"'] & \end{tikzcd}$$

Claim 4.8. $G_{g_1}(\Pi(g_1)) = g_1$ and $\Pi \circ G_{g_1} = \mathrm{Id}$.

Proof. Just from the diagram we see that

$$G_{g_1}(r_1 \mathrm{e}^{2i\pi\theta_1}) = G_{g_1} \circ p(1) = S_{g_1}(1) = g_1.$$

Recall that $\Pi(g_1) = r_1 \mathrm{e}^{2\pi i\theta_1}$ and therefore we have shown that $G_{g_1}(\Pi(g_1)) = g_1$. We now want to show that $\Pi \circ G_{g_1} = \mathrm{Id}$.

In order to do so, let us first determine the "inverse" of p. It is easy to check that, for $\mathrm{e}^{-2\pi i m} = r < 1$ and $\theta \in \mathbb{R}/\mathbb{Z}$ we have

$$p\left(\frac{m}{m_1} + i\frac{\theta_1 - \theta}{m_1}\right) = r\mathrm{e}^{2\pi i\theta}.$$

Then,

$$\Pi \circ G_{g_1}(r\mathrm{e}^{2\pi i\theta}) = \Pi \circ S_{g_1}\left(\frac{m}{m_1} + i\frac{\theta_1 - \theta}{m_1}\right) = \Pi(g),$$

where $g \in \mathcal{D}'_\alpha$ has a Herman ring H of modulus $m_1 \frac{m}{m_1} = m$ and twist $\theta_1 - m_1 \frac{\theta_1 - \theta}{m_1} = \theta$ (see Proposition 4.5). It then follows that $\Pi(g) = r\mathrm{e}^{2\pi i\theta}$ and hence $\Pi \circ G_{g_1} = \mathrm{Id}$. ■

Claim 4.9. G_{g_1} extends to 0 by $G_{g_1}(0) = (\mathrm{e}^{2i\pi\alpha}, 0)$.

Proof. By proposition 2.2, we know that

$$\mathcal{D}'_\alpha \subset \left\{(\lambda, b) \in \mathbb{C}^* \times \mathbb{C} \;\middle|\; 1/4 \leq |\lambda| \leq 4 \text{ and } |b| \leq 1\right\}.$$

Thus, by the removable singularity theorem, $G_{g_1} : \mathbb{D}^* \to \mathcal{D}'_\alpha$ extends holomorphically at 0.

We know that the modulus of the Herman ring H_δ of the map $G_{g_1}(\delta)$ is $\frac{1}{2\pi}\log\frac{1}{|\delta|}$. When $\delta \in \mathbb{D}^*$, the modulus of the Herman ring tends to $+\infty$. This Herman ring H_δ separates -1 which is mapped to 0 and $-b$ which is mapped to ∞. So, we have $b \to 0$ as $\delta \to 0$.

The map $\delta \mapsto b(\delta)$ is holomorphic, non constant and vanishes at $\delta = 0$. Therefore, there exists $\varepsilon > 0$ such that $b_1(\mathbb{D})$ contains the interval $[0, \varepsilon]$. When $b \in (0, 1/9)$, there is at most one parameter λ such that $(\lambda, b) \in \mathcal{D}'_\alpha$ (see Prop. 2.5). In addition, $\lambda = e^{2i\pi t} \in S^1$ and $(t, \sqrt{b})$ belongs to the Arnold tongue $\mathcal{T}_\alpha$. When $b \in \mathcal{T}_\alpha$ tends to 0, t tends to α, λ tends to $e^{2i\pi\alpha}$ and so, $G_{g_1}(0) = (e^{2i\pi\alpha}, 0)$. ■

Step 7. The map G_{g_1} is surjective. In order to complete the proof of proposition 4.3, it is now enough to show that G_{g_1} is surjective. Thus, we must prove that for all $g_1, g_2 \in \mathcal{D}'_\alpha$, there exists $\delta \in \mathbb{D}^*$ such that $g_2 = G_{g_1}(\delta)$. Thanks to step 3, it is enough to prove that there exist δ_1 and δ_2 such that $G_{g_1}(\delta_1) = G_{g_2}(\delta_2)$.

Again, if $G_{g_1}(\delta) = (\lambda_1(\delta), b_1(\delta))$, the map $\delta \mapsto b_1(\delta)$ is holomorphic (see Step 2), and vanishes at $\delta = 0$. Therefore, there exists $\varepsilon \in (0, 1/9)$ such that $b_1(\mathbb{D})$ contains the interval $[0, \varepsilon]$. Similarly, if $\varepsilon \in (0, 1/9)$ is small enough and $G_{g_2}(\delta) = (\lambda_2(\delta), b_2(\delta))$, then $b_2(\mathbb{D})$ contains the interval $[0, \varepsilon]$. Therefore, if we pick $b \in (0, \varepsilon]$, there exist $\delta_1, \delta_2 \in \mathbb{D}^*$ such that $b_1(\delta_1) = b_2(\delta_2) = b$. Now, since $b \in (0, 1/9)$, there is at most one parameter λ such that $(\lambda, b) \in \mathcal{D}'_\alpha$ (see Proposition 2.5). Moreover $\lambda = e^{2i\pi t} \in S^1$ and $(t, \sqrt{b})$ belongs to the Arnold tongue $\mathcal{T}_\alpha$. So $\lambda_1(\delta_1) = \lambda_2(\delta_2)$. Thus $G_{g_1}(\delta_1) = G_{g_2}(\delta_2)$ and the proof is completed. ■

We can then take $\mathcal{G}_\alpha = G_{g_1}$ and this concludes the proof of Proposition 4.3.

Corollary 4.10. *The map $\mathcal{G}_\alpha$ induces a continuous injection from $[0, 1)$ to $\mathcal{T}_\alpha$. In particular, the only maps in $\mathcal{D}'_\alpha$ with twist parameter 0 are those corresponding to the Arnold tongue $\mathcal{T}_\alpha$.*

4.2. **Conclusion of the proof of Theorem A.** Observe that the map $(\lambda, a) \mapsto (\lambda, a^2)$ provides a ramified covering of degree 2 from $\mathcal{D}_\alpha$ to $\mathcal{D}'_\alpha$, ramified at $(e^{2i\pi\alpha}, 0)$ and above $(e^{2i\pi\alpha}, 0)$. Thus, the map $\mathcal{G}_\alpha : \mathbb{D} \to \mathcal{D}'_\alpha$ lifts to a map $\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\mathcal{F}_\alpha"] \arrow[d, "\delta\mapsto\delta^2"'] & \mathcal{D}_\alpha \arrow[d, "(\lambda,a)\mapsto(\lambda,a^2)"] \\ \mathbb{D} \arrow[r, "\mathcal{G}_\alpha"'] & \mathcal{D}'_\alpha \end{tikzcd} \tag{2}$$

The map $\mathcal{F}_\alpha$ is an isomorphism between $\mathbb{D}$ and $\mathcal{D}_\alpha$, it maps 0 to $(e^{2i\pi\alpha}, 0)$ and the modulus of the Herman ring of $f_{\mathcal{F}_\alpha(\delta)}$ is

$$\frac{1}{2\pi}\log\frac{1}{|\delta|^2} = \frac{1}{\pi}\log\frac{1}{|\delta|}.$$

Let us set $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$. Since $\lambda(\delta) = \lambda(-\delta)$, we have

$$\frac{\partial\lambda}{\partial\delta}(0) = 0.$$

Moreover, we claim that

$$\frac{\partial a}{\partial\delta}(0) = r_\alpha$$

This is a consequence of the following proposition. The analogue for the complex standard family has been proved by Fagella, Seara and Villanueva in [FSV].

Proposition 4.11. *Let us fix a Bruno number α, and for $a \in (0,1/3)$ small enough, let $t_a \in \mathbb{R}/\mathbb{Z}$ be the unique angle such that (t_a, a) belongs to the Arnold tongue $\mathcal{T}_\alpha$. Then, as $a \to 0$, the modulus m_a of the Herman ring of $f_{e^{2i\pi t_a},a}$ satisfies*

$$m_a = \frac{1}{\pi}\log\frac{r_\alpha}{a} + o(1).$$

Proof. Let us again work with the family $g_{\lambda,b}$ with $b = a^2$. In the whole proof, we consider that (λ, b) varies in the Arnold disk $\mathcal{D}'_\alpha$. More precisely, we assume that

$$(\lambda, b) = (\lambda(\delta), b(\delta)) = \mathcal{G}_\alpha(\delta), \quad \text{with} \quad \delta \in \mathbb{D}.$$

Observe that when $\delta \to 0$, the maps $g_{\lambda,b}$ converge to P_α uniformly on every compact subset of $\mathbb{C}^*$. In particular, one of the critical points of $g_{\lambda,b}$, let's say $\omega(\delta)$ converges to the critical point $\omega(0) = -1/2$ of P_α.

The maps

$$\delta \mapsto g^n_{\lambda(\delta),b(\delta)}(\omega(\delta))$$

are all well defined and holomorphic for $\delta \in \mathbb{D}$. Moreover, they define a holomorphic motion of the orbit of the critical point $\omega(\delta)$, parametrized by $\delta \in \mathbb{D}$. By the Mañe-Sad-Sullivan λ-lemma [MSS], this holomorphic motion extends to a holomorphic motion of the closure of this critical orbit.

Let $B(\delta)$ be the outer boundary component of the Herman ring of $g_{\lambda,b}$ for $\delta \neq 0$ and the boundary of the Siegel disk for $\delta = 0$. Moreover, let $\Delta(\delta)$ be the bounded component of $\mathbb{C} \setminus B(\delta)$. Since $B(\delta)$ is contained in the closure of the orbit of $\omega(\delta)$, we see that $B(\delta)$ moves holomorphically with respect to $\delta \in \mathbb{D}$. As a consequence, $(\Delta(\delta), 0)$ depends continuously on $\delta \in \mathbb{D}$ for the Carathéodory topology. Thus, its conformal radius at 0 varies continuously with respect to $\delta \in \mathbb{D}$.

Let us now assume that $\delta \in (0,1)$, and thus, that $b(\delta) > 0$ (see Corollary 4.10). Then, the Herman ring is symmetric with respect to the circle centered at 0 with radius $\sqrt{b}$. It follows that the modulus $m(\delta)$ of the Herman ring is equal to twice the modulus of the annulus bounded by this circle and $B(\delta)$. As $\delta \to 0$, we have $b \to 0$ and the conformal radius of $\Delta(\delta)$ tends to r_α. It follows from lemma 4.12 below that

$$m(\delta) = 2 \cdot \frac{1}{2\pi}\log\frac{r_\alpha}{\sqrt{b}} + o(1).$$

Using the relation $b = a^2$ and $m_a = m(\delta)$, we get

$$m_a = \frac{1}{\pi}\log\frac{r_\alpha}{a} + o(1).$$

■

Lemma 4.12. *There exist a function $h : [0,1/4) \to \mathbb{R}$ with $h(x) = \mathcal{O}(x)$, such that for any topological disk $D \subset \mathbb{C}$ containing 0 and any $\varepsilon \in (0, \mathrm{rad}(D)/4)$, the set $U_\varepsilon = D \setminus \overline{\mathbb{D}}_\varepsilon$ is an annulus of modulus m_ε with*

$$h\left(\frac{\varepsilon}{\mathrm{rad}(D)}\right) \leq m_\varepsilon - \frac{1}{2\pi}\log\left(\frac{\mathrm{rad}(D)}{\varepsilon}\right) \leq 0.$$

Proof. Rescaling if necessary, we may suppose that $\mathrm{rad}(D) = 1$. Note that when $\varepsilon < 1/4$, it follows from Koebe's 1/4-theorem that U_ε is an annulus. Let $\phi : (\mathbb{D}, 0) \to (D, 0)$ be a conformal representation. By Koebe's distorsion theorem, for all $z \in \mathbb{D}$,

$$\frac{|z|}{(1+|z|)^2} \leq |\phi(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Therefore, $\gamma_\varepsilon = \phi^{-1}(C(0,\varepsilon))$ is contained in the annulus $\{r_\varepsilon < |z| < R_\varepsilon\}$ with $r_\varepsilon, R_\varepsilon \in (0,1)$ defined by

$$\varepsilon = \frac{r_\varepsilon}{(1-r_\varepsilon)^2} = \frac{R_\varepsilon}{(1+R_\varepsilon)^2}.$$

The annulus bounded by γ_ε and S^1 is isomorphic to U_ε, and so,

$$\frac{1}{2\pi}\log\frac{1}{R_\varepsilon} \le m \le \frac{1}{2\pi}\log\frac{1}{r_\varepsilon}.$$

So, we can take

$$h(\varepsilon) = \frac{1}{2\pi}\log\frac{\varepsilon}{R_\varepsilon} = \mathcal{O}(\varepsilon).$$

We also see that

$$m \le \frac{1}{2\pi}\log\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon).$$

This last inequality can be improved as follows. Choose $\varepsilon' < \varepsilon$ small. The circle of radius ε subdivides the annulus $U_{\varepsilon'}$ into the two annuli U_ε and $\{z \mid \varepsilon' < |z| < \varepsilon\}$. By the Grötzsch inequality, we have

$$\frac{1}{2\pi}\log\frac{\varepsilon}{\varepsilon'} + m_\varepsilon \le m_{\varepsilon'} = \frac{1}{2\pi}\log\frac{1}{\varepsilon'} + \mathcal{O}(\varepsilon').$$

We get the required inequality by letting ε' tend towards 0. ■

We now want to conclude from Prop. 4.11 the fact that $\dfrac{\partial a}{\partial \delta}(0) = r_\alpha$. Remember that for $\delta \in \mathbb{D}$ and $(\lambda, a) = \mathcal{F}_\alpha(\delta)$, we have

$$m_a = \frac{1}{\pi}\log\frac{1}{|\delta|}.$$

Thus, when $\delta \in (0,1)$, we have

$$\frac{1}{\pi}\log\frac{1}{\delta} = \frac{1}{\pi}\log\frac{r_\alpha}{a} + o(1) = \frac{1}{\pi}\log\left(\frac{r_\alpha}{a}(1+o(1))\right).$$

So, as $\delta \to 0$, we get

$$\frac{1}{\delta} = \frac{r_\alpha}{a}(1+o(1)).$$

In particular, we see that $a = \delta r_\alpha + \delta o(1) = \delta r_\alpha + o(\delta)$, which proves that

$$\frac{\partial a}{\partial \delta}(0) = r_\alpha$$

and completes the proof of Theorem A.

5. Proof of theorem B

Part (a) follows from $\partial a/\partial\delta(0) = r_\alpha \neq 0$.

Part (b) is obtained as follows. Since $\delta \mapsto a(\delta)$ is holomorphic at 0, we know that

$$a(\delta) = r_\alpha\delta + \mathcal{O}(\delta^2).$$

Moreover, by following the commutative diagram (2) we have that $\mathcal{F}_\alpha(-\delta) = (\lambda(\delta), -a(\delta))$ and therefore $a(\delta)$ is an odd function and

$$a(\delta) = r_\alpha\delta + \mathcal{O}(\delta^3).$$

By the inverse function theorem it follows that

$$\delta(a) = \frac{a}{r_\alpha} + \mathcal{O}(a^3).$$

In particular, as $a \to 0$, we have

$$\mathrm{e}^{\pi m_a} = \frac{1}{\delta(a)} = \frac{r_\alpha}{a} + \mathcal{O}(a).$$

6. The complex standard family.

We will now explain how to adapt the results to the complex standard family

$$f_{\lambda,a}(z) = \lambda z \mathrm{e}^{\frac{a}{2}(z-1/z)}.$$

In this section, we will no longer consider cubic rational maps and quadratic polynomials. We will therefore feel free to use the same notations as in the rest of the article to denote similar but different maps and objects.

The complex standard family has been extensively studied in the case $\lambda = \mathrm{e}^{2i\pi\alpha} \in S^1$ and $a \in \mathbb{R}$: it lifts via $Z \mapsto z = \mathrm{e}^{2i\pi Z}$ to the Arnold family

$$F_{\alpha,a}(Z) = Z + \alpha + \frac{a}{2\pi}\sin(2\pi Z).$$

For $\alpha \in \mathbb{R}$ and $a \in (-1,1)$, these maps restrict to analytic diffeomorphisms of $\mathbb{R}/\mathbb{Z}$.

Let us now assume that λ and a are complex parameters, and that $f_{\lambda,a}$ has a fixed Herman ring $H \subset \mathbb{C}^*$ with rotation number α. By the maximum modulus principle, H separates the essential singularities 0 and ∞. Following [G], we can perform Shishikura's surgery in order to obtain an entire mapping fixing 0 and having a Siegel disk with rotation number α around 0. By construction, this map does not vanish, except at 0 and may be normalized to have a critical point at -1. In fact, Geyer shows that the resulting map is given by

$$E_\alpha(z) = \mathrm{e}^{2i\pi\alpha} z \mathrm{e}^z,$$

and moreover, that E_α has a Siegel disk around 0 if and only if α is a Bruno number. Therefore a map $f_{\lambda,a}$ may have a fixed Herman ring only when α is a Bruno number.

Definition 6.1. Given a Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, we let Δ_α be the Siegel disk of the entire mapping E_α and we let $\mathcal{D}_\alpha$ be the set of parameters $(\lambda, a) \in \mathbb{C}^* \times \mathbb{C}$ such that $f_{\lambda,a}$ has a fixed Herman ring with rotation number α. We shall call $\mathcal{D}_\alpha$ the *Arnold disk of rotation number* α.

As in the case of cubic rational maps, to any map $f \in \mathcal{D}_\alpha$ we can associate two coordinates: the modulus of the Herman ring and a twist coordinate defined via the surgery construction (see Section 3.

Also, the change of coordinates $z \mapsto -z$ conjugates $f_{\lambda,a}$ to $f_{\lambda,-a}$ and it is thus useful to introduce a new family

$$g_{\lambda,b} : w \mapsto \lambda w \mathrm{e}^{w - b/(4w)}.$$

The map $f_{\lambda,a}$ is conjagate to the map g_{λ,a^2} via $z \mapsto w = az/2$, i.e.,

$$f_{\lambda,a}(z) = \frac{2}{a} g_{\lambda,a^2}\left(\frac{a}{2}z\right).$$

Note that $g_{\lambda,0}$ is the entire mapping $w \mapsto \lambda w \mathrm{e}^w$.

Definition 6.2. Given a Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, let $\mathcal{D}'_\alpha$ be the set of parameters (λ, b) such that $g_{\lambda,b}$ has a fixed Herman ring with rotation number α.

By convention, we consider that for $\lambda = \mathrm{e}^{2i\pi\alpha}$ and $b = 0$, the entire mapping $g_{\lambda,b} = E_\alpha$ has a Herman ring $\Delta_\alpha \setminus \{0\}$ of infinite modulus. As in the case of the cubic rational maps studied in this article, one can show the following proposition.

Proposition 6.3. *For any Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, there is an isomorphism $\mathcal{G}_\alpha : \mathbb{D} \to \mathcal{D}'_\alpha$ with $\mathcal{G}_\alpha(0) = (\mathrm{e}^{2i\pi\alpha}, 0)$. Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $g_{\mathcal{G}_\alpha(\delta)}$ is equal to* $\dfrac{1}{2\pi}\log\dfrac{1}{|\delta|}$.

The only difference in the proof appears when one shows that a certain map $G : \mathbb{D}^* \to \mathcal{D}'_\alpha$ has a removable singularity at 0 (this corresponds to Claim 4.9). Indeed, we do not know whether the Arnold disks $\mathcal{D}'_\alpha$ are compactly contained in $\mathbb{C}^* \times \mathbb{C}$.

Remark 6.4. This raises the following questions: are the Siegel disks Δ_α bounded in $\mathbb{C}$? Are the Arnold disks $\mathcal{D}_\alpha$ and $\mathcal{D}'_\alpha$ bounded in $\mathbb{C}^* \times \mathbb{C}$?

In order to overcome this difficulty, we set $G(\delta) = (\lambda(\delta), b(\delta))$ for $\delta \in \mathbb{D}^*$ and argue as follows: First, as in the case of the cubic rational maps, the modulus of the Herman ring tends to ∞ as $\delta \to 0$.. The Herman ring separates the critical points which are the roots of the equation $4z^2 + 4z + b(\delta) = 0$. It follows that as $\delta \to 0$, we have $b(\delta) \to 0$ (one of the critical points tends to 0 and the other tends to -1). To show that G has a removable singularity at $\delta = 0$, it is sufficient to show that when $\delta \to 0$, $\lambda(\delta)$ remains uniformly bounded away from 0 and ∞. This is a consequence of the following lemma and the fact that $b(\delta) \underset{\delta\to 0}{\longrightarrow} 0$.

Lemma 6.5. *Suppose $|a| < 1$ and $(\lambda, a) \in \mathcal{D}_\alpha$, then*

$$e^{-|a|} \le |\lambda| \le e^{|a|}.$$

Proof. Suppose $(\lambda, a) \in \mathcal{D}_\alpha$, $|a| < 1$ and $|\lambda| < e^{-|a|}$. Then the image $\Gamma = f_{\lambda,a}(S^1)$ of the unit circle is an $\mathbb{R}$-analytic Jordan curve in $\mathbb{D}$, and $f_{\lambda,a} : S^1 \to \Gamma$ is a diffeomorphism (this is easily seen by working in the lifted $F_{\omega,a}(Z) = Z + \omega + \frac{a}{2\pi}\sin(2\pi Z)$ with $\mathrm{Im}(\omega) > 0$ and $\lambda = e^{2i\pi\omega}$). We then get a contradiction as in the proof of Prop. 2.5. Similarly, we cannot have $|\lambda| > e^{|a|}$. ■

With the help of proposition 6.3, we can now proceed exactly as in the case of cubic rational maps, and we obtain the following results.

Theorem A'. *For any Bruno number $\alpha \in \mathbb{R}/\mathbb{Z}$, the set $\mathcal{D}_\alpha$ is a Riemann surface isomorphic to the unit disk and there is an isomorphism $\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha$ such that*

$$\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0) \quad \text{and} \quad \mathcal{F}'_\alpha(0) = (0, 2r_\alpha),$$

where r_α is the conformal radius of the entire mapping $E_\alpha(z) = e^{2i\pi\alpha} z e^z$. Moreover, for any $\delta \in \mathbb{D}$, the modulus of the Herman ring of $f_{\mathcal{F}_\alpha(\delta)}$ is equal to $\dfrac{1}{\pi}\log\dfrac{1}{|\delta|}$.

Theorem B'. *Assume α is a Bruno number. Then,*

(a) *the Arnold disk can be locally parameterized by a in a neighborhood of $(e^{2i\pi\alpha}, 0)$ (i.e., it is locally the graph of a holomorphic map $a \mapsto \lambda(a)$)*
(b) *as $|a| \to 0$, the modulus m_a of the Herman ring of $f_{\lambda(a),a}$ satisfies*

$$e^{\pi m_a} = \frac{2r_\alpha}{|a|} + \mathcal{O}(a).$$

References

[A] V.I. Arnol'd, *Small denominators. I: Mappings of the circumference onto itself*, Am. Math. Soc., Transl., II. Ser. 46, 213-284 (1965); translation from Izv. Akad. Nauk SSSR, Ser. Mat. 25, 21-86 (1961).

[Br] A.D. Bruno, *Analytic forms of differential equations*, Trans. Mosc. Math. Soc. **25** (1971).

[Bu] X. Buff, *Virtually Repelling Fixed Points*, Publicacions Matemàtiques **47**, 195-209 (2003).

[BE] X. Buff & A. Epstein, *A parabolic Yoccoz inequality*, Fund. Math.**172**, 249–289 (2002) .

[D] A. Denjoy, *Sur les courbes définies par les équations differentielles à la surface du tore*, J. Math. Pure et Appl. **11**, série 9, 333–375 (1932).

[EMc] C. Earle & C.T. McMullen, *Quasiconformal isotopies*, Holomorphic Functions and Moduli I, 143–154, Springer-Verlag, MSRI publications vol. 10, (1988).

[FG] N. Fagella, L. Geyer, *Surgery on Herman rings of the complex standard family*, Ergod. Th. & Dynam. Sys., **23**, 493–508 (2003).

[FSV] N. Fagella, T.M. Seara & J. Villanueva, *Asymptotic Size of Herman Rings of the Complex Standard Family by Quantitative Quasiconformal Surgery*, To appear in Ergod. Th. & Dynam. Sys.

[G] L. GEYER, *Siegel discs, Herman rings and the Arnold family*, *Trans. Amer. Math. Soc.* **353:9** (2001), 3661–3683 (electronic).

[H] C. HENRIKSEN, *Holomorphic Dynamics and Herman rings*, MS Thesis, Technical Univ. of Denmark, (1997).

[dMvS] W. DE MELO, S. VAN STRIEN, *One-Dimensional Dynamics*, Springer-Verlag, (1991).

[McS] C. MCMULLEN, D. SULLIVAN *Quasiconformal homeomorphisms and dynamics. III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math., **135**, No.2, 351–395 (1998).

[MSS] R. MAÑE, P. SAD & D.P. SULLIVAN, *On the dynamics of rational maps*, Ann. Sci. Éc. Norm. Sup., Paris, 16:193-217, (1983).

[P] H. POINCARÉ, *Sur les courbes défines par des équations différentielles*, J. Math. Pures et Appl., 4^{eme} série, 1:167–244, (1885).

[R] E. RISLER, *Linéarisation des perturbations holomorphes des rotations et applications*, Mém. Soc. Math. Fr., **77**, (1999).

[Sh1] M. SHISHIKURA, *On the quasiconformal surgery of rational functions*, Ann. Sci. Éc. Norm. Sup., **vol** 20, 1-29, (1987).

[Sh2] M. SHISHIKURA, *Connectivity of the Julia set and fixed point.* Preprint, Institut des Hautes Etudes Scientifiques, IHES/M/90/37 (1990).

[Y] J.C. YOCCOZ, *Petits diviseurs en dimension 1*, S.M.F., Astérisque **231** (1995).

UNIVERSITÉ PAUL SABATIER, LABORATOIRE EMILE PICARD, 118, ROUTE DE NARBONNE, 31062 TOULOUSE CEDEX, FRANCE

E-mail address: buff@picard.ups-tlse.fr

DEPT. DE MAT. APLICADA I ANALISI, UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08007 BARCELONA, SPAIN

E-mail address: fagella@maia.ub.es

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 525 E UNIVERSITY AVENUE, ANN ARBOUR, MI 48109-1109, USA

E-mail address: geyer@umich.edu

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF DENMARK, MATEMATIKTORVET, BUILDING 303, DK-2800 KGS. LYNGBY, DENMARK

E-mail address: chris@mat.dtu.dk