SCROLLS AND QUARTICS

by

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§ 1 INTRODUCTION

In [S1] Saint-Donat shows how to apply a theorem of Del Pezzo and Bertini (quoted as theorem 1 below) to recover the main result of [XXX] concerning the projective classification of codimension two cubic varieties. In this paper we show how the same theorem, and some related results, can be used to produce an "enumeration" of quartic varieties somewhat more explicit than that given by Swinnerton-Dyer in [S2]. Our main result essentially says that a codimension 2 quartic variety which is contained in a unique quadric is rationally ruled, so that, by a theorem of Bertini, must be the projection of a quartic scroll (see theorems 5 and 6 below for complete statements).

§ 2 NOTATIONS AND PRELIMINARIES

Let $\mathbb{P}^n$ be the n-dimensional projective space over an algebraically closed field of charateristic different from 2. Given a subset S of $\mathbb{P}^n$ we write $\langle S\rangle$ to denote the linear span of S. Unless otherwise stated we consider irreducible varieties $V \subseteq \mathbb{P}^n$ that are not contained in any hyperplane (i.e., such that $\langle V\rangle = \mathbb{P}^n$) and set $d = \dim(V)$, $g = \deg(V)$.

As it is well known we have $g \geqslant n - d + 1$. In case $g = n - d + 1$ we say that V is a *minimal degree variety*.

We say that V is a *normal rational scroll of type* $S = S(n_1, \ldots, n_d)$ (or just a scroll) if V is the image of the projectivized bundle of $\mathcal{O}(n_1) + \ldots + \mathcal{O}(n_d)$ over $\mathbb{P}^1$ under the complete linear system $|\mathcal{O}(1)|$. We always will assume $n_1 \geqslant \ldots \geqslant n_d$. The embedding dimension of $S(n_1,\ldots,n_d)$ is $n = n_1 + \ldots + n_d + d - 1$ and its degree is $n_1 + \ldots + n_d = n - d + 1$, so that scrolls are irreducible minimal degree varieties. If $d = 1$ then $n_1 = n$ and $S(n)$ is a normal rational curve in $\mathbb{P}^n$.

A more concrete description of $S(n_1, \ldots, n_d)$ is as follows (cf. [H1]). Take independent linear spaces $L_1, \ldots, L_d$ in $\mathbb{P}^n$ such that $L_1 + \ldots + L_d = \mathbb{P}^n$, and let $n_i = \dim(L_i)$. Suppose $n_1 \geqslant \ldots \geqslant n_d$. For each i such that $n_i \geqslant 1$, let C_i be any normal rational curve of degree n_i in L_i (in particular this implies that $\langle C_i \rangle = L_i$) and choose any isomorphism $h_i: \mathbb{P}^1 \to C_i$. If $n_i = 0$, so that L_i is a point, set $C_i = L_i$ and write $h_i: \mathbb{P}^1 \to C_i$ to denote the constant map. Then the union of the linear spaces $R_t = \langle h_1(t), \ldots, h_d(t) \rangle$, when t varies in $\mathbb{P}^1$, sweeps out an $S(n_1, \ldots, n_d)$, and conversely, any scroll can be obtained in this way. Using this description of scrolls we see that there exists a projective system of coordinates X_{ij}, $1 \leqslant i \leqslant d$, $0 \leqslant j \leqslant n_i$, such that the ideal of S is generated by the 2×2 minors of the matrix $A = (A_1 | \ldots | A_d)$, where

$$A_i = \begin{pmatrix} X_{i0} & X_{i1} & \cdots & X_{i,n_i-1} \\ X_{i1} & X_{i2} & \cdots & X_{i,n_i} \end{pmatrix}.$$

In particular, two scrolls $S(n_1, \ldots, n_d)$ and $S(m_1, \ldots, m_d)$ ara projectively equivalent if and only if $n_i = m_i$ for $i = 1, \ldots, d$. From these equations it also follows that $S(1, \overset{q+1}{\ldots}, 1)$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^q$ in $\mathbb{P}^{2q+1}$.

A surface V in $\mathbb{P}^5$ will be called a *Veronese surface* V_2^4 (or just a V_2^4) if it is projectively equivalent to the image of $\mathbb{P}^2$ in $\mathbb{P}^5$ under the 2-fold Veronese map.

A variety V is said to be *ruled* when it is the clousure in the Zariski topology of $\mathbb{P}^n$ of an ∞^1 family of $(d-1)$-dimensional linear spaces, which will be called *generators* or *rulings* of the variety. A set of linear spaces is said to be an ∞^1 *family* if they are the linear spaces corresponding to the points of an algebraic curve on a Grassmannian variety. Scrolls are clearly ruled.

§ 3 SOME TOOLS

In this section we quote a few results. The first, due to Del Pezzo [P], Bertini [B], and J. Harris [H1], gives the classification, up to projective equivalence, of minimal degree varieties (cf. also [X], where a rather elementary proof is included).

1. *Any irreducible minimal degree variety of $\mathbb{P}^n$ belongs to precisely one of the following three classes:*

   (i) *Scrolls*

   (ii) *Quadrics of rank not less than five*

   (iii) *A Veronese surface or a cone over a Veronese surface.*

Next result takes care of the set-theoretic structure of the linear sections of a scroll (see [X, 3 and 4]).

2. *If the intersection of a scroll* S *and a linear space* L *is irreducible, then this intersection is itself a scroll. Moreover, the intersection* $L \cap S$ *is irreducible if* L *cuts the rulings of* S *in linear spaces of constant dimension. Finally, a hyperplane that does not contain any ruling of* S *cuts* S *along an irreducible scroll* $\bar{S}$ *such that* $\deg(\bar{S}) = \deg(S)$.

The last two results we quote can be found in [B].

3. *Let* $V \subset \mathbb{P}^n$ *be a ruled variety of degree* $g = n \quad d + 2$ *such that the* ∞^1 *family of rulings in rational. Then given a point* P *in* $\mathbb{P}^{n+1}$ *outside* $\mathbb{P}^n$ *there exists a scroll* S *of degree* g *in* $\mathbb{P}^{n+1}$ *such that* V *is the projection of* S *from* P.

4. *Let* V *be a surface containing* ∞^2 *conics on it. Then* V *is a* V_2^4 *or a projection of a* V_2^4.

§ 4 CODIMENSION THREE QUARTICS

The codimension of a quartic variety in $\mathbb{P}^n$ can be one, two, or three, because the degree is bounded below by codimension + one. And codimension three quartics have minimal degree. Therefore by 1 we see that any codimension three quartic belongs to one of the following types:

5. S (1,1,1,1); S (2,1,1); S (2,2), S (3,1), V_2^4; S (4);
or a come over one of the preceding types.

Applying theorem 2 we see that *any* irreducible hyperplane section of S (1,1,1,1) is a S (2,1,1); that *any* irreducible hyperplane section of S (2,1,1) is a S (2,2) or a S (3,1), and that *any* irreducible hyperplane section of the latter types is a S (4). Notice that also the irreducible hyperplane sections of V_2^4 are of type S (4).

§ 5 CODIMENSION TWO QUARTICS

A condimension two quartic will be called of the *first kind* if there is more than one quadric containing it, of the *second kind* if there is exactly one quadric containing it, and of the *third kind* if there are no quadrics that contain it. With this terminology we have:

6. *Let* V *be a codimension two quartic in* $\mathbb{P}^n$. *Then*
   (i) *If* V *is of the first kind, then* V *is contained precisely in a pencil of quadrics.*
   (ii) *If* V *is of the second kind, then* V *is rationally ruled, so that by* 3 *it is the projection of a codimension three quartic scroll.*
   (iii) *If* V *is of the third kind, then* V *is the projection of a* V_2^4 *or a cone over such a projection.*

Before proving this theorem we are going to deal with two auxiliary results.

§ 6 TWO LEMMAS

First let us introduce some notation. Given $V \subset W \subset \mathbb{P}^n$, let $I_{V,W} \subset \mathcal{O}_W$ be the sheaf of ideals of functions vanishing on V. We set

$$i_{V,W}(r) = \dim H^0(W, I_{V,W}(r)) \quad 1,$$

where (r) denotes the opperation of r-fold twisting, that is, of tensoring with $\mathcal{O}_W(r)$. If $r = 2$ and W is linear, then $i_{V,W}(2)$ is the dimension of the linear system of quadrics of W that contain V.

7. *Let* V' *be a hyperplane section of a variety* $V \subset \mathbb{P}^n$, *and let* H *be the corresponding hyperplane. Then*

$$i_{V,\mathbb{P}^n}(2) \leqslant i_{V',H}(2).$$

Proof: Consider the map

$$H^0(\mathbb{P}^n, I_{V,\mathbb{P}^n}(2)) \to H^0(H, I_{V',H}(2))$$

given by restriction. The inequality will follow if we show that this map is a monomorphism. Let F be a quadratic homogeneous polynomial in the kernel. Then F is divisible by H (we denote a hyperplane and its equation with the same letter), say $F = H \cdot H'$, where H' is linear. Since F vanishes on V this implies that V is contained in the zero-set of H', since the zero-set of H is a hyperplane and V is not contained in any hyperplane by our basic assumptions. Thus the zero-set of H' is $\mathbb{P}^n$, and so H' is zero, hece F is also zero. QED.

8. *Let* $V \subset \mathbb{P}^n$ *have dimension* $d \geqslant 2$. *Let* H_i, $i = 1, 2$, *be hyperplanes such that* $\langle V_i \rangle = H_i$ *and* $\deg(V_i) = \deg(V)$, *where* $V_i = H_i \cap V$. *If* $i_{V_i, H_i}(2) \geqslant 1$ *and* $i_{H_1 \cap H_2 \cap V, H_1 \cap H_2}(2) = 1$, *then* $i_{V, \mathbb{P}^n}(2) = i_{V_1, H_1}(2) = i_{V_2, H_2}(2) = 1$.

Proof: By 7 we see on one hand that $i_{V_i, H_i}(2) = 1$, $i = 1, 2$, and on the other that in order to prove that $i_{V, \mathbb{P}^n}(2) = 1$ it is enough to show that $i_{V, \mathbb{P}^n}(2) \geqslant 1$. Let $Q_t \subset H_1$, $t \in \mathbb{P}^1$, be the pencil of quadrics that contain V_1. Then for each t there exists a unique quadric $Q'_t \subset H_2$ which contains V_2 and such that $Q'_t \cap H_1 = Q_t \cap H_2$, because the map

$$H^0(H_i, I_{V_i, H_i}(2)) \to H^0(H_1 \cap H_2, I_{H_1 \cap H_2 \cap V, H_1 \cap H_2}(2))$$

given by restriction is an isomorphism (is monomorphism and both members have dimension 2). Now, for fixed t, there exists a pencil of quadrics $Q^*_s \subset \mathbb{P}^n$, $s \in \mathbb{P}^1$, which contain Q_t and Q'_t. In this pencil there exists a unique member Q^*_t which goes through a fixed general point x of V. In order to see that $i_{V, \mathbb{P}^n}(2) \geqslant 1$ it is enough to show that $V \subset Q^*_t$. But if V were not contained in Q^*_t then $\deg(Q^*_t \cap V) > \deg(V_1) + \deg(V_2) = 2 \cdot \deg(V)$, which would contradict Bezout's theorem. QED.

§ 7 PROOF OF THEOREM 6 (i)

We shall proceed by induction on the dimension d of V. Assume first that $d = 1$, so that $n = 3$. Then since V is contained in two quadrics it follows that V is a complete intersection of two quadrics. If H is a general plane then $V' = H \cap V$ are four points in H no three of which are colinear. Thus $i_{V', H}(2) = 1$. Then we have $1 \leqslant i_{V, \mathbb{P}^3}(2) \leqslant i_{V', H}(2) = 1$ and so $i_{V, \mathbb{P}^3}(2) = 1$.

Let $d \geqslant 2$. Take two general hyperplanes H_1 and H_2 and set $V_i = H_i \cap V$, $i = 1, 2$. Then $\langle V_i \rangle = H_i$, $\deg(V_i) = \deg(V)$, and $\langle H_1 \cap H_2 \cap V \rangle = H_1 \cap H_2$. By 7 and the inductive hypothesis we see that

$$i_{H_1 \cap H_2 \cap V, H_1 \cap H_2}(2) = i_{V_1, H_1}(2) = i_{V_2, H_2}(2) = 1$$

(notice that $i_{H_1 \cap H_2 \cap V, H_1 \cap H_2}(2) = 1$ even when $d = 2$, because then $H_1 \cap H_2 \cap V$ consists of four points of $H_1 \cap H_2$ in general position). By 8 we conclude that $i_{V, \mathbb{P}^n}(2) = 1$. QED.

As a corollary we have:

9. *Let* $V \subset \mathbb{P}^n$ *be a codimension two quartic variety with* $d \geqslant 2$. *Let* L *be a*

generic linear space such that $3 \leqslant \dim(L) \leqslant n-1$. *Then if* $L \cap V$ *is of the first kind, so is* V.

Proof: By decending induction on $m = \dim(L)$. If $m = n-1$, let H_1 and H_2 be two general hyperplanes, so that if $V_i = H_i \cap V$, $i = 1, 2$, then V_i is of the first kind. Then $V' = H_1 \cap H_2 \cap V$ is also of the first kind. By 8, V is of the first kind as well.

Let now $m < n-1$, and let L' be a general $(m+1)$-dimensional linear space through L. Set $V' = L' \cap V$. Then V' is of the first kind because its generic hyperplane section is. By descending induction V itself is of the first kind. QED.

§ 8 PROOF OF THEOREM 6 (ii)

Again we will proceed by induction. Assume first that $d = 1$, so that V is a quartic curve in $\mathbb{P}^3$ contained in a unique quadric. Then V is rational ([H2, IV, 6.4.2]). This proves the case $d = 1$.

To proceed the induction, however, we need to be slightly more precise. We show that the condition that the curre V is contained in a unique quadric Q is equivalent to say that V *is non-singular and rational*, in which case Q *is also non-singular*. In fact, if V is a singular quartic curve in $\mathbb{P}^3$ then it has only one double point, say x. Let Q' be the cone over V with vertex x. Then Q' is a *quadric cone* and $V \subset Q'$. Therefore V is a complete intersection of two quadrics ([H2, V, Ex. 2.9]). So if V is contained in a unique quadric Q then V must be non-singular, and Q also must be non-singular.

Now let $d = 2$, so that V is a quartic surface in $\mathbb{P}^4$ contained in a unique quadric Q. This quadric must be singular, otherwise V would be a complete intersection, by a theorem of Klein ([H2, II, Ex. 6.5]). Consider two independent general hyperplanes H_0 and H_1 and set $V_i = H_i \cap V$, $Q_i = H_i \cap Q$, $i = 0,1$. If $L = H_0 \cap H_1$, then $L \cap V$ consists of four points in general position in L, so that $i_{L \cap V, L}(2) = 1$. This and lemma 8 imply that $i_{V_i,H_i}(2) = 0$. Therefore, by the case of curves in $\mathbb{P}^3$ explained above, V_i is a non-singular quartic curve in H_i contained in a unique quadric of H_i, which itself is non-singular. Since $V_i \subset Q_i$, this unique quadric must coincide with Q_i. From this it follows that Q is a cone over Q_0 with vertex at a point, say x_0. If H_t, $t \in \mathbb{P}^1$, is the pencil of hyperplanes through L, then $Q_t = H_t \cap Q$ is a non-singular quadric in H_t except for the hyperplane $L + x_0$, that we may assume is H_∞. The projection of H_t, $t \neq \infty$, onto H_0 with center x_0 maps Q_t isomorphically onto Q_0. If the two classes of rulings on Q_0 are denoted R_0^1 and R_0^2, then we will write R_t^1 and R_t^2 to denote the two classes of rulings on Q_t corresponding respectively to R_0^1 and R_0^2 under the above projection. By what we said before, $V_t = H_t \cap V$ is a non-singular rational quartic curve on Q_t for all t but finitely many. Therefore we can assume that V_t is of type (1, 3) with repect to the classes of rulings R_t^1 and R_t^2 (if it were of type (3,1) we would change the roles of the two classes).

Now consider a generic point z_0 on V_0 and let $R_0 \in R_0^1$ be the unique ruling through z_0. Let $R_t \in R_t^1$ be the ruling on Q_t corresponding to R_0. Then R_t intersects V_t, for all t but finitely many, at a single point z_t. The closure the set $\{z_t\}$ is a curve C which is nothing but $R \cap V$, where R is the plane $R_0 + x_0$. To prove that V is rationally ruled it is enough to show that C is a line. In order to see this notice that the rulings R_t pass through a fixed point a_0 of R, namely $a_0 = R_0 \cap L$, that C does not go through a_0 (for V does not go through a_0 either, since z_0 is generic on V_0), and that $R_t \cap C$ is a point counted once for general t (the intersection taken inside R). Thus deg (C) = 1.

Suppose then that d $\geqslant 3$ and assume that the result is true for lower dimensions. If H is a general hyperplane then $i_{H \cap V, H}(2) = 0$ by corollary 9. By induction $H \cap V$ is rationally ruled. And since dim $(H \cap V) \geqslant 2$ this implies that V itself is rationally ruled (cf. the prooof of theorem 2 in [X]). QED.

§ 9 PROOF OF THEOREM 6 (iii)

Let $V \subset \mathbb{P}^n$ be a codimension two quartic variety not contained in any quadric. Then d $\geqslant$ 2, for as we have seen before any quartic curve in $\mathbb{P}^3$ lies on at least one quadric.

Assume first that d = 2. Then V is a quartic surface in $\mathbb{P}^4$. Let L be a general plane in $\mathbb{P}^4$, so that $L \cap V$ consists of four points of L, no three of which are colinear, and hence $i_{L \cap V, L}(2) = 1$. Let H_t, $t \in \mathbb{P}^1$, be the pencil of hyperplanes going through L and set $V_t = H_t \cap V$. For general $t_1 \in V_t$ it is a smooth rational quartic curve in H_t, by 9. So V_t lies on a unique quadric Q_t of H_t, and Q_t is non-singular. Now it happens that V_t is contained in a unique quadric Q_t of H_t *for all* t, for if V_s, for some $s \in \mathbb{P}^1$, were contained in a pencil of quadrics of H_s, then this pencil would cut out on L the pencil of conics through $L \cap V$, and so, again, V would be contained in a quadric. Moreover, the map $t \mapsto Q_t \cap L$ gives a bijection between $\mathbb{P}^1$ and the pencil of conics on L that have $L \cap V$ as base points. Let $s \in \mathbb{P}^1$ be such that $Q_s \cap L$ is a pair of lines. Then V_s can not be irreducible (otherwise it would be rational and non-singular, Q_s would be non-singular and so L would be tangent to Q_s, which would imply that three points on $L \cap V$ would lie on a line). Also, V_s can not have multiple components, since if it did, then $L \cap V$ also would. Since V_s in contained in a single quadric, it must be a pair of conics. This implies that V contains ∞^2 conics. In fact, let $G = Gr_{4,2}$ be the Grassmannian of planes in $\mathbb{P}^4$, let C be the variety of conics on V and let $I \subset G \times C$ be the correspondence given by $(s, x) \in I$ if and only if there exists a hyperplane H such that H contains the planes L_s corresponding to $s \in G$ and the conic c_x corresponding to $x \in C$. Let $f: I \to G$ and $g: I \to C$ be the maps given by projection. By what we said above, the generic fiber of f is 0-dimensional, so that dim (I) = 6. On the other hand the fibers of g are 4-dimensional. It turns out that C is 2-dimensional. By 4 we conclude that V is a projection of a V_2^4 from a point.

Now assume $d \geqslant 3$. It is enough to show that V is a cone, because then if V' is a hyperplane section by a huperplane H not going through the vertex of V then $i_{V',H}(2) = 1$ so that by induction V' is a cone over a projection of a V_2^4, hence V itself is a cone over al projection of a V_2^4.

In order to see that V is a cone we again proceed by induction. So let us assume that d = 3. Consider the system $H \subseteq \mathbb{P}^{5*}$ of hyperplanes H such that $H \cap V$ is contained in at least one quadric of H. Then H is a hyperplane, so that all such hyperplanes H go through a fixed point x_0. In fact, to see that H is a hyperplane, or that deg $(H) = 1$, we show that a general line of $\mathbb{P}^{5*}$ cuts H just once. In other words, we want to see that if L is a generic codimension two linear space in $\mathbb{P}^5$ then there exists a hyperplane H through L such that $H \cap V$ is contained in a quadric of H, and that this H is unique. Now $L \cap V$ must be a non-singular rational curve, by 9. So $L \cap V$ is contained in a unique quadric Q of L. This uniqueness implies that there exists at most one H through L such that $H \cap V$ is contained in a quadric of H (otherwise we would again construct a quadric containing V). So we need only show that there exists at least one such H. Let $P \subset L$ be a general plane inside L, so that $P \cap V$ consists of four points on P no three of which are colinear. Now let L' be another codimension two linear space through P. If L' is general, $L' \cap V$ is a non-singular rational curve in L' and hence contained in a unique quadric Q' of L'. Due to the fact that there are ∞^2 spaces L', we can select an L' in such a way that $Q' \cap P = Q \cap P$. Then the hyperplane $H = L + L'$ satisfies the claim.

Now we show that V is a cone with vertex x_0. Let H be a general hyperplane through x_0. Then $H \cap V$ is contained in a unique quadric Q_H of H. Therefore $V_H = H \cap V$ is ruled (the rulings are lines). It is enough to show that all rulings go through x_0. To see this, let $L \subset H$ be a general codimension two linear space among those such that $x_0 \in L$. Then $V_L = L \cap V$ $(= L \cap V_H)$ is a quartic curve without multiple components (to see this take a general plane $P \subset L$ and consider $P \cap (L \cap V_H) = P \cap V$). Since $x_0 \in H$, V_L is contained in at least a quadric of L. If V_L were contained in only one quadric of L then again we would be able to construct a quadric containing V. So there exists at least a pencil of quadrics of L which contain V_L. These quadrics are all singular (otherwise V_L would be the intersection of two non-singular quadrics of L, which would contradict the fact that for a general codimension two linear space L*, $L^* \cap V$ is a non-singular rational quartic curve, for V_L is of type (2,2) on a non-singular quadric through it, and $L^* \cap V$ is of type (1, 3) on the unique quadric of L^* containing it). Then all quadrics of L that contain V_L are cones and have a common vertex, so that V_L consists of four concurrent lines, which are rulings of V_H. Let z be the common point to the lines of V_L. If z were not x_0 then a constant count involving the L's through x_0 would show that V contains ∞^4 lines, which is impossible because V has dimension three and is not a linear space.

Now let $d \geq 4$. Then 9 says that no generic linear section $L \cap V$ of V is of the first kind if the linear space L has dimension not less than three. And if $\dim(L) \geq 4$ then $L \cap V$ can not be of the second kind either, because if it were we could construct a quadric containing V taking two general hyplerplane sections V_1 and V_2, say by H_1 and H_2, and observing that the quadrics Q_1 and Q_2 of H_1 and H_2, respectively, that contain V_1 and V_2 lie on a pencil of quadrics of $\mathbb{P}^n$ *because* Q_1 and Q_2 *coincide* on $H_1 \cap H_2$.

So by induction a general hyperplane section of V is a cone. And from this it follows that V itself is a cone. QED.

§ 10 REMARKS

In what follows if $V \subset \mathbb{P}^n$ is a variety, πV denotes any projection into $\mathbb{P}^{n-1}$ from a point outside V and $\pi' V$ any such projection from a point outside the line chord locus of V.

10. (i) *Any codimension two quartic of the second kind belongs to precisely one of the following types*

$\pi S(1,1,1,1)$; $\pi S(2,1,1)$; $\pi S(2,2)$, $\pi S(3,1)$; $\pi' S(4)$;

or a cone over one of the preceding types. And conversely, any of these types is a codimension two quartic of the second king.

(ii) *Any codimension two quartic of the third kind is of type* πV_2^4 *or a cone over a* πV_2^4. *And conversely, any of these types is a codimension two quartic of the third kind.*

Proof: The direct part of (ii) is just 6 (iii). And the direct part of (i) follows from 6 (ii) and 5. Notice that since a projection of S (4) from a point on its line chord lucus is a singular quartic, which is of the first kind, we have to replace $\pi S(4)$ by $\pi' S(4)$.

In case (i) we have seen that a non-singular rational quartic curve in $\mathbb{P}^3$ is of the second king. Therefore any $\pi' S(4)$ is of the second kind. Now all other types in (i) *are ruled* and of dimension not less that two; hence they can not be of the first or of the third kind. On the other hand they are quartics because if S is a quartic scroll of codimension c then π S has codimension c − 1 and so deg (π S) $\geqslant$ c = deg (S) − 1 = 3; but π S can not have degree 3 if we project from a point outside S ([M, 5.5]). Similarly, any $\pi\ V_2^4$ is a quartic surface; since it is non-ruled and contains ∞^2 conics it must be of the thirk kind. In fact any quartic surface of the first kind does not contain ∞^2 conics, for if two quadrics in $\mathbb{P}^4$ contain ∞^2 conics in common then they coincide. It follows that any cone over any $\pi\ V_2^4$ also is of the third kind. QED.

The preceding proof tells us a little more:

11. *The three classes of codimension two quadrics*

(i) *Complete intersections of two quadrics,*

(ii) *Ruled quartics (excluding projections of* S (4) *from a point on its line chord locus, or cones over such),*

(iii) $\pi\ V_2^4$ *or cones over a* $\pi\ V_2^4$.

are pairwise disjoint. Consequently the reciprocals of 6 (i), 6 (ii) *and* 6 (iii) *are also true.*

§ 11 NON-SINGULAR QUARTICS

If we exclude hypersurfaces and codimension two quartics of the first kind then the remaining non-singular quartics are grouped as follows:

12. (i) *Any non-singular codimension three quartic belongs to one of the following types:*

S (1, 1, 1, 1); S (2, 1, 1); S (2, 2), S (3, 1), V_2^4; S (4)

(ii) *Any non-singular codimension two quartic which is not of the first kind is either a* π' S (4) *or a* $\pi'\ V_2^4$.

Proof: (i) follows from 5 and the fact that a scroll S = S $(n_1, \ldots, n_d)$, $n_1 \geqslant \ldots \geqslant n_d$, is non-singular if and only if $n_d \geqslant 1$, and that S $(n_1, \ldots, n_d, 0)$ is a cone over S $(n_1, \ldots, n_d)$. And (ii) follows immediately from (ii) and next result. QED.

13. *The line chord locus of a scroll* S = S $(n_1, \ldots, n_d)$ *in* $\mathbb{P}^n$ *is equal to* $\mathbb{P}^n$ *if and only if*

$n_1 = 3$ *and* $n_2 \leqslant 1$

or

$n_1 \leqslant 2$, $n_2 \leqslant 2$ *and* $n_3 \leqslant 1$.

Proof: Given a variety V let $L_1 V$ denote the line chord locus of V. Thus $L_1 V = pr_{|P^n}(Z_V)$, where $Z_V \subset V \times V \times \mathbb{P}^n$ is the closure of the set of triples (x, y, z) such that $x \neq y$ and $z \in \langle x, y \rangle$. Now in order to find out whether $L_1 S = \mathbb{P}^n$ one can assume that S is not a cone, since L_1 commutes with the formation of cones, as it is easily checked. Thus we can suppose that $n_d \geqslant 1$.

Assume that $L_1 S = \mathbb{P}^n$. Then $n \leqslant 2d + 1$, because $\dim(Z_S) = 2d + 1$. But since $n = n_1 + \ldots + n_d + d - 1$ we see that $n_1 + \ldots + n_d \leqslant d + 2$, or, equivalently, that $(n_1 - 1) + \ldots + (n_d - 1) \leqslant 2$. Since $n_i - 1 \geqslant 0$, we see that either $n_1 = 3$ and $n_2 = \ldots = n_d = 1$ or $n_1 \leqslant 2$, $n_2 \leqslant 2$ and $n_3 = \ldots = n_d = 1$.

Now in order to see the converse first observe that $L_1 S(2) = \mathbb{P}^2$, that $L_1 S(3) = \mathbb{P}^3$ ([W, § 33]), and, we claim, that $L_1 S(2,2) = \mathbb{P}^5$. Indeed, let L_1 and L_2 be the planes given by $X_3 = X_4 = X_5 = 0$ and $X_0 = X_1 = X_2 = 0$ respectively. Let $V_1 \subset L_1$ be the conic $X_1^2 - X_0 X_2 = 0$ and $V_2 \subset L_2$ the conic $X_4^2 - X_3 X_5 = 0$. Let $h_1 : \mathbb{P}^1 \to V_1$ be the map given by $h_1(t) = (1, t, t^2, 0, 0, 0)$ and $h_2 : \mathbb{P}^1 \to V_2$ the map given by $h_2(t) = (0, 0, 0, 1, t, t^2)$. Thus the line $L_t = \langle h_1(t), h_2(t) \rangle$ sweeps out a S (2,2) and any S (2,2) is projectively equivalent to it. Let $x = (x_0, x_1, x_2, x_3, x_4, x_5)$ be a general point of $\mathbb{P}^5$ and set $x' = (x_0, x_1, x_2, 0, 0, 0)$, $x'' = (0, 0, 0, x_3, x_4, x_5)$. For any $t \in \mathbb{P}^1$, let $t' \in \mathbb{P}^1$ be the unique point such that $h_1(t)$, $h_1(t')$ and x' are colinear. A computation shows that $t' = (x_2 - tx_1) / (x_1 - tx_0)$. Then $h_2(t)$, $h_2(t')$ and x'' are colinear if and only if

$$\begin{vmatrix} 1 & t & t^2 \\ 1 & t' & t'^2 \\ x_3 & x_4 & x_5 \end{vmatrix} = 0,$$

which is a quadratic equation in t. Thus there exists $t \in \mathbb{P}^1$ such that both triads $h_1(t)$, $h_1(t')$, x' and $h_2(t)$, $h_2(t')$, and x'' are colinear. Consider the 3-space $L = L_t + L_{t'}$. By construction x' and x'' are in L. Therefore x is also in L and hence there exists a point $a \in L_t$ and a point $a' \in L_{t'}$ such that x is in the line $a + a'$. This shows that $x \in L_1 S(2,2)$, and proves the claim.

Now in order to see that the conditions in our statement are sufficient we only need to apply recursively the following lemma:

Let $n_d = 1$ *and let* $S' = S(n_1, \ldots, n_{d\ \ 1})$. *If* $L_1\ S' = \mathbb{P}^m$, *where*

$m = n_1 + \ldots + n_{d-1} + d\ \ 2$, *then* $L_1\ S = \mathbb{P}^n$.

Proof: We use the notations explained in section 2. Since $n = n_1 + \ldots + n_d + d\ \ 1$, we see that if $n_d = 1$ then $m = n - 2$. Let z be a general point of $\mathbb{P}^n$ and let z' be its projection to $\mathbb{P}^{n-2}$ taking the line $L = L_d$ as center of projection. Since $L_1\ S' = \mathbb{P}^{n-2}$, there exist distinct points x, x' ϵ S' such that z' ϵ $\langle x, x' \rangle$. Let R and R' be the rulings fo S going through x and x' respectivley. Let y and y' be the points at which R and R' meet L. Then z belongs to the 3–space $\langle x, y, x', y' \rangle$, because L + z = L' + z'. Therefore there exists a ϵ R and a' ϵ R' such that z ϵ $\langle a, a' \rangle$, so that z ϵ L_1 S. QED.

ACKNOWLEDGEMENTS

I would like to thank D. Eisenbud for his teaching, understanding and encouragement granted to me through his kind correspondence.

Some key ideas for the proof of 6 (iii) are borrowed from Swinnerton-Dyer's beautiful paper [S2], which also has provided me with much inspiration throughout this work.

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