ON L^p-SOLUTIONS OF THE LAPLACE EQUATION AND ZEROS OF HOLOMORPHIC FUNCTIONS

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1. Introduction

The problem we solve in this paper is to characterize, in a smooth domain Ω in $\mathbb{R}^n$ and for $1 \leq p \leq \infty$, those positive Borel measures on Ω for which there exists a subharmonic function $u \in L^p(\Omega)$ such that $\Delta u = \mu$.

The motivation for this question is mainly for $n = 2$, in which case it is related with problems about distributions of zeros of holomorphic functions: if $\{a_n\}_{n=1}^{\infty}$ is a sequence in $\Omega \subset \mathbb{C}$ with no accumulation points in a simply connected domain Ω, and $\mu = 2\pi \sum_n \delta_{a_n}$, then all solutions u of $\Delta u = \mu$ are of the form $u = \log |f|$, with f holomorphic vanishing exactly on the points a_n. Thus our results give the characterization of the zero sequences of holomorphic functions with $\log |f| \in L^p(\Omega)$. A related class had been considered by Beller ([Bel1]).

In section 2 we consider first $\Omega = \mathbb{R}^n$. In this case, it is easily seen by looking at the behavior at ∞ that the problem only makes sense for $n \geq 3$ and $p > n/(n-2)$, and that it is equivalent to the description of the measures μ on $\mathbb{R}^n$ having a Newtonian potential in $L^p(\mathbb{R}^n)$. This was achieved by Muckenhoupt and Wheeden, and we give an alternative proof as well. The results and the methods of proofs serve as a guideline for the bounded case.

In section 3 we solve the problem for a bounded $\mathcal{C}^\infty$ domain Ω in $\mathbb{R}^n$. Among other characterizations we find that $\Delta u = \mu$ has a solution in $L^p(\Omega)$ if and only if the fractional maximal function

$$\mu^*(x) = \sup_{r \leq \lambda\delta(x)} r^{2-n} \mu(B(x,r))$$

belongs to $L^p(\Omega)$. Here $\delta(x)$ is the distance from x to $\partial\Omega$ and λ is a fixed constant, $0 < \lambda < 1$. The case $p = 1$ was previously known ([Hei], [DH]) in dimension 2, and amounts to saying that

$$\int_\Omega \delta(x)^2 \, d\mu(x) < +\infty.$$

We also consider at the end the case $p = \infty$.

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In section 4 we look closely to the case $\Omega = \mathbb{D}$, the unit disc in $\mathbb{C}$. We find here a linear solution operator $u = \mathbb{K}[\mu]$ of $\Delta u = \mu$ with $u \in L^p(\Omega)$, and relate the problem with others, such as Toeplitz operators, Carleson measures for Bergman spaces, and the bilaplacian.

In section 5 we particularize to the motivating question, the zeros of holomorphic functions. It turns out that the condition characterizing the zero sequences $\{a_n\}$ of functions with $\log|f| \in L^p(\mathbb{D})$ depends both on the radii and the angular disposition of the points a_n, as in many other classes of functions. We find a dyadic expression of it and also we find the best condition on the sequence of radii $r_n = |a_n|$. A probabilistic version is also considered.

As a final remark, we point out that we have bound ourselves to the situation described —subharmonic functions and positive measures— in view of the applications we had in mind, zeros of holomorphic functions. It is quite probable that our results extend to measures or distributions, and also replacing the laplacian by a general strongly elliptic operator, but we have not tried to do so.

Another generalization, replacing the Laplace equation $\Delta u = \mu$ by the equation $i\partial\bar\partial u = \theta$ in $\mathbb{C}^n$ and with applications to zero varieties of holomorphic functions of several variables, appears in [Ort].

2. Subharmonic functions in $L^p(\mathbb{R}^n)$

In this section we shall characterize the laplacians of subharmonic functions in $L^p(\mathbb{R}^n)$. For general facts on subharmonic functions that we will use, see for instance [Hay].

First note that if h is harmonic in $L^p(\mathbb{R}^n)$, the mean-value property implies that $h \equiv 0$. Hence any subharmonic function u in $L^p(\mathbb{R}^n)$ is completely determined by its laplacian $\mu = \Delta u$, a positive Borel measure in $\mathbb{R}^n$. The sub-mean value property implies that any such u is non-positive, hence we can consider in each ball $B(0,R)$ the Green decomposition

$$u(x) = h_R(x) + \int_{|y|\le R} G_R(x,y)\,d\mu(y), \quad |x| < R.$$

Here h_R is the least harmonic majorant of u in $B(0,R)$ and $G_R(x,y)$ is the Green function for the ball $B(0,R)$, given by

$$G_R(x,y) = \begin{cases} c_n\left\{|x-y|^{2-n} - R^{n-2}|y|^{2-n}|x-y'|^{2-n}\right\}, & n > 2 \\ \\ c_2 \log\dfrac{|x-y'|\,|y|}{R|x-y|}, & n = 2 \end{cases}$$

with $y' = yR^2|y|^{-2}$. Clearly $u(x) \le h_R(x) \le h_{R'}(x) \le 0$ if $R \le R'$. The limit $h(x) = \lim\limits_{R\to\infty} h_R(x)$, the least harmonic majorant of u in $\mathbb{R}^n$, is then in $L^p(\mathbb{R}^n)$, hence is identically zero. Thus we conclude that

$$u(x) = \lim_{R\to\infty}\int_{|y|\le R} G_R(x,y)\,d\mu(y).$$

It is immediate to check that for $n > 2$, $\lim_{R\to\infty} G_R(x, y) = c_n|x - y|^{2-n}$, while this limit is infinite for $n = 2$. By the monotone convergence theorem, this means that there are no non-zero subharmonic functions in $L^p(\mathbb{R}^2)$ and that for $n > 2$ they are exactly those Newtonian potentials

$$u(x) = c_n \int_{\mathbb{R}^n} |x - y|^{2-n}\, d\mu(y)$$

belonging to $L^p(\mathbb{R}^n)$.

Now note that unless μ is identically zero, $|u(x)|$ dominates $|x|^{2-n}$ for large $|x|$ and so p must be greater than $n/(n-2)$. In conclusion, we have seen that the question only makes sense for $n > 2$ and $p > n/(n - 2)$ and that it is equivalent to the description of the positive measures μ for which the Riesz potential of order 2 is in $L^p(\mathbb{R}^n)$. We use the notation

$$T_\alpha\mu(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha-n}\, d\mu(y)$$

for the *Riesz potential of order* α, $0 < \alpha < n$. We consider the *maximal fractional function* of μ, defined by

$$\mu^*_\alpha(x) = \sup_{t>0} \frac{\mu(B(x,t))}{|B(x,t)|^{1-\frac{\alpha}{n}}} = c_n \sup_{t>0} t^{\alpha-n}\mu(B(x,t)).$$

Note that μ^*_α is pointwise dominated by $T_\alpha\mu$. Next theorem was proved by Muckenhoupt and Wheeden in [MW] for absolutely continuous measures. We will give an alternative proof for it contains some concepts that will be used later.

Theorem 2.1. *Let μ be a positive measure in $\mathbb{R}^n$, $n > 2$, $1 < p < \infty$ and α such that $0 < \alpha < n$. Then $T_\alpha\mu \in L^p(\mathbb{R}^n)$ if and only if $\mu^*_\alpha \in L^p(\mathbb{R}^n)$.*

Proof. Write $T_\alpha\mu$ in the form

$$T_\alpha\mu(x) = (n - \alpha) \int_0^\infty t^{\alpha-n-1}\mu(B(x,t))\, dt.$$

Consider in $\mathbb{R}^{n+1}_+$, with coordinates (x,t), $x \in \mathbb{R}^n$, $t > 0$, the product measure $d\tilde{\mu}(x,t) = d\mu(x) \otimes t^{\alpha-1}\, dt$. Then we can write

$$T_\alpha\mu(x) = (n - \alpha) \int_{\Gamma(x)} t^{-n} d\tilde{\mu}(y,t) \stackrel{\text{def}}{=} A(\tilde{\mu})(x)$$

where $\Gamma(x)$ is the cone $\{(y,t) : |x - y| < t\}$. By [CMS, § 6], $A(\tilde{\mu})$ belongs to $L^p(\mathbb{R}^n)$, $1 < p$, simultaneously with the function

$$C(\tilde{\mu})(x) = \sup \left\{ \frac{\tilde{\mu}(Q(B))}{|B|},\, x \in B \right\}.$$

Here $Q(B)$ denotes the Carleson window over the ball B. (We point out that this definition of A, C does not coincide with that of [CMS]). In our case $\tilde{\mu}(Q(B)) \simeq \mu(B)|B|^{\alpha/n}$, whence $C(\tilde{\mu})(x) \sim \mu^*_\alpha(x)$. $\square$

3. Subharmonic functions in $L^p(\Omega)$

In this section we will characterize, for a smooth bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, the positive (Borel) measures μ on Ω such that the Poisson equation

$$\Delta\mu = \mu$$

has a (subharmonic) solution $u \in L^p(\Omega)$, $1 \leq p \leq \infty$.

3.1. For $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary of Ω. For a positive measure μ on Ω and a fixed λ, $0 < \lambda < 1$, we consider local versions of the fractional maximal function and Newtonian potentials in the previous section. These are

$$\mu_\lambda^*(x) = \sup_{r \leq \lambda\delta(x)} \frac{\mu(B(x,r))}{r^{n-2}}$$

and

$$T_\lambda\mu(x) = \int_{B(x,\lambda\delta(x))} |x-y|^{2-n}\, d\mu(y)$$

$B(x,r)$ being the ball of radius r centered at x.

Theorem 3.1. *The following are equivalent for a positive measure μ on Ω and $1 \leq p < \infty$:*

(a) *There is a subharmonic function $u \in L^p(\Omega)$ such that $\Delta u = \mu$.*
(b) *$T_\lambda\mu \in L^p(\Omega)$ for some $0 < \lambda < 1$.*
(c) *$\mu_\lambda^* \in L^p(\Omega)$ for some $0 < \lambda < 1$.*

In case $p = 1$, an equivalent condition is

(d) *$\int_\Omega \delta(x)^2\, d\mu(x) < +\infty$.*

In particular, conditions (b), (c) do not depend on λ. The theorem should be compared with Theorem 2.1.

3.2. It is trivial that (b) implies (c), because $\mu_\lambda^* \leq T_\lambda\mu$. An application of Fubini's theorem shows that up to constants the integral in (d) is the L^1-norm of $T_\lambda\mu$, using that $\delta(x) \simeq \delta(y)$ whenever $|x-y| \leq \lambda\delta(x)$. Obviously

$$\mu_\lambda^*(x) \gtrsim \mu(B(x,\lambda\delta(x))\delta(x)^{2-n}.$$

and so $\mu_\lambda^* \in L^1(\Omega)$ implies again by Fubini's theorem that (d) holds. Therefore (b), (c) and (d) are all equivalent for $p = 1$. We will now prove that $\mu_\lambda^* \in L^p(\Omega)$, $1 < p < \infty$ implies $T_\tau\mu \in L^p(\Omega)$ for some τ. The first step is a local version of Theorem 2.1.

Lemma 3.2. *For a positive measure μ on $B(0,8\varepsilon)$ and $1 < p < \infty$*

(1)
$$\int_{B(0,\varepsilon)} \left(\int_{|x-y|\leq\varepsilon} |x-y|^{2-n}\, d\mu(y) \right)^p dA(x) \leq C_p \int_{B(0,3\varepsilon)} \left(\sup_{r\leq 5\varepsilon} \frac{\mu(B(x,r))}{r^{n-2}} \right)^p dA(x).$$

Proof. We use the same method as in Theorem 2.1. The inner integral of the left-hand side equals

$$(n-2)\int_0^{\varepsilon} t^{1-n}\mu(B(x,t))\,dt.$$

We consider in the half-space $\mathbb{R}^{n+1}_+$ with coordinates (x,t) the product measure $d\tilde{\mu}(x,t) = \mathbb{1}_{B(0,2\varepsilon)}(x)\,d\mu(x)\otimes \mathbb{1}_{(0,\varepsilon)}(t)t\,dt$, so that the previous integral equals

$$\int_{\Gamma(x)} t^{-n} d\tilde{\mu}(x,t) = A(\tilde{\mu})(x), \quad x\in B(0,\varepsilon).$$

By the result of [CMS] already used in Theorem 2.1, the left-hand side of (1) is bounded by a constant times the L^p-norm of $C(\tilde{\mu})$ in $\mathbb{R}^n$. Now, for $|x|\le 3\varepsilon$

$$C(\tilde{\mu})(x) \simeq \sup_{r\le 5\varepsilon} \frac{\mu(B(x,r))}{r^{n-2}},$$

hence the integral of $C(\tilde{\mu})^p$ over $|x|\le 3\varepsilon$ is already bounded by the right hand side of (1). For $|x|>3\varepsilon$,

$$C(\tilde{\mu})(x) \lesssim \frac{\mu(B(0,2\varepsilon))\varepsilon^2}{|x|^n}$$

and hence

$$\int_{|x|\ge 3\varepsilon} (C(\tilde{\mu})(x))^p\,dA(x) \lesssim \varepsilon^{n-p(n-2)}\mu(B(0,2\varepsilon))^p,$$

which is also bounded by the right hand side. □

Assume $\tau < 1/8$ and consider Ω covered by the family of balls $\{B(x,\tau\delta(x)), x\in \Omega\}$. By Besicovitch covering lemma (see [Fed 2.8.14] for instance), there is a finite set of families $B_1,\dots,B_N$ of balls in the covering whose total union is also Ω and such that each family consists of disjoint balls. Then, writing B_x for $B(x,\tau\delta(x))$, B'_x for $B(x,3\tau\delta(x))$,

$$\int_{\Omega} T^p_{\tau}\mu(x)\,dA(x) \le \sum_{i=1}^{N}\sum_{B_x\in B_i}\int_{B_x} T^p_{\tau}\mu(x)\,dA(x)$$

which by Lemma 3.2 is dominated by

$$\sum_{i=1}^{N}\sum_{B_x\in B_i}\int_{B'_x} \mu^{*p}_{5\tau}(x)\,dA(x).$$

We claim now that, for each family B_i, the balls B'_x cover each point of Ω at most a finite number (depending only on n) of times. Indeed, if $y\in B'_x$ then $m^{-1}\delta(y)\le\delta(x)\le m\delta(y)$ for some constant $m=m(\tau)$, whence the balls $B_x\in B_i$, such that $y\in B'_x$ have volume $\simeq \delta(y)^n$, are disjoint and are all of them in the ball $B(y,(3\tau+m)\delta(y))$, and the claim follows. It is then clear that the last expression is bounded by

$$C(p,\tau)\int_{\Omega} \mu^{*p}_{5\tau}(x)\,dA(x).$$

Therefore, $\mu^*_{\lambda}\in L^p(\Omega)$ implies $T_{\tau}\mu\in L^p(\Omega)$ for τ small enough.

3.3. We will prove now that (a) implies (b). For $r < \delta(x)$ consider the Green decomposition of u in $B(x,r)$

$$\frac{1}{\sigma_n r^{n-1}} \int_{|x-y|=r} u(y)\, d\sigma(y) - u(x) = \int_{|x-y|<r} G_r(x,y)\, d\mu(y).$$

Here σ_n is the surface measure of the unit sphere and $G_r(x,y)$ is Green's function for the ball $B(x,r)$. Multiply the above by nr^{n-1}, integrate between 0 and $R \le \delta(x)$, and divide by R^n. The result is, for $n \ge 3$

$$\begin{aligned} &\frac{n}{\sigma_n R^n} \int_{|x-y|\le R} u(y)\, dA(y) - u(x) = \\ &= \frac{1}{(n-2)\sigma_n} \int_{|x-y|\le R} |x-y|^{2-n} \left\{ 1 - \frac{|x-y|^n}{R^n}\left(1-\frac{n}{2}\right) - \frac{n}{2}\frac{|x-y|^{n-2}}{R^{n-2}} \right\} d\mu(y) \end{aligned}$$

and for $n = 2$

$$\begin{aligned} &\frac{1}{\pi R^2} \int_{|x-y|\le R} u(y)\, dA(y) - u(x) = \\ &\qquad = \frac{1}{2\pi} \int_{|x-y|\le R} \left\{ \log \frac{R}{|x-y|} + \frac{1}{2}\left(\frac{|x-y|^2}{R^2} - 1 \right) \right\} d\mu(y). \end{aligned}$$

The functions $1 + \left(\frac{n}{2} - 1\right) x^n - \frac{n}{2} x^{n-2}$ and $\log \frac{1}{x} + \frac{1}{2}(x^2-1)$ are positive decreasing functions in $(0,1)$. Therefore for each $\tau < 1$ there is $c(\tau)$ such that

$$\frac{1}{|B(x,R)|} \int_{B(x,R)} u(y)\, dA(y) - u(x) \ge c(\tau) \int_{|x-y|\le \tau R} |x-y|^{2-n}\, d\mu(y).$$

Writing $\lambda\delta(x) = \tau R$ with $R = \sqrt{\lambda}\delta(x)$, it follows that for (a) $\Rightarrow$ (b) it is enough to prove that for a fixed $\tau < 1$, $u \in L^p(\Omega)$ implies that the average of u over $B_x = B(x, \tau\delta(x))$ belongs to $L^p(\Omega)$ as well. To see this we use Holder's inequality

$$\int_\Omega \left| \frac{1}{\delta(x)^n} \int_{B_x} u(y)\, dA(y) \right|^p dA(x) \le \int_\Omega \frac{1}{\delta(x)^n} \int_{B_x} |u(y)|^p\, dA(y)\, dA(x).$$

If $y \in B_x$, then $\delta(x) \simeq \delta(y)$ and Fubini theorem finishes the proof that (a) implies (b).

3.4. Next we prove that (d) implies (a) in case $p = 1$. More precisely we want to show that whenever μ is a positive measure such that

$$\int_\Omega \delta(x)^2\, d\mu(x) = M < +\infty$$

then there is $u \in L^1(\Omega)$ such that $\Delta u = \mu$ and $\|u\|_1 \le CM$, where C is a constant depending only on Ω. The dual statement of this is contained in next theorem

Theorem 3.3. *For any test function Φ in Ω,*

$$|\Phi(x)| \leq C\delta(x)^2 \|\Delta\Phi\|_\infty, \quad x \in \Omega.$$

These are indeed equivalent statements. Assume Theorem 3.3 holds and let μ be as above. Define a linear functional Λ as follows

$$\Lambda(\Delta\Phi) = \int_\Omega \Phi(x)\, d\mu(x)$$

Λ is well defined, and by Theorem 3.3

$$|\Lambda(\Delta\Phi)| \leq CM\|\Delta\Phi\|_\infty.$$

By Hahn-Banach theorem, Λ extends to a continuous functional on the space $C_c(\Omega)$ of continuous functions with compact support, with norm $\leq CM$. By Riesz's representation theorem, we conclude that there is a measure σ on Ω, with total variation $\|\sigma\| \leq CM$ such that

$$\Lambda(\Delta\Phi) = \int_\Omega \Delta\Phi(x)\, d\sigma(x).$$

This means that $\Delta\sigma = \mu$ in the weak sense, and then σ must be in fact absolutely continuous $d\sigma = u\, dx$, with $\|u\|_1 \leq CM$ and $\Delta u = \mu$. It is immediate to check that this argument can be reversed.

Proof of Theorem 3.3. Let $G(x, y)$ denote the fundamental solution for Δ in $\mathbb{R}^n$, $G(x, y) = c_n|x - y|^{2-n}$ if $n \geq 3$, $G(x, y) = c_2 \log|x - y|$ in $\mathbb{R}^2$, so that

$$\Phi(x) = \int_\Omega G(x, y)\Delta\Phi(y)\, dA(y).$$

Since

$$|\Phi(x)| \leq \|\Delta\Phi\|_\infty \int_\Omega |G(x, y)|\, dA(y) \leq C\|\Delta\Phi\|_\infty$$

it is enough to prove the inequality for points x close to $\partial\Omega$. Let U be a tubular neighborhood of $\partial\Omega$, that is, there is a well-defined projection $p : U \to \partial\Omega$ of class C^∞, $p(x)$ is the unique closest point to x, and the line between x and $p(x)$ is orthogonal to $\partial\Omega$ at $p(x)$. Let $z = 2p(x) - x$ be the symmetric point of x with respect $\partial\Omega$.

The idea is that in the integral representation of Φ above, one can subtract to $G(x, y)$ any harmonic function h in Ω because these annihilate $\Delta\Phi(y)$, by Green's theorem. Fixed x, we take as $h(y)$ the second-order Taylor polynomial of $G(\cdot, y)$ at z, evaluated at x,

$$h(y) = G(z, y) + 2\delta(x)\frac{\partial}{\partial n}G(z, y) + 2\delta(x)^2\frac{\partial^2}{\partial n^2}G(z, y).$$

Here n is the unit direction from z to x. Thus we have

$$|\Phi(x)| \leq \|\Delta\Phi\|_\infty \int_\Omega |G(x,y) - h(y)|\, dA(y)$$

If $|x-y| \geq 3\delta(x)$, by Taylor's formula,

$$|G(x,y) - h(y)| \lesssim \frac{\delta(x)^3}{|x-y|^{n+1}}$$

and hence the contribution of $\Omega \backslash B(x, 3\delta(x))$ in the last integral is dominated by $\delta(x)^2$. If $|x-y| \leq 3\delta(x)$, we write

$$|G(x,y) - h(y)| \leq |G(x,y) - G(z,y)| + 3\delta(x)|\nabla_z G(z,y)| + 5\delta(x)^2 |\nabla_z^2 G(z,y)|.$$

This is bounded by $c|x-y|^{2-n}$ if $n > 2$, and by $c + \log \frac{3\delta(x)}{|x-y|}$ if $n = 2$. In both cases the integral over $B(x, 3\delta(x))$ is estimated by $\delta(x)^2$. □

The same proof will go through if for any point x close to the boundary, we can find a point z outside the open set Ω, such that the distance of z to the boundary of Ω is comparable to $|z-x|$. This holds, for instance, if Ω satisfies a uniform exterior cone condition. H. Shapiro [Sha], has let us known that it is possible to get the same result under very mild regularity assumptions on the boundary of Ω.

3.5. To complete the proof of Theorem 3.1, we finally prove that (b) implies (a) when $p > 1$. Let Φ be a test function supported in $|y| < 1$, equal to one on $|y| < 1/2$, $0 \leq \Phi \leq 1$; with $0 < \tau < 1/3$, $0 < \tau < \lambda$ small enough to be defined later, we define an "approximate solution" by

$$K(\mu)(x) = \int_{\mathbb{R}^n} \Phi\left(\frac{x-y}{\tau\delta(x)}\right) K(x,y)\, d\mu(y).$$

Here $K(x,y)$ is the fundamental solution for Δ in $\mathbb{R}^n$, $K(x,y) = G(x,y) = c_n|x-y|^{2-n}$ if $n > 2$; if $n = 2$ we need to modify it by a function $H(x,y)$

$$K(x,y) = G(x,y) - H(x,y) = c_2 \log(x-y) - H(x,y)$$

where $H(x,y)$ is harmonic in x, equal to $G(x,y')$ for y close to $\partial\Omega$, y' being the symmetric of y with respect to $\partial\Omega$, as in the previous paragraph. Another choice, in all dimensions, is $K(x,y) = G_\Omega(x,y)$ the Green's function for Ω, if Ω is smooth enough. In any event,

$$\Delta K(\mu) = \mu + R(\mu)$$

with

$$R(\mu)(x) = \int_{\mathbb{R}^n} \left\{ 2\nabla_x \Phi\left(\frac{x-y}{\tau\delta(x)}\right) \cdot \nabla_x K(x,y) + \Delta_x \Phi\left(\frac{x-y}{\tau\delta(x)}\right) K(x,y) \right\} d\mu(y).$$

We claim now that under the assumption (b), both $K(\mu)$ and $\delta^2 R(\mu)$ are in $L^p(\Omega)$. This is clear if $n > 2$, because both are pointwise estimated by $T_\lambda \mu$. If $n = 2$, both are pointwise estimated by

$$\mu(B(x, \tau\delta(x))) + \int_{B(x,\tau\delta(x))} \log \frac{\tau\delta(x)}{|x-y|}\, d\mu(y)$$

and so it is enough to prove that the function defined by this last integral, call it $\Psi(x)$, is in $L^p(\Omega)$. By Holder's inequality and Fubini's theorem

$$\int_\Omega \Psi(x)^p\, dA(x) \le \int_\Omega \left\{ \int_E \left(\log \frac{\tau\delta(x)}{|x-y|} \right)^p \mu(B(x,\tau\delta(x)))^{p-1}\, dA(x) \right\} d\mu(y).$$

The inner integral is over the set $E = \{x : y \in B(x, \tau\delta(x))\}$. With $\varepsilon = \frac{2\tau}{1-\tau}$ one has $B(x, \tau\delta(x)) \subset B(y, \varepsilon\delta(y))$ if $x \in E$, and a computation in polar coordinates as in the previous section gives

$$\int_{B(y,\varepsilon\delta(y))} \left(\log \frac{\varepsilon\delta(y)}{|x-y|} \right)^p dA(x) \simeq \delta(y)^2.$$

Therefore

$$\int_\Omega \Psi(x)^p\, dA(x) \lesssim \int_\Omega \mu(B(y, \varepsilon\delta(y)))^{p-1} \delta(y)^2\, d\mu(y).$$

Now we can just repeat the same argument: this last integral is comparable to

$$\int_\Omega \mu(B(y,\varepsilon\delta(y)))^{p-1} \left\{ \int_{B(y,\varepsilon\delta(y))} dA(x) \right\} d\mu(y) =$$
$$= \int_\Omega \left\{ \int_F \mu(B(y,\varepsilon\delta(y)))^{p-1}\, d\mu(y) \right\} dA(x),$$

the set F being now $\{y : x \in B(y, \varepsilon\delta(y))\}$. Choosing ε such that $\frac{2\varepsilon}{1-\varepsilon} = \lambda$, $B(y, \varepsilon\delta(y)) \subset B(x, \lambda\delta(x))$ for $y \in F$ and hence the above is bounded by

$$\int_\Omega \mu(B(x, \lambda\delta(x)))^p\, dA(x) = \int_\Omega T_\lambda \mu(x)^p\, dA(x) < +\infty.$$

It is then enough to prove that given a function f $(= K(\mu))$ with $\delta^2 f \in L^p(\Omega)$ there is $u \in L^p(\Omega)$ such that $\Delta u = f$ in the distribution sense, with

$$\int_\Omega |u|^p\, dA \le M \int_\Omega \delta^{2p} |f|^p\, dA$$

the constant M depending only on Ω. In the same way as before it is enough to prove the dual statement that follows:

Theorem 3.4. *For $1 < q < \infty$, there is a constant M depending on q and Ω such that for any test function in Ω*

$$\int_\Omega |\Phi(x)|^q \delta(x)^{-2q}\, dA(x) \le M \int_\Omega |\Delta\Phi(x)|^q\, dA(x).$$

Proof of Theorem 3.4. As in Theorem 3.3 it is enough to estimate the contribution in the integral of a tubular neighborhood U of $\partial\Omega$, because $\|\Phi\|_q \lesssim \|\Delta\Phi\|_q$. For $x \in U$, we write

$$|\Phi(x)| \le \int_0^{\delta(x)} |\nabla^2\Phi(p(x) + tn(x))| t\, dt$$

and use Hardy's inequality to obtain

$$\int_U |\Phi(x)|^q \delta(x)^{-2q}\, dA(x) \lesssim \int_U |\nabla^2\Phi(y)|^q\, dA(y).$$

Therefore

$$\int_\Omega |\Phi(x)|^q \delta(x)^{-2q}\, dA(x) \lesssim \int_\Omega |\nabla^2\Phi|^q\, dA(x)$$

and now it is enough to observe that for test functions Φ, $\|\nabla^2\Phi\|_q$ is comparable to $\|\Delta\Phi\|_q$ ([Ste, p. 59]). $\square$

We remark that along the proof we have seen that for $n = 2$ an equivalent condition to (b) is

$$\int_\Omega \mu(B(x, \lambda\delta(x)))^{p-1} \delta(x)^2\, d\mu(x) < +\infty.$$

3.6. The L^∞-version of Theorem 3.1 is essentially known, at least for the disk in $\mathbb{R}^2$ (see [Ber]). For completeness, we generalize now it to a general Ω in $\mathbb{R}^n$. We want to characterize the laplacians of bounded subharmonic functions in Ω. Let $G_\Omega(x, y)$ and $P_\Omega(x, y)$ denote the Green's function and the Poisson kernel for the domain Ω. Precise estimates are known for both G_Ω and P_Ω, using the comparison method with suitable balls tangent to $\partial\Omega$. For the Poisson kernel

$$P_\Omega(x, y) \sim \frac{\delta(x)}{|x - y|^n} \quad x \in \Omega,\ y \in \partial\Omega.$$

Fixed $\lambda < 1$,

$$\begin{aligned} |G_\Omega(x, y)| &\sim |x - y|^{2-n} \left(\log \frac{\lambda\delta(x)}{|x - y|} \text{ if } n = 2\right), && |x - y| \le \lambda\delta(x) \\ |G_\Omega(x, y)| &\sim \frac{\delta(x)\delta(y)}{|x - y|^n}, && |x - y| \ge \lambda\delta(x). \end{aligned}$$

For a reference, see [Swe] and the references given there.

Since every bounded subharmonic function u in Ω has a Green decomposition

$$u(x) = \int_{\partial\Omega} u(y)P_\Omega(x,y)\,d\sigma(y) + \int_\Omega G_\Omega(x,y)\,d\mu(y) \quad \mu = \Delta u$$

and the first integral on the right is obviously bounded, the question is obviously equivalent to the description of the positive measures μ on Ω having bounded Green potential; using the estimate for $G_\Omega(x,y)$ in $B(x,\lambda\delta(x))$ it is easy to check that for this part an equivalent condition is

$$\int_0^{\lambda\delta(x)} \frac{\mu(B(x,r))}{r^{n-1}}\,dr \le C, \quad x \in \Omega. \tag{2}$$

This is a condition on the mass of μ "inside Ω". For the mass near $\partial\Omega$, the condition is

$$\int_{|x-y|\ge\lambda\delta(x)} \frac{\delta(x)\delta(y)}{|x-y|^n}\,d\mu(y) \le C, \quad x \in \Omega. \tag{3}$$

At this point, we recall the definition of a *Carleson measure* on Ω. A positive measure $d\nu$ on Ω is called Carleson if

$$\nu(B(x,r)\cap\Omega) \le Cr^{n-1}, \quad x \in \partial\Omega,\ r > 0.$$

A simple doubling argument (see details in [Gar] for the unit disk) shows that (3) holds if and only if $d\nu = \delta(y)d\mu(y)$ is a Carleson measure. In conclusion one has

Theorem 3.5. *There exists a bounded subharmonic function u in Ω such that $\Delta u = \mu$ if and only if $\delta(y)d\mu$ is a Carleson measure and (2) holds.*

4. The unit disk situation

In this section $\mathbb{D}$ stands for the unit disk in the complex plane. We have proved in Theorem 3.1 that for a positive Borel measure μ on $\mathbb{D}$ a solution $u \in L^p(\mathbb{D})$, $1 \le p < \infty$ of $\Delta u = \mu$ exists if and only if the function μ^*

$$\mu^*_\lambda(z) = \mu(B(z,\lambda(1-|z|))), \quad z \in \mathbb{D}$$

belongs to $L^p(\mathbb{D})$, independently of $0 < \lambda < 1$. In this section we gather some remarks and precisions about this result.

4.1. First we will see that the solution u can be explicitly exhibited by means of an integral operator. Among all possible solutions of $\Delta u = \mu$ let us look at the one with minimal L^2 norm. For smooth v, the orthogonal decomposition of v in $L^2(\mathbb{D})$ as a sum of an harmonic function in $L^2(\mathbb{D})$ and an $L^2(\mathbb{D})$ function orthogonal to the harmonic subspace is $v = \mathbb{P}[v] + \mathbb{K}[\Delta v]$ with

$$\begin{aligned}\mathbb{P}v(z) &= \frac{1}{\pi}\int_{\mathbb{D}} v(\zeta)\left\{\frac{1}{(1-\bar\zeta z)^2} + \frac{1}{(1-\zeta\bar z)^2} - 1\right\} dA(\zeta)\\ \mathbb{K}[\mu](z) &= \int_{\mathbb{D}} K(z,\zeta)\, d\mu(\zeta)\end{aligned}$$

where

$$K(z,\zeta) = \frac{1}{2\pi}\left\{\log\left|\frac{\zeta - z}{1-\bar\zeta z}\right|^2 + \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar\zeta z|^2} + |z|^2\left(\frac{1-|\zeta|^2}{|1-\bar\zeta z|}\right)^2\right\}. \tag{4}$$

The kernel $K(z,\zeta)$ satisfies the estimate

$$|K(z,\zeta)| \lesssim \frac{(1-|\zeta|^2)^2}{|1-\bar\zeta z|^2}\left(1 + \log\left|\frac{1-\bar\zeta z}{\zeta - z}\right|\right), \quad z,\zeta \in \mathbb{D}. \tag{5}$$

A regularization argument shows then that the decomposition $v = \mathbb{P}[v] + \mathbb{K}[\mu]$ with $\mu = \Delta v$, holds for an arbitrary subharmonic function $v \in L^1(\mathbb{D})$. Hence $\mathbb{K}[\mu]$ is the solution of $\Delta u = \mu$ with minimal $L^2(\mathbb{D})$-norm (see [And, p. 147], [Pas] for all these facts). For $1 < p < \infty$, $p \neq 2$ something similar can be said.

Lemma 4.1. *The kernel $|1 - z\bar\zeta|^{-2}$ defines a bounded integral operator in $L^p(\mathbb{D})$, $1 < p < \infty$.*

Proof. This is an easy consequence of Schur's lemma (see [Rudin, Prop. 1.4.10] for details). □

The lemma implies that the projection $\mathbb{P}$ is bounded in $L^p(\mathbb{D})$, $1 < p < \infty$ and therefore that $\Delta u = \mu$ has a solution in $L^p(\mathbb{D})$, $1 < p < \infty$, if and only if $\mathbb{K}[\mu] \in L^p(\mathbb{D})$.

So, the next result is a restatement of Theorem 3.1 in the particular case of the disc, but nevertheless, we will give a short direct proof.

Theorem 4.2. *Assume $1 < p < \infty$ and let μ be a positive Borel measure such that $\mu^*_\lambda \in L^p(\mathbb{D})$. Then $\mathbb{K}[\mu] \in L^p(\mathbb{D})$.*

Proof. The estimate of K above implies, for all ε, $0 < \varepsilon < 1$

$$|\mathbb{K}[\mu](z)| \lesssim \int_{D(2,\varepsilon(1-|z|))} \log\left|\frac{1-\bar\zeta z}{\zeta - z}\right| d\mu(\zeta) + \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^2}{|1-\bar\zeta z|^2}\, d\mu(\zeta) \overset{\text{def}}{=} u_1(z) + u_2(z).$$

The term u_1 is estimated as Ψ was in section 3.5, choosing ε in a suitable way. For u_2 we use that $|1-\overline{w}z|^{-2}$ is a subharmonic function so that by the submean value property

$$\frac{(1-|\zeta|^2)^2}{|1-\bar\zeta z|^2} \lesssim \int_{D(\zeta,\varepsilon(1-|\zeta|))} \frac{dA(w)}{|1-\overline{w}z|^2}$$

and hence

$$u_2(z) \lesssim \int_{\mathbb{D}} \left\{ \int_{D(\zeta,\varepsilon(1-|\zeta|))} \frac{dA(w)}{|1-\overline{w}z|^2} \right\} d\mu(\zeta) = \int_{\mathbb{D}} \frac{\mu(E_w)}{|1-\overline{w}z|^2}\, dA(w)$$

with $E_w = \{\zeta = w \in D(\zeta, \varepsilon(1-|\zeta|))\}$. Again choosing ε small enough we get $E_w \subset D(w, \lambda(1-|w|))$, whence

$$u_2(z) \lesssim \int_{\mathbb{D}} \frac{\mu_\lambda^*(w)}{|1-\overline{w}z|^2}\, dA(w)$$

and the result follows from Lemma 4.1. $\square$

The previous argument breaks down for $p = 1$, for Lemma 4.1 does not hold in this case. To proceed in an analogous way for $L^1(\mathbb{D})$ one needs to consider the solution of $\Delta u = \mu$ which is minimal in the weighted Hilbert space $L^2(\mathbb{D}, (1-|\zeta|^2)\, dA)$. An analogous decomposition formula $v = \mathbb{P}_0[v] + \mathbb{K}_0[\Delta v]$ holds, where now the integral operators $\mathbb{P}_0$, $\mathbb{K}_0$ have kernels $P_0(z,\zeta)$, $K_0(z,\zeta)$ satisfying the estimates (see [And], [Pas])

$$\begin{aligned} |P_0(z,\zeta)| &\lesssim \frac{1-|\zeta|^2}{|1-\overline{z}\zeta|^3} \\ |K_0(z,\zeta)| &\lesssim \frac{(1-|\zeta|^2)^3}{|1-\overline{\zeta}z|^3} \left(1 + \log\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right|\right). \end{aligned}$$

Then $\mathbb{P}_0$ is bounded in $L^1(\mathbb{D})$. Again $\mathbb{K}_0[\mu]$ can be estimated by $u_1(z) + u_2(z)$, with u_1 as before and now

$$u_2(z) = \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^3}{|1-\overline{\zeta}z|^3}\, d\mu(\zeta)$$

from which if follows that $\mathbb{K}_0[\mu] \in L^1(\mathbb{D})$ if $\int_{\mathbb{D}} (1-|\zeta|^2)^2\, d\mu(\zeta) < +\infty$.

4.2. Next we shall show that there is a finite energy interpretation of the result in case $p = 2$. With the previous notations,

$$\int_D \mathbb{K}[\mu]^2(z)\, dA(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} H(\zeta, y)\, d\mu(\zeta)\, d\mu(\eta)$$

with

$$H(\zeta,\eta) = \int_{\mathbb{D}} K(z,\zeta) K(z,\eta)\, dA(z), \quad \zeta, \eta \in \mathbb{D}.$$

Next lemma is essentially known ([Lig], [Bell]):

Lemma 4.3. *The function $H(\zeta,\eta)$ is the Green function for the bilaplacian, which is ([Gara, p. 272])*

$$H(\zeta,\eta)=\frac{1}{16\pi}\left\{|\zeta-\eta|^2\log\left|\frac{\zeta-\eta}{1-\bar\zeta\eta}\right|^2+(1-|\zeta|^2)(1-|\eta|^2)\right\}.$$

Proof. Call $G(\zeta,\eta)$ the Green function for the bilaplacian, given by the right-hand term above. As such, $\Delta^2_\eta G(\zeta,\eta)$ is the delta mass δ_ζ at ζ, and $G(\zeta,\cdot)$ vanishes to first order on $\partial\mathbb{D}$. We will prove that $K(z,\zeta)=\Delta_z G(z,\zeta)$. Then, by Green's theorem,

$$\begin{aligned}H(\zeta,\eta)&=\int_{\mathbb{D}}\Delta_z G(z,\zeta)\Delta_z G(z,\eta)\,dA(z)=\int_{\mathbb{D}}G(z,\zeta)\Delta_z^2 G(z,\eta)\,dA(z)=\\&=\int_{\mathbb{D}}G(z,\zeta)\,d\delta_\eta(z)=G(\eta,\zeta)=G(\zeta,\eta)\end{aligned}$$

as wanted. That $K(\eta,\zeta)=\Delta_\eta G(\eta,\zeta)$ can be checked either by direct computation or else by showing that the properties of G imply that

$$u(z)=\int_{\mathbb{D}}\Delta_z G(z,\zeta)\Phi(\zeta)\,dA(\zeta)$$

gives a solution to $\Delta u=\Phi$ which is orthogonal to harmonic functions. Of course that $H=G$ can be as well checked by direct computation. $\square$

Using that

$$1-\left|\frac{\zeta-z}{1-\bar\zeta z}\right|^2=\frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar\zeta z|^2}$$

it is immediately seen that

$$H(\zeta,\eta)\simeq\frac{(1-|\zeta|^2)(1-|\eta|^2)^2}{|1-\bar\zeta\eta|^2}.$$

Therefore we conclude that

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|\zeta|^2)^2(1-|\eta|^2)^2}{|1-\bar\zeta\eta|^2}\,d\mu(\zeta)\,d\mu(\eta)<+\infty$$

is the necessary and sufficient condition for the existence of $u\in L^2(\mathbb{D})$ with $\Delta u=\mu$. Its equivalence with the condition $\mu^*_\lambda\in L^2(\mathbb{D})$ can be checked directly as well.

4.3. In this subsection we relate our problem at hand, laplacians of subharmonic functions in $L^p(\mathbb{D})$, with two other classical questions in function theory in the disc, Toeplitz operators and Carleson measures for Bergman spaces.

For a finite measure $d\nu$ in $\mathbb{D}$, the *Toeplitz operator* associated to ν is the operator

$$T_\nu f(z)=\int_{\mathbb{D}}B(z,w)f(w)\,d\nu(w),\quad z\in\mathbb{D}.$$

Here $B(z,w)=(1-z\overline{w})^{-2}$ is the Bergman kernel for $\mathbb{D}$. The *Berezin symbol* of T_ν is the function

$$\tilde\nu(z)=\int_{\mathbb{D}}\frac{(1-|\zeta|^2)^2}{|1-z\overline{w}|^4}\,d\nu(w).$$

Theorem 4.4. *For a finite Borel measure $d\nu$ in $\mathbb{D}$ and $1 \le p < \infty$, the following statements are equivalent:*

(a) *For $0 < \lambda < 1$, the function $z \mapsto (1-|z|)^{-2}\nu(D(z, \lambda(1-|z|)))$ belongs to $L^p(\mathbb{D})$.*
(b) *There is a subharmonic function $u \in L^p(\mathbb{D})$ such that $(1-|z|^2)^2\Delta u = \nu$.*
(c) *The measure ν is a Carleson measure for the Bergman space $L^q_a(\mathbb{D})$ with q the conjugate exponent of p, that is*

$$\int_{\mathbb{D}} |f|\, d\nu \le C\|f\|_q, \quad f \in L^q_a(\mathbb{D}).$$

(d) *T_ν extends to an operator in the Bergman space $L^2_a(\mathbb{D})$ belonging to the Schatten class S_p.*
(e) *$\tilde{\nu} \in L^p(\mathbb{D})$.*

The equivalence between (a) and (b) is our Theorem 3.1. The equivalence between (a) and (c) can be found in [Lue1] and the equivalence between (a), (d) and (e) in [Lue2] and [Zhu].

4.4. For later reference we point out a dyadic expression of the condition $\mu^*_\lambda \in L^p(\mathbb{D})$. Let us call T_{kj}, $k = 0, 1, \dots; j = 1, \dots, 2^k$ the usual dyadic regions

$$T_{kj} = \{z = re^{i\theta} : 1 - 2^{-k} < r \le 1 - 2^{-k-1}, (j-1)2^{1-k}\pi < \theta \le j2^{1-k}\pi\}$$

and let n_{kj} denote the mass $\mu(T_{kj})$ of μ in T_{kj}. Then using the independence in λ, the statement $\mu^*_\lambda \in L^p(\mathbb{D})$ amounts to say that

$$\sum_{k=0}^{\infty} 2^{-2k} \sum_{j=1}^{2^k} n_{kj}^p < +\infty.$$

5. Zeros of holomorphic functions

In this section we apply the previous results to the problem of describing the zero sequences of certain spaces of holomorphic functions in $\mathbb{D}$. For $1 \le p < \infty$ we will consider the class

$$\mathcal{A}_p(\mathbb{D}) = \{f : \log|f| \in L^p(\mathbb{D})\},$$

the Beller class

$$\mathcal{B}_p(\mathbb{D}) = \{f : \log^+|f| \in L^p(\mathbb{D})\},$$

and finally the still wider class

$$\mathcal{W}_p(\mathbb{D}) = \left\{ f : \int_0^1 \left(\int_0^{2\pi} \log^+|f(re^{it})|\, dt \right)^p dr < +\infty \right\}.$$

We have then the obvious inclusions $\mathcal{A}_p(\mathbb{D}) \subset \mathcal{B}_p(\mathbb{D}) \subset \mathcal{W}_p(\mathbb{D})$ for each $p \ge 1$. Note that $\mathcal{A}_1(\mathbb{D}) = \mathcal{W}_1(\mathbb{D})$, because as a consequence of the submean value property the negative part of a subharmonic function is controlled by the positive part in L^1. By the same reason $\log^+|f|$ can be replaced by $|\log|f||$ in the definition of $\mathcal{W}_p(\mathbb{D})$.

The connections with the problem $\Delta u = \mu$ is well known and classic: if $Z = \{a_n\}$ is a sequence with no accumulation points in $\mathbb{D}$ and $\mu = 2\pi \sum_n \delta_{a_n}$, the fact that an holomorphic function f vanishes exactly at Z is equivalent to say that $\Delta \log|f| = \mu$.

5.1. For a sequence $Z = \{a_j\}$ in $\mathbb{D}$ with no accumulation points in $\mathbb{D}$, we call n_{jk} the number of points of Z in the dyadic region T_{kj}, and $N_k = \sum_{j=1}^{2^k} n_{jk}$ the number of points of Z in the annulus $1 - 2^{-k} < |z| \le 1 - 2^{-k-1}$, counting multiplicities.

Theorem 5.1. *Z is the zero sequence of a function $f \in \mathcal{A}_p(\mathbb{D})$ if and only if*

$$\sum_{k=0}^{\infty} 2^{-2k} \sum_{j=1}^{2^k} n_{kj}^p < +\infty \tag{6}$$

and it is the zero sequence of $f \in \mathcal{W}_p(\mathbb{D})$ if and only if

$$\sum_{k=0}^{\infty} 2^{-(p+1)k} N_k^p < +\infty. \tag{7}$$

Proof. The first part is the restatement of Theorem 3.1 in subsection 4.4. We prove now the necessity of condition (7).

Let $f \in \mathcal{W}_p(\mathbb{D})$ and assume without loss of generality that $f(0) = 1$. Set $\mu = 2\pi \sum_n \delta_{a_n}$, $N(r) = \mu\{|z| \le r\}$. By Jensen's formula, for r close to 1

$$N(r)(1-r) \simeq \int_0^r \log \frac{r}{|\zeta|}\, d\mu(\zeta) = \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta$$

and hence,

$$\int_0^1 N(r)^p (1-r)^p \, dr < +\infty$$

which is easily seen to be equivalent to (7).

For the sufficiency, we will prove that the solution $u = \mathbb{K}[\mu]$ of $\Delta u = \mu$, considered in the previous section, satisfies

$$\int_0^1 \left(\int_0^{2\pi} u^+(re^{it})\, dt \right)^p dr < +\infty.$$

Note that the estimate (5) of the kernel $K(z,\zeta)$ of $\mathbb{K}$ implies that $\mathbb{K}[\mu]$ is a subharmonic function in $L^1_{\mathrm{loc}}(\mathbb{D})$ whenever μ satisfies

$$\int_{\mathbb{D}} (1-|\zeta|)^2\, d\mu(\zeta) < +\infty, \text{ or } \sum_k 2^{-2k} N_k < +\infty,$$

and that (7) implies this (as corresponds to the inclusion $\mathcal{W}_p(\mathbb{D}) \subset \mathcal{W}_1(\mathbb{D}) = \mathcal{A}_1(\mathbb{D})$). Moreover (4) implies

$$u^+(z) \lesssim \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}z|^2}\, d\mu(\zeta)$$

because $\log s + 1 - s \le 0$ for $s \le 1$. Then

$$\int_0^{2\pi} u^+(re^{it})\, dt \lesssim \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^2}{(1-r|\zeta|)}\, d\mu(\zeta).$$

The right-hand side is comparable to

$$\int_{|\zeta|\le r} (1-|\zeta|)\, d\mu(\zeta) + \frac{1}{1-r}\int_{r<|\zeta|<1} (1-|\zeta|)^2\, d\mu(\zeta).$$

In terms of the distribution function $N(r)$, we finally obtain

$$\int_0^{2\pi} u^+(re^{it})\, dt \lesssim \int_0^r N(t)\, dt + N(r)(1-r) + \frac{1}{1-r}\int_r^1 N(t)(1-t)\, dt$$

and the result follows using Hardy's and Holder's inequalities. $\square$

The problem of describing the zero sequences of $\mathcal{B}_p(\mathbb{D})$ remains open ([Bel1], [Bel2], [BH]). This is a harder problem than the ones we deal with, mainly because it is a nonlinear problem. Note the different character of conditions (6) and (7). While (7) only depends on the radii of the points of Z, (6) also depends on their angular distribution. We can see directly that (6) is stronger than (7), because by Jensen's inequality for convex functions,

$$N_k^p (2^{-k})^{p-1} = 2^k \left(2^{-k} \sum_{j=1}^{2^k} n_{kj} \right)^p \le \sum_{j=1}^{2^k} n_{kj}^p.$$

The only way to have equality above is that all n_{kj} are equal, that is, the points in Z are uniformly distributed in the angle. Only in this case conditions (6), (7) are the same. In particular, (7) is the necessary and sufficient condition on a sequence of radii r_n for the existence of $f \in \mathcal{A}_p(\mathbb{D})$ (or $f \in \mathcal{B}_p(\mathbb{D})$) with zeros a_n, $r_n = |a_n|$. Next we will make this even more precise.

5.2. Let $\{r_n\}$ be a fixed increasing sequence of radii. Let $\theta_n(w)$ be a sequence of independent random variables, all uniformly distributed in $[0, 2\pi]$. We consider the random sequence $\{a_n\} = Z(w)$ with $a_n(w) = r_n e^{i\theta_n(w)}$ and ask about the probability that $Z(w)$ be a zero sequence of $\mathcal{A}_p(\mathbb{D})$ or $\mathcal{B}_p(\mathbb{D})$. Questions of this type have been considered in other classes of functions in [Coc], [Bom], [Leb], [Rud], [Hor], [Mas] and [BH].

Theorem 5.2. *If the sequence $\{r_n\}$ satisfies (7), then $Z(w)$ is the sequence of zeros of a function $f \in \mathcal{A}_p(\mathbb{D})$ with probability 1.*

Proof. It is enough to prove that the random variable

$$X(w) = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} n_{kj}(w)^p 2^{-2k}$$

has finite mean, by Theorem 5.1. Here n_{kj} is the random variable that counts the number of points $a_n = r_n e^{i\theta_n(w)}$ in T_{kj}, which has binomial law $B(N_k, 2^{-k})$ for each $j = 1, \dots, 2^k$. Then $E(n_{kj}^p) \lesssim \sum_{l=1}^{p} (2^{-k} N_k)^l$ and

$$E(X) \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \sum_{l=1}^{p} (2^{-k})^{l+1} N_k^l \simeq$$
$$\simeq \sum_{l=1}^{p} \int_0^1 N(r)^l (1-r)^l \, dr \lesssim \int_0^1 N(r)^p (1-r)^p \, dr < +\infty. \quad \square$$

5.3. Finally we make some remarks about properties of the zero sequences of functions in $\mathcal{A}_p(\mathbb{D})$ and $\mathcal{B}_p(\mathbb{D})$. It is immediate from the characterization in Theorem 5.1 that those of $\mathcal{A}_p(\mathbb{D})$ are *hyperbolically stable*, in the sense that whenever $\{a_n\}$ is such a sequence, and

$$\left| \frac{a_n - b_n}{1 - \bar{a}_n b_n} \right| < \delta, \quad 0 < \delta < 1$$

then $\{b_n\}$ is also a zero sequence of $\mathcal{A}_p(\mathbb{D})$. It is also clear that a subsequence of a zero sequence of $\mathcal{A}_p(\mathbb{D})$ is also a zero sequence of $\mathcal{A}_p(\mathbb{D})$. Even though the characterization of those of $\mathcal{B}_p(\mathbb{D})$ is unknown, this later property can still be proved using an idea from [Lue3].

Theorem 5.3. *Every subsequence of a zero sequence of $\mathcal{B}_p(\mathbb{D})$ is also a zero sequence of $\mathbb{B}_p(\mathbb{D})$.*

Proof. We will show that if μ is a positive Borel measure on $\mathbb{D}$ for which there is u, $u^+ \in L^p(\mathbb{D})$, with $\Delta u = \mu$, and $0 \leq \nu \leq \mu$, then there is v such that $\Delta v = \nu$

and $v^+ \in L^p(\mathbb{D})$. Recall from subsection 4.1 the decomposition $v = \mathbb{P}[v] + \mathbb{K}[\Delta v]$, in particular

$$v(0) = \frac{1}{\pi}\int_{\mathbb{D}} v(\zeta)\,dA(\zeta) + \frac{1}{2\pi}\int_{\mathbb{D}} \{\log|\zeta|^2 + 1 - |\zeta|^2\}\Delta v(\zeta)\,dA(\zeta).$$

Fixed $z \in \mathbb{D}$, we apply this to $v = u\circ\varphi_z$, where φ_z is the automorphism of $\mathbb{D}$ permuting z with 0, and change variables in the integrals. The result is

$$u(z) = Q[u](z) + \frac{1}{2\pi}\int_{\mathbb{D}} \left\{\log\left|\frac{\zeta - z}{1-\bar{\zeta}z}\right|^2 + \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar{\zeta}z|^2}\right\} d\mu(\zeta)$$

with

$$Q[u](z) = \frac{1}{\pi}\int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{\zeta}z|^2}\right)^2 u(\zeta)\,dA(\zeta).$$

Subtracting from $u = \mathbb{P}[u] + \mathbb{K}[\Delta u]$, and using (4), we see that

$$Q[u](z) = \mathbb{P}[u](z) + \int_{\mathbb{D}} |z|^2\left(\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|}\right)^2 d\mu(\zeta).$$

We take as solution of $\Delta v = \nu$ the function $v = \mathbb{K}[\nu] + \mathbb{P}[u]$. From (4),

$$v(z) \lesssim \int_{\mathbb{D}} |z|^2\left(\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|}\right)^2 d\nu(\zeta) + \mathbb{P}[u]$$

and therefore $v(z) \lesssim Q[u](z)$. Then

$$v^+(z) \lesssim \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{\zeta}z|^2}\right)^2 u^+(\zeta)\,dA(\zeta).$$

But Schur's lemma can be applied as well to the kernel above, and since $u^+ \in L^p(\mathbb{D})$ we obtain that $v^+ \in L^p(\mathbb{D})$, as required. $\square$

References

And. M. Andersson, *Solution formulas for the $\partial\bar{\partial}$-equation and weighted Nevanlinna classes in the polydisc*, Bull. Soc. Math. France **109** (1985), 135–154.

Bel1. E. Beller, *Zeros of A^p functions and related classes of analytic functions*, Israel J. Math. **22** (1975), 68–80.

Bel2. E. Beller, *Factorization for non-Nevanlinna classes of analytic functions*, Israel J. Math. **27** (1977), 320–330.

Bell. S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J. **29** (1982), 123–128.

Ber. B. Berndtsson, *$\bar{\partial}_b$ and Carleson type inequalities*, Lecture Notes in Math. **1276**, 42–55.

BH. E. Beller and C. Horowitz, *Zero sets and random zero sets in certain function spaces*, J. Analyse Math. **64** (1994), 203–217.

Bom. G. Bomash, *A Blaschke-type product and random zero sets for Bergman spaces*, Ark. Mat. **30** (1992), 45–60.

CMS. R. R. Coiffman, Y. Meyer and E. M. Stein, *Some new function spaces and their application to harmonic analysis*, J. Funct. Anal. **62** (1985), 304–335.

Coc. C. Cochran, *Random Blaschke products*, Trans. Amer. Math. Soc. **322** (1990), 731–755.

DH. S. A. Dautov and G. M. Henkin, *Zeros of holomorphic functions of finite order and weighted estimates for solutions of $\bar\partial$-problem*, Math. Sbornik **107** (1978), 163–174.

Fed. H. Federer, *Geometric measure theory*, Springer-Verlag, New York, N.Y., 1969.

Gar. J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, N.Y:, 1981.

Gara. P. R. Garabedian, *Partial differential equations*, second edition, Chelsea Publishing Company, New York, N.Y., 1986.

Hei. A. Heilper, *The zeros of functions in Nevanlinna's area class*, Israel J. Math. **34** (1979), 1–11.

HK. W. K. Hayman and P. B. Kennedy, *Subharmonic functions*, Academic Press, London, 1976.

Hor. C. Horowitz, *Some conditions on Bergman spaces zero sets*, J. Analyse Math. **62** (1994), 323–348.

Leb. E. Leblanc, *A probabilistic zero condition for the Bergman space*, Michigan Math. J. **37** (1990), 427–436.

Lig. E. Ligocka, *On the reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball of $\mathbb{R}^n$*, Studia Math. **87** (1987), 23–32.

Lue1. D. Luecking, *Multipliers of Bergman spaces into Lebesgue spaces*, Prod. Edinburgh Math. Soc. (2) **29** (1986), 125–131.

Lue2. D. Luecking, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal. **73** (1987), 345–368.

Lue3. D. Luecking, *Zero sequences for Bergman spaces*, To appear in Complex Variables.

Mas. X. Massaneda, *Random sequences with prescribed radii in the unit ball*, Preprint (1995).

MW. B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.

Ort. J. Ortega-Cerdà, *Zero sets of holomorphic functions in the bidisc*, Preprint (1995).

Pas. D. Pascuas, *Zeros i interpolació en espais de funcions holomorfes del disc unitat*, PhD. Thesis, Universitat Autònoma de Barcelona (March, 1988).

Rudin. W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer-Verlag, New York, 1980.

Rud. R. Rudowicz, *Random interpolating sequences with probability one*, Bull. London Math. Soc. **26** (1994), 160–164.

Sha. H. Shapiro, *Quasi-balayage and apriori inequalities*, In preparation.

Ste. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.

Swe. G. Sweers, *Positivity for a strongly coupled elliptic system*, J. Geom. Anal. **4** (1994), 121–142.

Zhu. K. Zhu, *Operator theory in the function spaces*, Marcel-Dekker, New York, 1990.

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