A NECESSARY AND SUFFICIENT CONDITION FOR A VECTOR FIELD TO BE KILLING

by

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ABSTRACT

Lynge (see (4)), proved that X vector field on a compact and connected manifold M, is Killing in a Riemannian metric g, if and only if there exist k complete vector fields $X_1, \ldots, X_k$; such that: 1) The flows of the X_i are all periodic, 2) $[X_i, X_j] = 0$, $i, j = 1, \ldots, k$, 3) $X = \Sigma_{i=1}^{k} c_i X_i$, c_i real numbers.

We study a generalisation of this result in no-compact manifolds, obtaining: If X is a complete vector field on M, ϕ_t its uniparametric group; then X is Killing in a Riemannian metric g, if and only if : a) ϕ_t acts freely and properly, or b) there exist k complete vector fields in Lynge-conditions. We use some results of proper actions, contained in (3) and (4).

We generalise this result to proper and free actions of Lie groups.

Finally we study the Riemannian metrics, in which a vector field in a)-conditions can be Killing.

§ 1. INTRODUCTION

We need some results of convergence in the group of isometries of a connected Riemannian manifold, and we list these briefly.

LEMMA 1.1. — (See (2). LEMMA 3, pag. 48).

Let $\{\phi_n\}$ be a sequence of isometries such that $\{\phi_n(p)\}$ converges for some point $p \in M$. Then there is a subsequence $\{\phi_{n_k}\}$, such that $\{\phi_{n_k}(x)\}$ converges for each $x \in M$.

LEMMA 1.2. — (See (2). LEMMA 4 pag. 48).

Assuming that $\{\phi_n\}$ *is a sequence of isometries such that* $\{\phi_n(x)\}$ *converges for each* $x \in M$. *We define* $\phi(x) = lim. \{\phi_n(x)\}$ *for each* $x \in M$. *Then* ϕ *is an isometry.*

LEMMA 1.3. — (See (2). LEMMA 5 pag. 48).

Let $\{\phi_n\}$ *be a sequence of isometries, and* ϕ *an isometry. If* $\{\phi_n(x)\}$ *converges to* $\phi(x)$, *for every* $x \in M$, *then the convergence is uniform on every compact subset of* M.

From this it follows:

LEMMA 1.4.

Let $\{\phi_n\}$ *be a sequence of isometries, and* ϕ *an isometry.* $\{\phi_n\}$ *converges to* ϕ *with respect to the compact-open topology if and only if* $\{\phi_n\}$ *converges to* ϕ *uniformly on every compact subset of* M.

With the aid of these results we can study:

§ 2. NECESSARY CONDITIONS FOR X TO BE KILLING

Let X be a Killing vector field in (M, g). We suppose X to be complete (we can suppose from now on, that M is complete).

Let ϕ_t be the uniparametric group of $X.F : R \times M \to M$, $F(t, p) = = \phi_t(p)$. If H is the adherence of $\{\phi_t\}$ in G (isometries group of (M, g)), H is a closed and abelian Lie subgroup of G, so $H = R^q \times T^k$.

It can be shown:

THEOREM 2.1.

$H = R$ *or* $H = T^k$.

PROOF. — If $H = R^q \times T^k$, we can consider $\{A_1, ..., A_q, B_1, ..., B_k\}$ a basis of the Lie algebra of H, $L(H)$, such that all the Lie brackets are zero, $exp.\, tA_i$ are closed uniparametric groups which are diffeomorfic to R, and $exp.\, tB_r$ are closed uniparametric groups diffeomorfic to S^1.

$\phi_t = exp.\, t(\Sigma_{i=1}^{q} a^i A_i + \Sigma_{r=1}^{k} b^r B_r)$, and $exp.\, t(\Sigma_{i=1}^{q} a^i A_i)$ is a closed uniparametric group diffeomorfic to R, so $q = 1$ or $q = 0$.

If $q = 1$, $H = R \times T^k$, and taking $\{A; B_1, ..., B_k\}$ basis of $L(H)$ in the same conditions that above; it is very easy to see that *exp*. $t(A + \Sigma_{r=1}^{k} b^r B_r)$ is a closed uniparametric group diffeomorfic to R, so if $q = 1$, $k = 0$.

When $H = T^k$, we are in Lynge-conditions, and this case is studied in (4), but we think it is interesting to note:

THEOREM 2.2. — (See (1)).

If the X-orbit for some point $p \in M$ is closed, then $H = T^k$, with $k \leqslant dim. M$.

PROOF. — Consider $\{\phi_t(p)\}$, $t \in R$, which is a compact submanifold of M diffeomorfic to S^1. So for every sequence $\{\phi_{t_n}(p)\}$, there is a subsequence $\{\phi_{\alpha_n}(p)\}$ which is convergent. Lemma 1.1. says that there exists a subsequence of $\{\alpha_n\}$, $\{\beta_n\}$, such that $\{\phi_{\beta_n}(x)\} \to \phi(x)$ for each $x \in M$. Because Lemmas 1.2, 1.3, 1.4. $\{\phi_{\beta_n}\}$ converges to ϕ in the compact-open topology, so H is compact and as H is abelian, $H = T^k$, with $k \leqslant dim. M$.

COROLLARY 2.1.

Let X be a Killing vector field, with an orbit that is closed. Then there exists at least a Killing vector field with periodic flow.

For more results on Killing vector fields with closed orbits, see (1).

From now on, suppose $H = R$. A is a basis of the Lie algebra of H, and A^* is the fundamental vector fieid on M associated to A, $X = A^*$.

From Th. 2.2 it follows that ϕ_t acts freely on M. Taking $F_p = F | \{p\} \times R$, we have:

PROPOSITION 2.1.

For each $p \in M$, F_p is an embedding from R into M.

PROOF. — If it was not true, for some $p \in M$, there would be $t_0 \in R$, and $\varepsilon > 0$, such that $F_p(t_0 - \varepsilon, t_0 + \varepsilon)$ would no to be open in $F_p(R)$, so for each open neighborhood U of $\phi_{t_0}(p)$, it would be $\phi_{t_n}(p) \in U$, with $t_n \notin (t_0 - \varepsilon, t_0 + \varepsilon)$.

So we can consider a sequence $\phi_{t_n}(p)$, which converges to $\phi_{t_0}(p)$, with $t_n \notin (t_0 - \varepsilon, t_0 + \varepsilon)$ $\forall n$.

It follows from Lemmas 1.1, 1.2, 1.3 that there exists a subsequence $\{\phi_{s_n}\}$, with $\{\phi_{s_n}\} \to \phi \in G$, with $\phi(p) = \phi_{t_0}(p)$, ϕ must be some ϕ_t, and as F_p is injective, $\phi = \phi_{t_0}$, so $\{\phi_{s_n}\} \to \phi_{t_0}$, with $s_n \notin (t_0 - \varepsilon, t_0 + + \varepsilon)$, which is wrong.

As H is an isometries group, H acts properly, (see (3)).

§ 3 Sufficient conditions

In hypothesis (b), Lynge has proved the existence of a metric g, in which X is a Killing vector field; for another proof see (6).

Before proving that (a) is a sufficient condition, we can observe that if ϕ_t is an uniparametric group, such that it acts freely and properly, then F_p is an embedding for each $p \in M$.

Consider $F_p : R \times \{p\} \to M$, which is proper. We have to prove that one can not have a sequence $\{t_n\}$ with $t_n \notin (t_0 - \varepsilon, t_0 + \varepsilon)$, for a given $\varepsilon > 0$, such that $\{\phi_{t_n}(p)\} \to \phi_{t_0}(p)$.

Take the compact $Q = \{\phi_{t_n}(p)\}_{n \in N} \cup \{\phi_{t_0}(p)\}$. $F_p^{-1}(Q) = \{(t_n, p)\}_{n \in N} \cup \cup \{(t_0, p)\}$, is a compact, and $\{(t_n, p)\}$ is a sequence, so there is a subsequence $\{\alpha_n\}$ of $\{t_n\}$, with $\{\alpha_n\} \to t_k = t_0$, and $\{\phi_{\alpha_n}(p)\} \to \phi_{t_k}(p) = \phi_{t_0}(p)$ which is wrong.

Now we are going to prove the results we annonced at the beginning.

Theorem 3.1.

Let X be a complete vector field in the connected manifold M. ϕ_t its uniparametric group; then X is Killing in a Riemannian metric g, if and only if :(a) ϕ_t acts freely and properly, or (b) there exist k-complete vector fields in Lynge-conditions.

Proof. — From Th. 2.1, we know that (a) and (b) are necessary conditions, we have only to prove that (a) is a sufficient condition:

It is known (see (3) and (6)), that in this case we can construct a tubular neighborhood for each orbit. For instance if we take $p \in M$, it is possible to obtain an open cubic coordinate neighborhood centered at p, U, with coordinates $(x^1, \ldots, x^n)$, such that the slices $\{x^2 = k^2, \ldots, x^n = k^n\}$, are the integral curves of X. And if $D = U \cap \cap \{(x^i) | x^1 = 0\}$, $F|R \times D \colon R \times D \to M$, $F(t, (0, x^2, \ldots, x^n)) = \phi_t\,(0, x^2, \ldots, x^n)$ is an embedding.

It results then that $(M, M/H, H)$ is a principal bundle and as $H = R$, this bundle is trivial (see (5)), so $M = (M/H) \times H$.

Taking a Riemannian metric in M/H, g'. And taking dt^2 in H, we have that $g = g' \oplus dt^2$, is a Riemannian metric in M in which X is Killing.

If we consider a Lie group $H \cong R^k$, which acts freely and properly on M, in the same form as above we obtain: $M = (M/H) \times H$, and we can give to M a Riemannian metric in which H acts as a group of isometries.

It is known that when a group of transformations acts properly, one can construct a Riemannian metric, such that the group acts isometrically. But we think interesting to study this when this action is also free, giving the construction of a Riemannian metric, such that the group acts isometrically.

THEOREM 3.2.

Let H be a connected Lie group, which acts freely and properly on M.

There exists a Riemannian metric g, such that H acts isometrically.

PROOF. — As above we have $(M, M/H, H)$, principal bundle, but now it is not necessarilly trivial.

In M/H we take a locally finite cover $\{U_i\}_{i \in I}$, such that $\pi^{-1}(U_i) \cong$ $\cong U_i \times H$, and take a partition of unity subordinate to it $\{\phi_i\}_{i \in I}$.

Consider g_i a Riemannian metric in each U_i, and Ψ a left invariant metric on H. Now consider in $\pi^{-1}(U_i)$ the Riemannian metric $\pi^* g_i \oplus \Psi = \bar{g}_i$.

Then $\Sigma_{i \in I} (\pi^* \phi_i) \bar{g}_i$ is a Riemannian metric in M, such that H acts isometrically on (M, g).

To end this section we observe that we have begun with a Lie transformations group H, and this group is a subset of G (isometries group of the manifold). We shall see that H is a closed Lie subgroup of G, so the initial topology in H is the compact-open one.

PROPOSITION 3.1.

Let H be a connected Lie group, which acts properly and effectively on M.

Then H is a closed Lie subgroup of G.

PROOF. — $L(H)$ (Lie algebra of H), determines $L(H)^*$ Lie algebra of fundamental vector fields associated to $L(H)$. Let $i(M)$ be the Lie algebra of Killing vector fields, $L(H)^* \subset i(M)$, and $i(M) = L(G)$. So $L(H)$ is a subalgebra of $L(G)$.

If we have an abstract subgroup H of G, and a subspace $\mathfrak{H}$ of $L(G)$, and a neighborhood U of O in $L(G)$, diffeomorphic by exponential map to a neighborhood of e in G, V, such that $exp.\ (U \cap \mathfrak{H}) = H \cap V$, then H with the relative topology is a Lie subgroup of G. The initial differentiable structure must be diffeomorphic to this one, because if A is a subset of M, such that in the relative topology has a differentiable structure, (A, i) being a submanifold of M, there exists a unique differentiable structure such that (A, i) is a submanifold.

For these results on submanifolds, see (7).

We need an effective action, to be sure that H is a submanifold of G.

APPLICATION.

We know (see (2), pag. 250), that if M is a connected Riemannian manifold, whose Ricci tensor field is S and it is negative definite everywhere on M, and if the length of a Killing vector field X, attains a relative maximum at some point of M; then X vanishes identically on M.

Then if X is a Killing vector field, with some orbit not relatively compact, we know that ϕ_t acts freely and properly, and if S is negative definite everiwhere, then $M/\{\phi_t\}$ can not be compact.

§ 4. STUDY OF RIEMANNIAN METRICS

In this section we assume that X is a complete vector field, ϕ_t its uniparametric group, which acts freely and properly.

We shall give all the Riemannian metrics in which X can be Killing.

We know that $M = R \times N$, being $N = M/\{\phi_t\}$. All the vector fields in M can be expressed in the form (λ, Y); $\lambda \in T(R)$, $Y \in T(N)$.

If X be Killing in (M, g), $g((\lambda, Y), (\mu, Z))_{(t, n)} = g((\lambda, Y), (\mu, Z))_{(0, n)}$.

Let g' be the restriction of g to $T(N)$; then:

$g((\lambda, Y), (\mu, Z))_{(0, n)} = \lambda \mu g((1, 0), (1, 0))_{(0, n)} + \mu g((1, 0), (0, Y))_{(0, n)} + \lambda g((1, 0), (0, Z))_{(0, n)} + g'(Y, Z)_n$.

$g((1, 0), (0, Z))_{(0, n)}$ is linear in Z, therefore it is $w(Z)$, when w is a 1-form in N.

$g((1, 0), (1, 0))_{(0, n)} = f(n)$, it defines a differentiable function on N.

We have: $g((\lambda, Y), (\mu, Z))_{(t, n)} = (\lambda \mu) f(n) + \lambda w(Z)_n + \mu w(Y)_n + g'(Y, Z)_n$.

For g to be a Riemannian metric we need $f(n) > 0$, $\forall n \in N$, g' a Riemannian metric in N, and $g((\lambda, Y), (\lambda, Y))_{(t, n)} = \lambda^2 f(n) + 2\lambda w(Y) + g'(Y, Y)_n > 0$, for all $(\lambda, Y) \neq (0, 0)$.

So $w(Y)^2|_n < f(n) g'(Y, Y)|_n$, and as both expressions are two-order homogeneous; without it being a restriction, we can consider $S(N)$ to be the unit sphere in $T(N)$ for g', and take $Y \in S(N)$; the result is:

$$w(Y)_n^2 < f(n).$$

So the set of Riemannian metrics in which X be Killing, can be obtained taking g' any Riemannian metric in N. Considering w any 1-form in N, the function $g(n) = Sup. \{w(Y)^2; Y \in S(N_n)\}$, and taking any differentiable function $f: N \to R^+$, with $f(n) > g(n)$, $\forall n \in N$; then $g((\lambda, Y), (\mu, Z))_{(t, n)} = (\lambda \mu) f(n) + \lambda w(Z)_n + \mu w(Y)_n + g'(Y, Z)_n$, is the Riemannian metric determined by g', w, and f in which X is Killing.

Therefore:

THEOREM 4.1.

The set of all Riemannian metrics, such that X be Killing, is completely determined by g' and w.

Observe that in the points where f takes a critical value, the X-orbits are geodesics.

Finally we observe that if $M = M_1 \times M_2$ (Riemannian product), and X is a Killing vector field in (a) condition, $X = X_1 + X_2$, with $X_i \in i(M_i)$, $i = 1, 2$. It is very easy to see that X_1 or X_2 must be in (a) condition, because if it was not so, the orbis of X_1 and X_2, should be relatively compact, therefore the X-orbits should also be relatively compact.

REFERENCES

(1) CURRÁS BOSCH, CARLOS «*Isometrías infinitesimales con órbitas cerradas*». Actas VI Jornadas Matemáticas Hispano-Lusitanas (to appear).

(2) KOBAYASHI, S.-NOMIZU, K. «*Foundations of Differential Geometry* (I)» Interscience (1963).

(3) KOSZUL, J. L. «*Lectures on groups of transformations*» Tata Institute (1965).

(4) LYNGE, W. C. «*Sufficient conditions for periodicity of a Killing vector field*». Proc. Amer. Math. Soc. 38 (3), 614-616. May 1973.

(5) STEENROD, N. «*The topology of fibre bundles*». Princ. Univ. Press 1951.

(6) YAU, S. T. «*Remarks on the group of isometries of a Riemannian manifold*». Topology 16 (3) 1977.

(7) WARNER, F. W. «*Foundations of Differentiable Manifolds and Lie groups*». S. F. 1971.

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