

# LOCALIZATION OF NILPOTENT FIBRE MAPS

by

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## 1. Introduction.

The theory of localization first introduced in topology by D. Sullivan [6] has been exploited by many topologists, e.g. A.K. Bousfield - D.M. Kan [1], P. Hilton - G. Mislin - J. Roitberg [2], and works particularly well in the homotopy category of nilpotent spaces, i.e. spaces whose Postnikov system admits a principal refinement.

Now let  $f : E \rightarrow B$  be a fibre map with  $E, B$  and the fibre  $F$  connected CW-complexes. Then  $\pi_1 E$  acts on the homotopy groups  $\pi_n F$ ,  $n \geq 1$ , and we say [2] that  $f$  is nilpotent if  $\pi_1 E$  acts nilpotently on all the homotopy groups of  $F$ . This is the case, for example, if  $E$  and  $B$  are nilpotent. If  $f$  is a nilpotent fibre map then the fibre  $F$  is nilpotent. The conversely is true only if  $\pi_1 B$  operates nilpotently on the homology groups of  $F$  [3].

Nilpotent fibre maps turns out to be the right relativization of nilpotent spaces, because their Moore-Postnikov systems admit principal refinements ([2] th. II.2.4).

In [4] P. Hilton has developed a localization theory for the corresponding group-theoretical situation, that of group extensions  $N \rightarrow G \twoheadrightarrow Q$  such that  $G$  acts nilpotently on  $N$ .

In this paper we introduce a  $P$ -localization theory (where  $P$  is a given family of rational primes) for nilpotent fibre maps which generalizes both the existing theory of localization of nilpotent spaces and the localization theory for relative groups of [4]. These ideas had already been treated in the semisimplicial category in [1] (with a slight different definition of nilpotent action).

In fact in §2 we prove an universal property which shows in particular the uniqueness of the localization of a nilpotent fibre map and in §3 we construct this licalization in a similar way as is done in [2] for the absolute case.

I wish to thank P. Hilton for suggesting this work to me as well as for some helpful conversations.

2. An universal property.

We work in the pointed homotopy category of connected CW-complexes. We say that a fibre map  $f: E \rightarrow B$  with connected fibre  $F$  is nilpotent if  $\pi_1 E$  acts nilpotently on  $\pi_n F$ ,  $n \geq 1$ . Recall that if  $w$  is an action of a group  $G$  on  $N$  the lower central  $w$ -series of  $N$  is defined by

$$\Gamma_w^0 N = N, \quad \Gamma_w^i N = \langle \alpha(xb)\alpha^{-1}b^{-1}; x \in G, \alpha \in N, b \in \Gamma_w^{i-1} N \rangle, \quad i > 0;$$

and we say that  $G$  acts nilpotently on  $N$  if  $\Gamma_w^c N = \{1\}$  for  $c$  sufficiently large.

Let now  $F \rightarrow E \xrightarrow{f} B$  be a nilpotent fibre map and let  $P$  be an arbitrary collection of rational primes. A  $P$ -localization of  $f$  is a map  $\beta: f \rightarrow f_{(P)}$  over  $B$ , where  $f_{(P)}: E_{(P)} \rightarrow B$  is a fibre map with fibre  $F_{(P)}$  and the induced map  $F \rightarrow F_{(P)}$  is a  $P$ -localization:

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{f} & B \\ e \downarrow & & \beta \downarrow & \searrow f_{(P)} & \downarrow \\ F_{(P)} & \longrightarrow & E_{(P)} & \longrightarrow & B \end{array}$$

*Theorem 2.1.* Let  $F \rightarrow E \xrightarrow{f} B$  be a nilpotent fibre map and let  $\beta: f \rightarrow f_{(P)}$  a  $P$ -localization of  $f$ . For every nilpotent fibre map  $\tilde{f}: \tilde{E} \rightarrow B$  with  $P$ -local fibre  $\tilde{F}$ ,

$$[f_{(P)}, \tilde{f}]_B \xrightarrow{\beta^*} [f, \tilde{f}]_B$$

is a bijection, where  $[ , ]_B$  stands for the set of homotopy classes of maps over  $B$ .

*Proof:* Given  $\tau \in [f, \tilde{f}]_B$  we will show that there is an unique  $\delta \in [f_{(P)}, \tilde{f}]_B$  such that  $\beta^* \delta = \tau$ , i.e. rendering commutative the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\tau} & \tilde{E} \\ \beta \downarrow & \nearrow \delta & \downarrow \tilde{f} \\ E_{(P)} & \xrightarrow{f_{(P)}} & B \end{array}$$

Let

$$\dots \rightarrow Y_c \xrightarrow{q_c} Y_{c-1} \rightarrow \dots \rightarrow Y_0 = B$$

be a principal refinement of the Moore-Postnikov factorisation of  $\bar{f}$ . Each

$$q_c: Y_c \rightarrow Y_{c-1}$$

is a principal fibre map with fibre an Eilenberg-MacLane space  $K(G_c, n)$  where  $G_c$  are (abelian) groups obtained as quotients  $\Gamma_w^i \pi_n \bar{F} / \Gamma_w^{i+1} \pi_n \bar{F}$  ( $w$  the action of  $\pi_1 \bar{F}$  on  $\pi_n \bar{F}$ ). These groups are trivial  $\pi_1 \bar{F}$ -modules: for  $n \geq 2$  it is clear; for  $n = 1$  use that  $w$  is characterized by  $i_*(\xi \cdot \alpha) = (i_* \xi) \cdot \alpha$ ,  $\xi \in \pi_1 \bar{F}$ ,  $\alpha \in \pi_1 \bar{F}$ ,  $i: \bar{F} \rightarrow \bar{E}$  the inclusion, and that  $\pi_1 \bar{F}$  acts on itself by conjugation.

The obstruction to the lifting of  $f_{(P)}$  to  $Y_1$  lies in  $H^2(E_{(P)}, E; G_1)$  and the higher obstructions to each  $Y_c$  are subsets of  $H^n(E_{(P)}, E; G_c)$ . Furthermore, at each step  $c$ , two lifting maps  $\delta_c'$  and  $\delta_c''$  of the same  $\delta_{c-1}$  are homotopic relative to  $\delta_{c-1}$  if a certain cocycle of  $H^{n-1}(E_{(P)}, E; G_c)$  vanishes. Hence the existence and uniqueness of  $\delta$  follow from Lemmata 2.2 and 2.3 below, which ensure the vanishing of all groups  $H^n(E_{(P)}, E; G_c)$ .

*Lemma 2.2.* Let  $A$  be a  $P$ -local abelian group. Then, for all  $n \geq 0$ ,

$$H^n(E_{(P)}, E; A) = 0.$$

*Proof:* We look at the induced homomorphism  $\beta^*$  in the Serre spectral sequences of  $f$  and  $f_{(P)}$

$$\begin{array}{ccc} H^p(B; \mathcal{J}C^q(I_p; A)) & \xrightarrow{\quad} & H^{p+q}(E_{(P)}; A) \\ \downarrow e^* & & \downarrow \beta^* \\ H^p(B; \mathcal{J}C^q(I; A)) & \xrightarrow{\quad} & H^{p+q}(E; A) \end{array}$$

Since  $e: I \rightarrow I_p$  is a  $P$ -localization, by the universal coefficient theorem and [2] 1.2.9 and 1.1.8

$$H^n(I_p; A) \xrightarrow{e^*} H^n(I; A)$$

is an isomorphism in each dimension. Thus we get an isomorphism at the  $E_2$  level of the Serre spectral sequences and the lemma follows.

*Lemma 2.3.* The groups  $G_c$  are  $P$ -local.

*Proof:* Since  $\pi_n \bar{F}$  are  $P$ -local nilpotent groups  $\Gamma_w^i \pi_n \bar{F}$  are  $P$ -local by theorem 2.1 of [5]. Hence  $G_c$  are  $P$ -local as quotients of  $P$ -local nilpotent groups.

*Corollary 2.4.* Let  $F \rightarrow E \xrightarrow{f} B$  be a nilpotent fibre map. If  $\beta : f \rightarrow f_{(P)}$  is a  $P$ -localization with  $f_{(P)}$  nilpotent then  $f_{(P)}$  and  $\beta$  are uniquely determined up to an homotopy equivalence.

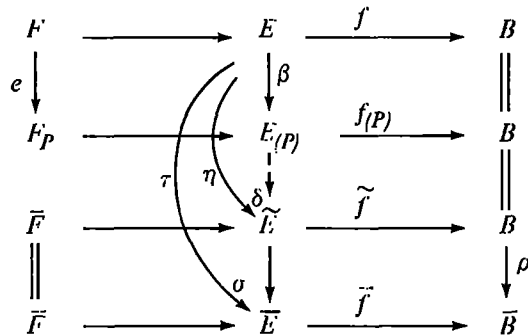
*Proof:* Let  $\bar{\beta} : \bar{f} \rightarrow \bar{f}$  be another  $P$ -localization of  $f$  with  $\bar{f}$  nilpotent. By theorem 2.1 there exist  $\delta_1 : \bar{f} \rightarrow f_{(P)}$  and  $\delta_2 : f_{(P)} \rightarrow \bar{f}$  such that  $\bar{\beta} = \delta_1 \beta$  and  $\beta = \delta_2 \bar{\beta}$ ; thus  $\beta = \delta_2 \delta_1 \beta$  and  $\bar{\beta} = \delta_1 \delta_2 \bar{\beta}$  and therefore  $\delta_2 \delta_1 \simeq id$  and  $\delta_1 \delta_2 \simeq id$  by the uniqueness in theorem 2.1.

*Corollary 2.5.* Let  $F' \rightarrow E \xrightarrow{f} B$  be a nilpotent fibre map and let  $\beta : f \rightarrow f_{(P)}$  a  $P$ -localization. For every nilpotent fibre map  $\bar{f} : \bar{E} \rightarrow \bar{B}$  with  $P$ -local fibre  $\bar{F}$  and every map  $\rho : B \rightarrow \bar{B}$ ,

$$[f_{(P)}, \bar{f}]_{\rho} \xrightarrow{\beta^*} [f, \bar{f}]_{\rho}$$

is a bijection.

*Proof:* Consider the diagram



where  $\tilde{f}$  is induced by  $\rho$ . Every  $\tau \in [f, \tilde{f}]_\rho$  induces a map  $\eta \in [f, \tilde{f}]_B$  and we can apply theorem 2.1 to ensure the existence of  $\delta \in [f_{(P)}, \tilde{f}]_B$  such that  $\delta \beta = \eta$ . Hence  $\beta^*(\sigma \delta) = \tau$ . Observe that the nilpotency of  $f$  follows from that of  $\tilde{f}$ .

A standard argument shows the uniqueness of  $\delta$ .

3. Existence of a P-localization

Let  $f: E \rightarrow B$  be a nilpotent fibre map and let

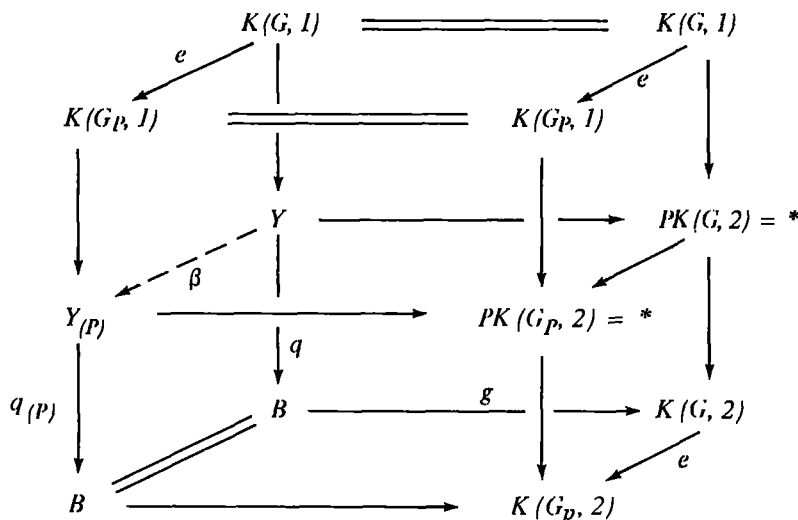
$$\dots \rightarrow Y_c \xrightarrow{q_c} Y_{c-1} \rightarrow \dots \rightarrow Y_1 \xrightarrow{q_1} Y_0 = B$$

be a principal refinement of its Moore-Postnikov system. We will construct a P-localization of  $f$  by induction on the height of the system.

First suppose we have a principal fibre map

$$K(G, 1) \rightarrow Y \xrightarrow{q} B$$

induced by a map  $g: B \rightarrow K(G, 2)$ . We define  $q_{(P)}: Y_{(P)} \rightarrow B$  as the principal fibre map induced by  $e_*g: B \rightarrow K(G_P, 2)$  where  $e$  is a P-localization



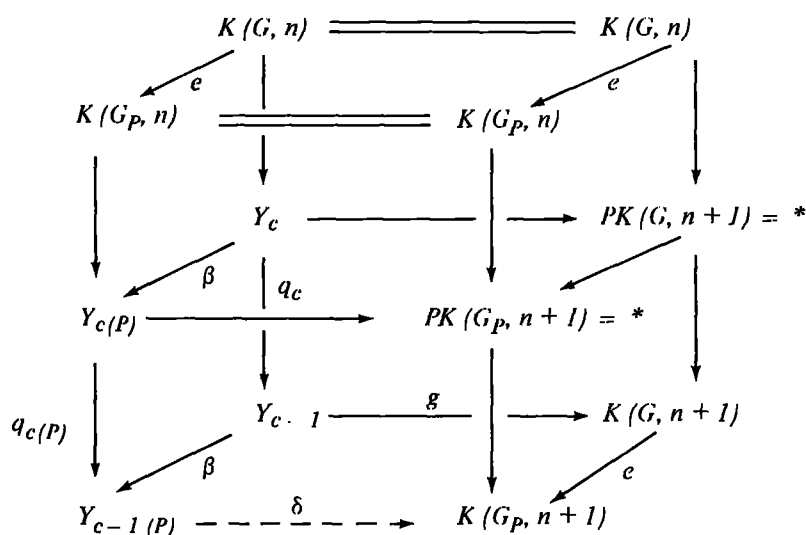
Let  $\beta: Y \rightarrow Y_{(P)}$  be induced by  $q: Y \rightarrow B$  and  $Y \rightarrow PK(G_P, 2)$ . From the commutativity of the above diagram it follows that the map  $K(G, 1) \rightarrow K(G_P, 1)$  induced on the fibres by  $\beta$  is a P-localization.

Let  $s_{c-1} = q_1 \cdots q_{c-1}: Y_{c-1} \rightarrow B$  and assume now that we have already defined  $s_{c-1}(P): Y_{c-1}(P) \rightarrow B$  and also  $\beta: Y_{c-1} \rightarrow Y_{c-1}(P)$  over  $B$  inducing on the fibres a  $P$ -localization.

If  $q_c: Y_c \rightarrow Y_{c-1}$  is induced by  $g: Y_{c-1} \rightarrow K(G, n+1)$  we define  $q_c(P): Y_c(P) \rightarrow Y_{c-1}(P)$  as the principal fibre map induced by a

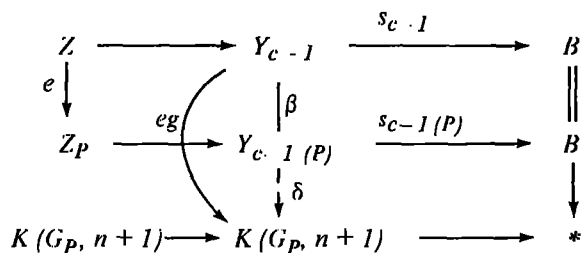
$$\delta: Y_{c-1}(P) \rightarrow K(G_P, n+1)$$

such that  $\delta \beta = e g$ , where  $e$  is a  $P$ -localization. See diagram below.

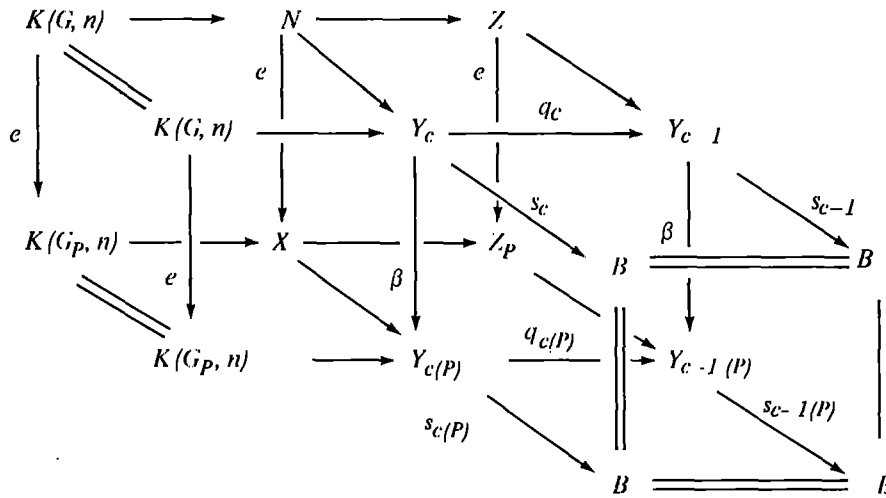


We can argue as in the first step in order to obtain a map  $\beta: Y_c \rightarrow Y_c(P)$  inducing a  $P$ -localization on the fibres.

The existence and uniqueness of  $\delta$  follow then applying Corollary 2.5 to



It remains only to see that  $\beta: s_c \rightarrow s_{c(P)}$  induces a  $P$ -localization on the fibres. Consider the diagram of fibre maps



We may apply now the homotopy exact sequence to the back squares to conclude that  $\pi_i \mathcal{N} \rightarrow \pi_i \mathcal{X}$  is a  $P$ -localization for all  $i$ . Hence  $\mathcal{N} \rightarrow \mathcal{X}$  is a  $P$ -localization.

If the refined principal Moore-Postnikov system is finite the above procedure yields the desired  $P$ -localization. In the general case there is a weak homotopy equivalence  $E \rightarrow \varprojlim Y_c$  over  $B$ . Let  $E_{(P)}$  be the geometric realization of the singular complex of  $\varprojlim Y_{c(P)}$ . Then there is a map  $\beta: E \rightarrow E_{(P)}$  such that the diagram

$$\begin{array}{ccc}
 E & \longrightarrow & \varprojlim Y_c \\
 \beta \downarrow & & \downarrow \varprojlim \beta \\
 E_{(P)} & \longrightarrow & \varprojlim Y_{c(P)}
 \end{array}$$

is homotopy-commutative over  $B$ .

Let  $F_c$  be the fibre of  $s_c: Y_c \rightarrow B$ . The fibre maps  $\varprojlim Y_c \rightarrow B$  and  $E \rightarrow B$  have weak homotopy equivalent fibres  $\varprojlim F_c$  and  $F$  respectively. Analogously

$\varprojlim F_{c(P)}$  and  $F_P$  are weak homotopy equivalent. Moreover the induced map  $\varprojlim F_c \rightarrow \varprojlim F_{c(P)}$  is a  $P$ -localization. Thus

$$\begin{array}{ccccc}
 F & \longrightarrow & E & \xrightarrow{f} & B \\
 \downarrow e & & \downarrow \beta & & \parallel \\
 F_P & \longrightarrow & E_{(P)} & \xrightarrow{f_{(P)}} & B
 \end{array}$$

is a  $P$ -localization.

Note that  $\dots \rightarrow Y_{c(P)} \rightarrow Y_{c-1(P)} \rightarrow \dots \rightarrow B$  is a principal refinement of the Moore-Postnikov system of  $f_{(P)}$ ; the fibre map  $f_{(P)}$  is therefore nilpotent.



## REFERENCES

- [1] A.K. Bousfield-D.M. Kan: "*Homotopy limits, completions and localizations*", Lect. Notes in Math. 304, Springer 1972.
- [2] P. Hilton - G. Mislin - J. Roitberg: "*Localization of nilpotent groups and spaces*", North-Holland Math. Studies 15, 1975.
- [3] P. Hilton: "*On G-spaces*", Bol. Soc. Bras. Mat. (1976), 65-73.
- [4] P. Hilton: "*Relative nilpotent groups*", Proc. Conf. Cat. Alg. and Top., Carleton Univ. Lect. Notes in Math. 915, Springer (1982), 136-147.
- [5] P. Hilton: "*Nilpotent groups and abelianization*", Quaestiones Mathematicae (1982), to appear.
- [6] D. Sullivan: "*Geometric topology, part I: Localization, periodicity and Galois symmetry*", MIT, June 1970, (mimeographed notes).

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