BEURLING-TYPE DENSITY THEOREMS FOR WEIGHTED L^p SPACES OF ENTIRE FUNCTIONS

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1. Introduction

In [1], Berndtsson and one of us identified Beurling-type density conditions for sampling and interpolation in certain generalized Fock spaces. The purpose of the present work is to solve an essential problem left open in that paper.

The setting is as follows. Let a subharmonic function ϕ be given, whose Laplacian satisfies $0 < m \leq \Delta\phi(z) \leq M$ (m, M positive constants) for all $z \in \mathbb{C}$. We denote by $\mathcal{F}^p_\phi$ ($1 \leq p \leq \infty$) the set of all entire functions f for which $fe^{-\phi}$ belongs to $L^p(\mathbb{C})$. Our objective is to prove that the density conditions which in [1] were shown to be sufficient, are also *necessary*, for a sequence to be sampling or interpolating for $\mathcal{F}^p_\phi$. (We defer the definitions of sampling and interpolating sequences to the next section.)

The arguments which have previously allowed such strict density theorems (see [12,13,9]), all rest on a scheme developed by Beurling in his study of sampling and interpolation of bandlimited functions [2]. Some of these arguments seem indispensable and will be used again. We shall need some additional new tricks, but the main novelty of this paper is probably our way of adapting Beurling's approach to a setting seemingly different from the previous cases, which all involved holomorphic spaces with a suitable *shift invariance*. A prime example is the classical Fock space, which we obtain by setting $\phi(z) = \alpha|z|^2$ ($\alpha > 0$). In [12], a key role was played by the translation operator T_ζ, defined for every $\zeta \in \mathbb{C}$ by

$$(T_\zeta f)(z) = (T^\alpha_\zeta f)(z) = e^{\alpha 2\bar{\zeta} z - \alpha|\zeta|^2} f(z - \zeta).$$

A crucial property of T_ζ used in Beurling's scheme is that it acts isometrically in $\mathcal{F}^p_{\alpha|z|^2}$.

Clearly, the spaces $\mathcal{F}^p_\phi$ do not enjoy this sort of group invariance, but we have found that the translation group can be brought into action in much the same way. Namely, we define a translation operator T_ζ for every $\zeta \in \mathbb{C}$ acting isometrically from $\mathcal{F}^p_\phi$ to a *different* space $\mathcal{F}^p_{\phi_\zeta}$. The two functions ϕ and ϕ_ζ are related by the equation

$$\Delta_z(\phi(z - \zeta) - \phi_\zeta(z)) = 0,$$

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and we have uniform bounds on the growth of the functions ϕ_ζ. In the classical case, it so happens that we may choose $\phi = \phi_\zeta$ for every ζ. This "translation invariance" is what allows us to adapt Beurling's scheme (as modified in [12]) to the present setting.

By giving a complete description of sampling and interpolation in $\mathcal{F}^p_\phi$ in terms of densities only, our paper completes a line of research which was started in [12] and [14], and continued in [9] and [1]. However, it leaves us with the problem of understanding why some Banach spaces of entire functions allow strict density theorems, and others do not, like certain spaces of Paley-Wiener-type (see [10] and also [3] and [8]). This is a question which awaits a careful study, although the two papers [9] and [10] shed some light on it.

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2. Definitions and main results

We set $\Delta = \partial^2/\partial z \partial \bar{z}$, which differs from the standard convention for the Laplace operator Δ by a factor 4. We write $f \lesssim g$ whenever there is a constant K such that $f \leq Kg$, and $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$. Throughout this paper, we assume a subharmonic function ϕ is fixed, satisfying

$$0 < m \leq \Delta\phi(z) \leq M \tag{1}$$

for all $z \in \mathbb{C}$, where m and M are positive constants.

Let $d\sigma$ denote Lebesgue area measure on $\mathbb{C}$. We define

$$\|f\|^p_{\phi,p} = \int_{\mathbb{C}} |f|^p e^{-p\phi}\, d\sigma$$

for $p < \infty$, and $\|f\|_{\phi,\infty} = \sup_z |f(z)|e^{-\phi(z)}$. If $\Gamma = \{\gamma_n\}$ is a sequence of distinct points from $\mathbb{C}$, we set

$$\|f|\Gamma\|^p_{\phi,p} = \sum_n |f(\gamma_n)|^p e^{-p\phi(\gamma_n)}$$

for $p < \infty$, and $\|f|\Gamma\|_{\phi,\infty} = \sup_n |f(\gamma_n)|e^{-\phi(\gamma_n)}$, where we permit f to be a function defined on some set containing Γ. We say that the sequence Γ is *sampling* for $\mathcal{F}^p_\phi$ if we have

$$\|f|\Gamma\|_{\phi,p} \simeq \|f\|_{\phi,p}$$

for $f \in \mathcal{F}^p_\phi$. We say that Γ is *interpolating* for $\mathcal{F}^p_\phi$ if, for every sequence of values $a(\gamma_n) = a_n$ such that $\|a|\Gamma\|_{\phi,p} < +\infty$, there exists a solution $f \in \mathcal{F}^p_\phi$ to the interpolation problem $f(\gamma_n) = a_n$ for every $\gamma_n \in \Gamma$. The space of sequences a such that $\|a|\Gamma\|_{\phi,p} < \infty$ will sometimes be denoted by ℓ^p_ϕ.

In order to describe sampling and interpolating sequences, we introduce lower and upper uniform densities. A sequence is called *uniformly separated* if the infimum of the distances between distinct points is strictly positive. For a fixed Γ, we denote by $n(z, r)$ the number of points of the sequence Γ in $D(z, r)$, which is the disc of center z and radius r. The *lower uniform density of* Γ *with respect to* ϕ is defined as

$$D_\phi^-(\Gamma) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{n(z,r)}{\int_{D(z,r)} \Delta\phi},$$

and the *upper uniform density of* Γ *with respect to* ϕ is

$$D_\phi^+(\Gamma) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{n(z,r)}{\int_{D(z,r)} \Delta\phi}.$$

The purpose of the present paper is to prove the necessity of the following two density conditions.

Theorem 1. *A sequence* Γ *is sampling for* $\mathcal{F}_\phi^p$ *if and only if it contains a uniformly separated subsequence* Γ' *satisfying* $D_\phi^-(\Gamma') > 2/\pi$ *and in addition, when* $1 \le p < \infty$*, it is a finite union of uniformly separated sequences.*

Theorem 2. *A sequence* Γ *is interpolating for* $\mathcal{F}_\phi^p$ *if and only if it is uniformly separated and satisfies* $D_\phi^+(\Gamma) < 2/\pi$*.*

The *positive* part of Theorems 1 and 2 was proved by Berndtsson and the first named author in [1]. (Strictly speaking, only the cases $p = 2$ and $p = \infty$ were explicitly covered there, but the sufficiency for general p are easy consequences of these special cases.) Similar conditions for interpolation have also been considered by Grishin and Russakovskii in a related work [6].

Before turning to the proofs, a few remarks are in order.

We could have allowed a slightly more general function ϕ, satisfying instead of (1) that there is an $R > 0$ such that $\inf_{z\in\mathbb{C}} \int_{D(z,R)} \Delta\phi > 0$, and that the Laplacian $\Delta\phi$ has bounded local logarithmic potential, i.e.

$$\sup_{z\in\mathbb{C}} \int_{D(z,1)} \log \frac{1}{|z-\zeta|} \Delta\phi(\zeta) < +\infty.$$

The second condition ensures that if we pick $\psi(z) = 1/(\pi R^2) \int_{D(z,R)} \phi \, d\sigma$, then $|\phi - \psi| \le C$. Thus, the spaces $\mathcal{F}_\phi^p$ and $\mathcal{F}_\psi^p$ are the same. Moreover the first and second condition imply that $0 < m \le \Delta\psi \le M$, and so there is no loss of generality in making the a priori assumption (1).

On the other hand, we may assume that $\Delta\phi$ is not just a bounded function, but also uniformly Lipschitz. If it were not, we could take as before

$$\psi(z) = \frac{1}{\pi R^2} \int_{D(z,R)} \phi \, d\sigma.$$

Now, $\Delta\psi$ is uniformly Lipschitz since $\Delta\phi$ is bounded and $\mathcal{F}_\phi^p = \mathcal{F}_\psi^p$. The density conditions of Theorems 1 and 2 are not affected by replacing ϕ by ψ. Thus from now on, we will assume that $\Delta\phi$ is uniformly Lipschitz.

In our definition of $D_\phi^-(\Gamma)$ and $D_\phi^+(\Gamma)$, we have confined ourselves to a count of points from Γ in all possible discs. It follows from considerations due to Landau [7] that these limits are unchanged if we replace the discs $D(z,R)$ by dilations and translations of an arbitrary compact set whose boundary has measure 0.

We end this section by sketching the plan of the proofs. The next section contains some preliminary results and estimates. In particular, we prove some basic pointwise estimates corresponding to those associated with the names Phragmén-Lindelöf and Bernstein in the classical situation, and draw the conclusion that we may restrict ourselves to considering only uniformly separated sequences. Section 4 develops a tool analogous to that used by Beurling in [2], namely, we extend the notion of translation invariance, as already indicated, and correspondingly the notion of compactwise limits of sequences. In Section 5, we use a simple and powerful idea due to Ramanathan and Steger [11] to prove a "comparison lemma", saying essentially that an interpolating sequence has density smaller than any sampling sequence. Then using some specific examples of sampling and interpolating sequences, we obtain nonstrict density conditions. Section 6 contains a direct proof of the strict inequality $D^+_\phi(\Gamma) < 2/\pi$ for interpolation, by means of certain manipulations with Jensen's formula. Finally, in Section 7, a combination of Beurling-type arguments and the nonexistence of sequences being both sampling and interpolating (as follows from Sections 5 and 6) leads to the strict inequality for sampling as well.

3. Preliminary results and basic properties of $\mathcal{F}^p_\phi$

Our assumption about ϕ does not imply any restriction on the growth of the functions in $\mathcal{F}^p_\phi$. However, we may assume that functions in $\mathcal{F}^p_\phi$ are of order 2. To see this, consider

$$\phi_0(z) = \frac{2}{\pi}\int_{\mathbb{C}} \{\log|1-\frac{z}{w}| + \mathrm{Re}[\frac{z}{w}+\frac{1}{2}\frac{z^2}{w^2}]\Omega(w)\}\Delta\phi(w)d\sigma(w),$$

where

$$\Omega(z) = \begin{cases} 1, & |z| \geq 1 \\ 0, & |z| < 1. \end{cases}$$

We have $\Delta\phi = \Delta\phi_0$, and so the difference between ϕ and ϕ_0 is a harmonic function, whence there exists an entire function h such that $f \mapsto f\exp h$ maps $\mathcal{F}^p_\phi$ isometrically onto $\mathcal{F}^p_{\phi_0}$. From the representation of ϕ_0 we obtain the estimate

$$|\phi_0(z)| \lesssim 1 + |z|^2\log^+|z|,$$

and this along with Lemma 1 below yields the order 2 condition on functions in $\mathcal{F}^p_{\phi_0}$.

In the proof of Theorem 1, we will need the following estimate for $\bar\partial$ due to Christ [4]:

Theorem A. *If ϕ satisfies the condition $m \leq \Delta\phi \leq M$, then for any $1 \leq p \leq \infty$ the equation $\bar\partial u = f$ has a solution u in $\mathbb{C}$ such that*

$$\|ue^{-\phi}\|_{L^p(\mathbb{C})} \lesssim \|fe^{-\phi}\|_{L^p(\mathbb{C})}.$$

We will also need an estimate of the decay far from the diagonal of the Bergman kernel in the spaces $\mathcal{F}^2_\phi$. This estimate is also obtained by Christ in [4]:

Theorem B. *Let $K(z,\zeta)$ denote the Bergman kernel of the space $\mathcal{F}^2_\phi$, i.e. for any $f \in F^2_\phi$,*

$$f(z) = \int_{\mathbb{C}} K(z,\zeta) f(\zeta) e^{-2\phi(\zeta)}\, d\sigma(\zeta).$$

Then there is an $\varepsilon > 0$ such that K satisfies

$$|K(z,\zeta)| \lesssim e^{\phi(z)+\phi(\zeta)-\varepsilon|\zeta - z|}.$$

If we denote by L^p_ϕ the space of all functions f such that $fe^{-\phi} \in L^p(\mathbb{C}, d\sigma)$, and define the projection operator as

$$P_\phi f(z) = \int_{\mathbb{C}} f(\zeta) K(z,\zeta) e^{-2\phi(\zeta)}\, d\sigma(\zeta),$$

then we get as a corollary of Theorem B:

(1) The operator P_ϕ projects L^p_ϕ onto $\mathcal{F}^p_\phi$ for all $1 \le p \le \infty$.
(2) The dual of $\mathcal{F}^p_\phi$ is $\mathcal{F}^q_\phi$ for any $1 \le p < \infty$, where $1/p + 1/q = 1$. Moreover the dual of $\mathcal{F}^{\infty,0}_\phi$ is $\mathcal{F}^1_\phi$, where $\mathcal{F}^{\infty,0}_\phi$ is the closed subspace of $\mathcal{F}^\infty_\phi$ consisting of functions f such that $|f(z)|e^{-\phi(z)} \to 0$ as $z \to \infty$.

We will need several times the following pointwise estimates, playing the role of the Phragmén-Lindelöf and Bernstein inequalities in Beurling's situation:

Lemma 1. *If f belongs to $\mathcal{F}^p_\phi$, then*

$$|f(z)| \lesssim \|f\|_{\phi,p} e^{\phi(z)},$$

$$|\nabla(|f|^r e^{-r\phi})(z)| \lesssim \|f\|^r_{\phi,p}$$

for all $r > 0$, provided $f(z) \neq 0$.

Proof. We start by decomposing ϕ in the disc $D = D(z,1)$ as

$$\phi(w) = \phi(z) + u(w) + \int_D (G(w,\eta) - G(z,\eta))\Delta\phi(\eta)\, d\sigma(\eta), \tag{2}$$

where G is the Green function of the disc $D(z,1)$ and u is a harmonic function in the disc such that $u(z) = 0$. Since $\Delta\phi$ is bounded, we know then that

$$|\phi(w) - \phi(z) - u(w)| \le K.$$

Since u is harmonic, there is a holomorphic function $h \in \mathcal{H}(D)$ such that $|e^h| = e^u$, whence

$$|f(z)|^p = |f(z)e^{-h(z)}|^p \le \frac{1}{\pi}\int_D |f|^p e^{-pu} d\sigma.$$

Therefore, we obtain

$$|f(z)|^p e^{-p\phi(z)} \lesssim \int |f(w)|^p e^{-pu(w)-p\phi(z)} d\sigma \lesssim \int |f(w)|^p e^{-p\phi(w)} d\sigma \le \|f\|^p_{\phi,p}.$$

To prove the inequality for the gradient we begin as follows:

$$|\nabla(|f|^r e^{-r\phi})| = \left|\frac{r}{2}|f|^{r-2} f' \bar{f} e^{-r\phi} - r|f|^r e^{-r\phi}\frac{\partial\phi}{\partial z}\right| \lesssim \|f\|_{\phi,p}^{r-1}\left|f' e^{-\phi} - 2fe^{-\phi}\frac{\partial\phi}{\partial z}\right|.$$

Moreover, because of (2) we know that $|\partial\phi/\partial w - \partial u/\partial w| < K$. Thus

$$|\nabla|f|^r e^{-r\phi}| \lesssim \|f\|_{\phi,p}^r + \|f\|_{\phi,p}^{r-1}\left|f' e^{-\phi} - 2fe^{-\phi}\frac{\partial u}{\partial w}\right|. \tag{3}$$

We pick as before $h \in \mathcal{H}(D)$ such that $h(z) = 0$ and $\operatorname{Re} h = u$. By the Cauchy inequalities we get

$$\left|\frac{\partial(fe^{-h})}{\partial w}(z)\right| \leq \frac{1}{2\pi}\int_{|w-z|=1} |f(w)|e^{-u(w)}|dw| \lesssim \|f\|_{\phi,p}.$$

Since $h' = 2\partial u/\partial z$, we finally obtain

$$e^{-\phi}\left|f' - 2f\frac{\partial u}{\partial z}\right| \lesssim \|f\|_{\phi,p},$$

which along with (3) yields the desired inequality. □

The same proof also gives the following result: If $\Gamma = \{\gamma_n\}$ is a finite union of uniformly separated sequences, then

$$\|f|\Gamma\|_{\phi,p}^p \lesssim \sum_n \int_{D(\gamma_n,1)} |f|^p e^{-p\phi} d\sigma \lesssim \|f\|_{\phi,p}^p \tag{4}$$

for all $f \in \mathcal{F}_\phi^p$ ($p < \infty$), and if $\Gamma = \{\gamma_n\}$ is a uniformly separated sequence, then

$$\sum_n |\nabla(|f|^p e^{-p\phi})(\gamma_n)| \lesssim \|f\|_{\phi,p}^p \tag{5}$$

for all $f \in \mathcal{F}_\phi^p$ ($p < \infty$), and

$$\sup_n |\nabla(|f|e^{-\phi})(\gamma_n)| \lesssim \|f\|_{\phi,\infty}$$

for all $f \in \mathcal{F}_\phi^\infty$.

We establish next some basic necessary conditions for sampling and interpolation.

Proposition 1. *If $p < \infty$, we have*

$$\|f|\Gamma\|_{\phi,p} \lesssim \|f\|_{\phi,p}$$

for all $f \in \mathcal{F}_\phi^p$ if and only if Γ can be expressed as a finite union of uniformly separated sequences.

Proof. One of the directions is (4). For the converse implication, assume that the inequality is true, but that the sequence is not a finite union of uniformly separated sequences. Then for every n and $\varepsilon > 0$, there will be a point $z \in \mathbb{C}$ such that there are n points in an ε neigborhood of z. One can easily construct a function such that $\|f\|_{\phi,p} < C$ (where C is independent of z) and $f(z) = e^{\phi(z)}$. This is so because a single point is trivially an interpolating sequence for this space (see [1]). But $\|f|\Gamma\|_{\phi,p}$ will tend to ∞ when $n \to \infty$ because of the inequality on the gradient in Lemma 1, which is a contradiction. □

The inequality for the gradient in Lemma 1 also yields the following two propositions, the first of which is an immediate consequence of that inequality.

Proposition 2. *If Γ is a sampling sequence for $\mathcal{F}^p_\phi$, then there exists a uniformly separated subsequence $\Gamma' \subset \Gamma$ which is also sampling for $\mathcal{F}^p_\phi$.*

Before stating the corresponding result for interpolating sequences, recall that, by the closed graph theorem, there is a constant M such that for every $\{a_n\}$ under the compatibility condition, the interpolating function f can be chosen so that $\|f\|^p_{\phi,p} \le M^p \sum |a_n|^p e^{-p\phi(\gamma_n)}$. The smallest such constant M is called the *interpolation constant* of Γ and will be denoted by M_Γ.

Proposition 3. *If $\Gamma = \{\gamma_n\}$ is an interpolating sequence for $\mathcal{F}^p_\phi$, then Γ is uniformly separated.*

Proof. We know that

$$\Big| |f(z)|e^{-\phi(z)} - |f(w)|e^{-\phi(w)} \Big| \lesssim \|f\|_{\phi,p}|z-w|.$$

Moreover, for any two distinct points γ_n and γ_m of Γ we can find a function $f \in \mathcal{F}^p_\phi$ such that $\|f\|_{\phi,p} < M_\Gamma$, $f(\gamma_n) = e^{\phi(\gamma_n)}$ and $f(\gamma_m) = 0$. Consequently, we get $1 \lesssim M_\Gamma|\gamma_n - \gamma_m|$. $\square$

So from now on, we may assume that all sequences Γ are uniformly separated.

4. A translation operator and the concept of compactwise limits

Suppose ϕ is a subharmonic function such that $\Delta\phi < M$ and $\Delta\phi$ satisfies a uniform Lipschitz condition. For every $\zeta \in \mathbb{C}$ we define

$$\phi_\zeta(z) = \frac{2}{\pi}\int_{\mathbb{C}} \{\log|1-\frac{z}{w}| + \mathrm{Re}[\frac{z}{w} + \frac{1}{2}\frac{z^2}{w^2}]\Omega(w)\}\Delta\phi(w-\zeta)d\sigma(w),$$

where as above

$$\Omega(z) = \begin{cases} 1, & |z| \ge 1 \\ 0, & |z| < 1. \end{cases}$$

As explained at the beginning of Section 3, we may assume without loss of generality that $\phi = \phi_0$. We observe that ϕ_ζ satisfies

$$\Delta_z\phi_\zeta(z) = \Delta_z\phi(z-\zeta)$$

for all $z \in \mathbb{C}$, and

$$|\phi_\zeta(z)| \le C(1 + |z|^2 \log^+ |z|),$$

with C independent of ζ. These two conditions along with the assumption that $\Delta\phi$ is uniformly Lipschitz imply that ϕ_ζ and $\Delta\phi_\zeta$ are a normal family in the topology of uniform convergence over compacts. Moreover, the operator T_ζ defined as

$$(T_\zeta f)(z) = e^{q(z,\zeta)}f(z-\zeta),$$

satisfies $\|T_\zeta f\|_{\phi_\zeta,p} = \|f\|_{\phi,p}$ for some polynomial $q(z,\zeta)$ of degree 2 in z. This relation shows in particular that if a sequence Γ is sampling for $\mathcal{F}^p_\phi$, then $\Gamma - \zeta$ is sampling for $\mathcal{F}^p_{\phi_\zeta}$ with the same constants in the sampling inequalities.

A sequence Q_j of closed sets converges strongly to Q, denoted $Q_j \to Q$, if $[Q, Q_j] \to 0$; here $[Q, R]$ denotes the Fréchet distance between two closed sets Q and R, i.e., $[Q, R,]$ is the smallest number t such that $Q \subset \{z : \ d(z, R) \le t\}$ and $R \subset \{z : \ d(z, Q) \le t\}$, where $d(\cdot,\cdot)$ denotes Euclidean distance in $\mathbb{C}$. Q_j converges compactwise to Q, denoted $Q_j \rightharpoonup Q$, if for every compact set D $(Q_j \cap D) \cup \partial D \to (Q \cap D) \cup \partial D$.

Suppose that associated with a sequence of sets Γ_j there is a sequence of subharmonic functions ϕ_j. If $\Gamma_j \rightharpoonup \Lambda$, $\psi = \lim_{j\to\infty} \phi_j$, and $\Delta\psi = \lim_{j\to\infty} \Delta\phi_j$ (uniformly on compacts), we write $(\Gamma_j, \phi_j) \rightharpoonup (\Lambda, \psi)$ and say that (Γ_j, ϕ_j) *converges compactwise* to (Λ, ψ). For a closed set Γ and subharmonic function ϕ satisfying (1), we let $W(\Gamma, \phi)$ denote the collection of pairs (Λ, ψ) such that $(\Gamma - a_j, \phi_{a_j}) \rightharpoonup (\Lambda, \psi)$ for some sequence a_j. By the Arzela-Ascoli theorem, $W(\Gamma, \phi)$ is compact in the sense that every sequence of elements $(\Gamma_j, \phi_j) \in W(\Gamma, \phi)$ has a subsequence converging compactwise to some element in $W(\Gamma, \phi)$.

Lemma 3. *If Γ is a uniformly separated sampling sequence for $\mathcal{F}^p_\phi$ and $(\Gamma - a_j, \phi_{a_j}) \rightharpoonup (\Lambda, \psi)$, then Λ is a sampling sequence for $\mathcal{F}^p_\psi$.*

Proof. Consider first the case $1 \le p < \infty$, and suppose the lemma is false. This assumption implies that for every $\varepsilon > 0$ we can find an $f \in \mathcal{F}^p_\psi$ such that $\|f\|_{\psi,p} = 1$ and $\|f|\Lambda\|_{\psi,p} \le \varepsilon$. We intend to prove the lemma by finding a j and produce another function, say f_j, with the same properties with respect to ϕ_{a_j} and the sequence $\Gamma - a_j$. This we do as follows:

Choose an $R > 0$ so large that $\int_{|z|>(R-3)} |f|^p e^{-p\psi} d\sigma < \varepsilon^p$. Take a cutoff function χ, such that $0 \le \chi \le 1$, $\chi(z) = 1$ if $|z| < R-1$, $\chi(z) = 0$ if $|z| > R$ and $|\nabla\chi(z)| \le 2$. Set $h = \chi f$, and choose a j such that $\|h|(\Gamma - a_j)\|_{\phi_{a_j},p} \le 2\varepsilon$, $\|h\|_{\phi_{a_j},p} \ge 1/2$, and

$$e^{-\phi_{a_j}(z)} \le 2e^{-\psi(z)}$$

for $|z| < R$. Then h has the desired properties, but it is not holomorphic. We may, however, correct it by solving the $\bar\partial$ equation $\bar\partial u = f\bar\partial\chi$. Because

$$\int_{\mathbb{C}} |f\bar\partial\chi|^p e^{-p\phi_{a_j}} d\sigma \le (4\varepsilon)^p,$$

we may apply Theorem A, and so there is a solution u with $\|u\|_{\phi_{a_j},p} \lesssim \varepsilon$. We split $\Gamma - a_j$ into two sequences β and β', where β consists of the points from $\Gamma - a_j$ which lie in the corona $R - 2 < |z| < R + 1$, and $\beta' = (\Gamma - a_j) \setminus \beta$. Since u is holomorphic outside $R - 1 < |z| < R$, we may apply the first inequality of (4) to obtain $\|u|\beta'\|_{\phi_{a_j},p} \lesssim \varepsilon$. On the other hand, $\int_{R-3<|z|<R+2} |h - u|^p e^{-p\phi_{a_j}} d\sigma \lesssim \varepsilon$, and since $h - u$ is holomorphic, we may again apply the first inequality of (4), now to $h - u$ in the corona $R - 2 < |z| < R + 1$. Using also that $\|h|_\beta\|_{\phi_{a_j},p} \lesssim \varepsilon$, we obtain $\|h - u|(\Gamma - a_j)\|_{\phi_{a_j},p} \lesssim \varepsilon$. It follows that $f_j = h - u$ is the desired function.

Consider now the case $p = \infty$. A sequence Λ is a sampling sequence for $\mathcal{F}^\infty_\psi$ if and only if the same sequence Λ is sampling for $\mathcal{F}^{\infty,0}_\psi$. In fact, Λ is a sampling sequence for $\mathcal{F}^\infty_\psi$ if the restriction operator $R : \mathcal{F}^\infty_\psi \to \ell^\infty_\psi$ is injective and with closed range. Equivalently, that means that the operator $R^* : \ell^1_\psi \to \mathcal{F}^1_\psi$ defined as $R^*(\{a_n\}) = \sum a_n K(z, \lambda_n) e^{-2\psi(\lambda_n)}$ is surjective, and again by duality this is equivalent to saying that $R^{**} : \mathcal{F}_{\psi}^{\infty,0} \to \ell_{\psi}^{\infty,0}$ is injective and of closed range, which in turn amounts to saying that Λ is a sampling sequence for $\mathcal{F}_{\psi}^{\infty,0}$.

Therefore, if the lemma is false for $p = \infty$, then for every $\varepsilon > 0$ there exists a function $f \in \mathcal{F}_{\psi}^{\infty,0}$ such that $\|f\|_{\psi,\infty} = 1$ and $\|f|\Lambda\|_{\psi,\infty} \leq \varepsilon$. We then repeat the construction of the function f_j, which was done above. The point at which we use that $f \in \mathcal{F}_{\psi}^{\infty,0}$ instead of $\mathcal{F}_{\psi}^{\infty}$, is at the beginning of the construction, when we need to assure the existence of a big $R > 0$ such that $\sup_{|z|>R-1} |f|e^{-\psi} \leq \varepsilon$. □

Lemma 3 yields the following characterization of sampling sequences for $\mathcal{F}_{\phi}^{\infty}$.

Theorem 3. *A sequence Γ is sampling for $\mathcal{F}_{\phi}^{\infty}$ if and only if every pair $(\Lambda, \psi) \in W(\Gamma, \phi)$ has the property that Λ is set of uniqueness for $\mathcal{F}_{\psi}^{\infty}$.*

Proof. One of the directions is just Lemma 3 since any sampling sequence is a uniqueness set. The other direction follows by the same argument as in the proof of Theorem 3 in [2], p. 345. □

5. A comparison lemma of Ramanathan and Steger, and nonstrict density conditions for sampling and interpolation

We begin by stating the Ramanathan-Steger comparison lemma [11] for our spaces.

Lemma 4. *Let I be an interpolating sequence for $\mathcal{F}_{\phi}^{2}$ and S a uniformly separated sampling sequence. Then $D_{\phi}^{+}(I) \leq D_{\phi}^{+}(S)$ and $D_{\phi}^{-}(I) \leq D_{\phi}^{-}(S)$.*

Proof. If S is a sampling sequence, then $\{k(z,s) = K(z,s)e^{-\phi(s)}\}_{s\in S}$ is a frame in $\mathcal{F}_{\phi}^{2}$, where K is the Bergman kernel. This means that we have

$$\|f\|^2 \simeq \sum_{s\in S} |\langle k(z,s), f(z)\rangle|^2$$

for $f \in \mathcal{F}_{\phi}^{2}$. A consequence is that any $f \in \mathcal{F}_{\phi}^{2}$ can be written in at least one way as $f = \sum_{s\in S} c_s k(z,s)$, with square-summable coefficients c_s. The unique representation minimizing the ℓ^2-norm of the coefficients is given by

$$f(z) = \sum_{s\in S} \langle \widetilde{k}(w,s), f(w)\rangle k(z,s),$$

where $\widetilde{k}(z,s)$ is the dual frame of $k(z,s)$. It follows that the dual frame of $\widetilde{k}(z,s)$ is $k(z,s)$, so that

$$f(z) = \sum_{s\in S} \langle k(w,s), f(w)\rangle \widetilde{k}(z,s) = \sum_{s\in S} f(s)e^{-\phi(s)}\widetilde{k}(z,s).$$

We refer to [5] for proofs and more information about frames.

On the other hand, if we denote by H the closed linear span in $\mathcal{F}_{\phi}^{2}$ of the functions $k(z,i) = K(z,i)e^{-\phi(i)}$, $i \in I$, then I is an interpolating sequence if and only if the system $\{k(z,i)\}$, $i \in I$ is a Riesz basis for H.

Now, for any given $z \in \mathbb{C}$ and $R, \rho > 0$, we introduce the following two finite dimensional subspaces of $\mathcal{F}_{\phi}^{2}$. Denote by W_S the subspace generated by the functions $\widetilde{k}(w,s)$, $s \in D(z, R+\rho) \cap S$, and by W_I the subspace generated by the $k(w,i)$,

$i \in D(z,R) \cap I$. (One should think of R as much bigger than ρ.) We define P_S and P_I to be the orthogonal projections of $\mathcal{F}^2_\phi$ onto W_S and W_I, respectively. Consider the operator $T = P_I P_S$ defined from W_I to W_I. We are going to estimate the trace of this operator in two different ways.

To begin with,

$$\text{tr } (T) \le \text{rank} W_S \le \#\{D(z, R+\rho) \cap S\}.$$

On the other hand,

$$\text{tr } (T) = \sum_{i \in I \cap D(z,R)} \langle T(k(w,i)), P_I k^*(w,i)\rangle,$$

where $\{k^*(w,i)\}$ is the dual basis of $k(w,i)$ in H. Since $T = P_I P_S$, then for any $i \in I \cap D(z,R)$,

$$\langle T(k(w,i)), P_I k^*(w,i)\rangle = \langle k(w,i), k^*(w,i)\rangle + \langle P_S(k(w,i)) - k(w,i), P_I k^*(w,i)\rangle,$$

whence

$$\text{tr } (T) \ge \#\{i \in I \cap D(z,R)\}(1 - \sup_i |\langle P_S(k(w,i)) - k(w,i), k^*(w,i)\rangle|).$$

Recall that we normalized $k(w,i)$ so that $\|k(w,i)\| \simeq 1$. Therefore $\|k^*(w,i)\| \simeq 1$ too. Thus if we can show that $\|P_S(k(w,i)) - k(w,i)\| \le \varepsilon$ for a sufficiently big ρ, we have proved that for every $\varepsilon > 0$ there exists a ρ such that for all large R, the following inequality holds

$$(1-\varepsilon)\#\{I \cap D(z,R)\} \le \#\{S \cap D(z, R+\rho)\}.$$

This estimate implies the desired inequalities for the densities.

It remains to be shown that $\|P_S(k(w,i)) - k(w,i)\| \le \varepsilon$. To this end, we note that since S is a sampling sequence, $k(w,i) = \sum_{s\in S} k(s,i)\widetilde{k}(w,s)e^{-\phi(s)}$. Therefore,

$$\|P_S(k(w,i)) - k(w,i)\| \le \sum_{|s-i|>\rho} e^{-\phi(s)}|k(s,i)|\|\widetilde{k}(w,s)\| \le \sum_{|s-i|>\rho} e^{-\varepsilon|i-s|},$$

where the last inequality follows from Theorem B. The sum on the right side of this inequality is smaller than ε if ρ is big enough, since S is a uniformly separated sequence. $\square$

We obtain now nonstrict density conditions for sampling and interpolation. The inequality in the interpolation case will not be used later, because we will give a direct proof of the strict inequality in the next section. But the proof is the same, and it comes for free.

Corollary 1. *If Γ is a uniformly separated sampling sequence for $\mathcal{F}^2_\phi$, then $D^-_\phi(\Gamma) \ge 2/\pi$. If Γ is an interpolating sequence for $\mathcal{F}^2_\phi$. Then $D^+_\phi(\Gamma) \le 2/\pi$.*

Proof. Assume that Γ is a sampling sequence for $\mathcal{F}^2_\phi$. For every $\varepsilon > 0$, we may construct a sequence Λ with $D^+(\Lambda) = D^-(\Lambda) = 2/(\pi + \varepsilon)$ as follows. Partition the plane $\mathbb{C}$ into strips of the form $k-1 < \operatorname{Re} z \le k,\ k \in \mathbb{Z}$. Each of the strips can be partitioned into rectangles R_j such that $\int_{R_j} \Delta\phi = (\pi+\varepsilon)/2$. The sides of the rectangles R_j will be bounded above and below by some constants since $\Delta\phi$ is bounded above and below. Thus if we take as Λ a uniformly separated sequence consisting of one point from each of these rectangles, it is easy to check that it has the desired density. Now Λ is an interpolating sequence because $D_\phi^-(\Lambda) < 2/\pi$. (This is Theorem 2b of [1].) Thus by the comparison lemma, we have just proved that $D_\phi^-(\Gamma) \ge D_\phi^+(\Lambda) = 2/(\pi+\varepsilon)$. Since ε was arbitrary, we have the nonstrict inequality.

For Γ an interpolating sequence, we proceed analogously. This time we use Theorem 2a of [1] saying that Λ is interpolating for $\mathcal{F}_\phi^2$ whenever $D_\phi^+(\Lambda) < 2/\pi$. □

Proposition 4. *If Γ is a sampling sequence for $\mathcal{F}_\phi^p$ ($1 \le p \le \infty$), then Γ is sampling for $\mathcal{F}^2_{\phi-\varepsilon|z|^2}$ for all sufficiently small $\varepsilon > 0$.*

Proof. If Γ is a sampling sequence for $\mathcal{F}_\phi^\infty$, then there is a sequence of functions $\{g(z,\gamma)\}_{\gamma\in\Gamma}$ such that for every $\varepsilon > 0$ and $f \in \mathcal{F}_\phi^{\infty,0}$,

$$e^{-\phi(z)} f(z) = \sum_{\gamma\in\Gamma} f(\gamma) g(z,\gamma) e^{-\phi(\gamma)},$$

and $\sum |g(z,\gamma)| \le K$ uniformly in z. This is so by a duality argument, because $\{f(\gamma)\}_{\gamma\in\Gamma} \mapsto f(z)e^{-\phi(z)}$, with $f \in \mathcal{F}_\phi^{\infty,0}$, is a bounded linear functional on a closed subspace of $\ell_\phi^{\infty,0}$, whose norm we can bound independently of z. (We repeat here an argument from [13], p. 36.) We apply this representation formula to the function $f(w)e^{2\varepsilon w\bar{z}-2\varepsilon|z|^2}$, and get

$$e^{-\phi(z)} f(z) = \sum_{\gamma\in\Gamma} f(\gamma) e^{-\phi(\gamma)-2\varepsilon|z|^2+2\varepsilon\gamma\bar{z}} g(z,\gamma)$$

for every $f \in \mathcal{F}^2_{\phi-\varepsilon|z|^2}$. By the Cauchy-Schwarz inequality, we obtain

$$|f(z)|^2 e^{-2\phi(z)} \le K \sum_{\gamma\in\Gamma} |f(\gamma)|^2 e^{-2\phi(\gamma)-4\varepsilon|z|^2+4\varepsilon\operatorname{Re}(\gamma\bar{z})} |g(z,\gamma)|.$$

Integrating against $e^{2\varepsilon|z|^2}$, we obtain the sampling inequality

$$\int_{\mathbb{C}} |f|^2 e^{-2\phi+2\varepsilon|z|^2} d\sigma \lesssim \sum_{\gamma\in\Gamma} |f(\gamma)|^2 e^{-2\phi(\gamma)+2\varepsilon|\gamma|^2}.$$

Finally, suppose that Γ is a sampling sequence for $\mathcal{F}_\phi^p$ ($1 \le p < \infty$). This implies, by an argument similar to that leading to Lemma 3, that if $(\Gamma - a_j, \phi_{a_j}) \rightharpoonup (\Lambda, \psi)$, then (Λ, ψ) is also a sampling sequence for $\mathcal{F}_\psi^p$. Since $\mathcal{F}^\infty_{\psi-\varepsilon|z|^2} \subset \mathcal{F}_\psi^p$, it follows that Λ is a uniqueness set for $\mathcal{F}^\infty_{\psi-\varepsilon|z|^2}$, and thus by Theorem 3, it follows that Γ is a sampling sequence for $\mathcal{F}^\infty_{\phi-\varepsilon|z|^2}$. As we have just seen, this means that it is then a sampling sequence for $\mathcal{F}^2_{\phi-\varepsilon'|z|^2}$ for any $\varepsilon' > \varepsilon$. □

Joining Corollary 1 and Proposition 4, we obtain the following corollary.

Corollary 2. *If Γ is a uniformly separated sampling sequence for $\mathcal{F}_\phi^p$ ($1 \le p \le \infty$), then $D_\phi^-(\Gamma) \ge 2/\pi$.*

We could have made a similar transition to general $1 \le p \le \infty$ in the interpolation case, but this is not needed in view of the next section.

6. Strict necessary conditions for interpolation

In order to get necessary conditions for interpolation, we will use Jensen's formula to control the number of points of an interpolating sequence contained in some disc. If we want to improve the inequality obtained by this procedure to get a strict inequality for the density, we can proceed in the following way. We perturb the original sequence by fitting into the disc under consideration more points than were present from the original sequence. If we know that the perturbed sequence is still an interpolating sequence with controlled interpolation constant, we apply Jensen's formula again and get better estimates.

Thus we begin with a stability result. Unfortunately, we cannot prove directly that the perturbed sequence is interpolating, except in the L^2 setting. We must increase slightly the function space, but this will do.

We say that Γ is a *linear interpolating sequence for* $\mathcal{F}^p_\phi$ if it is an interpolating sequence and moreover there is a bounded linear operator solving the interpolation problem. (It will follow from Theorem 2 that all interpolating sequences are in fact linear interpolating sequences.) We shall proceed in two steps, the first of which is the following lemma.

Lemma 5. *If Γ is an interpolating sequence for $\mathcal{F}^p_\phi$ with interpolation constant M_Γ, then for all $\varepsilon > 0$, Γ is a linear interpolating sequence for $\mathcal{F}^p_{\phi+\varepsilon|z|^2}$ with interpolation constant bounded by K/ε^α, where K and α are positive constants depending only on M_Γ and p.*

Proof. We begin by proving that Γ is an interpolating sequence for $\mathcal{F}^\infty_{\phi+\varepsilon|z|^2}$ with interpolation constant bounded by M_Γ/ε^2.

For every $\gamma_m \in \Gamma$ there is a function $f_m \in \mathcal{F}^p_\phi$ such that $f_m(\gamma_m) = 1$, $f_m(\gamma) = 0$, for all $\gamma \in \Gamma$, $\gamma \neq \gamma_m$) and with controlled norm $\|f_m\|_{\phi,p} \le M_\Gamma e^{-\phi(\gamma_m)}$. Thus by Lemma 1, $|f_m(z)| \le M_\Gamma e^{-\phi(\gamma_m)+\phi(z)}$. Now, for any sequence of values b_m such that $b_m \le e^{\phi(\gamma_m)+\varepsilon|\gamma_m|^2}$, consider the function $g = \sum b_m g_m$, where $g_m = f_m e^{2\varepsilon(\bar\gamma_m z - |\gamma_m|^2)}$. The function g satisfies $g(\gamma_m) = b_m$ and moreover

$$|g(z)| \le M_\Gamma e^{\phi(z)+\varepsilon|z|^2} \sum e^{2\varepsilon\Re(\bar\gamma_m z) - \varepsilon|\gamma_m|^2 - \varepsilon|z|^2}.$$

Thus $|g(z)| \lesssim M_\Gamma/\varepsilon^2 e^{\phi(z)+\varepsilon|z|^2}$.

Now we will assume that Γ is interpolating for $\mathcal{F}^\infty_\phi$ and check that in such a case it is a linear interpolating sequence for $\mathcal{F}^p_{\phi+\varepsilon|z|^2}$. As before we know that there are functions f_m such that $f_m(\gamma_n) = \delta_{mn}$ and $|f_m| \le M_\Gamma e^{-\phi(\gamma_m)+\phi(z)}$. We take $g_m = f_m e^{2\varepsilon(\bar\gamma_m z - |\gamma_m|^2)}$ and for any b_m that verify $\sum |b_m|^p e^{-p\phi(\gamma_m) - p\varepsilon|\gamma_m|^2} \le 1$, we consider $g = \sum b_m g_m$. Then

$$\begin{aligned}\int_{\mathbb{C}} |g|^p e^{-p\phi - p\varepsilon|z|^2} d\sigma &\le M_\Gamma^p \int_{\mathbb{C}} \Big|\sum |b_m| e^{-\phi(\gamma_m) - \varepsilon|\gamma_m|^2} e^{-\varepsilon|z-\gamma_m|^2}\Big|^p d\sigma(z) \le \\ &\le \int_{\mathbb{C}} \Big(\sum |b_m|^p e^{-p\phi(\gamma_m) - p\varepsilon|\gamma_m|^2 - \varepsilon|z-\gamma_m|^2}\Big) \Big(\sum e^{-\varepsilon|z-\gamma_m|^2}\Big)^{p-1} d\sigma(z) \le \\ &\le K'/\varepsilon^\alpha \sum |b_m|^p e^{-p\phi(\gamma_m) - p\varepsilon|\gamma_m|^2}. \quad \square\end{aligned}$$

The second step is to prove that a small perturbation of an interpolating sequence is still interpolating.

Lemma 6. *If $\Gamma = \{\gamma_n\}$ is a linear interpolating sequence for $\mathcal{F}^p_\phi$, then there is a $\delta > 0$ such that any other sequence $\Gamma' = \{\gamma_n{}'\}$ which is δ^p-close to Γ (i.e. $|\gamma_n - \gamma_n{}'| < \delta^p$ for all n) is interpolating with $M_{\Gamma'} \leq \frac{M_\Gamma}{1-\delta M_\Gamma}$.*

Proof. Using (5) we prove that if Γ and Γ' are δ^p-close (with δ^p smaller than the separation constant of Γ), then

$$\sum_{n=0}^{\infty} \Big| |f(\gamma_n)|^p e^{-p\phi(\gamma_n)} - |f(\gamma_n{}')|^p e^{-p\phi(\gamma_n{}')} \Big| \lesssim \delta^p \|f\|^p_{\phi,p}.$$

In order to prove that Γ' is interpolating, we pick an arbitrary sequence of values $\{a_n'\}$ such that $\sum |a_n'|^p e^{-p\phi(\gamma_n')} < 1$. We must construct a function f such that $f(\gamma_n{}') = a_n'$ and $\|f\|_{\phi,p} \leq K$. We know that there exist g_n such that $g_n(\gamma_n) = e^{+\phi(\gamma_n)} a_n' e^{-\phi(\gamma_n')}$, $g_n(\gamma_m) = 0$ for $n \neq m$ and the function $g = \sum g_n$ verifies $\|g\|_{\phi,p} \leq M_\Gamma$. Moreover, we have that

$$\sum_{n=0}^{\infty} \Big| |a_n'|^p e^{-p\phi(\gamma_n')} - |g(\gamma_n{}')|^p e^{-p\phi(\gamma_n{}')} \Big| \lesssim \delta^p M_\Gamma^p.$$

Hence we can pick λ_n of modulus 1 in such a way that if we define $f_1 = \sum \lambda_n g_n$, it still verifies $\|f_1\|_{\phi,p} \leq M_\Gamma$, and moreover

$$\sum_{n=0}^{\infty} |a_n' - f_1(\gamma_n{}')|^p e^{-p\phi(\gamma_n{}')} \lesssim \delta^p M_\Gamma^p.$$

If we pick δ very small ($|\delta^p M_\Gamma^p| < 1$), we have almost interpolated the sequence a_n' at the points γ_n'. We can iterate the process and make a new correction. Take as new values $a_{n2}' = a_n' - f_1(\gamma_n')$, and construct a function f_2 such that $\|f_2\|_{\phi,p} \lesssim M_\Gamma(\delta M_\Gamma)$ and

$$\sum_{n=0}^{\infty} |a_{n2}' - f_2(\gamma_n{}')|^p e^{-p\phi(\gamma_n{}')} \lesssim (\delta^p M_\Gamma^p)^2.$$

Continuing this process inductively, we obtain a sequence $\{f_n\}$ such the function $f = \sum_n f_n$ interpolates the desired values a_n' at the points γ_n' and its norm is bounded by

$$\|f\|_{\phi,p} \lesssim M_\Gamma \sum_{n\geq 1} (\delta M_\Gamma)^{n-1} = \frac{M_\Gamma}{1-\delta M_\gamma}. \quad \square$$

We proceed to check what happens when we add a point to the sequence. We already know that if we place the new point very close to the original sequence, the interpolating constant must be very big. Even if we keep the new point far from the original sequence, it is not obvious that the new sequence is interpolating or whether we have a uniform control on the interpolation constant. The next lemma addresses this point. We get a uniform control on the interpolation constant if we allow a slight change of the function space. This is enough for our purpose.

Lemma 7. *If Γ is an interpolating sequence and $\gamma \in \mathbb{C}$ verifies $d(\gamma, \gamma_n) > \delta > 0$ for all points $\gamma_n \in \Gamma$, then the sequence $\Gamma' = \Gamma \cup \{\gamma\}$ is interpolating for $\mathcal{F}^p_\psi$ ($\psi = \phi + \varepsilon|z-\gamma|^2$) with interpolation constant smaller than K/ε^α, where K depends only on Γ and δ.*

Proof. We shall prove that for any γ under the hypothesis of the lemma there is a function f such that $f(\gamma) = e^{\phi(\gamma)}$, $f(\gamma_n) = 0$ for all $\gamma_n \in \Gamma$ and $\|f\|_{\psi,p} \leq K/\varepsilon^\alpha$. To begin with, we take a function g such that $\|g\|_{\phi,p} \leq K$, $g(\gamma) = e^{\phi(\gamma)}$. This function does not vanish on Γ, but because Γ is an interpolating sequence, we may construct a function $h \in \mathcal{F}^p_\phi$ such that $h(\gamma_n) = g(\gamma_n)/(\gamma - \gamma_n)$. We apply (4) to get

$$\sum e^{-p\phi(\gamma_n)} |g(\gamma_n)|^p / |\gamma - \gamma_n|^p \lesssim \|g\|^p_{\phi,p}/\delta^p,$$

and so $\|h\|_{\phi,p} \lesssim \|g\|_{\phi,p}$. We take then

$$f(z) = g(z) + (z - \gamma)h(z),$$

and since $|z-a|^p e^{-\varepsilon|z-a|^2} < \varepsilon^{-p/2}$, we get the desired estimate. □

Now we have all the elements needed for a good estimate of the density. We are going to prove the following inequality:

$$\int_2^R \frac{n(z,s)}{s}\, ds \leq (1 - \delta/R)\frac{2}{\pi}\int_0^R \frac{\int_{D(z,s)}(\Delta\phi + \varepsilon)}{s}\, ds + K \log\frac{1}{\varepsilon}, \tag{6}$$

where δ and K are fixed constants, R is arbitrarily big and ε arbitrarily small.

We consider two different cases. To begin with, assume that z is close to the original sequence Γ. By Lemma 5 and 6, we may replace one of the points of the sequence by z to obtain a new interpolating sequence, which we again denote by Γ. We may construct a function $f \in \mathcal{F}^p_\psi$ ($\psi(\zeta) = \phi(\zeta) + \varepsilon|\zeta|^2$) such that $f(z) = e^{\psi(z)}$, $f(\lambda) = 0$ for $\lambda \in \Gamma$, $\lambda \neq z$ and $\|f\|_{\psi,p} \leq K/\varepsilon^\alpha$. If we apply Jensen's formula to f, we get

$$\int_1^R \frac{n(z,s)}{s}\, ds \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(z + Re^{i\theta})|\, d\theta - \psi(z).$$

Since $\log|f(z+Re^{i\theta})| \leq \psi(z + Re^{i\theta}) + K\log 1/\varepsilon$, we obtain

$$\int_1^R \frac{n(z,s)}{s}\, ds \leq \frac{2}{\pi}\int_0^R \frac{\int_{D(z,s)}(\Delta\phi + \varepsilon)}{s}\, ds + K\log\frac{1}{\varepsilon},$$

where K depends on the interpolation and separation constants. We want to obtain a strict inequality and up to now we only have an inequality for the density. But we have only used that we could move the point z and still get an interpolating sequence. We know more: We may move all the points of the sequence Γ and still get an interpolating sequence with control on the interpolation constant. In particular, we may move the points towards the point z a small distance δ and still get

$$\int^R \frac{n(z,s+\delta)}{s}\, ds \leq \frac{2}{\pi}\int^R \frac{\int_{D(z,s)}(\Delta\phi + \varepsilon)}{s}\, ds + K\log\frac{1}{\varepsilon}.$$

If we change variables $s+\delta=t$, we obtain

$$\int_2^R \frac{n(z,t)}{t}\,dt \le \frac{2}{\pi}\int_0^R (1-\delta/R)\frac{\int_{D(z,s)}(\Delta\phi+\varepsilon)}{s}\,ds + K\log\frac{1}{\varepsilon}.$$

We deal next with discs with centers far from the original sequence. If z is such a point, consider the function $f\in\mathcal{F}^2_\psi$ ($\psi(w)=\phi(w)+\varepsilon|z-w|^2$) given by Lemma 7 such that $f(z)=e^{\phi(z)}$ and $f(\gamma_n)=0$ for all $\gamma_n\in\Gamma$ and $\|f\|_{\psi,p}\le C/\varepsilon^\alpha$. We apply again Jensen's formula to this particular function and obtain

$$\int_1^R \frac{n(z,s)}{s}\,ds \le \frac{2}{\pi}\int_0^R \frac{\int_{D(z,s)}(\Delta\phi+\varepsilon)}{s}\,ds + K\log\frac{1}{\varepsilon}.$$

Now we may, as before, perturb the original sequence Γ and move the points towards z to obtain (6).

We may pick $\varepsilon>0$ very small. Take for instance $\varepsilon=1/R^2$. Now we use that $\Delta\phi>m$ and get

$$\int_2^R \frac{n(z,s)}{s}\,ds \le \frac{2}{\pi}\int_0^R \frac{\int_{D(z,s)}\Delta\phi}{s}\,ds + (2K+1)\log R - \delta m R.$$

Consequently there is a very big r and a positive τ such that

$$\int_2^r \frac{n(z,s)}{s}\,ds \le \frac{2}{\pi}\int_0^r \frac{\int_{D(z,s)}\Delta\phi}{s}\,ds - \tau,$$

for all $z\in\mathbb{C}$. If we make a convolution of this last inequality with the characteristic function of a very large disk, we get

$$n(z,R)/R^2 \le \frac{2\int_{D(z,R)}\Delta\phi}{\pi R^2} - \tau'$$

for all z and for all sufficiently large R. This inequality implies the strict density condition $D^+_\phi(\Gamma)<2/\pi$.

7. Strict necessary conditions for sampling

We begin with the case $p=\infty$, which can be dealt with in the same way as in [12]. The following lemma is crucial.

Lemma 8. *If Γ is a sampling sequence for $\mathcal{F}^\infty_\phi$, then for all sufficently small $\varepsilon>0$ Γ is sampling for $\mathcal{F}^\infty_{\phi+\varepsilon|z|^2}$.*

Proof. Suppose the lemma is false. Then we can find a sequence $\varepsilon_j\to 0$ and corresponding sequences z_j and $f_j\in\mathcal{F}^\infty_{\phi+\varepsilon_j|z|^2}$ such that $\|f_j\|_{\phi+\varepsilon_j|z|^2,\infty}=1$, $|f_j(z_j)|e^{-\phi(z_j)-\varepsilon_j|z_j|^2}\ge 1/2$, and $\|f_j|\Gamma\|_{\phi+\varepsilon_j|z|^2,\infty}\le\varepsilon_j$. It is clear that a subsequence of $(\Gamma-z_j,\phi_{z_j}+\varepsilon_j|z|^2)$ will converge compactwise to some element $(\Lambda,\psi)\in W(\Gamma,\phi)$, and correspondingly that a subsequence of $T_{z_j}f_j$ will converge uniformly on compacts to a nontrivial function $f\in\mathcal{F}^\infty_\psi$ vanishing on Λ. This contradicts Theorem 3, and so we have proved the lemma. □

Lemma 8 and Corollary 2 now combine to prove the necessity for $p = \infty$.

For $1 \leq p < \infty$, we use that the removal of a point from a sampling sequence yields either a sampling sequence or a sequence which is not a set of uniqueness, as follows from the open mapping theorem. If the latter alternative holds when one point is removed from Γ, it is clear that it holds when any other point is removed. But this is impossible, because it would have implied that the sampling sequence Γ were also interpolating, contradicting Theorem 2 and Corollary 2. So the removal of a point from a sampling sequence yields another sampling sequence.

Suppose now that Γ is sampling for $\mathcal{F}^p_\phi$, $1 \leq p < \infty$. Then by Lemma 3, for every $(\Lambda, \psi) \in W(\Gamma, \phi)$, Λ is sampling for $\mathcal{F}^p_\psi$. By what was just observed, there can be no function from $\mathcal{F}^\infty_\psi$ vanishing on Λ. For if a function f has this property and $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, then

$$\frac{f(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} \in \mathcal{F}^p_\psi,$$

contradicting the fact that $\Lambda \setminus \{\lambda_1, \lambda_2, \lambda_3\}$ is sampling for $\mathcal{F}^p_\psi$. So we have proved that Γ is sampling for $\mathcal{F}^\infty_\phi$ by virtue of Theorem 3. This completes the proof.

8. Final remarks

It may be observed that all of the above arguments work also for $0 < p < 1$, except the proof of Lemma 3. It seems more than likely, though, that both Theorem 1 and 2 hold for $0 < p < 1$.

It should also be noted that Theorems 1 and 2 have counterparts in the unit disc for weighted Bergman spaces of the type treated in [1]; see [13] for the case of standard radial weights. The same methods apply, but the arguments are easier at some places: One does not need the manipulations with Jensen's formula as in Section 6; the Ramanathan-Steger comparison lemma does not apply, but one can instead make a simple direct use of Blaschke products as in [13] .

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