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Treball de Fi de Grau

On the Discovery of non-Euclidean Geometry

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Contents

Abstract

The main goal of this work is to investigate the historical transition from Euclidean to non-Euclidean geometry, to understand what the motivation of such a transition was and to understand to the best of my abilities how it was achieved. This will be done by reviewing the relevant authors' original work and the correspondence between some of them.

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Foreword

Geometry, as its name from Ancient Greek indicates (*geo-* "earth" and *-metron* "measurement"), was born as a necessary tool to measure portions of land as back as the second millennium BC in Mesopotamia and Egypt. With the arrival of Classical Greece, the properties and rules of thumb that had been being used began to be written more rigorously. The pinnacle of such formalization is undoubtedly Euclid of Alexandria (c. 300 BC), who wrote the world-renowned *Elements*. This thirteen-book treatise contains axioms, definitions, formal propositions and proofs, the rigor of which would not be surpassed until the nineteenth century. Until then, it was considered that all educated people had read the *Elements*. However, as we shall see later on, there was a specific point in it that would be questioned for centuries. Said controversy was to be the culprit behind restless hours of dedication by various mathematicians throughout the years. And it is precisely that same restlessness that fueled some of the greatest contributions the mathematical world has ever seen: non-Euclidean geometry.

In this work, I set out to investigate the history of how this seemingly never-ending nightmare led mathematicians to discover non-Euclidean geometry in the 19th century. I will do so by investigating the relevant authors' original work and correspondence, highlighting the most important points, as well as delving into some of the literature concerning this topic. I am aware that the vast majority of the technical mathematical content that is involved in this discussion is well beyond what has been covered in this degree. For that reason, I will focus on the historical importance of the subject while presenting it in a mathematical frame. I am also aware of the fact that I am highly influenced by the Western perspective of history. Therefore, although I try to mention the exceptions I am aware of, I cannot say that I have taken into account non-Western history. With the limited time and resources that I had, here follows my work.

Chapter 1

A controversial postulate

Euclidean geometry remained barely untouched from 300BC to approximately 1800, with a few exceptions. Omar Khayyam (1048-1131) already touched upon Euclid's geometry in his *Discussion of Difficulties in Euclid (Risâla fî sharh mâ ashkala min musâdarât Kitâb 'Uglîdis)*, whose ideas were shared by Giovanni Saccheri (1667-1733) – although it is unknown whether Saccheri knew of Khayyam's work. René Descartes (1596-1650) unified Euclidean geometry and algebra in his Analytic Geometry (1637), thus introducing coordinates. Georg Mohr (1640-1697) proved in 1672 that any Euclidean construction done with a compass and straightedge can be done using solely a compass. This result is known as the Mohr-Mascheroni theorem because Mohr's work did not receive much attention until Lorenzo Mascheroni (1750-1800) rediscovered it in 1797.

During this period there were several concerns occupying the mathematicians, such as squaring the circle and Euclid's fifth postulate. We shall deal with the latter. Let us remember the five postulates found on Euclid's Book I of the *Elements*:

1. Let it have been postulated to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side)

To see what the fifth postulate means, consider the following figure.

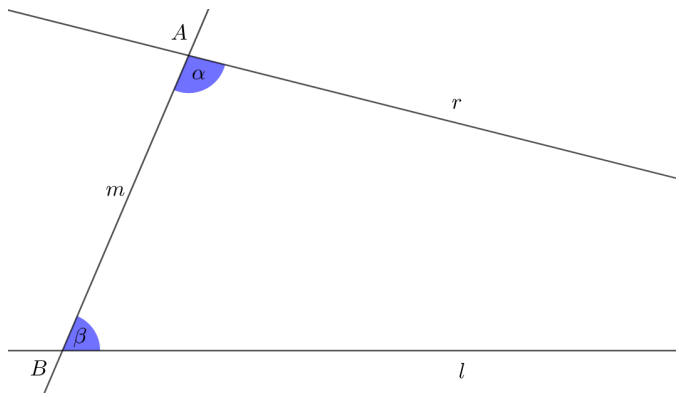


Figure 1.1: Euclid's formulation of the fifth postulate

Let l and r be two straight lines crossed by a third, s. Let α and β be the angles formed at the points A and B. The postulate says that, assuming that α and β add up to less than two right angles, then if we extend l and r far enough on the side where α and β are (on the right in our case) the lines will eventually meet.

The idea behind the axioms in Euclid's world-renowned *Elements* was that they were supposed to be self-evident truths about physical space, but the fifth postulate seemed somewhat more complicated. No one believed it to be false, yet it was not as compelling as the other axioms. Even at first glance the fifth postulate looks more intricate than the others, or at least it hints that more explanation is in order. Euclid himself did not refer to this postulate until he had proved all the theorems he could without it. Several approaches were made throughout the centuries to prove the fifth postulate with the other four axioms but they all resulted in vain. For example, a doctoral thesis by the German Georg Klügel (1739-1812) recorded 28 attempts on the postulate in 1763. Adrien-Marie Legendre (1752-1833) also had a remarkable history of attempts on the postulate. He had no doubt that it was true, but it felt to him absurd to assume when it could be proved – or so he thought.

No matter how careful mathematicians tried to be, there always appeared some unproved truth they overlooked, almost as if they were trying to fit a slightly oversized carpet in a room and a corner kept going out of place. They did succeed in replacing the fifth postulate with equivalent statements. The most famous reformulation of the postulate is quite probably the following, which is known as Playfair's Axiom[1]:

[1]Playfair, John (1846). *Elements of Geometry.* W. E. Dean.

> In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

Here are some other examples:

1. Two parallel lines are equidistant. (Posidonius, first century BC)
2. If a line intersects one of two parallels, it also intersects the other. (Proclus, 410-485 AD)
3. Given a triangle, we can construct a similar triangle of any size whatever. (Wallis, 1616-1703)
4. The sum of the angles of a triangle is equal to two right angles. Legendre(1752-1833) [2]
5. Three non-collinear points always lie on a circle. (Farkas Bolyai, 1775-1856)

So why even consider the fifth postulate? What justifies it? Perhaps how intuitive it is. It is rather easy to imagine what would happen to the intersection of l and r in the previous figure as the sum of α and β approach two right angles. Or perhaps it is how useful it is. As we have seen, it is possible to reformulate the postulate as statements involving distance or angles, and even as statements that have nothing to do with parallels, such as Playfair's Axiom. For instance, let us prove the fourth reformulation with Playfair's Axiom.

Given a triangle ABC with angles α, β and γ, we will show that $\alpha + \beta + \gamma = \pi$, or as it would have been worded at the time, the sum of all three angles equals two right angles. Consider the following figure.

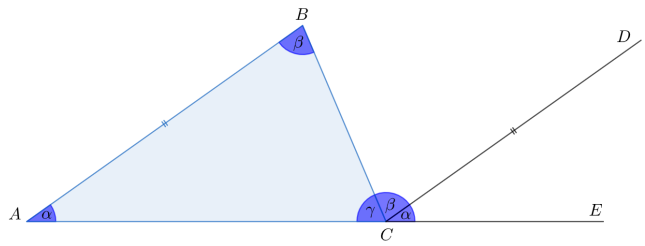


Figure 1.2

We first extend the line AC to an arbitrary point E. Then we draw *the* (using Playfair's Axiom) line through C parallel to AB. It is clear that the sum of the

[2] Although this statement was known to Euclid (see *Elements*, Book I, Proposition 32), Legendre was the first to present it as possible substitute for the postulate

angles at C is equal to two right angles. Now, since $AB \parallel CD$, BCD equals ABC (that is, β) and DCE equals BAC (that is, α). And thus we have arrived to the fact that $\alpha + \beta + \gamma$ is equal to two right angles.

Saccheri, being the logician that he was, in his book *Euclides ab omni naevo vindicatus* (1733) (Euclid Freed of Every Flaw), dared to deny the fifth postulate in the hopes that it would lead him to a contradiction, thus proving it by *reductio ad absurdum*. However, the results were different. He constructed a quadrilateral $ABCD$ by drawing a base AB of finite length and two sides of equal length perpendicular to the base. The angles formed at C and D are called summit angles. This quadrilateral is referenced as Saccheri's quadrilateral, or Khayyam-Saccheri's quadrilateral since Khayyam had previously used it too. Since the fifth postulate is equivalent to the fact that the sum of the angles of a triangle is equal to two right angles, Saccheri considered two hypotheses, one in which the angles add up to more than two right angles, and one in which they add up to less than two right angles. The first led him to the conclusion that straight lines are of finite length, which contradicts the second postulate, so Saccheri rejected it. This case was to be picked up later by Bernhard Riemann (1826-1866) and Arthur Cayley (1821-1895) as the basis of *elliptic geometry*, in which both the second and fifth postulates are rejected. The second hypothesis not only did not lead to a contradiction, but led to non-intuitive results such as that the area of all triangles is bounded by some constant. Saccheri concluded: "*the hypothesis of the acute angle is absolutely false; because it is repugnant to the nature of straight lines*". As we shall see later on, this hypothesis was studied further, and Saccheri's results were to become theorems of *hyperbolic geometry*. The advantage of the quadrilateral is that both cases come up as possible answers to the question "What are the summit angles?" If they are right angles, then the existence of the quadrilateral is equivalent to Euclid's postulate. If they are acute, we are led to hyperbolic geometry. And if they are obtuse, we are led to elliptic or spherical geometry.

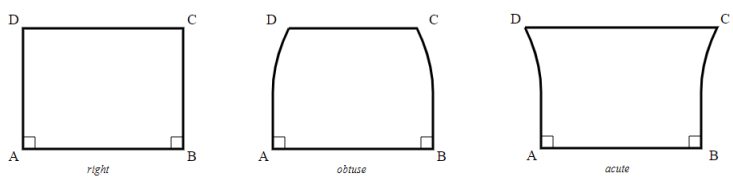


Figure 1.3: Different cases of Saccheri's quadrilateral

As it is known, no major branch of mathematics is only the fruition of a single individual's work but rather a conjunction of efforts and perspectives. At best we

can mention some mathematicians depending on what we mean by 'non-Euclidean geometry'. If we refer to the realization that there can be geometries alternative to Euclid's, then Klügel and Johann Heinrich Lambert (1728-1777) deserve the credit. If we refer to the technical development of a system of axioms that are able to give an alternative to Euclid's fifth postulate, then the credit belongs mostly to Saccheri, at least restricting our view to the Western World. Since space is just as accurately described with both Euclidean and non-Euclidean geometries, it is clear that the former cannot be the intrinsic system a priori. The first to put this thought forward was Carl Friedrich Gauss (1777-1855).

Chapter 2

Gaussian curvature and the Theorema Egregium

Carl Friedrich Gauss (Brunswick, 30th April 1777 - Göttingen, 23rd February 1855) was one of the most influential figures in the history of mathematics. Gauss studied many fields such as algebra, number theory, astronomy, magnetism, geodesy, geometry, statistics and optics. Although its contents do not interfere with the subject at hand, it is worth mentioning his *magnum opus*, *Disquisitiones arithmeticae* (Arithmetical investigations).

Gauss in 1840

It is not quite clear what Gauss' opinion of the fifth postulate was since he never made a full statement and we cannot draw solid conclusions from the scattered remarks he made in his correspondence to fellow mathematicians. Gauss and the Hungarian mathematician Farkas Bolyai (1775-1856) became good friends when studying together in Göttingen. After Farkas went back to Hungary in 1798, Gauss wrote to him in hopes of staying in contact and Bolyai keeping him updated on his work on the fifth postulate. Gauss wrote:

> Only, the path which I have chosen does not lead to the goal that one seeks, and which you assure me you have achieved, but rather makes the truths of geometry doubtful.

Undaunted, in 1804 Farkas sent him his work of the previous three years, only for Gauss to find a crucial error and dismiss it.

Gauss was also critic of Legendre's arguments. In 1817, Gauss' hope started to get smaller as he wrote to the astronomer Wilhelm Olbers (1758-1840):

> I am becoming more and more convinced that the necessity of our geometry cannot be proved... Perhaps only in another life will we attain another insight into the nature of space, which is unattainable to us now. Until then we must not place geometry with arithmetic, which is purely a priori, but rather in the same rank as mechanics.

and smaller...

> My conviction that we cannot base geometry a priori has, if anything, become even stronger. (Letter to Friedrich Wilhelm Bessel (1784-1846) in 1829)

and even smaller as he stated that he would not publish his ideas in his lifetime, and surely enough he did not.

In 1827 Gauss presented to the Göttingen Royal Society a paper divided in 29 articles, named *Disquisitiones generales circa superficies curvas*[1]. This paper contained some ideas that would result very useful in the developing of geometry. In the paper, Gauss often uses an auxiliary sphere of radius 1 (he would call it *the* sphere). In order to offer some context on the type of matters Gauss discussed in this article, let us highlight some of the points.

In article 2, he brings together some propositions that were frequently used:

> I. The angle between two intersecting straight lines is measured by the arc between the points on the sphere which correspond to the directions of the lines
>
> II. The orientation of any plane whatever can be represented by the great circle on the sphere, the plane of which is parallel to the given plane.
>
> III. The angle between two planes is equal to the spherical angle between the great circles representing them, and, consequently, is also measured by the arc intercepted between the poles of these great circles. And, in like manner, the angle of inclination of a straight line

[1] C.F. Gauss, *General Investigations of Curved Surfaces* (1827).

> to a plane is measured by the arc drawn from the point which corresponds to the direction of the line, perpendicular to the great circle which represents the orientation of the plane.

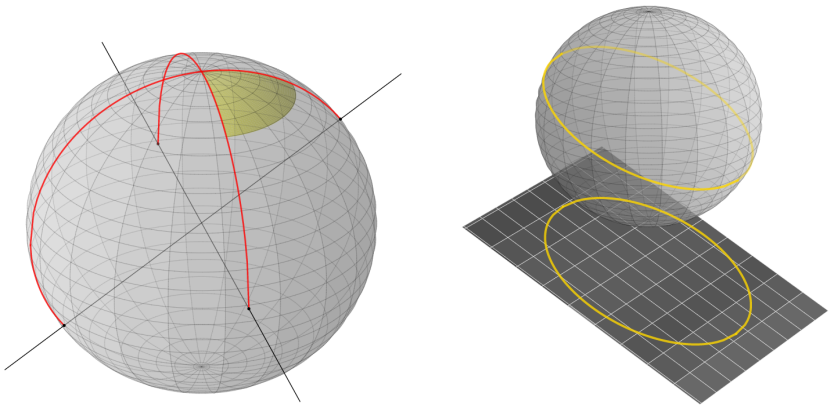


Figure 2.1: Interpretation of propositions 1 and 2

We can also find the formula for the angle formed by two arcs on the sphere:

THEOREM. *If L, L', L'', L''' denote four points on the sphere, and A the angle which the arcs $LL', L''L'''$ make at their point of intersection, then we shall have*

$$cosLL'' \cdot cosL'L''' - cosLL''' \cdot cosL'L'' = sinLL' \cdot sinL''L''' \cdot cosA$$

Article 3 defines continuous curvature as follows, and states that he shall only study surfaces with continuous curvature:

> A curved surface is said to possess continuous curvature at one of its points A, if the directions of all the straight lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing through A. This plane is said to *touch* the surface at the point A. If this condition is not satisfied for any point, the continuity of the curvature is here interrupted, as happens, for example, at the vertex of a cone. The following investigations will be restricted to such surfaces, or to such parts of surfaces, as have the continuity of their curvature nowhere interrupted. We shall only observe now that the methods used to determine the position of the tangent plane lose their meaning at singular points, in which the continuity of the curvature is interrupted, and must lead to indeterminate solutions.

If we wish to study the tangent plane to a point A of coordinates x, y, z on the surface, it is more convenient to do so by studying the direction of the normal vector pointing to A. This direction is represented by a point L on the auxiliary sphere, of coordinates X, Y and Z. If we denote the coordinates of a point A' infinitely close to A by $x + dx, y + dy, z + dz$, the following equality is obtained

$$Xdx + Ydy + Zdz = 0$$

Gauss proceeds to consider three different methods for defining a curved surface.

Method 1 (Zeros of a function). It consists in expressing the coordinates of the points on the surface as zeros of a function W, thus we would have $S = \{W = 0\}$. If we assume the differential of W to be

$$dW = Pdx + Qdy + Rdz$$

then points on the surface satisfy

$$Pdx + Qdy + Rdz = 0$$

and since $X^2 + Y^2 + Z^2 = 1$, the coordinates of the normal vector are either

$$X = \frac{P}{\sqrt{P^2 + Q^2 + R^2}}, \qquad Y = \frac{Q}{\sqrt{P^2 + Q^2 + R^2}}, \qquad Z = \frac{R}{\sqrt{P^2 + Q^2 + R^2}}$$

or

$$X = \frac{-P}{\sqrt{P^2 + Q^2 + R^2}}, \qquad Y = \frac{-Q}{\sqrt{P^2 + Q^2 + R^2}}, \qquad Z = \frac{-R}{\sqrt{P^2 + Q^2 + R^2}}$$

Method 2 (Parametrization). It expresses the coordinates in the form of functions of two variables, p, q. Supposing

$$\begin{aligned} dx &= a\ dp + a'\ dq, \\ dy &= b\ dp + b'\ dq, \\ dz &= c\ dp + c'\ dq \end{aligned}$$

and setting

$$\Delta = \sqrt{(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}$$

we shall have either

$$X = \frac{bc' - cb'}{\Delta}, \qquad Y = \frac{ca' - ac'}{\Delta}, \qquad Z = \frac{ab' - ba'}{\Delta}$$

or

$$X = \frac{cb' - bc'}{\Delta}, \qquad Y = \frac{ac' - ca'}{\Delta}, \qquad Z = \frac{ba' - ab'}{\Delta}$$

Method 3 (Graph of a function). It is associated to the second method, in this case one of the coordinates, z, say, is expressed as a function of the other two. Setting

$$dz = t\,dx + u\,dy$$

we shall have either

$$X = \frac{-t}{\sqrt{1+t^2+u^2}}, \qquad Y = \frac{-u}{\sqrt{1+t^2+u^2}}, \qquad Z = \frac{1}{\sqrt{1+t^2+u^2}}$$

or

$$X = \frac{t}{\sqrt{1+t^2+u^2}}, \qquad Y = \frac{u}{\sqrt{1+t^2+u^2}}, \qquad Z = \frac{-1}{\sqrt{1+t^2+u^2}}$$

where each set of two solutions found in each method refers to opposite points on the sphere.[2]

In article 6 we find Gauss' definition of what would be later named the *Gauss map*. Although Gauss himself did not name this map – let alone naming it Gauss map – we will write it as N since that is the standard modern notation for it. This is how it was introduced:

> Just as each definite point on the curved surface is made to correspond to a definite point on the sphere, by the direction of the normal to the curved surface which is transferred to the surface of the sphere, so also any line whatever, or any figure whatever, on the latter will be represented by a corresponding line or figure on the former.

Thus, if we denote a surface by S and the sphere of radius one centered at the origin by $\mathbb{S}^2$, we have

$$N\colon S \to \mathbb{S}^2$$

where $N(p)$ is a unit vector orthogonal to the surface at p, namely the normal vector to S at p.

Given a finite part of a curved surface A, Gauss defines the *total* or *integral curvature*, which is the area of the image of A under N, that is, $|N(A)|$. Notice how this concept is initially defined for a set and not for a single point. Gauss goes on to define what we would now call *Gaussian curvature* for a point on the surface:

[2] A comprehensive derivation of these results can be found in Gauss' paper *Disquisitiones generales circa superficies curvas*

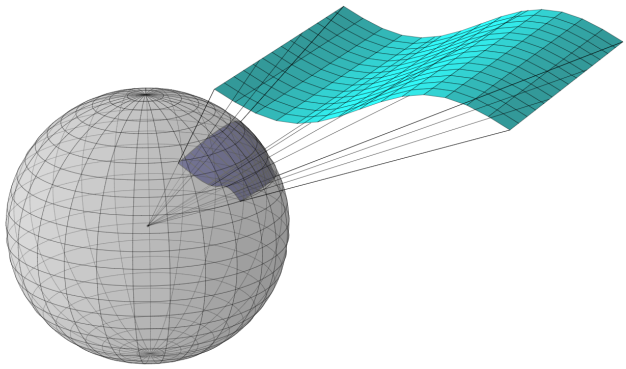


Figure 2.2: A surface and its image under N

> From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the *measure of curvature*. The latter refers to a *point* of the surface, and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself [...]

which would be, for $p \in A$

$$k(p) = \lim_{|A| \to p} \frac{|N(A)|}{|A|}$$

and thus the previously defined integral curvature can be expressed as $\int k \; d\sigma$, where $d\sigma$ denotes the element of area.

In article 7 Gauss sets out to find a formula to express the measure of curvature k for any point of a curved surface, which will eventually lead him to the *Theorema Egregium*. Along the way, Gauss proves that the curvature is equal to the product of principal curvatures, which was already known by Euler (1707-1783). Following the previously introduced notation, Gauss sets:

$$\begin{aligned} a^2 + b^2 + c^2 &= E \\ aa' + bb' + cc' &= F \\ a'^2 + b'^2 + c'^2 &= G \end{aligned}$$

and finds the following formula for the curvature:

$$4(EG-F^2)k = E\left(\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial q} - 2\frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial q} + \left(\frac{\partial G}{\partial p}\right)^2\right)$$
$$+F\left(\frac{\partial E}{\partial p}\cdot\frac{\partial G}{\partial q} - \frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial q}\cdot\frac{\partial F}{\partial q} + 4\frac{\partial F}{\partial p}\cdot\frac{\partial F}{\partial q} - 2\frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial p}\right)$$
$$+G\left(\frac{\partial E}{\partial p}\cdot\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial p}\cdot\frac{\partial F}{\partial q} + \left(\frac{\partial E}{\partial q}\right)^2\right) - 2(EG-F^2)\left(\frac{\partial^2 E}{\partial q^2} - 2\frac{\partial^2 F}{\partial p\cdot\partial q} + \frac{\partial^2 G}{\partial p^2}\right)$$

The remarkable trait of this equality is that k is expressed in terms of only E, F, G and their partial derivatives up to second order, which takes us to the *Theorema Egregium*. Quoting Gauss in article 12:

> Since we always have
>
> $$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\cdot dq + G\,dq^2,$$
>
> it is clear that
>
> $$\sqrt{E\,dp^2 + 2F\,dp\cdot dq + G\,dq^2}$$
>
> is the general expression for the linear element on the curved surface. The analysis developed in the preceding article thus shows us that for finding the measure of curvature there is no need of finite formulæ, which express the coordinates x, y, z as functions of the indeterminates p, q; but that the general expression for the magnitude of any linear element is sufficient. Let us proceed to some applications of this very important theorem.
>
> Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates x, y, z, will correspond a definite point of the latter surface, whose coordinates are x', y', z'. Evidently x', y', z' can also be regarded as functions of the indeterminates p, q, and therefore for the element $\sqrt{dx'^2 + dy'^2 + dz'^2}$ we shall have an expression of the form
>
> $$\sqrt{E'\,dp^2 + 2F'\,dp\cdot dq + G'\,dq^2}$$
>
> where E', F', G' also denote functions of p, q. But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically
>
> $$E = E', \quad F = F', \quad G = G'.$$

> Thus the formula of the preceding article [the one given before this quote] leads of itself to the remarkable
>
> THEOREM. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

Gauss states that at that time, geometers restricted their attention to surfaces developable upon a plane, and that his theorem shows in one big swing that this is only a particular case for which the curvature at every point is equal to zero. In the following eight articles Gauss studies geodesics and after some calculations and manipulations of the coefficients E, F and G, he proves that if α, β and γ are the internal angles of a geodesic triangle T on a surface whatever, then the following equality holds:

$$\alpha + \beta + \gamma - \pi = \int_T k \, d\sigma \tag{2.1}$$

This is how he stated it:[3]

> The excess over 180° of the sum of the angles of a triangle formed by shortest lines on a concavo-concave surface, or the deficit from 180° of the sum of the angles of a triangle formed by shortest lines on a concavo-convex curved surface, is measured by the area of the part of the sphere which corresponds, through the directions of the normals, to that triangle, if the whole surface of the sphere is set equal to 720 degrees.

Notice how on the Euclidean plane any triangle is flat (with 0 curvature), and thus we get $\alpha+\beta+\gamma = \pi$, as we have shown before. However, on a surface with constant curvature $k = 1$, like the sphere, this formula states that the area of any triangle is equal to the difference between the sum of the angles and π. In particular, since the area of a triangle is always greater than zero, we obtain that on a sphere the sum of the angles is always greater than two right angles. Conversely, on a surface with constant negative curvature, the sum of the angles is always smaller than two right angles.

Thus, for surfaces of constant curvature (which are not very common), we can say the following from 2.1:

- If $k = 0$, $\alpha + \beta + \gamma = \pi$, and the surface is developable

[3] General Investigations of Curved Surfaces, page 29

- If $k > 0$, $\alpha + \beta + \gamma > \pi$, and the surface is applicable to a sphere of radius $1/\sqrt{k}$
- If $k < 0$, $\alpha + \beta + \gamma < \pi$, and the surface is applicable to a pseudosphere (*)

(*): A surface of constant negative curvature was not known until Eugenio Beltrami (1835-1900) in 1868 discovered the pseudosphere [4], a revolution surface obtained by spinning a tractrix, which can be expressed as

$$y = \pm\ h\ log\ \frac{h + \sqrt{h^2 - x^2}}{x} - \sqrt{h^2 - x^2}$$

for $x < h$ where $h > 0$. In this case, the curvature is $k = -\frac{1}{h^2}$.

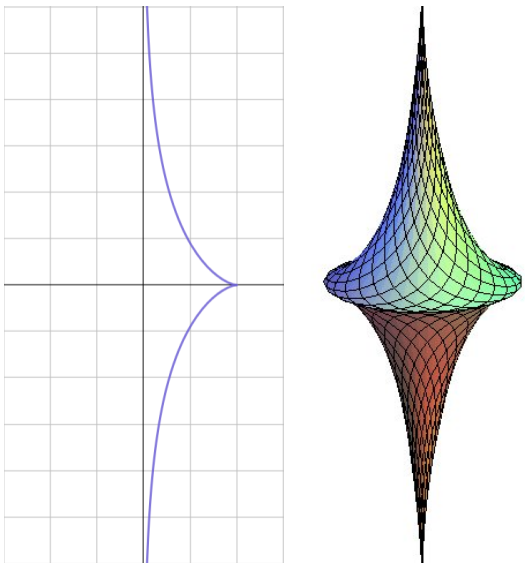


Figure 2.3: A tractix and a pseudosphere

[4]Beltrami, Eugenio (1868). *Saggio sulla interpretazione della geometria non euclidea* (Treatise on the interpretation of non-Euclidean geometry). Gior. Mat. 6: pp 248–312.

Chapter 3

János Bolyai and Lobachevsky

János Bolyai[1] (Kolozsvár, 15th December 1802 – Neu Markt am Mieresch, 27th January 1860) was the son of Zsuzsanna Benkő and the well-known mathematician Farkas Bolyai. Instructed by his father, János read the first six books of the *Elements* by Euclid at an early age. He also read Euler's *Algebra* and attended lectures at the Evangelical Reformed College where Farkas became a professor. János also knew Latin and was considered a great violinist. He was not fond of poetry, unlike his father, who thought János was too inclined to study. From 1818 to 1823 János studied at the Royal Engineering Academy in Vienna for military service. He went on to work for the Austrian Army as an engineer for ten years until 1833, when he decided that he had already had enough of the military service and retired. In his time in the army he developed an interest for the fifth postulate. In Vienna János met Carl Száz, who gave him the idea to introduce parallels in the way that he did, which we will see later on.

Portrait of János Bolyai

By 1820, he began to suspect that the reason that he was failing to prove the parallel postulate might be due to the fact that the postulate was in fact not true. This is where he went the opposite direction and attempted to show that there could be a geometry independent of the parallel postulate. He wrote to his father to announce this, saying:

> One must do no violence to nature, nor model it in conformity to any blindly formed chimera; that on the other hand, one must regard nature

[1] Also referred to as Johann or Johannes Bolyai

> reasonably and naturally, as one would the truth, and be contented only with a representation of it which errs to the smallest possible extent

His father, alarmed, replied:

> You must not attempt this approach to parallels. I know this way to the very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labours; my creations are far better than those of others and yet I have not achieved complete satisfaction... I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind. Learn from my example: I wanted to know about parallels, I remain ignorant, this has taken all the flowers of my life and all my time from me. [2]

and yet again:

> I admit that I expect nothing from the deviation of your lines.It seems to me that I have been in those regions; that I have travelled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date to this time. I thoughtlessly risked my life and happiness – *aut Caesar aut nihil* [either Caesar or nothing]

Happily the son did not listen to his father, and on November 3, 1823 he could write to say that he was succeeding:

> I am determined to publish a work on parallels as soon as I can put it in order, complete it, and the opportunity arises. I have not ye made the discovery but the path that I am following is almost certain to lead to my goal, provided this goal is possible. I do not yet have it but I have found things so magnificent that I was astounded. It would be an eternal pity if these things were lost as you, my dear father, are bound to admit when you see them. All I can say now is that I have created a new and different world out of nothing. All that I have sent you thus far is like a house of cards compared with a tower. I am as convinced now that it will bring me no less honor, as if I had already discovered it.

[2] Stäckel. *Wolfgang und Johann Bolyai*, 81.

His father advised him to publish his results as soon as possible, as an appendix to a work on geometry that he had been writing for some time. János later commented that:

> He advised me that, if I was really successful, then there were two reasons why I should speedily make a public announcement. Firstly because the ideas might easily pass to someone else who would then publish them. Secondly there is some truth in this, that certain things ripen at the same time and then appear in different places in the manner of violets coming to light in early spring. And since all scientific striving is only a great war and one does not know when it will be replaced by peace one must win, if possible; for here pre-eminence comes to him who is first.

When János visited his father in February 1825, he was unable to convince him, worried as he was about an arbitrary constant that entered the formulae his son had found. As late as 1829, they continued to disagree about what the younger man had done, and his father continued to advise János not to waste his life like a hundred geometers before him had. Finally, they agreed to publish it anyway. The two-volume work, entitled *Tentamen juventutem studiosam in Elementa Matheosis purae* (Essay on the Elements of Mathematics for Studious Youths) was published by the College in Maros-Vásérhely in 1832, seven years after János had finished his work. That is why János' most important work is named *Appendix*, because it was one to his father's book. Some sources, however, refer to his work as the title Bolyai had chosen: *The science of absolute space.*

A copy was sent to Gauss, who replied on March 6, 1832:

> If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. Indeed the majority of people have not clear ideas upon the questions of which we are speaking, and I have found very few people who could regard with any special interest what I communicated to them on this subject. To be able to take such an interest it is first of all necessary to have devoted careful thought to the real nature of what

> is wanted and upon this matter almost all are most uncertain. On the other hand it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared the trouble, and I am very glad that it is just the son of my old friend, who takes precedence of me in such a remarkable manner. [3]

Farkas was pleased that the great geometer had endorsed his son's discoveries, but János was upset. It took him a decade until he believed his father in that he did not confide his ideas to Gauss, just as Farkas had warned him that someone might. For a long time father and son did not speak to each other, because of the *Appendix* and because Farkas did not approve of János living unmarried with a woman with whom he had had three children. Their relationship eventually resumed until Farkas died in 1856. János ended his relationship around the same time and died shortly after, in 1860.

A new type of parallels

Bolyai explored the possibilities with his new definitions and focused on what was compatible with the parallel postulate and what was not. He developed an interest to what he called *absolute geometry* or *absolute space*, which is a collection of results that are true no matter if we consider the parallel postulate to be true or false, these results hold true in both cases.

In the *Appendix*, Bolyai introduced the notion of parallel lines as follows:
Take any line l and a point outside of said line P. Now draw a line through P intersecting l, and let R be the intersection point, like so:

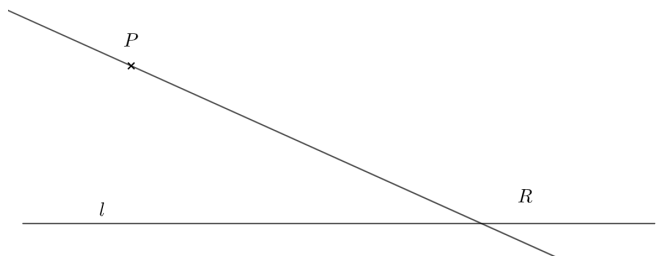


Figure 3.1

Moving the intersection R towards infinity (we can think of it as moving infinitely to the right on the picture), there is a limit situation in which the two lines do

[3] Gauss. *Werke, VIII*, 220-224

not intersect each other anymore. Doing the same on the other direction, we obtain two limit lines, x and y, which are parallel to the initial line l. These limit lines x and y are called *limiting parallel* (sometimes called *critically parallel* or just *parallel*). Bolyai negates Euclid's fifth postulate in considering an infinity of

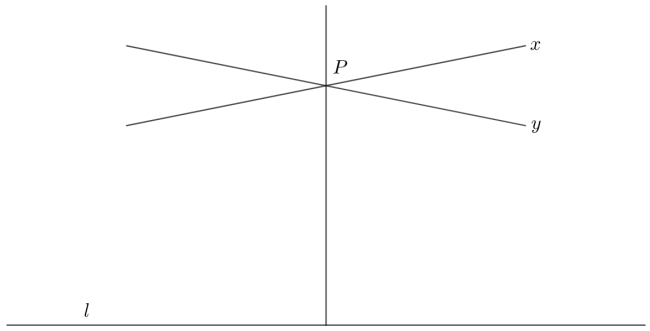


Figure 3.2

parallel lines to l, all of them encompassed in the angle formed by x and y, which, by construction, do not intersect l. These lines are usually called *hyperparallel*, they are represented by dashed lines on the figure. If d is the perpendicular line to l through P, then θ is the *angle of parallelism*. It is worth noting that Bolyai's

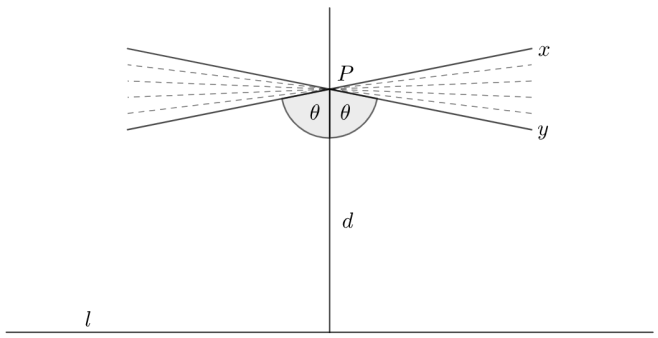
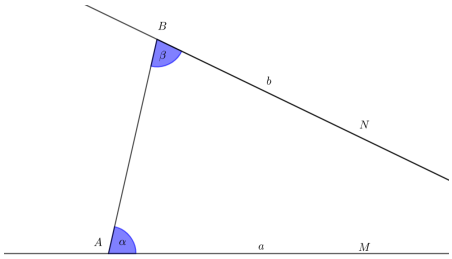


Figure 3.3

concept of line is different from the common in spite of how these lines are represented on the figures.

Bolyai then established some basic properties of parallel lines based on this new definition. In particular, he showed that if a and b are parallel and A is a fixed point on a, then there is a unique point B on b such that the angles $MAB = \alpha$ and $NBA = \beta$ are equal:



At this point Bolyai made an important switch into three dimensions. He was the first one to do so, as previous investigations of the parallel postulate were all in two dimensions. His justification is something along the lines of induced geometry. For instance, when we think of a sphere, we usually think of it embedded in real three-dimensional space. If we were to consider the distance between two points on the sphere, we really mean the distance measured along the surface of the sphere – no tunnels allowed – and we take the same distance we had in Euclidean space. Likewise, Bolyai started with non-Euclidean three-dimensional space and introduced a special surface F (presumably from the German word for surface, *Fläche*), obtained in the following manner (see figure 3.4. He took a straight line a with a point on it, A, and in any plane containing the line a he considered all parallel lines to a. On each parallel line, b, he located the point B such that the

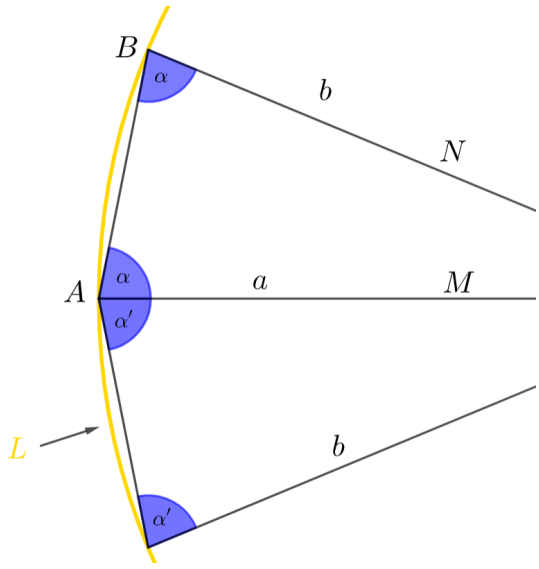


Figure 3.4: Bolyai's L-curve construction

angles MAB and NBA are equal (their size depends on the position of the point B). This produced a curve, which he denoted L (from the German *Linie*), in the plane containing the lines a and b, and then, as the plane through a is varied, the

surface F. If the parallel postulate is true, L is just a straight line and F is just the plane through A perpendicular to a. If it is not true, F has the shape of a bowl, being the result of rotating L about the axis a. Bolyai went on to prove as many results as he could, noting which were absolute and which were not. One of these results is that all the lines b parallel to the axis a meet F at right angles. This leads to think of F as a sphere of infinite radius just like Gauss might have had in mind.

Bolyai next considers the formulae that relate the sides and angles of a triangle on the (Euclidean) sphere. In particular, he showed that they too are absolute theorems. It meant that he could also use spherical trigonometry in the new setting.

To find the appropriate trigonometric formulae in a geometry where the parallel postulate is false Bolyai began with the following construction (see figure 3.5). It consists of a straight line AB of length y, meeting a line AM at right angles and a line BN at an acute angle u, where the lines AM and BN are parallel. The length y determines the angle u and vice versa. As we showed before, the angle u is the angle of parallelism corresponding to the length y, which is usually written as $u = \Pi(y)$. Bolyai set out to find an expression on y as a function of u.

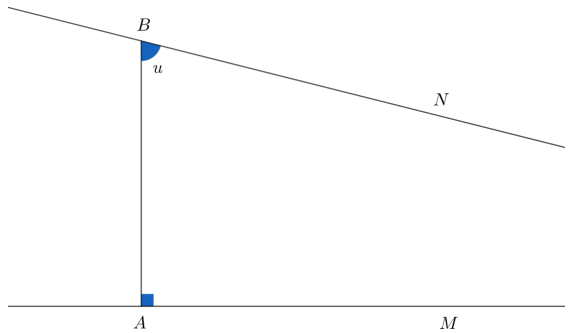


Figure 3.5: Construction of the angle of parallelism

Bolyai's method is long and does not add much to what we are trying to discuss. For the sake of completeness, here is a short summary that can be found in Jeremy J. Gray's *János Bolyai, non-Euclidean geometry and the nature of space*, pages 64-66.

> Bolyai first considered an arbitrary point C on the line a and the L-curve through it. He called this L-curve L'. He showed that the ratio $AB : CD$ is independent of AB and depends only on the length

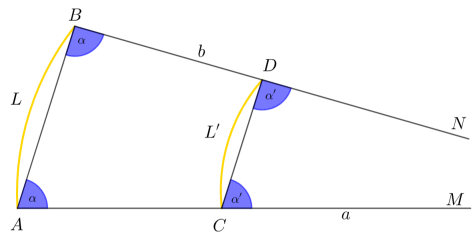


Figure 3.6

$AB = x$. He denoted this ratio X and set himself the task of evaluating it. He gave his answer in the form a formula relating u and Y, where Y is the same function of y that X is of x, and u is the angle of parallelism corresponding to the length y. He showed by a simple scaling argument that given lengths x and x' the corresponding X and X' satisfy $Y^{1/y} = X^{1/x}$, from which it follows, although Bolyai did not say so immediately, that $X = e^{kx}$ for some arbitrary constant k.

To find X he considered the curve which is everywhere equidistant from a straight line. In Euclidean geometry this is another straight line, but if the parallel postulate is false it will not be. Bolyai imagined it was swept out by the tip of a line segment that moves perpendicular to the given line. In the following figure, the triangle ABC is supposed to slide rigidly along the line a, with the edge AC remaining always perpendicular to a. The point C draws the equidistant curve c to the line a. The segment BD is another position of the segment AC, so the point D lies on the curve c. The angles u and v are as shown in figure 3.7; $u = CAD$ and $v = ADB$.

Bolyai showed that the ratio of the length of the line segment AB to

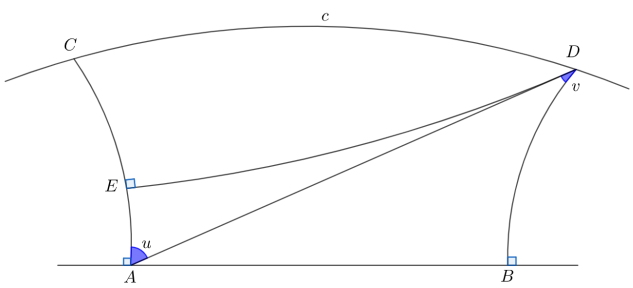


Figure 3.7

the length of the segment CD of the curve c is equal to $sin(v)/sin(u)$

by dropping the perpendicular DE from D to AC and first applying his trigonometric formulae to the triangles AED and ABD. Then he used a limiting argument to find the ratio of the lengths AB and CD. He observed that since the ratio AB/CD is a constant (it depends only on the height AC, not the width AB) this ratio can be evaluated in the limit as AC moves off infinitely far, when the angle u tends to a right angle and the angle v to the angle of parallelism.

However, he also showed that the ratio X he was interested in before was equal to $sin(u)/sin(v)$ (see figure below). From this he could deduce the result he wanted: the angle of parallelism is given by the formula $Y = cot(u/2)$. If we use $Y = e^{ky}$, we get $sinh\ y = cot\ u$, or we can re-write it as $sinh\ y = cot\ \Pi(y)$.

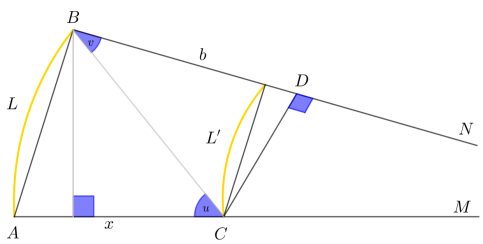


Figure 3.8

Although the proof is quite technical and I will not include it here, Bolyai proved that he could square the circle. Or in his geometry, at least. It can be found in Jeremy J. Gray's *János Bolyai, non-Euclidean Geometry and the Nature of Space*, pp 69-75.

Nikolai Lobachevsky

The immediate response to Bolyai's work was poor. Which is not hard to believe considering that it was published as an appendix to a two-volume work in Latin written by a Hungarian who was far from being well-known. And the fact that Gauss chose not to draw any public attention despite claiming to be fond of Bolyai's work did not help either. Nikolai Ivanovich Lobachevsky (1792-1856) sadly went through something similar. In 1829, as a professor at the University of Kazan, he published the first of several articles in the *Kazan Messenger* where he described a geometry equivalent to Euclid's. Lobachevsky did not fancy the concepts of "line", "surface" and "position". He thought of them as vague and obscured an preferred to think that geometry is about bodies and their motion.

Lobachevsky thought of the connection with the parallax of stars, thus suggesting that the real space we move in is non-Euclidean, but this argument did not take him far. In 1815, at the early age of 22, he was already working on the parallels. Several attempts at proving Euclid's fifth postulate have been found in notes for his lectures. In 1823 he started working on a geometry independent of the fifth postulate. Unlike Bolyai, however, he focused on proving as many results as he could rather than to think of which results would work if the parallel postulate was true too. To be more specific, he considered a geometry where two parallels to a given line can be drawn through a point, an where the sum of the angles of a triangle is less than two right angles.

Portrait of Lobachevsky

Here is a brief explanation of his method[4]:

Lobachevsky considers a pencil with vertex A and a straight line BC in the plane of the pencil, but not belonging to it. Let AD be the line of the pencil which is perpendicular to BC, and AE that perpendicular to AD. In the Euclidean system this latter line is the *only* line which does not intersect BC. In the geometry of Lobachevsky there are other lines of the pencil through A which do not intersect BC. The non-intersecting lines are separated from the intersecting lines by the two lines h, k (see figure 3.9), which do not meet h, k. These lines, which Lobachevsky calls parallels, have each a definite direction of parallelism. The line h of the figure is the parallel to the right and k to the left. He also defined the angle of parallelism of the length AD. Thus far everything is quite analogous to Bolyai's version. That notwithstanding Lobachevsky introduced two new figures to find his trigonometric formulae: the *horocycle* (a circle of infinite radius, just like Gauss had suggested) and the *horosphere* (a sphere of infinite radius), which in Euclidean geometry correspond simply to the straight line and plane, respectively. He then proves that his geometry on the horosphere was equivalent to the Euclidean geometry (which Bolyai knew too). As long as the angle of parallelism is concerned, Lobachevsky also found that $tan\frac{\Pi(x)}{2} = a^{-x}$.

[4]Lobachevsky, *Geometrische Unterschungen zur Theorie der Parallelinien*, 1840

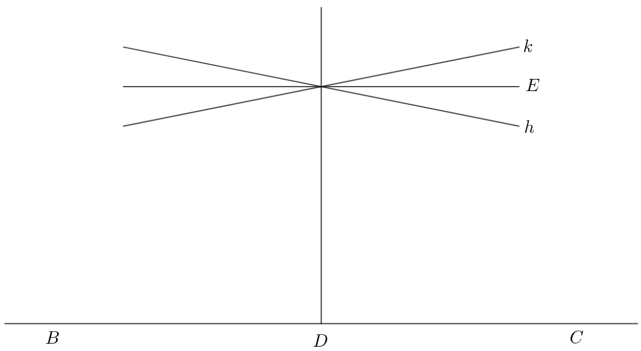


Figure 3.9

Thus it seems fair to say that Bolyai and Lobachevsky created the same geometry independently – all be it with some minor particularities and slightly different styles. For instance, Lobachevsky had the habit of naming everything he came across, whereas Bolyai limited himself to giving a letter to each object, for which Gauss criticized him. Lobachevsky's work did not see much success until he sent a copy to Gauss, who immediately acclaimed his work and in 1842 had him made a corresponding member of the Göttingen Academy of Sciences. This was all the success Lobachevsky would see in his lifetime.

Chapter 4

Consolidation of non-Euclidean geometry

Although neither Gauss, Bolyai or Lobachevsky showed that a contradiction would never be found – they just never found one –, the overall conviction was that they were right. Gauss' death prompted mathematicians to read what he had written but never published. Bolyai and Lobachevsky's work was translated (the latter's was already in French and German) so their work began to spread, keeping the interest on non-Euclidean geometry alive in spite of Gauss, Lobachevsky and Bolyai's deaths in 1855, 1856 and 1860 respectively.

The mathematician who first had the idea that Gauss missed was a student of his, Bernhard Riemann (1826-1866). In 1854 he presented his *Habilitationsschrift* (the qualification that permitted one to teach at a German university) at Göttingen, with Gauss present. It was titled *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the hypotheses which underlie geometry)[1]. Although rather vaguely, Riemann took Gauss' idea of curvature and took it further. The remarkable thing of Gauss' *Theorema egregium* and curvature is that they were intrinsic results: they depended only on measurements taken on the surface itself, without stepping back into the ambient space. Riemann's idea was to focus on the concept of curvature and to argue that

Portrait of Riemann

[1] A translated transcription of the lecture can be found at `http://www.emis.de/classics/Riemann/WKCGeom.pdf`

geometry was fundamentally about two types of problem: the intrinsic properties of a surface, and the ways in which a surface can be mapped into another space. He thus introduced the concept of metric and of what we now call Riemannian manifolds.

Riemann broke with the idea that geometry is about undefined but clear ideas such as point and line. He much preferred to speak in terms of point and distance, thus making a line simply an aggregate of points that satisfy a certain equation involving distances. So it turns out that there are infinitely many geometries in each dimension – or infinitely many ways of measuring, i.e., metrics.

According to several sources[2], Riemann's ideas circulated slowly. They were published for the first time in 1868, two years after his death. The Italian mathematician Eugenio Beltrami (1835-1900) got word of Riemann's work, which motivated him to publish – against his supervisor's recommendation – his ow account of non-Euclidean geometry. With, as we shall see, measuring on a surface with no underlying space was achieved, thus giving mathematicians for the first time an impeccable proof of the existence of non-Euclidean geometry.

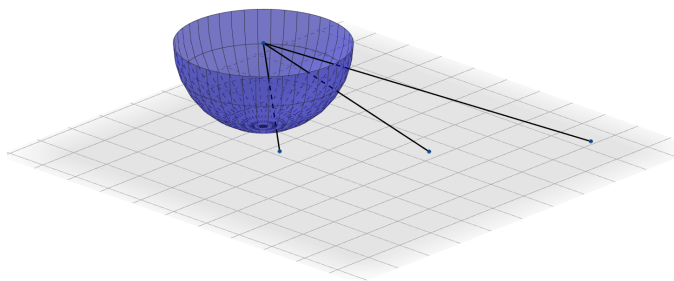


Figure 4.1: Geodetic projection

Beltrami took the geodetic map from a hemisphere onto a plane, which is obtained by imagining a source of light at the center of the hemisphere and mapping each point on the hemisphere to its corresponding shadow on the plane. This can be done by drawing the ray from the center to any point and taking its intersection with the plane (see figure 4.1). This map conveniently maps geodesics onto geodesics (hence the name), namely great circles onto straight lines. Regarding distances, they get distorted. A pair of points on the sphere will end up much further apart if they are near the equator than if they are near the south pole, even if they are the same Euclidean distance from each other.

[2]Gray [14] and Coxeter [16]

Beltrami took the expression of this map and replaced the term R^2 by the term $-R^2$. It turned out to make sense, but only inside a disc of radius R. The disc was, in a way, a map of something. What it was a map of Beltrami was not sure of. Had this happened before Riemann, the hunt for whatever it is a map of might have kept Beltrami from moving forward. Luckily, this was not the case. Just like Riemann did not need a space into which to put the surface that he was measuring, Beltrami did not need an application of his disc either. It was, therefore, a geometry, in the new sense of the term.

Let us briefly explain how the Beltrami disc works (see figure 4.2). Points of non-Euclidean space appear as points inside the disc. Points on the boundary circle are not part of the map and do not represent anything. Non-Euclidean straight lines appear as straight lines in the disc. Several lines through the point P are shown. The line b meets the line a, and the line c is parallel to a – they meet on the boundary of the disc. The lines d and d' are examples of lines through P that do not meet a. The curvature of the surface is $-1/R^2$.

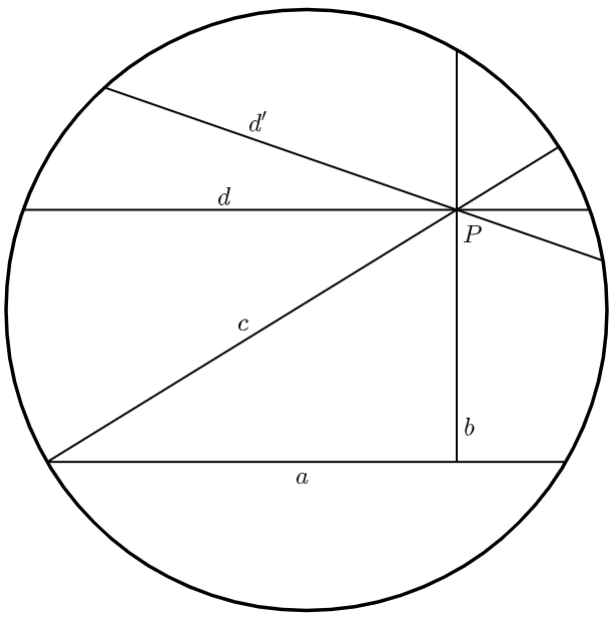


Figure 4.2: Beltrami disc

The Beltrami disc solved the problem once and for all, but it is certainly not easy to use. The next development is probably that of Felix Klein (1849-1925). He realized that the Beltrami disc had the same idea behind it as projective geometry.

Klein later published the Erlangen Program in his *Vergleichende Betrachtungen über neuere geometrische Forschungen* (Comparative considerations about recent geometry investigations) in 1872. In it, he offered a unifying frame for all geometries through group theory. The relations between geometries could be explained by how subgroups of a symmetry groups are related to each other. For instance, Euclidean geometry turns out to be more restrictive than affine geometry, which in turn is more restrictive than projective geometry. Klein was also the one who conceded the names of elliptic and hyperbolic geometry. Here is a rough idea of what that classification looks like:

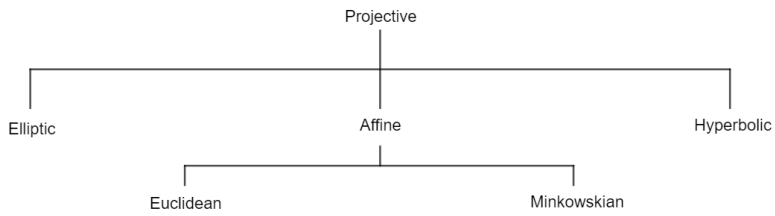


Figure 4.3: Classification of geometries

An even more useful new way of seeing the Beltrami disc occurred to French mathematician Henri Poincaré (1854-1912) in June 1880, when he was changing buses as part of a field trip (he was a trainee mining engineer at the time). He had been studying a complicated network of triangles inside a disc, and to study them he had applied a transformation that straightened out the sides. This is how he recalled the event many years later, in 1908: [3]

> At that moment I left Caen where I then lived, to take part in a geological expedition organized by the École des Mines. The circumstances of the journey made me forget my mathematical work; arrived at Coutances we boarded an omnibus for I don't know what journey. At the moment when I put my foot on the step the idea came to me, without anything in my previous thoughts having prepared me for it; that the transformations I had made us of... were identical with those of non-Euclidean geometry. I did not verify this, I did not have the time for it, since scarcely had I sat down in the bus than I resumed a conversation already begun, but I was entirely certain at once. On returning to Caen I verified the result at leisure to salve my conscience.

[3] Henri Poincaré. *Science et méthode* (1908), pp. 51-52.

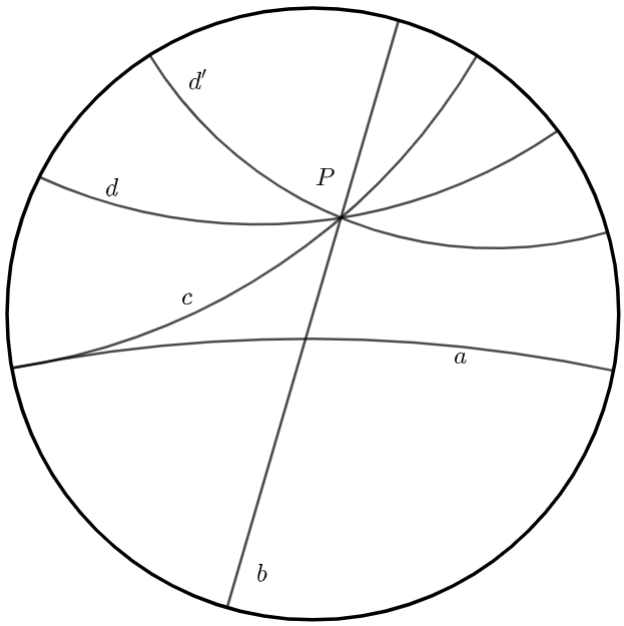


Figure 4.4: Poincaré disc

Let us have a look at what the Poincaré disc looks like (see figure 4.4). Like the Beltrami disc , Euclidean points are mapped onto points, and points on the boundary represent nothing. Non-Euclidean straight lines appear either as straight lines in the disc perpendicular to the boundary circle, or as arcs of circles in the disc and perpendicular to the boundary circle. Several lines through the point P are shown. The line b meets the line a, and the line c is parallel to a – they meet on the boundary of the disc. The lines d and d' are examples of lines through P that do not meet a.

The Poincaré disc model of non-Euclidean geometry is the one generally used today. One of its advantages is that in it angles appear drawn to the correct size. So in the figure 4.5, all triangles have the same angles, which are either $\pi/2$, $\pi/3$ or $\pi/7$.

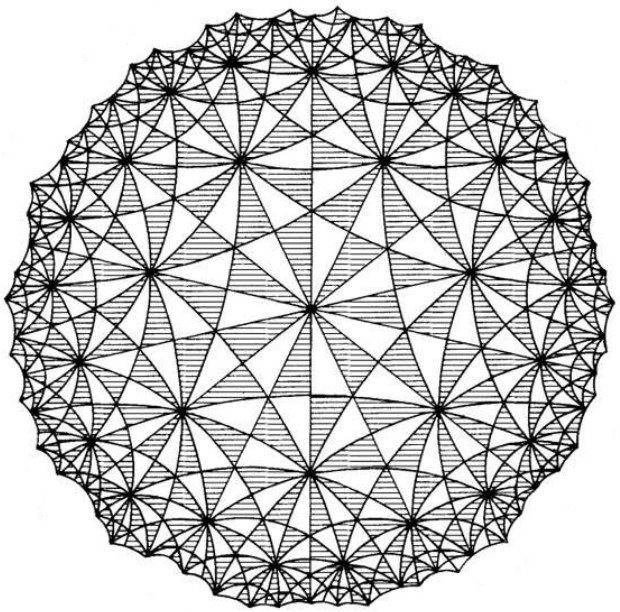


Figure 4.5: Tessellation {2, 3, 7} of the Poincaré disc

Afterword

In this work, we have navigated the thread going from early nineteenth century – where achieving anything consistent without the parallel postulate seemed unattainable and yet the frustration of being unable to prove it did not shrink – to the last fourth of that century – where the Pandora Box had been opened and all types of geometries poured in, and even a classification of them was settled. We have also seen how they thought by inspecting both their works and reading some of their letters.

This is, in my opinion, a unique period of time in the history of mathematics. Until about 1800, all mathematicians were convinced that Euclidean geometry was the correct model of physical space. The rise of non-Euclidean geometry obliged mathematicians to revise their understanding of the nature of mathematics and its relation to the physical world. Who knows if this time could have been even more productive had Gauss been more open about his ideas, since he published very little of what he knew. An explanation for this habit may lie on his own aphorism *pauca sed matura* (little but ripe). It does not mean that there is no room for the mundane (Gauss actually published routine calculations), but that it is best to speak when the thought has matured enough. Even if these hypothetical publications by Gauss would not have meant much, it may have given the right idea to someone else.

And then, what is the answer to the question that has been lingering for so long? Is the physical space we live in Euclidean or non-Euclidean? From what I have gathered, we do not know yet. According to the General Theory of Relativity, astronomical space has positive curvature locally (wherever there is matter), but we cannot tell whether the curvature of "empty" space is exactly zero or has a very small positive or negative value.

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