On the Gorenstein property of the diagonals of the Rees algebra

**Olga Lavila-Vidal** and **Santiago Zarzuela**

Departament d’Àlgebra i Geometria, Universitat de Barcelona,
Gran Via 585, E-08007 Barcelona, (Spain)

E-mail: lavila@cerber.mat.ub.es
zarzuela@cerber.mat.ub.es

Dedicated to the memory of Fernando Serrano

**Abstract**

Let \( Y \) be a closed subscheme of \( \mathbb{P}^{n-1}_k \) defined by a homogeneous ideal \( I \subseteq A = k[X_1, \ldots, X_n] \), and \( X \) obtained by blowing up \( \mathbb{P}^{n-1}_k \) along \( Y \). Denote by \( I_c \) the degree \( c \) part of \( I \) and assume that \( I \) is generated by forms of degree \( \leq d \). Then the rings \( k[[I^c_e]] \) are coordinate rings of projective embeddings of \( X \) in \( \mathbb{P}^{N-1}_k \), where \( N = \dim_k(I^c_e) \) for \( c \geq de+1 \). The aim of this paper is to study the Gorenstein property of the rings \( k[[I^c_e]] \). Under mild hypothesis we prove that there exist at most a finite number of diagonals \( (c, e) \) such that \( k[[I^c_e]] \) is Gorenstein, and we determine them for several families of ideals.

**1. Introduction**

Let \( Y \) be a closed subscheme of \( \mathbb{P}^{n-1}_k \) defined by a homogeneous ideal \( I \subseteq A = k[X_1, \ldots, X_n] \), and \( X \) obtained by blowing up \( \mathbb{P}^{n-1}_k \) along \( Y \). Denote by \( I_c \) the degree \( c \) part of \( I \) and assume that \( I \) is generated by forms of degree \( \leq d \). Then the rings \( k[[I^c_e]] \) are coordinate rings of projective embeddings of \( X \) in \( \mathbb{P}^{N-1}_k \), where \( N = \dim_k(I^c_e) \) for \( c \geq de+1 \) (see [3], [2], [9]).

Among the projective varieties obtained in this way we have the Room surfaces, which have been studied in detail by A. Geramita and A. Gimigliano in [5]. These

* Supported by a grant FPI from Ministerio de Educación y Ciencia.
† Partially supported by DGICYT PB94-0850.
surfaces are obtained by blowing-up $\mathbb{P}_k^2$ along \((d+1)\) points, \(d \geq 2\), which do not lie on any curve of degree \(d-1\), and then embedding in $\mathbb{P}_k^{2d+2}$. See also [6] and [7] for other results about embedded rational surfaces obtained by blowing up a set of points in $\mathbb{P}^2$.

Recently, the study of the Cohen-Macaulay property of the rings $k[(I^c)_e]$ has received much attention. Considering the Rees algebra $R_A(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$ endowed with a natural bigrading, one can obtain the above rings as diagonals of $R_A(I)$. A useful strategy consists in assuming the Cohen-Macaulay property of $R_A(I)$ and then to look for which diagonals inherit this property, see for instance A. Simis, N.V. Trung and G. Valla [19], A. Conca, J. Herzog, N.V. Trung and G. Valla [2] and O. Lavila–Vidal [17]. In particular it is known that if $R_A(I)$ is Cohen-Macaulay there are infinitely many pairs $(c,e)$ such that $k[(I^c)_e]$ is Cohen-Macaulay ([17], Theorem 4.5).

Here we are interested in the (quasi) Gorenstein property of the rings $k[(I^c)_e]$. Recall that the $a$-invariant of a positively graded ring $T$ over a local ring $T_0$ is defined as $a(T) = \max \{i | [H^d_M(T)]_i \neq 0\}$, where $M$ is the maximal homogeneous ideal of $T$ and $d = \dim T$. Assuming that $T$ has a canonical module $K_T$, $T$ is said to be quasi-Gorenstein if there exists a graded isomorphism $K_T \cong T(a)$ with $a = a(T)$, and Gorenstein if in addition $T$ is Cohen-Macaulay.

Under appropriate hypothesis we are able to determine for which pairs $(c,e)$ the ring $k[(I^c)_e]$ is quasi-Gorenstein. In order to state the result assume that $I$ is minimally generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ respectively, and put $d = d_r \geq \ldots \geq d_1$. Suppose $n \geq r \geq 2$ and $c \geq de + 1$. Let $G_A(I) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ be the form ring of $I$. Then we prove the following:

**Theorem** (Theorem 2.8) 

Assume $ht(I) \geq 2$, $\dim(A/I) > 0$, and $G_A(I)$ is Gorenstein. Set $a = -a(G_A(I))$. Then $k[(I^c)_e]$ is quasi-Gorenstein if and only if $\frac{c}{e} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^c)_e]) = -l_0$.

This result can be applied to several families of ideals. In particular, to any complete intersection ideal (extending in this way a result by A. Conca et al. in [2] for the case $r = 2$) and to the ideal generated by the maximal minors of a generic matrix. Note also that under the assumptions of the above theorem there are at most a finite number of rings $k[(I^c)_e]$ which are quasi-Gorenstein. We show that this holds in general:

**Proposition** (Proposition 3.1) 

There exist at most a finite number of diagonals $(c,e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein.
For a real number $x$, let us denote by $\lceil x \rceil = \min \{ m \in \mathbb{Z} \mid m \geq x \}$. Assuming that the Rees algebra is Cohen-Macaulay we can give upper bounds for the diagonals $(c,e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein:

**Proposition** (Proposition 3.2)

Assume that $ht(I) \geq 2$ and $R_A(I)$ is Cohen-Macaulay. Let $a = -a(G_A(I))$. If $k[(I^c)_e]$ is quasi-Gorenstein, then $e \leq a - 1$ and $c \leq n$. If $\dim(A/I) > 0$ then $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{e} = l \in \mathbb{Z}$. In particular, if $a = 1$ there are no diagonals $(c,e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein.

We also prove a converse of Theorem 2.8 by showing that, under some restrictions, the existence of a diagonal $(c,e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein implies that $G_A(I)$ is Gorenstein. Denoting by $l(I)$ the analytic spread of an ideal $I$, we have:

**Theorem** (Theorem 3.3)

Assume that $R_A(I)$ is Cohen-Macaulay, $ht(I) \geq 2$, $l(I) < n$ and $I$ is equigenerated. If there exists a diagonal $(c,e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.

Finally, by using a variation of Proposition 3.2, we study the case of the Room surfaces. We show that the only Room surface which is Gorenstein is the del Pezzo sestic surface in $\mathbb{P}^6$, so recovering that well known result (see [5], Example 1).

Throughout the paper we shall use the following notation: $A = k[X_1,...,X_n]$ will denote the usual polynomial ring with coefficients in a field $k$, and $I \subset A$ a homogeneous ideal minimally generated by forms $f_1,...,f_r$. We put $d = d_1 \geq ... \geq d_1$, $u = \sum_{j=1}^r d_j$. If $d_1 = d_2 = \ldots = d_r$, we say that $I$ is equigenerated. Let us consider the Rees algebra of $I$: $R_A(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$ endowed with the $\mathbb{N}^2$-grading given by $R_A(I)_{(i,j)} = (I^j)_i t^i$. Let $S = k[X_1,...,X_n,Y_1,...,Y_r]$ be the polynomial ring with the $\mathbb{N}^2$-grading obtained by giving deg $X_i = (1,0)$ for $i = 1,...,n$, deg $Y_j = (d_j,1)$ for $j = 1,...,r$. Then $R_A(I)$ can be seen in a natural way as a bigraded $S$-module.

For any pair of positive integers $\Delta = (c,e)$ and any bigraded $S$-module $L = \bigoplus_{(i,j)} L_{(i,j)}$ we may consider $L_\Delta := \bigoplus_{s \in \mathbb{Z}} L_{(c,s,e)}$ which is a graded module over the graded ring $S_\Delta := \bigoplus_{s \geq 0} S_{(c,s,e)}$. We call these modules the *diagonals of $L$ and $S$ along $\Delta$*. We shall always assume that $e > 0, c \geq de + 1$. It is then known ([2], Section 1) that $S_\Delta$ is Cohen-Macaulay with dim $S_\Delta = n + r - 1$, $R_A(I)_\Delta \cong k[(I^c)_e]$ and dim $k[(I^c)_e] = n$. 


Let $T$ be a positively bigraded $d$-dimensional ring defined over a local ring, and denote by $\mathcal{M}$ the maximal homogeneous ideal of $T$. The bigraded $a$-invariant of $T$ is then defined by $a(T) = (a_1, a_2)$, where $a_j = \max \{n_j \mid n = (n_1, n_2) \in \mathbb{Z}^2, [H^d_{\mathcal{M}}(T)]_n \neq 0\}$.

2. The case of ideals whose form ring is Gorenstein

Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring introduced before and $\Delta = (c, e)$. Applying the diagonal functor, $S^\Delta$ is always a Cohen-Macaulay ring. We begin this section by showing that, on the contrary, $S^\Delta$ is Gorenstein only for a finite number of diagonals. Furthermore, we may determine them.

**Proposition 2.1**

$S^\Delta$ is Gorenstein if and only if $r = n + u = l \in \mathbb{Z}$. Then $a(S^\Delta) = -l$.

**Proof.** Let $T = S^\Delta = \bigoplus_{s \geq 0} U_s$, where $U_s$ is the $k$-vector space generated by the monomials $X_1^{\alpha_1} \ldots X_n^{\alpha_n} Y_1^{\beta_1} \ldots Y_r^{\beta_r}$ with $\alpha_i, \beta_j \geq 0$ satisfying the equations (⋆)

$$\sum_{i=1}^n \alpha_i + \sum_{j=1}^r d_j \beta_j = cs$$

$$\sum_{j=1}^r \beta_j = es.$$

By [2], Lemma 3.1 and local duality, $K_T = \bigoplus_{s \geq 1} V_s$ with $V_s$ the $k$-vector space generated by the monomials $X_1^{\alpha_1} \ldots X_n^{\alpha_n} Y_1^{\beta_1} \ldots Y_r^{\beta_r}$, and $\alpha_i > 0, \beta_j > 0$ which satisfy (⋆). Since $T$ is Cohen-Macaulay, $T$ is Gorenstein if and only if $K_T \cong T(a(T))$. Assume first that $r = n + u = l \in \mathbb{Z}$. Then, multiplication by $X_1 \ldots X_n Y_1 \ldots Y_r \in T_l$ induces an isomorphism $T \cong K_T(l)$ and so $T$ is Gorenstein with $a(T) = -l$.

To prove the converse set $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_r)$ with $\alpha_i, \beta_j > 0$ and assume the contrary. This means that $(1, 1)$ is not a solution of (⋆) for any $s$. On the other hand, the set of solutions of (⋆) for some $s$ is partially ordered by means of $(\alpha, \beta) \leq (\gamma, \rho) \iff \alpha_i \leq \gamma_i, \beta_j \leq \rho_j, \forall i, j$. Then one can easily check that for any $i, j$ there exists a solution of (⋆) for some $s$ such that $\alpha_i = \beta_j = 1$. This implies the existence of at least two minimal solutions, and so $T$ is not Gorenstein. □

**Remark 2.2.** Note that the number of minimal elements in the set of solutions of the system (⋆) coincides with the type of $S^\Delta$. It is not difficult to see that if $S^\Delta$ is not Gorenstein, then its type is $\geq r$. 
This result leads to the question of when there exist diagonals \((c,e)\) such that \(k[(I^c)_e]\) be quasi-Gorenstein, and how one can determine them.

Our answer will be partially based on the following proposition which links the diagonal of the canonical module of \(R_A(I)\) to the canonical module of the diagonal of \(R_A(I)\). It is stated and proved for complete intersection ideals in [2], Proposition 4.5 but in fact the same statement and proof are valid in general. We include the proof for completeness.

**Proposition 2.3**

\[KR_A(I)_\Delta = (KR_A(I))_\Delta.\]

**Proof.** Let us denote by \(T = S_\Delta\) and \(R = R_A(I)\). Consider a presentation of \(R\) as \(S\)-module

\[0 \to C \to S \to R \to 0\]

which leads to the bigraded exact sequence of local cohomology modules

\[0 \to H^{n+1}_m(S) \to H^{n+2}_m(C) \to H^{n+2}_m(S) \to 0,\]

where \(m_S\) is the maximal homogeneous ideal of \(S\).

Similarly, we get the graded exact sequence

\[0 \to H^n_{m_T}(R_\Delta) \to H^{n+1}_{m_T}(C_\Delta) \to H^{n+1}_{m_T}(T) \to 0,\]

where \(m_T\) is the maximal homogeneous ideal of \(T\).

On the other hand, by [2], Theorem 3.6 we have a commutative diagram

\[\begin{array}{cccc}
0 & \to & H^{n+1}_m(R)_\Delta & \to & H^{n+2}_m(C)_\Delta & \to & H^{n+2}_m(S)_\Delta & \to & 0 \\
\varphi^n_R \uparrow & & \varphi^{n+1}_C \uparrow & & \varphi^{n+1}_S \uparrow \\\n0 & \to & H^n_{m_T}(R_\Delta) & \to & H^{n+1}_{m_T}(C_\Delta) & \to & H^{n+1}_{m_T}(T) & \to & 0 \\
\end{array}\]

where \(\varphi^{n+1}_C, \varphi^{n+1}_S\) are isomorphisms, and so \(\varphi^n_R\) also is an isomorphism. Therefore \(H^n_{m_T}(R_\Delta) \cong H^{n+1}_{m_T}(R_\Delta)\) and we get

\[K_{R_\Delta} = \text{Hom}_k(H^n_{m_T}(R_\Delta), k) = \text{Hom}_k(H^{n+1}_{m_T}(R_\Delta), k)\]

\[= \text{Hom}_k(H^{n+1}_{m_T}(R), k)_\Delta = (K_R)_\Delta. \quad \Box\]

**Remark 2.4.** The hypothesis \(n \geq r \geq 2\) fixed in the introduction is only used in this paper to prove Proposition 2.3, and of course its applications. Nevertheless, the
isomorphism $K_{R_A(I)}(\Delta) = (K_{R_A(I)})\Delta$ is also valid if $n, r \geq 2$, $I$ is equigenerated and $R_A(I)$ is Cohen-Macaulay. To prove this, set $R = R_A(I)$ and assume $r > n$ (if $n \geq r$ we may apply Proposition 2.3). Let

$$0 \to D_{r-1} \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$$

be the $\mathbb{Z}^2$-graded minimal free resolution of $R$ over $S$. For every $p$, $D_p$ is a direct sum of $S$-modules of the type $S(a, b)$. Denote by $\overline{b}$ the maximum of the $-b$’s which appear in the resolution. Since $R$ is Cohen-Macaulay, we get from [17], Lemmas 3.6 and 3.7 that $\overline{b} = -1 + r$. On the other hand, from [2], Lemmas 3.1 and 3.3 (note that hypothesis $n \geq r$ is not used there) we have that $H_{mS}^r(S(a, b)\Delta)_s \neq 0$ if and only if $(b+r)d - u - a \leq s \leq \frac{b-r}{e}$, hence $s < 0$. Also by [17], Proposition 4.1 $-a \geq -bd$ and so $(b+r)d - u - a = bd - a \geq 0$. So we get $H_{mT}^r((D_p)_\Delta) = 0$ for all $p$, and by [2], Lemma 3.1 that $H_{mT}^i((D_p)_\Delta) = 0$ for all $n < i < n + r - 1$ and that $\varphi_{D_p}^{n+r-1}$ is an isomorphism for all $p$. By [2], Lemma 1.7 we then have $\varphi_R^i, \varphi_B^i$ are isomorphisms for all $i > n$, and the same proof as in Proposition 2.3 shows that $K_{R_\Delta} = (K_R)\Delta$.

This means that all the results we are going to prove are also valid if $n, r \geq 2$, $I$ is equigenerated and $R_A(I)$ is Cohen-Macaulay.

In view of Proposition 2.3 any information on the bigraded structure of $K_{R_A(I)}$ will be of interest. Let $B$ be a $d$-dimensional local ring, $d \geq 1$, which has a canonical module $K_B$ and $I \subset B$ an ideal of positive height such that $R_B(I)$ is Cohen-Macaulay. In [21], Theorem 2.2 it is given a description of $K_{R_B(I)}$ in terms of a filtration of submodules of $K_B$. Assume now that $B = \bigoplus_{n \geq 0} B_n$ is a positively graded ring of positive dimension over a local ring $B_0$, which has a canonical module $K_B$. Let $I \subset B$ be a homogeneous ideal of positive height. Then, the Rees algebra $R_B(I)$ has a bigraded structure by means of $[R_B(I)]_{(i, j)} = (I^j)_i t^j$ for all $i, j \geq 0$. We also have a bigraded structure on the form ring by means of $[G_B(I)]_{(i, j)} = (I^j)_i/(I^{j+1})_i$ for all $i, j \geq 0$.

Then, the proof of [21], Theorem 2.2 may be “bigraded” and we thus obtain a description of the bigraded structure of $K_{R_B(I)}$. Namely, we get:

**Theorem 2.5**

With the notation above assume that $R_B(I)$ is Cohen-Macaulay. Then there exists a homogeneous filtration $\{K_m\}_{m \geq 0}$ of $K_B$ and isomorphisms of bigraded modules such that

$$K_{R_B(I)} \cong \bigoplus_{(l, m), m \geq 1} [K_m]_l,$$

$$K_{G_B(I)} \cong \bigoplus_{(l, m), m \geq 1} [K_{m-1}]_l/[K_m]_l.$$
Several other results of [2] may also be “bigraded”. In particular [21], Lemma 4.1 which makes precise when the canonical module of the Rees algebra has the expected form. Recall that $K_{R_B(I)}$ has the expected form if

$$K_{R_B(I)} \cong Bt \oplus Bt^2 \oplus \ldots \oplus Bt^l \oplus It^{l+1} \oplus I^2t^{l+2} \oplus \ldots,$$

for some $l \geq 0$. This definition was introduced by J. Herzog, A. Simis and W. Vasconcelos in [14]. We still use the same notation and again omit the proof.

**Corollary 2.6**

Assume $R_B(I)$ is Cohen-Macaulay and $G_B(I)$ is quasi-Gorenstein. Let $a(G_B(I)) = (-b, -a)$ be the bigraded $a$-invariant of $G_B(I)$. Then $K_B \cong B(-b)$ and

$$K_{R_B(I)} = \bigoplus_{(l,m), m \geq 1} [I^{m-a+1}]_l^{-b},$$

where $I^n = B$ if $n \leq 0$.

Note that $-a$ coincides with the usual $a$-invariant of $G_B(I)$. By Ikeda-Trung’s criterion [16] it is always negative if $R_B(I)$ is Cohen-Macaulay, and it has been calculated in many cases (see for instance [13], [10]). As for $b$, it is clear that under the hypothesis of Corollary 2.6 we get $-b = a(B)$. It is then also easy to compute the bigraded $a$-invariant of $R_B(I)$. Namely, we get that if $a = 1$ then $a(R_B(I)) = (-d_1 + a(B), -1)$, and if $a > 1$ then $a(R_B(I)) = (a(B), -1)$.

**Remark 2.7.** Assume that $B = A = k[X_1, \ldots, X_n]$ and $I$ is a complete intersection ideal. Then, the Eagon-Northcott complex provides a $\mathbb{Z}^2$-graded minimal free resolution of $R_A(I)$. Following the proof of Yoshino [24] it is possible to see that

$$K_{R_A(I)} = J((r-2)d_1 - n, -1)$$

with $J = (f_1^{r-2}, f_1^{r-2} t_1, \ldots, f_1^{r-2} t_r, R_A(I))$.

Observe that in this case $a(G_A(I)) = (-n, -r)$ and by Corollary 2.6

$$K_{R_A(I)} = \bigoplus_{(l,m), m \geq 1} [I^{m-r+1}]_l^{-n}.$$

A straightforward computation shows that, in fact, multiplication by $f_1^{r-2}$ provides an explicit isomorphism

$$\bigoplus_{(l,m), m \geq 1} [I^{m-r+1}]_l^{-n} \cong J((r-2)d_1 - n, -1).$$
Let us now assume that $I \subset A = k[X_1, \ldots, X_n]$ is a homogeneous ideal whose form ring is Gorenstein. We are now ready to prove the main result of this section determining the possible quasi-Gorenstein diagonals of $R_A(I)$. We use the same notation as before, and note that in this case $b = -a(A) = n$. Then we get:

**Theorem 2.8**

Assume $ht(I) \geq 2$, $\dim(A/I) > 0$, and $G_A(I)$ is Gorenstein. Then $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a_k((I^e)_c) = -l_0$.

**Proof.** Let $R = R_A(I)$. Recall that $R_\alpha = k[(I^e)_c] = \bigoplus_{l \geq 0}(I^e)_l$. Note that $R$ is Cohen-Macaulay by using a result of Lipman [18, Theorem 5]. By now applying Corollary 2.6, $K_R = \bigoplus_{(l,m),m \geq 1}[I^{m-a+1}]_{l-n}$, so that by Proposition 2.3 we get $K_R = (K_R)_\alpha = \bigoplus_{l \geq 1}[I^{cl+1}]_{cl-n}$. Let $l_0 = \min \{ l \in \mathbb{Z} \mid l \geq \frac{n}{c} \}$, $s = a - 1 - e l_0$.

We shall now distinguish three cases.

If $s = 0$, then the first non-zero component of $K_R = [I^{cl_0+1}]_{cl_0-n} = A_{cl_0-n}$, so that if $R_\Delta$ is quasi-Gorenstein $cl_0 - n = 0$ and we get that $l_0 = \frac{n}{c} = \frac{a-1}{e}$ and $a(R_\Delta) = -l_0$. Conversely, if $l_0 = \frac{n}{c} = \frac{a-1}{e}$ then $[K_R]_{l_0+m} = [I^{cl_0+a+1}]_{cl_0+cm-n} = [I^{em}]_{cm} = [R_\Delta]_{m}$ for all $m$ and so $R_\Delta$ is quasi-Gorenstein.

If $s < 0$, let $l_1 = \min \{ l \mid cl - a + 1 > 0, cl - n \geq d_1(cl - a + 1) \}$. Then $l_1 \geq l_0$ and the first non-zero component of $K_R = [I^{cl_0+1}]_{cl_0-n}$. In particular, $a(R_\Delta) = -l_1$. Assume $R_\Delta$ is quasi-Gorenstein. Then $K_R \cong R_\Delta(-l_1)$ and so $[K_R]_{l_1} \cong k$. This implies that $cl_1 - n = d_1(cl_1 - a + 1)$: If $cl_1 - n - d_1(cl_1 - a + 1) > 0$ we may choose two linearly independent forms $g,h \in A_r$ such that $g_1^{el_1-a+1}, h_1^{el_1-a+1} \in [I^{el_1-a+1}]_{cl_1-n} \cong k$, which is a contradiction. From the isomorphism one gets that $K_R = k$ is generated by $f_1^{el_1-a+1}$ as $R_\Delta$-module. Now let $f_j \notin rad(f_1)$ (it exists because $ht(I) \geq 2$), and choose $m$ such that $m(c - d_j e) > d_j - d_1$ and there exists $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, f_1) = 1$. Then $f_1^{el_1-a}f_j^{em+1}f \in [I^{el_1-a+1+em}]_{d_1(em+1)+cm} = f_1^{el_1-a+1}[I^{em}]_{cm}$, and we get $f_j^{em+1}f \in (f_1)$ which is a contradiction.

If $s > 0$, the first non-zero component of $K_R = [I^{cl_0+1}]_{cl_0-n} = A_{cl_0-n}$, so if $R_\Delta$ is quasi-Gorenstein we get $cl_0 - n = 0$. Furthermore, for all $m \geq 1$ we have $[K_R]_{l_0+m} = [I^{s+em}]_{cl_0-n+cm} = [I^{s+em}]_{cm} \cong [I^{em}]_{cm}$. Since $s > 0$ and $[I^{em}]_{cm} \subset [I^{s+em}]_{cm}$ this isomorphism is possible if and only if $[I^{em}]_{cm} = [I^{s+em}]_{cm}$. Now choose $X_i$ such that $X_i \notin rad(I)$ (it always exists because $ht(A/I) > 0$) and $m$ with $cm - s \geq 1$. For any $j$ consider $F_j = X_i^{\alpha_j} f_j^{em-s}$ where $\alpha_j = cm - d_j(em-s) = (c-d_j e)m + d_j s \geq 1$, and assume $[I^{em}]_{cm} = [I^{s+em}]_{cm}$. Then $F_j \in [I^{em}]_{cm}$ and so $X_i^{\alpha_j} f_j^{em-s} \in I^{em}$. Now let $f_1^{em} \ldots f_r^{em}$ such that
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$\alpha_1 + \ldots + \alpha_r \geq r(em-s)$. This implies that there exists $l$ with $c_l \geq em-s$ and so $X_i^{\alpha_1}f_i^{\alpha_1} \ldots f_i^{\alpha_r} = X_i^{\alpha_1}f_i^{em-s}f_i^{\alpha_1} \ldots f_i^{\alpha_i-em+s} \ldots f_i^{\alpha_r} \in I^{c_1+\ldots+\alpha_r+s}$, since $\alpha_1 \geq \alpha_i$ for all $i$. Thus we get $X_i^\alpha I^h \subset I^{h+s}$ for $h \gg 0$, which implies that $X_i^\alpha \in I^s \subset I$ since $R_A(I)$ is Cohen-Macaulay. But this contradicts $X_i \notin \text{rad}(I)$ and so $R_\Delta$ cannot be quasi-Gorenstein. □

The remaining cases $ht(I) = 1, n$ in the above theorem are studied separately in the following remarks.

Remark 2.9. If $ht(I) = 1$ then $k[(I^e)_c]$ is never quasi-Gorenstein. In fact, by [21] Proposition 4.6, $a(G_A(I)) = -1$ and so $a = 1$. Following the same proof as in Theorem 2.8 we have that $s = -el_0 < 0$. On the other hand, since $ht(I) = 1$ we may write $I = gJ$, with $ht(J) \geq 2$, $J = (\overline{f_1}, \ldots, \overline{f_r})$ and $f_j = \overline{f}_j g$ for all $j$. The same argument as in Theorem 2.8 for the case $s < 0$ but taking $\overline{f}_j \notin \text{rad}(\overline{f}_1)$ and $f \in A_{d_1 + em - d_2 (em + 1)}$ such that $(f, \overline{f}_1) = 1$ leads to $\overline{f}_j^{em+1} \in (\overline{f}_1)$, which is a contradiction.

Remark 2.10. When $\dim(A/I) = 0$, the condition $\frac{n}{c} = \frac{a-1}{c} = l_0 \in \mathbb{Z}$ is sufficient but not necessary for $k[(I^e)_c]$ to be quasi-Gorenstein. For instance, let $A = k[X_1, X_2, X_3]$ and $I = (X_1, X_2, X_3)$. Set $R = R_A(I)$. Note that $a = -3$ and by Corollary 2.6 $K_R = \prod_{i,m \geq 1}[l^{m-2}]_{i-3}$. By taking the $(3,1)$-diagonal, $K_{R_\Delta} = \prod_{i \geq 1}[l-2]_{i-1} = \prod_{i \geq 1} A_{3(i-1)} = (\prod_{i \geq 0} A_{3i})(-1) = R_{\Delta}(-1)$ and so $R_\Delta = k[I_3]$ is quasi-Gorenstein. In this case, $n = a = 3$, $c = 1$ and $\frac{n}{c} = 1 \neq 2 = \frac{a-1}{c}$.

As a consequence of Theorem 2.8 we obtain the following result for the case of complete intersection ideals. It generalizes [2], Corollary 4.7 where the case of ideals generated by two elements was considered.

Corollary 2.11

Let $I \subset k[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by $r$ forms of degrees $d_1 \leq \ldots \leq d_r = d$, with $r < n$. Then for $c \geq de+1$, $k[(I^e)_c]$ is Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{c} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.

Proof. Since $a(G_A(I)) = -r$ we get by Theorem 2.8 that $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{c} = l_0 \in \mathbb{Z}$. But then $\sum_{j=1}^r d_j + (e-1)d - n \leq rd + ed - d - n = (r-1)d + de - n = c\frac{n}{c}d + de - n = n\left(\frac{ed-c}{c}\right) + de \leq de < c$, and by [2], Theorem 4.3, $k[(I^e)_c]$ is also Cohen-Macaulay and so Gorenstein. □
We may also study the ideals generated by the maximal minors of a generic matrix. We thank A. Conca for suggesting we consider this case.

**Example 2.12:** Let \( X = (X_{ij}), 1 \leq i \leq n, 1 \leq j \leq m \) be a generic matrix, with \( m \leq n \). Let us consider \( I \subset A = k[X_{ij}, 1 \leq i \leq n, 1 \leq j \leq m] \) the ideal generated by the maximal minors of \( X \), where \( k \) is a field of arbitrary characteristic. It is well-known (see [4]) that the Rees algebra \( R_A(I) \) is Cohen-Macaulay and the form ring \( G_A(I) \) is Gorenstein. Moreover, it has been proved by A. Conca (personal communication) that all the diagonals of \( R_A(I) \) are Cohen-Macaulay. Now we want to study the Gorenstein property of these rings. Note that \( I \) is an equigenerated ideal whose Rees algebra is Cohen-Macaulay, so one can apply Theorem 2.8. From the fact that \( I \) is generically a complete intersection, one can easily see that \( a(G_A(I)) = -ht(I) = -(n - m + 1) \). We shall distinguish two cases.

If \( m < n \), then \( k[(I^c)_c] \) is Gorenstein if and only if \( \frac{nm}{c} = \frac{n-m}{e} \in \mathbb{Z} \). So there is always at least one diagonal which is Gorenstein by taking \( c = nm, e = n - m \).

If \( m = n \), note that \( I \) is a principal ideal and so the Rees algebra is isomorphic to a polynomial ring. Then it is easy to prove that the only diagonal which is Gorenstein occurs when \( c = n(n + 1), e = 1 \).

### 3. Restrictions to the existence of Gorenstein diagonals. Applications

In the previous section we proved that under the assumptions of Theorem 2.8 there exist at most a finite number of diagonals \((c, e)\) such that \( k[(I^c)_c] \) is quasi-Gorenstein. Our next result shows that this holds in general.

**Proposition 3.1**

There exist at most a finite number of diagonals \((c, e)\) such that \( k[(I^c)_c] \) is quasi-Gorenstein.

**Proof.** Let \( R = R_A(I) \) and \( w_1, ..., w_m \in K_R \) a system of generators of \( K_R \) as \( R \)-module with \( \deg w_i = (\alpha_i, \beta_i) \) for all \( i \), and so \( K_R = \sum_{i=1}^{m} Rw_i \). Note that since \( R \) is a domain \( K_R \) is torsion free. For \( \Delta = (c, e) \) we then have by Proposition 2.3 that for all \( l \geq 1 \)

\[
[K_{R_{\Delta}}]_l = \sum_{i=1, ..., m, \ e-l-\beta_i \geq 0} [I^{e-l-\beta_i}]_{c-\alpha_i} w_i.
\]

If \( R_{\Delta} \) is quasi-Gorenstein there exists an integer \( l \) such that \( [K_{R_{\Delta}}]_l = k \) and so \( [I^{e-l-\beta_i}]_{c-\alpha_i} \neq 0 \) for some \( i \) (\( \star \)). We shall distinguish two cases.
Assume first that $I$ is an equigenerated ideal of degree $d$. Then condition (⋆) implies that $e I - \beta_i = 0$ and $e I - \alpha_i \geq 0$ or $e I - \beta_i > 0$ and $e I - \alpha_i \geq d(e I - \beta_i)$. If $e I - \beta_i = 0$, then $k = [K R_{\Delta}]_{l} \supset A_{e I - \alpha_i, w_i}$ and since $K_R$ is torsion-free we get $e I - \alpha_i = 0$. Hence $(c, e)$ satisfies $\frac{\beta_i}{c} = \frac{\alpha_i}{e} = l \in \mathbb{Z}$ and the statement holds. If $e I - \beta_i > 0$ then $k = [K R_{\Delta}]_{l} \supset [I^{e I - \beta_i}]_{e I - \alpha_i, w_i}$ which is impossible since $K_R$ is torsion free and $e I - \alpha_i \geq d(e I - \beta_i)$.

Assume now that $I$ is not equigenerated. Condition (⋆) implies that $e I - \beta_i = 0$ and $e I - \alpha_i \geq 0$ or $e I - \beta_i > 0$ and $e I - \alpha_i \geq d_1(e I - \beta_i)$. In the first case we may proceed as before to get the statement. In the second case we have that $k = [K R_{\Delta}]_{l} \supset [I^{e I - \beta_i}]_{e I - \alpha_i, w_i}$ and so $e I - \alpha_i = d_1(e I - \beta_i)$ and $d_1 < d_2$. Then $\alpha_i - d_1 \beta_i = e I - d_1 e \geq c - d_1 e \geq (d - d_1) e$ since $l \geq 1$ and $c \geq d e + 1 > d e$. Thus we obtain the inequality $e \leq \frac{\alpha_i - d_1 \beta_i}{d - d_1}$ and for each $e$, we have $e \leq d_1 e + \alpha_i - d_1 \beta_i$. In any case, these inequalities hold for at most a finite number of diagonals and so we get the result. □

If the Rees algebra $R_A(I)$ is Cohen-Macaulay we can also give bounds for the diagonals $(c, e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein.

**Proposition 3.2**

Assume that $h(I) \geq 2$ and $R_A(I)$ is Cohen-Macaulay. Let $a = -a(G_A(I))$. If $k[(I^c)_e]$ is quasi-Gorenstein, then $e \leq a - 1$ and $c \leq n$. Moreover, if $\dim(A/I) > 0$ then $\lceil \frac{a}{c} \rceil - 1 = \frac{a}{c} = l \in \mathbb{Z}$. In particular, if $a = 1$ there are no diagonals $(c, e)$ such that $k[(I^c)_e]$ is quasi-Gorenstein.

**Proof.** Set $R = R_A(I)$ and $G = G_A(I)$. By Theorem 2.5, there exists a homogeneous filtration $\{K_m\}_{m \geq 0}$ of $K_A$ such that $K_R \cong \bigoplus_{m \geq 1} K_m$ and $K_G \cong \bigoplus_{m \geq 1} K_{m-1}/K_m$. Bigrating the proof of [21], Corollary 2.5, we have that $K_m = \text{Hom}_A(I, K_{m+1})$ for every $m \geq 0$. Note that $K_A$ may be viewed as an ideal of $A$. Assume that $R_{\Delta}$ is quasi-Gorenstein. Then there is an integer $l_0$ such that $[K R_{\Delta}]_{l_0} = k$. By Proposition 2.3 we may find an element $f \in [K_{cl_0}]_{c l_0} = [K R_{(cl_0, cl_0)}], f \neq 0$, $K R_{\Delta} = R_{\Delta} f$.

**Claim.** $K_{cl_0} = A f$.

To prove the claim we first show that for any $g \in K_{cl_0}$, $g \neq 0$, then $\deg g \geq cl_0$. Assume the contrary: $\deg g = k < cl_0$. Then $[Ag]_{cl_0} = A_{cl_0} - k g \subset [K_{cl_0}]_{cl_0} \cong k$. But since $cl_0 - k > 0$, $\dim_k A_{cl_0} - k > 1$, so we get a contradiction.

Now let $g \in K_{cl_0}$. If $\deg g = cl_0$, then $g \in A f$ because $[K_{cl_0}]_{cl_0} \cong k$. Let us assume that $\deg g = k > cl_0$. Then, for each $l > 0$, $[I^{cl_0}]_{cl_0} f + [I^{cl_0}]_{(cl_0+l) - k} g \subset$
[K_{c(l_0+l)}]_{c(l_0+l)} \cong [I^d]_d as k-vector spaces, and so \([I^d]_{c(l_0+l)} - k g \subset [I^d]_d f\). Now let 
\(I^d = (F_1, \ldots, F_l)\) where \(F_i\) is a homogeneous polynomial of degree \(d\) for all \(i\), and set \(\alpha = c(l_0 + l) - k - \deg F_i\). Note that for \(l > 0\), \(\alpha \geq c(l_0 + l) - d - 1\) \((c - d)l + c l_0 - k > 0\) and we can find \(h \in A_\alpha\) such that \((h, f) = 1\). Then 
\(h g F_i \in [I^d]_{c(l_0+l)} - k g \subset [I^d]_d f \subset Af\) and we have that \(g F_i \in Af\) for all \(i\). Thus \(I^d g \subset (f)\) and writing \(g = d_\tilde{f}, f = d_\delta\) with \((\tilde{f}, \delta) = 1\) we get \(I^d \delta \subset \tilde{f}^d\). If \(g \notin Af\), then \(\tilde{f} \notin k\) and so \(I^d \subset (\delta)\) which is absurd because \(ht(I) \geq 2\).

Now, as grade(I) \(\geq 2\) we have \(K_m = K_{el_0}\) for all \(m \leq el_0\), which implies that 
\(K_A = K_{el_0}\) and so \(e \leq el_0 = n\). Furthermore, \(e \leq el_0 \leq \min \{m \mid K_m \nsubseteq K_{m-1}\} - 1 = a - 1\).

Finally assume that \(\dim(A/I) > 0\). We shall distinguish two cases. If \(e = 1\) we have that \(K_{el_0} \nsubseteq K_{el}\). If not, then \(l_e \cong [K_{el_0+1}]_{c(l_0+1)} = [A_f]_{c(l_0+1)} \cong A_e\) which is absurd if \(\dim(A/I) > 0\). Therefore \(a = l_0 + 1 = \frac{n}{e} + 1\). If \(e > 1\), let \(\Delta = (c, 1)\) and \(R = R(I^e)\). Note that \(R_{\Delta} = R_{\Delta}\) which is quasi-Gorenstein. Applying the case before we obtain that \(a(G_A(I^e)) = \frac{n}{e} + 1\). By [15], \(a(G_A(I^e)) = \left[\frac{n}{e}\right] = -\left[\frac{n}{e}\right] + 1\) and so \(\left[\frac{n}{e}\right] - 1 = \frac{n}{e} - 1 \in \mathbb{Z}\).

Let us denote by \(m\) the maximal homogeneous ideal of \(A\). Given a homogeneous ideal \(I \subseteq A\) we define the fiber cone of \(I\) as \(F_m(I) = \bigoplus_{n \geq 0} I^n/mI^n\). Then \(l(I) = \dim F_m(I)\) is called the analytic spread of \(I\). Note that if \(I\) is equigenerated in degree \(d\) the fiber cone of \(I\) is nothing but \(k[I_d]\).

Our next result shows that in some cases the existence of a diagonal \((c, e)\) such that \(k[I^e]_c\) is quasi-Gorenstein forces the form ring to be Gorenstein. It may be seen as a converse of Theorem 2.8 for those cases.

**Theorem 3.3**

Assume that \(R_A(I)\) is Cohen-Macaulay, \(ht(I) \geq 2\), \(l(I) < n\) and \(I\) is equigenerated. If there exists a diagonal \((c, e)\) such that \(k[I^e]_c\) is quasi-Gorenstein then \(G_A(I)\) is Gorenstein.

**Proof.** Let \(R = R_A(I), G = G_A(I)\) and \(\Delta = (c, e)\). Assume first that \(e = 1\). We have seen in the proof of Proposition 3.2 that there is a homogeneous filtration \(\{K_m\}_{m \geq 0}\) of \(K_A\) such that \(K_R \cong \bigoplus_{m \geq 1} K_m\) and \(K_G \cong \bigoplus_{m \geq 1} K_{m-1}/K_m\), and an integer \(l_0 = -a(R_\Delta)\) such that \(K_0 = \ldots = K_{l_0} = Af\), with \(f \in K_R\) and \(\deg f = cl_0\). It is then clear that for all \(m \geq 0\), \(I^m f \subseteq K_{l_0+m}\) and so \([I^m]_{cm} f \subseteq [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}\) since \(R_\Delta\) is quasi-Gorenstein. This implies that \([K_{l_0+m}]_{c(l_0+m)} = [I^m]_{cm}\).

We want to show that \(K_{l_0+m} = I^m f\) for all \(m \geq 0\). Suppose that there exists \(m_0\) such that \(I^{m_0} \nsubseteq K_{l_0+m_0}\). Then let \(g \in K_{l_0+m_0}, g \notin I^{m_0} f\) be a homogeneous
element of degree \( k \). Note that from the inclusion \( K_{l_0+\alpha_0} \subset K_{l_0} = A\overline{f} \) one has \( g = \overline{f}\overline{g} \) with \( \overline{g} \notin I^{m_0} \).

If \( k \geq c(l_0 + m_0) \) then for all \( m > m_0 \) we have \( I^{m_0}f + I^{m-m_0}g \subset K_{l_0+m} \) and so \( [I^m]_c f + [I^m-m_0]_{c(l_0+m)-k} g \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_c m \). Hence \( [I^m-m_0]_{c(l_0+m)-k} g \subset [I^m]_c m \). Let \( \lambda = c(l_0 + m) - k - d(m - m_0) = (c - d)m + cl_0 + dm_0 - k \). For \( m \gg 0 \) we have that \( \lambda > 0 \). Then, if \( A_{\overline{g}} I \in I^{m_0} \) we would have that \( \overline{g} \in (I^{m_0})^* = \{ p \in A \mid p \cdot m^{k} \subset I^{m_0}, \) for some \( k \} \), the saturation of \( I^{m_0} \). It is well-known that if \( G_A(I) \) is Cohen-Macaulay then \( \inf \{ \text{depth}(A/I^n) \} = \dim A - l(I) \) [4]. As \( l(I) < n \), we then get \( \overline{g} \in I^{m_0} \) which is a contradiction. So there exist \( \lambda > 0 \), \( h \in A_{\lambda} \) such that \( \overline{g}h \notin I^{m_0} \). On the other hand, \( \overline{g}h[I^m-m_0]_{d(m-m_0)} \subset \overline{g}[I^m-m_0]_{c(l_0+m)-k} \subset [I^m]_c m \). So by using that \( I \) is equigenerated we have that \( \overline{g}h \in (I^m : I^{m-m_0}) = I^{m_0} \), since \( R \) is Cohen-Macaulay. This is a contradiction.

If \( k < c(l_0 + m_0) \), let us write \( k = c(l_0 + m_0) - s \) with \( s > 0 \). Then \( A_{\overline{g}} g \subset [K_{l_0+m_0}]_{c(l_0+m_0)} = [I^{m_0}]f \), and \( g \in (I^{m_0})^* = I^{m_0} \) which, as before, is a contradiction.

Hence we have proved that \( K_{l_0+m} = I^m f \) for all \( m \geq 0 \) and \( K_R = f(At \oplus \ldots \oplus At^{s_0} \oplus It^{s_0+1} \oplus \ldots) \), i.e. \( K_R \) has the expected form. By [21], Theorem 4.2 this implies that both \( R_A(I^{l_0}) \) and \( G_A(I) \) are Gorenstein.

Finally assume \( c > 1 \), and denote by \( \overline{\Delta} = (c-1) \) and \( \overline{R} = R(I^c) \). Then \( \overline{R}_{\overline{\Delta}} = R_{\Delta} \) is quasi-Gorenstein and so there exists \( l_0 \) such that \( R_A(I^{l_0}) \) is Gorenstein. By [21], Theorem 4.2 this implies again that \( G_A(I) \) is Gorenstein. \( \square \)

**Example 3.4 (Room surfaces):** Let \( k \) be an algebraically closed field. Set \( t = \left( \frac{d+1}{2} \right) \), with \( d \geq 2 \). We are going to study the rational projective surfaces which arise as embeddings of blowing ups of \( \mathbb{P}^d_k \) at a set of \( t \) distinct points \( P_1, \ldots, P_t \) not contained in any curve of degree \( d - 1 \).

Let \( I \) be the ideal defining the set of points \( P_1, \ldots, P_t \). It can be easily seen that \( I \) is a homogeneous ideal equigenerated in degree \( d \). For each \( c \geq d+1 \), we obtain a surface by the embedding associated to \( I_c \). For \( c = d+1 \) the resulting surfaces are called Room surfaces. This has been proved by A. Geramita and A. Gimigliano that they are arithmetically Cohen-Macaulay. Assume \( d \geq 3 \). Gimigliano [8] proved that \( I_d \) also defines an embedding of this blow up in the projective space \( \mathbb{P}^d_k \) with defining ideal given by the \( 3 \times 3 \) minors of a \( 3 \times d \) matrix of linear forms, and that this ideal has a linear resolution that comes from the Eagon-Northcott complex. From this fact and applying [1], Example 3.6.15, one obtains that \( a(k[l_d]) = -1 \) and so by [20] the reduction number of \( I \) is \( r(I) = a(k[l_d]) + l(I) = -1 + 3 = 2 \). Moreover the analytic deviation of \( I \) is \( ad(I) = l(I) - ht(I) = 1 \) and \( I \) is generically a complete intersection ideal. So we may conclude by [11] that \( G_A(I) \) is Cohen-Macaulay and
hence by [10], Proposition 2.4, $a(G_A(I)) = r(I) - ht(I) - 1 = -1$. By Ikeda-Trung’s criterion, $R_A(I)$ is also Cohen-Macaulay. From Proposition 3.2 we get that there are not diagonals $(c, e)$ such that $k[(I^c)_e]$ is Gorenstein. In particular, $k[I_{d+1}]$ is not Gorenstein for $d \geq 3$.

If $d = 2$, by choosing the points to be $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, we have $I = (X_1X_2, X_1X_3, X_2X_3)$. Note that $I$ is an almost complete intersection ideal such that $A/I$ is Cohen-Macaulay. Moreover, it is easy to check that $\mu(I_p) \leq ht(p)$ for all prime ideals $p$. So one knows from [13] that $G_A(I)$ is Gorenstein and $a(G_A(I)) = -ht(I) = -2$. By Theorem 2.8, $k[(I^c)_e]$ is quasi-Gorenstein if and only if $\frac{3}{e} = \frac{1}{e} \in \mathbb{Z}$. So $(3, 1)$ is the only diagonal with the Gorenstein property. This corresponds to the del Pezzo sextic surface in $\mathbb{P}^6$.

References


