MULTIPLIERS FOR ENTIRE FUNCTIONS AND AN INTERPOLATION PROBLEM OF BEURLING

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Abstract. We characterize the interpolating sequences for the Bernstein space of entire functions of exponential type, in terms of a Beurling-type density condition and a Carleson-type separation condition. Our work extends a description previously given by Beurling in the case that the interpolating sequences are restricted to the real line. An essential role is played by a multiplier lemma, which permits us to link techniques from Hardy spaces with entire functions of exponential type. We finally present a characterization of the sampling sequences for the Bernstein space, also extending a density theorem of Beurling.

1. Introduction

In classical work on interpolation in Paley-Wiener-type spaces, one constructs functions with given values at points along the real line. If instead no a priori assumption is made about the location of the points at which one interpolates, the nature of the subject changes in an interesting way, and one is led to combine techniques from entire functions and Hardy spaces in a nontrivial manner, as shown for example in [**LS**].

In this paper, we solve such an extended version of an interpolation problem studied by Beurling [**B**, pp. 351–365], concerning the classical Bernstein space $\mathcal{B}_\tau$ ($\tau > 0$), i.e., the set of all entire functions of exponential type at most τ whose restrictions to the real line belong to $L^\infty(\mathbb{R})$. We say that a sequence Λ of distinct points in the complex plane is an *interpolating sequence for* $\mathcal{B}_\tau$ if the interpolation problem $f(\lambda_k) = a_k$ has a solution $f \in \mathcal{B}_\tau$ whenever the sequence $\{a_k\}$ satisfies the compatibility condition

$$\sup_k |a_k| e^{-\tau |\operatorname{Im} \lambda_k|} < +\infty.$$

This condition is natural because the Phragmén-Lindelöf principle implies

$$|f(z)| \leq \|f\|_{L^\infty(\mathbb{R})} e^{\tau |\operatorname{Im} z|}$$

for all functions $f \in \mathcal{B}_\tau$ and $z \in \mathbb{C}$. (This inequality also shows that $\mathcal{B}_\tau$ is complete with respect to the $L^\infty(\mathbb{R})$-norm.) Beurling gave a description of those interpolating sequences Λ which consist only of real numbers. We shall extend this characterization to arbitrary complex sequences.

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It is convenient to introduce the following distance function:

$$\sigma(z,\zeta) = \frac{|z-\zeta|}{1+|z-\bar{\zeta}|}.$$

We say that Λ is *separated* if there exists a number $\delta > 0$ such that $\sigma(\lambda_j, \lambda_k) \geq \delta$ whenever $j \neq k$. Moreover, Λ is said to satisfy the *two-sided Carleson condition* if for any disk D centered on the real axis and of radius $r(D)$, we have

$$\sum_{\lambda_k \in D\cap\Lambda} |\operatorname{Im} \lambda_k| \leq Cr(D),$$

with C independent of D. The smallest such constant C will be referred to as the *Carleson constant* of Λ.

In addition, we need an extension of Beurling's notion of upper uniform density. Suppose Λ is a separated sequence and let A be a positive number. Denote by $n_A^+(r)$ the maximum number of points from Λ to be found in a rectangle of the form $\{z = x+iy : \ t < x < t+r,\ |y| < A\}$, where t is any real number. The upper uniform density of Λ is defined to be

$$D^+(\Lambda) = \lim_{A\to\infty} \lim_{r\to\infty} \frac{n_A^+(r)}{r}.$$

Our main theorem is the following.

Theorem 1. *A sequence Λ is interpolating for $\mathcal{B}_\tau$ if and only if it is separated, satisfies the two-sided Carleson condition, and $D^+(\Lambda) < \tau/\pi$.*

It is plain to extend Theorem 1 to cover the Paley-Wiener spaces L^p_τ, $0 < p \leq 1$, in line with what is done in [**F**]. Only a partial extension can be made for Paley-Wiener spaces with $1 < p < \infty$, because of the existence of so-called complete interpolating sequences [**LS**].

The Carleson condition and the transition $A \to \infty$ in the definition of $D^+(\Lambda)$, which are relevant only if the numbers $|\operatorname{Im} \lambda_k|$ are unbounded, are the new ingredients as compared to Beurling's theorem. Neither the necessity nor the sufficiency of the conditions in Theorem 1 are straightforward consequences of the latter result, although the proof of the necessity (see Section 2 below) builds on a basic idea from Beurling's work.

The most difficult part and the main novelty of this paper is the proof of the sufficiency of the conditions in Theorem 1. The main idea is to "reduce" the original interpolation to a problem which is close to Carleson's classical interpolation problem for H^∞ [**C**]. While Carleson's original duality proof does not apply, two alternative techniques do: We may solve the problem via Jones' explicit solution of the $\bar{\partial}$ equation in a half-plane [**J**], or we may apply Earl's elementary method [**E**], which produces solutions in terms of interpolating Blaschke products; the Blaschke products of Earl's method are not entire functions, but they can be corrected in a proper way to give entire functions of desired kind. Both methods are intrinsically interesting, and will be presented below (see Sections 4 and 5).

The essential ingredient both in transforming the problem and in correcting the Blaschke products obtained by Earls's method is presented in Section 3: It is an elementary lemma on multipliers, playing a role similar to that of the famous multiplier theorem of Beurling and Malliavin [**BM**, **K**]. The applications to the interpolation rest on certain very precise estimates on the multipliers, which can be obtained by elementary means (contrasting the Beurling-Malliavin construction) due to the fact that our problem is invariant under real translations. No claim is made about originality; the construction underlying our multiplier lemma can certainly be found in different forms at many places in the literature.

Finally, to complete the picture, we present in Section 6 the solution of a corresponding sampling problem, which arises from Beurling's study of balayage of Fourier-Stieltjes transforms [**B**, pp. 341–350]. Our presentation here is brief, because the solution is to a large extent a straightforward extension of Beurling's theorem.

2. The necessity of the conditions in Theorem 1

We start by proving the necessity of separation and the Carleson condition. For any real number a we set

$$\mathbb{C}_a^+ = \{z \in \mathbb{C} : \ \operatorname{Im} z > a\}, \qquad \mathbb{C}_a^- = \{z \in \mathbb{C} : \ \operatorname{Im} z < a\}.$$

If Λ is an interpolating sequence for $\mathcal{B}_\tau$, then $\Lambda \cap \mathbb{C}_a^+$ is an interpolating sequence for the space $H^\infty(\mathbb{C}_a^+)$ of bounded analytic functions in $\mathbb{C}_a^+$. To see this, consider $\Lambda' = \mathbb{C}_a^+ \cap \Lambda$. For any bounded sequence $\{c_k\}$ set $a_k = c_k e^{-i\tau\lambda_k'}$. Clearly $\sup_k |a_k| e^{-\tau \operatorname{Im} \lambda_k'} < \infty$, and therefore there is a function $g \in \mathcal{B}_\tau$ such that $g(\lambda_k') = a_k$. The function $f = g e^{i\tau z}$ is bounded in $\mathbb{C}_a^+$ and solves $f(\lambda_k') = c_k$. Likewise, it is clear that $\Lambda \cap \mathbb{C}_a^-$ is an interpolating sequence for $H^\infty(\mathbb{C}_a^-)$. Therefore, by Carleson's interpolation theorem [**G**, p. 287], we may conclude that the sequence Λ is hyperbolically separated and verifies the (one-sided) Carleson condition in any half-plane $\mathbb{C}_a^+$ or $\mathbb{C}_a^-$. This statement is equivalent to the separation and two-sided Carleson condition of Theorem 1.

We turn to the proof of the necessity of the condition $D^+(\Lambda) < \tau/\pi$. Let S_A denote the strip $|\operatorname{Im} z| < A$. We begin by noting that the two-sided Carleson condition (or even the separation) implies the following upper bound on the density outside S_A: $D^+(\Lambda \setminus S_A) \leq K/A$, with K a positive constant. Our plan is to show that, on the other hand, for wide strips S_A we have

$$D^+(\Lambda \cap S_A) \leq \tau/\pi - C(\log A)/A \tag{1}$$

for some positive constant C. The density condition $D^+(\Lambda) < \tau/\pi$ will then follow by combining the two estimates for a sufficiently large A.

The proof of (1) will be based on two lemmas of Beurling [**B**, p. 353], as well as the Carleson condition. Before stating these lemmas, we need to recall that if Λ is interpolating, there exists a positive constant K such that $f(\lambda_k) = a_k$ can be solved with $\|f\| \leq K \sup_k |a_k| e^{-\tau|\operatorname{Im} \lambda_k|}$; this follows from an argument based on the open mapping theorem. The smallest such constant K is called the *interpolation constant* of Λ, and we denote it by $K(\Lambda)$. Moreover, for every $x \in \mathbb{R}$ we set $\rho(x, \Lambda) = \sup_\phi |\phi(x)|$, where ϕ ranges over all functions $\phi \in \mathcal{B}_\tau$ such that $\phi(\lambda) = 0$, $\lambda \in \Lambda$ and $|\phi(x)| \leq 1$, $x \in \mathbb{R}$. Beurling's lemmas, which we state without proofs, are the following.

PSfrag replacements

x

s

FIGURE 1. The squares

Lemma 1. *If $x \in \mathbb{R} \setminus \Lambda$, then $K(\Lambda \cup \{x\}) \le \big(1 + 2K(\Lambda)\big)/\rho(x, \Lambda)$.*

Lemma 2. *Given δ, k and τ, there is a constant $C = C(\delta, k, \tau)$ such that if Λ is an interpolating sequence for $\mathcal{B}_\tau$ with interpolating constant $K(\Lambda) \le k$ and if $x \in \mathbb{R}$ satisfies $\operatorname{dist}(x, \Lambda) \ge \delta$, then $\rho(x, \Lambda) \ge C$.*

Lemmas 1 and 2 imply that given an interpolating sequence Λ, we can add a finite number of real points to the sequence and still have an interpolating sequence. In addition, the new interpolation constant has an upper bound which is independent of the position of the new points, as long as there is a lower bound on the separating distance of the extended sequence.

Let Q be a positive integer such that Q exceeds the Carleson constant of Λ. For every $x \in \mathbb{R}$ we may add $Q+1$ real points to the sequence Λ which are at most at distance 1 from x, and such that Λ plus the new $Q+1$ points constitute an interpolating sequence. By keeping a lower bound on the separating distance, we get an upper bound on the interpolation constant, which is independent of x, and we note that trivially the Carleson constant remains bounded by Q. Lemma 2 shows that if the distance from x to Λ exceeds 1, we may build a function $f \in \mathcal{B}_\tau$ such that $f(x) = 1$ and $f(\lambda) = 0$ if $\lambda \in \Lambda$ or λ is one of the $Q+1$ extra points added, with $\|f\| \le M$ and M independent of x. If the distance from x to Λ is smaller than 1, we remove the point from Λ closest to x and add instead $Q+2$ points to Λ in the same way, at the cost of a controlled increase of the interpolation constant, after which an f with the same properties can be produced. Jensen's formula applied to f yields in either case

$$\int_1^A \frac{n_\Lambda(x,s)}{s}\, ds + (Q+1)\log A \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(x + Ae^{i\theta})|\, d\theta \le \frac{2\tau A}{\pi} + C,$$

where $n_\Lambda(x,s)$ is the number of points from Λ in a disk of center x and radius s, and C is independent of x and A. To make the computation in what follows easier, we replace this disk by a square as shown in Figure 1. We denote by $m_\Lambda(x,s)$ the number of points from Λ in this square of side-length $\sqrt{2}s$, and use the trivial inequality $m_\Lambda(x,s) \le n_\Lambda(x,s)$ to obtain

$$\int_1^A \frac{m_\Lambda(x,s)}{s}\, ds + (Q+1)\log A \le \frac{2\tau A}{\pi} + C.$$

We now convolve both sides of the inequality with the normalized characteristic function $(2R+2A)^{-1}\chi_{[-R-A,R+A]}(x)$, $R > A$, so that we get

$$\frac{1}{2R+2A}\int_{-R-A}^{R+A}\int_1^A \frac{m_\Lambda(x-t,s)}{s}\,ds\,dt \le \frac{2\tau A}{\pi} - (Q+1)\log A + C.$$

Changing the order of integration on the left-hand side, we obtain the estimate

$$\frac{1}{2R+2A}\int_{S_A(x,R)}\int_1^A 2\left(1-\frac{|\operatorname{Im} z|}{s}\right)^+ ds\,d\mu_\Lambda(z) \le \frac{2\tau A}{\pi} - (Q+1)\log A + C,$$

where $S_A(x,R) = \{z : \ |\operatorname{Re} z - x| < R,\ |\operatorname{Im} z| < A\}$ and μ_Λ is the counting measure of the sequence Λ. Therefore,

$$\begin{aligned}\frac{1}{2R+2A}&\int_{S_A(x,R)}(2A - 2(\log A+1)|\operatorname{Im} z|)\,d\mu_\Lambda(z)\\ &\le \frac{2\tau A}{\pi} - (Q+1)\log A + C + \frac{2}{2R+2A}\int_{S_1(x,R)} d\mu_\Lambda(z)\end{aligned}$$

The last term on the right-hand side is uniformly bounded because Λ is separated. Since Λ satisfies the Carleson condition with constant Q, we get

$$\frac{2A}{2R+2A}\int_{S_A(x,R)} d\mu_\Lambda(z) \le \frac{2\tau A}{\pi} - \log A + C',$$

which yields the desired estimate (1) for $D^+(\Lambda \cap S_A)$.

3. A lemma on multipliers

The following notation will be used repeatedly below: We write $f \lesssim g$ if there is a constant K such that $f \le Kg$, and $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$.

Lemma 3. *Suppose U is a subharmonic function of the form*

$$U(z) = \int_{-\infty}^{\infty}\left[\log|1-z/t| + \left(1-\chi_{[-1,1]}(t)\right)\operatorname{Re} z/t\right]m(t)\,dt + C,$$

$m(t) \simeq 1$ and C is any real constant. Then there exists an entire function F with a separated and real zero set $\Gamma = \{\gamma_k\} \not\ni 0$ such that

$$|F(z)|e^{-U(z)} \simeq \operatorname{dist}(z,\Gamma) \tag{2}$$

when $|\operatorname{Im} z| \le 1$, and otherwise

$$|\log F(z) - U(z) - iV(z)| \le \frac{\pi\|1/m\|_\infty}{|\operatorname{Im} z|} \tag{3}$$

for a suitably defined analytic branch of $\log F$ in respectively $\mathbb{C}_0^\pm$, where V is a harmonic conjugate of U in $\mathbb{C}\setminus\mathbb{R}$ such that $V(z) = -V(\bar z)$ for all $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof. Partition the real line into a sequence of disjoint intervals $I_j = [x_j, x_{j+1})$, $j \in \mathbb{Z}$, with $x_0 = 0$, such that

$$\int_{I_j} m(t)dt = 1$$

for all j, and choose $\gamma_j \in I_j$ so that

$$\gamma_j = \int_{I_j} tm(t)dt.$$

The condition $m(x) \simeq 1$ ensures that $|I_j| \simeq 1$ and that $\Gamma = \{\gamma_j\}$ is a separated sequence. Let δ_{γ_j} denote a point mass at the point γ_j, and set $\nu = \sum_j \delta_{\gamma_j}$, and $d\mu(t) = d\nu(t) - m(t)dt$. We choose F such that

$$\log|F(z)| = U(z) + \sum_j \int_{x_j}^{x_{j+1}} \log|1 - z/t|\, d\mu(t),$$

where the last sum is convergent, as will become clear shortly. Set $h(x) = \int_0^x d\mu(t)$ and $H(x) = \int_0^x h(t)\, dt$. Observe that

$$h(x_j) = H(x_j) = 0$$

for all j, by the construction of the sequence Γ, and consequently both h and H are bounded functions, or more precisely, $\|h\|_\infty \leq 1$ and $\|H\|_\infty \leq \max_j |I_j| \leq \|1/m\|_\infty$. Integrating twice by parts, we get

$$\int_{x_j}^{x_{j+1}} \log|1 - z/t|\, d\mu(t) = \operatorname{Re} \int_{x_j}^{x_{j+1}} \frac{H(t)}{(z-t)^2}\, dt - \int_{x_j}^{x_{j+1}} \frac{H(t)}{t^2}\, dt, \tag{4}$$

which immediately implies (2). We claim that (4) also implies (3). To see this, note that because F is an entire function with real zeros satisfying $\log|F(z)| = \log|F(\bar{z})|$, we may define $\operatorname{Im}\log F$ in respectively $\mathbb{C}_0^{\pm}$ in such a way that $\operatorname{Im}\log F(z) = -\operatorname{Im}\log F(\bar{z})$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore, since $V(z) = -V(\bar{z})$ (such a V exists because $U(z) = U(\bar{z})$), $\log F$ may be defined such that

$$\log F(z) - U(z) - iV(z) = \int_{-\infty}^{\infty} \frac{H(t)}{(z-t)^2}\, dt - \int_{-\infty}^{\infty} \frac{H(t)}{t^2}\, dt.$$

This identity implies (3). ■

We make now a first step towards solving the interpolation problem by using the multiplier lemma to construct a function vanishing on the sequence $\Lambda = \{\lambda_k = \xi_k + i\eta_k\}$ and enjoying appropriate estimates. For a given $A > 0$, set

$$\delta_A(x) = \frac{1}{\pi} \sum_{|\eta_k| \geq A} \frac{|\eta_k|}{(x - \xi_k)^2 + {\eta_k}^2}.$$

By the separation of Λ, it is clear that $\|\delta_A\|_\infty \to 0$ when $A \to \infty$. Set $4\varepsilon = \tau - \pi D^+(\Lambda)$. We fix A big enough such that $\|\delta_A\|_\infty < \varepsilon/\pi$.

We may assume for convenience that $\eta_k \neq 0$ for all k because Λ may be shifted vertically: Λ is interpolating if and only if $\Lambda + ia$ is interpolating, and the density and separation conditions of Theorem 1 are also stable under vertical shifts. The Carleson condition implies $\sum |\operatorname{Im}\lambda_k/\lambda_k^2| < \infty$, and we may therefore define

$$F_\Lambda(z) = \prod_{\lambda_k\in\Lambda}(1 - z/\lambda_k)e^{z\operatorname{Re}(1/\lambda_k)}.$$

We denote by $B^\pm$ the two Blaschke products of the sequences $\Lambda\cap\mathbb{C}_0^\pm$ in the upper and lower half-planes $\mathbb{C}_0^\pm$. Put

$$w_A(x) = \frac{1}{\pi}\sum_{|\eta_k|<A}\frac{|\eta_k|}{(x-\xi_k)^2+\eta_k^2},$$

and observe that $2\pi\big(w_A(x)+\delta_A(x)\big)dx$ is the Riesz measure of the subharmonic function $\Phi(z)$ defined as $\log|F_\Lambda/B^\pm|$ in the respective half-planes $\mathbb{C}_0^\pm$.

Set

$$w_{A,R}(x) = \frac{1}{R}\int_{-R}^{R} w_A(x+t)dt,$$
$$\delta_{A,R}(x) = \frac{1}{R}\int_{-R}^{R} \delta_A(x+t)dt$$

for $R>0$, and $4\varepsilon = \tau - \pi D^+(\Lambda)$, and recall that $\varepsilon>0$ by assumption. By the definition of $D^+(\Lambda)$, it is clear that for each $A>0$ we can find an $R>0$ so that $w_{A,R}(x)\le(\tau-3\varepsilon)/\pi$ for all x, and moreover $\delta_{A,R}(x)\le\varepsilon/\pi$ because of our choice of A. Now put $m=\tau-\varepsilon-\pi w_A-\pi\delta_A$, $m_R=\tau-\varepsilon-\pi w_{A,R}-\pi\delta_{A,R}$ and

$$U_{m_R}(z) = \int_{-\infty}^{\infty}\big[\log|1-z/t| + \big(1-\chi_{[-1,1]}(t)\big)\operatorname{Re} z/t\big]m_R(t)\,dt,$$

and analogously

$$U_m(z) = \int_{-\infty}^{\infty}\big[\log|1-z/t| + \big(1-\chi_{[-1,1]}(t)\big)\operatorname{Re} z/t\big]m(t)\,dt.$$

Then

$$\big|\Phi(z)+U_{m_R}(z)-(\tau-\varepsilon)|\operatorname{Im} z|\big|\lesssim 1.$$

Indeed, observe first that $|U_m - U_{m_R}|\lesssim 1$, due to the relation between $w_A+\delta_A$ and $w_{A,R}+\delta_{A,R}$. Moreover, $h(z)=\Phi(z)+U_m(z)-(\tau-\varepsilon)|\operatorname{Im} z|$ =Constant, because h is a harmonic function such that $h(z)\lesssim|z|^{1+\varepsilon}$ for all $\varepsilon>0$, $h(z)=h(\bar z)$, and h is bounded on the real axis.

We apply Lemma 3 to $U=U_{m_R}$ and define $G(z)=F(z)F_\Lambda(z)$. It follows that

$$|G(z)|\simeq e^{(\tau-\varepsilon)|\operatorname{Im} z|}|B^\pm(z)| \tag{5}$$

in the respective half-planes $\mathbb{C}_0^\pm$, if the distance from z to Γ is bounded from below.

We would like to keep a positive distance between Λ and Γ, and this can be achieved by a small perturbation of Γ. So if necessary, set

$$\tilde{F}(z) = F(z) \prod_{\gamma_k \in \Gamma} \frac{1 - z/(\gamma_k + i\delta_k)}{1 - z/\gamma_k},$$

where the δ_k are chosen so that there is a positive distance between Λ and $\{\gamma_k + i\delta_k\}$ and $|\delta_k| \le 1$ for all k. Then it follows from (5), the separation of Λ, and the fact that $B^{\pm}$ are interpolating Blaschke products, that the function $H_\Lambda(z) = \tilde{F}(z)F_\Lambda(z)$ satisfies

$$|H_\Lambda(z)| \simeq e^{(\tau-\varepsilon)|\operatorname{Im} z|}\sigma(z,\Gamma), \quad |H'_\Lambda(\lambda_k)| \simeq e^{(\tau-\varepsilon)|\operatorname{Im}\lambda_k|}/(1+|\operatorname{Im}\lambda_k|);$$

here the second estimate is a consequence of the first.

4. Solution of the interpolation problem I: Via Jones' solution of the $\bar\partial$ equation in a half-plane

We present now our first solution of the interpolation problem. It will be obtained via an explicit solution of the $\bar\partial$ equation in a half-plane due to Jones.

We say that a measure $\mu \in M(\mathbb{C})$ is a *two-sided Carleson measure* whenever there is a constant K such that $|\mu|(D) \le Kr(D)$, where D is any disk of radius $r(D)$ with its center in the real axis. The smallest such constant K is called the *Carleson constant* of μ.

Lemma 4. *Let $\mu \in M(\mathbb{C})$ be a compactly supported measure such that $e^{-\varepsilon|\operatorname{Im} z|}d\mu(z)$ is a two-sided Carleson measure. Then there is a solution u to $\bar\partial u = \mu$ such that*

$$\limsup_{z\to\infty} |u(z)|e^{-\varepsilon|\operatorname{Im} z|} < \infty \qquad \text{and} \qquad |u(x)| \le C\left(1 + \int_{|z-x|<1} \frac{d|\mu|(z)}{|x-z|}\right)$$

for any $x \in \mathbb{R}$, where C depends only on ε and on the Carleson constant of μ. In particular, it does not depend on the support of μ.

Proof. We split the measure μ into two measures $\mu_1 + \mu_2$, where the support of μ_1 is contained in the strip $|\operatorname{Im} z| \le 10$, and the support of μ_2 is contained in $|\operatorname{Im} z| \ge 10$. We solve the problems $\bar\partial u_1 = \mu_1$ and $\bar\partial u_2 = \mu_2$ separately. For the measure μ_1 we take as solution

$$u_1(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\sin\varepsilon(z-\zeta)}{\varepsilon(z-\zeta)^2}\, d\mu_1(\zeta).$$

Because the support of μ_1 is contained in $|\operatorname{Im} z| \le 10$, it is plain that u_1 has the properties:

$$\limsup_{z\to\infty} |u_1(z)|e^{-\varepsilon|\operatorname{Im} z|} < \infty \quad \text{and} \quad |u_1(x)| \le C\left(1 + \int_{|z-x|<1} \frac{d|\mu|(z)}{|x-z|}\right).$$

The problem lies then in solving $\bar\partial u_2 = \mu_2$ with the right estimates.

By hypothesis, $d\sigma = e^{i\varepsilon z}d\mu_2$ is a Carleson measure in the upper half plane $\mathbb{C}_0^+$. Thus there is a bounded solution v_1 to $\bar\partial v_1 = e^{i\varepsilon z}d\mu_2$ in $\mathbb{C}_0^+$. This solution can be given explicitly (see [**J**]) as

$$v_1(z) = \frac{2i}{\pi}\int_{\mathbb{C}^+} \frac{\operatorname{Im}\zeta}{(z-\bar\zeta)(z-\zeta)} \exp\left(\int_{0\le \operatorname{Im} w \le \operatorname{Im}\zeta}\left(\frac{i}{\zeta-\bar w} - \frac{i}{z-\bar w}\right) d|\sigma|(w)\right) d\sigma(\zeta).$$

It has the following properties: It decays as $|z|^{-2}$ when $|z| \to \infty$, and it is bounded in a strip of width 2 along the real axis with a bound K depending only on the Carleson constant of $e^{i\varepsilon z}d\mu_2$. Analogously, there is a solution v_2 to the equation $\bar\partial v_2 = e^{-i\varepsilon z}d\mu_2$ in the lower half-plane with similar estimates. We take as an approximate solution to $\bar\partial u_2 = \mu_2$ the function $u_2 = \phi(z)e^{i\varepsilon z}v_2(z)$ in the lower half-plane and $u_2 = \phi(z)e^{-i\varepsilon z}v_1(z)$ in the upper half-plane, where ϕ is a cut-off function which equals 0 in a strip of width one along the real axis and 1 outside a strip of width 2 along the real axis.

The function u_2 has the right growth, but it does not satisfy $\bar\partial u_2 = \mu_2$ because of the cut-off function. This can be remedied by correcting u_2 with the help of a new function u_3 satisfying $\bar\partial u_3 = \bar\partial u_2 - \mu_2$: Since $|\bar\partial u_2|$ is bounded by a constant times the characteristic function of two strips which lie along the real axis (containing the support of $\nabla\phi$), we choose

$$u_3(z) = \frac{1}{\pi}\int_{1<|\operatorname{Im}\zeta|<2} \frac{\sin\varepsilon(z-\zeta)}{\varepsilon(z-\zeta)^2}\bar\partial u_2(\zeta)\,dm(\zeta).$$

We conclude that the function $u = u_1 + u_2 - u_3$ satisfies the equation $\bar\partial u = \mu$ and has the desired growth properties. ■

We will now solve the problem $f(\lambda_k') = a_k$ for all finite subsequences $\Lambda' \subset \Lambda$ in such a way that the bounds on f are independent of the number of points from Λ'. Then by using a normal family argument, we will obtain a solution to the original interpolation problem for the infinite sequence Λ.

We use an extension argument which goes as follows (see [**A**]): We take the function H_Λ constructed in the previous section and solve the equation

$$\bar\partial u = \sum_{\lambda_k\in\Lambda'} \frac{a_k}{H_\Lambda'(\lambda_k)}\delta_{\lambda_k};$$

then $f = H_\Lambda u$ is a holomorphic function such that $f(\lambda_k) = a_k$ for all $\lambda_k \in \Lambda'$. Thus what we need is to find a solution u with the restriction $|u(x)| \le M/\operatorname{dist}(x,\Lambda')$ for $x \in \mathbb{R}$ and $\limsup_{|z|\to\infty} |u(z)|e^{-\varepsilon|\operatorname{Im} z|} < \infty$. This function u is obtained from Lemma 4, because the measure $\mu = \sum_{\lambda_k\in\Lambda'} \frac{a_k}{H_\Lambda'(\lambda_k)}\delta_{\lambda_k}$ satisfies the hypothesis of that lemma.

5. Solution of the interpolation problem II: Via Earl's interpolating Blaschke products

We turn now to the second solution of the interpolation problem. It is divided into two main steps. We begin by picking a big $A > 0$ as before, and solve $f(\lambda_k) = a_k$ with $\lambda_k \in S_A$ in a standard way, using essentially Lagrange interpolation. Then we solve the problem $g(\lambda_k) = a_k - f(\lambda_k)$ for $\lambda_k \notin S_A$ and $g(\lambda_k) = 0$ for $\lambda_k \in S_A$, using Earl's elementary proof of Carleson's theorem. In both steps, the multiplier lemma plays an essential role.

Set $\Lambda(A) = \Lambda \cap S_A$. We build the function $H_{\Lambda(A)}$ in the same way as we constructed H_Λ above. It has the following properties:

$$|H_{\Lambda(A)(z)}| \simeq e^{(\tau-\varepsilon)|\operatorname{Im} z|}\sigma\big(z,\Lambda(A)\big), \quad |H_{\Lambda(A)}'(\lambda_k)| \simeq e^{(\tau-\varepsilon)|\operatorname{Im}\lambda_k|}/(1+|\operatorname{Im}\lambda_k|)$$

for all z and $\lambda_k \in \Lambda(A)$.

The problem $f(\lambda_k)=a_k$ for $\lambda_k\in S_A$ is solved by means of the Lagrange-type formula

$$f(z)=\sum_{\lambda_k\in\Lambda(A)}\frac{a_k}{H'_{\Lambda(A)}(\lambda_k)}\frac{H_{\Lambda(A)}(z)}{z-\lambda_k}\frac{\sin\varepsilon(z-\lambda_k)}{\varepsilon(z-\lambda_k)}.$$

We next seek to solve the problem $h(\lambda_k)=b_k$, $\lambda_k\in\Lambda\setminus\Lambda(A)$, with $h\in\mathcal{B}_\varepsilon$, where

$$b_k=\big(a_k-f(\lambda_k)\big)/H_{\Lambda(A)}(\lambda_k).$$

Then $g=f+hH_{\Lambda(A)}$ will be a solution of the given interpolation problem.

We set $b_{k,0}=b_k$, and begin by solving $\varphi_0^+(\lambda_k)=b_{k,0}e^{i\varepsilon\lambda_k}$ in H^∞ of the upper half-plane by means of Earl's elementary method. This solution is $\varphi_0^+=K\sup(|b_k|e^{-\varepsilon|\operatorname{Im}\lambda_k|})B_0^+$, where B_0^+ is an interpolating Blaschke product whose zeros λ_k' satisfy $\sigma(\lambda_k,\lambda_k')\le 1/3$, $\operatorname{Im}\lambda_k\ge A$, and K is a positive constant depending only on the separation and Carleson constant of Λ (see [**G**, pp. 308–313]). It follows from Earl's proof that in fact

$$B_0^+(z)=\prod_{\operatorname{Im}\lambda_k\ge A}\frac{\overline{\lambda_k'}}{\lambda_k'}\left(\frac{z-\lambda_k'}{z-\overline{\lambda_k'}}\right).$$

We solve similarly $\varphi_0^-(\lambda_k)=b_ke^{-i\varepsilon\lambda_k}$ in H^∞ of the lower half-plane, and obtain then $\varphi_0^-=K\sup(|b_k|e^{\varepsilon-|\operatorname{Im}\lambda_k|})B_0^-$, where

$$B_0^-(z)=\prod_{\operatorname{Im}\lambda_k\le -A}\frac{\overline{\lambda_k'}}{\lambda_k'}\left(\frac{z-\lambda_k'}{z-\overline{\lambda_k'}}\right),$$

λ_k' satisfy $\sigma(\lambda_k,\lambda_k')\le 1/3$ and K is the same positive constant as above.

Now we correct the solution which is $e^{-i\varepsilon z}\varphi_0^+$ in the upper half-plane and $e^{i\varepsilon z}\varphi_0^+$ in the lower half-plane. Set

$$F_{\Lambda'}(z)=\prod_{|\operatorname{Im}\lambda_k|\ge A}(1-z/\lambda_k')e^{z\operatorname{Re}(1/\lambda_k')},$$

where $\Lambda'=\{\lambda_k'=\xi_k'+i\eta_k'\}$ is the union of the zero sets of the two Blaschke products $B_0^\pm$, and correspondingly

$$m_{\Lambda'}(x)=\frac{1}{\pi}\sum_{\lambda_k'\in\Lambda'}\frac{|\eta_k'|}{(x-\xi_k')^2+{\eta_k'}^2}.$$

We apply Lemma 3 to the function $U=\varepsilon|\operatorname{Im}z|-\log|F_{\Lambda'}/B^\pm|$ for which $m=\varepsilon/\pi-m_{\Lambda'}$, and set $h_0=K\sup(|b_k|e^{\varepsilon|\operatorname{Im}\lambda_k|})FF_{\Lambda'}$. We define $b_{k,1}=b_{k,0}-h_0(\lambda_k)$, and observe that it may be written as

$$b_{k,1}=b_{k,0}\big(1-F(\lambda_k)F_{\Lambda'}(\lambda_k)e^{\pm\varepsilon i\lambda_k}/B_0^\pm(\lambda_k)\big)$$

for $\lambda_k\in\mathbb{C}_0^\pm$. Then because of the special form of the two Blaschke products $B_0^\pm$, inequality (3) of Lemma 3 yields

$$|b_{k,1}/b_{k,0}|\le\exp\left(\frac{\pi^2}{A(\varepsilon-\pi\|m_{\Lambda'}\|_\infty)}\right)-1.$$

Now if A has been chosen so large that the right-hand side is bounded by, say $1/2$, this procedure can be iterated, i.e., we may solve inductively $\phi_n^{\pm}(\lambda_k) = b_{k,n}e^{\pm i\varepsilon\lambda_k}$ for $n = 1, 2, 3, \ldots$, and obtain a sequence $h_1, h_2, h_3, \ldots$ such that

$$h(z) = \sum_{n=0}^{\infty} h_n(z),$$

with convergence in $\mathcal{B}_\varepsilon$.

6. Extension of Beurling's sampling theorem

We say that $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ is a *sampling sequence* for $\mathcal{B}_\tau$ if the inequality

$$\sup_{z\in\mathbb{C}} |f(z)|e^{-\tau|\operatorname{Im} z|} \lesssim \sup_{\lambda\in\Lambda} |f(\lambda)|e^{-\tau|\operatorname{Im}\lambda|}$$

holds for all $f \in \mathcal{B}_\tau$. In this section, we shall explain how Beurling's characterization of real sampling sequences [**B**, p. 346] can be extended to the case of arbitrary complex sequences.

We introduce again an adequate notion of density. Suppose Λ is separated. Denote by $n_A^-(r)$ the minimum number of points from Λ to be found in a rectangle of the form $\{z = x + iy : \ t < x < t + r, \ |y| < A\}$, where t is any real number. The lower uniform density of Λ is defined to be

$$D^-(\Lambda) = \lim_{A\to\infty} \lim_{r\to\infty} \frac{n_A^-(r)}{r}.$$

Theorem 2. *A sequence Λ is sampling for $\mathcal{B}_\tau$ if and only if there exists a separated subsequence $\Lambda' \subset \Lambda$ such that $D^-(\Lambda') > \tau/\pi$.*

The sufficiency of this condition is proved by means of Jensen's formula and a characterization of sampling sequences in terms of sets of uniqueness for $\mathcal{B}_\tau$. We omit this part of the proof because it is in essence identical to Beurling's proof for real sequences (see [**B**, p. 346]).

We turn to the proof of the necessity. First we show how we can reduce our discussion to separated subsequences.

Lemma 5. *If Λ is a sampling sequence for $\mathcal{B}_\tau$, then there exists a separated subsequence $\Lambda' \subset \Lambda$ which is also a sampling sequence for $\mathcal{B}_\tau$.*

Proof. The lemma is a consequence of a basic Bernstein-type inequality for functions $f \in \mathcal{B}_\tau$:

$$\big||f(z)|e^{-\tau|\operatorname{Im} z|} - |f(w)|e^{-\tau|\operatorname{Im} w|}\big| \lesssim \sigma(z,w)\|f\|, \qquad \text{for } \sigma(z,w) < 1.$$

Suppose that $\operatorname{Im} z \geq 0$ and that $f(z) \neq 0$. We claim that then

$$\big|\nabla(|f(z)|e^{-\tau|\operatorname{Im} z|})\big| \lesssim |\nabla f e^{i\tau z}| \lesssim \|f\|/(1 + |\operatorname{Im} z|),$$

from which the Bernstein inequality follows. The first inequality is trivial, while the second follows by an application of Cauchy's formula to the bounded analytic function $f(z)e^{i\tau z}$ in the half-plane $\mathbb{C}_{-1}^+$. We argue similarly if $\operatorname{Im} z < 0$. ∎

We may assume from now on that Λ itself is separated. Since for any $x \in \mathbb{R}$ the operator $T_x : \mathcal{B}_\tau \to \mathcal{B}_\tau$ defined as $T_x f(z) = f(z-x)$ is an isometry, we may apply the same arguments as in [**B**, Thm 4, p 345] and conclude that if Λ is a sampling sequence, then it is also a sampling sequence for a slightly bigger space $\mathcal{B}_{\tau+\delta}$. Therefore, in order to prove the theorem it is enough to show that $D^-(\Lambda) \geq \tau$.

We finally prove the inequality $D^-(\Lambda) \geq \tau$. We assume to the contrary that $D^-(\Lambda) = \tau - \varepsilon$ for some $\varepsilon > 0$ and show that this leads to a contradiction. Set $S_r(t,R) = \{x+iy \in \mathbb{C} : \ t \leq x < t+R, \ |y| < r\}$ and

$$G(z) = \prod_{\lambda_k \in S_r(t,R)} \sin \frac{\pi(z-\lambda_k)}{R-2r},$$

for $R > 2r > 0$. Take x^* to be the maximum of $|G(x)|$ in the interval $[t+r, t+R-r)$, and note that x^* is in fact a global maximum of $|G(x)|$ since $|G(x)|$ is an $(R-2r)$-periodic function. Define

$$f(z) = \frac{G(z)}{G(x^*)} \frac{\sin \delta(z-x^*)}{\delta(z-x^*)},$$

where $\delta = \varepsilon/2$. By our assumption $D^-(\Lambda) = \tau - \varepsilon$, we can find a $t \in \mathbb{R}$ and $R > 2r$ such that G is of type less than $\tau - \delta$. By the Phragmén-Lindelöf principle, $|G(z)| \leq |G(x^*)| e^{(\tau-\delta)|\operatorname{Im} z|}$ for all z, and so $\|f\| = 1$ and

$$|f(\lambda_k)| e^{-\tau |\operatorname{Im} \lambda_k|} \leq \frac{1}{\delta r}.$$

Since r can be chosen arbitrarily big, we have reached a contradiction.

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