Master thesis in Pure and applied Logic

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Proof theory of the calculus of relations

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Contents

1 Introduction **1**

1.1 Historical introduction 1

1.2 Introduction to sequent calculus 4

1.3 Desideratum: cut-free 8

1.4 Desideratum: shared contexts 9

1.5 Some initial theorems: inversion, weakening, contraction 10

1.6 The intuitionistic variant and the classical logic variant 11

2 Extending the calculus **14**

2.1 Extending the calculus 14

2.2 Geometric axioms and rules 15

2.3 Inversion, substitution, weakening, contraction 20

2.4 Cut elimination 22

2.5 Barr's theorem 26

3 Non-geometric theories **28**

3.1 Introduction 28

3.2 Starting definitions 29

3.3 Skolem's result 33

3.4 Example 34

3.5 An extension of Skolem's extensions 35

4 Relation Algebra and Sequent Calculi **40**

4.1 Historical remarks 40
4.1.1 Relation algebra 42
4.1.2 Geometric axioms for RA 46
4.1.3 A sequent calculus for RA 50
4.2 Example of use 53
4.3 Simplifying the calculus 54
4.4 A comparison with the Maddux sequent calculus of relation algebras . 56
4.5 Conclusions 63

Bibliography **65**

Abstract

The calculus of relations is a rich discipline that spanned much of the history of symbolic logic, with contributions ranging from De Morgan to Tarski. Yet not much is known about the proof theory for the calculus of relations. For example, a sequent calculus developed by Maddux has the shortcoming of not being cut-free. In this work we will develop a sequent calculus that is cut-free. To do this we will use a procedure developed by Negri & von Plato to transform geometric theories into rules for a sequent calculus.

Keywords: Proof theory; Calculus of relations; Cut-elimination; Non-geometric theories.

Chapter 1

Introduction

1.1 Historical introduction

In the year 1900, David Hilbert published a list of twenty-three open problems in mathematics. Several of them became very influential for the development of 20th century mathematics, and some of them lack a definitive solution even today. Yet a not very well known fact is that in the year 2000 a note by Hilbert was discovered, mentioning a problem he never added to the list[1][2]:

> As a 24th problem of my Paris talk I wanted to pose the problem: criteria for the simplicity of proofs, or, to show that certain proofs are simpler than any others. In general, to develop a theory of proof methods in mathematics.

This note should elicit quite some interest, if not only to show the great insight that Hilbert had, since the discipline of proof theory, which proved many results that could be related to this 24th problem, would be born only 20 years after this note was written.

At the beginning of the 20th century the mathematical community was starting to develop the methods that we now consider to be central to the discipline of the foundations of mathematics. Many mathematical fields started to investigate how to well-formulate some research questions related to the axioms of the field itself. For example, even one of the most classical mathematical discipline, that of geometry, had unresolved questions relative to the independence of one of its axioms (the famous 5th postulate, dating back to Euclid).

And to answer such questions a systematic theory of how to obtain proofs from some starting axioms was needed. It was from this and similar challenges that the idea of a logical calculus gradually emerged. It was with the starting contributions from Frege, Peano, and others, that the discipline of metamathematics evolved. And for the first time it achieved to express mathematical statements purely with formulas, totally manipulable in a precise syntactic fashion. This work was later perfected by Russell in the first decade of the 20th century, thanks to him the principles of proof found by Frege where expressed in the clearer notation of Peano. And thus the seminal *Principia Mathematica* were born.

These formalizations played a central role in the research program outlined by Hilbert's 23 open problems. In particular, it was one of Hilbert's aims to use such formalization to show the consistency and completeness of arithmetic and analysis. Thanks to the results of Gödel, we know that this research program of Hilbert failed. The incompleteness theorems determined that it was impossible to obtain a complete formalization for arithmetic, and that the consistency of Peano arithmetic is unprovable.

This came at a great shock for the field, yet not all hopes of proving the consistency of arithmetic went lost. This was due to the work of Gentzen, and the foundations he gave to proof theory. Indeed one could claim that Gentzen's work gave some answers to Hilbert's 24th problem we cited at the beginning, since he developed the first successful theory about proof methods in mathematics.

First, through his work in natural deduction and sequent calculi systems, he pinpointed the subformula property as one of the key elements for the complexity and decidability of a derivation. The basic idea being that a "good" proof system would have the property that its derivations only have formulas that are contained in the conclusion to be derived. And so this property can be used to derive an upper-bound on the number of formulas that can possibly appear in the derivation, and thus inform questions regarding the complexity and decidability of a derivation. In particular Gentzen proved that its sequent calculus of first order logic had this property, though the famous result of cut-elimination (we will discuss this matter in more detail later on in the chapter).

Secondly, Gentzen further developed proof theory to not only cover logic, but also arithmetic. In his work he formalized Peano arithmetic derivations as proof trees constructed from the rules of inference[3]. He defined a procedure to attach an ordinal $\leq \epsilon_0$ to a proof, and showed that, while moving down the tree, the ordinals would get smaller. From this he deduced that if a contradiction was derivable, then this would cause an infinite descending chain of ordinals.

By Gödel's results we know that such a proof would not be provable in PA (indeed Gentzen proved it in primitive recursive arithmetic plus quantifier free transfinite induction up to ϵ_0). So Gentzen's proof would end up becoming the first "actual" arithmetical statement not provable in PA (since Gödel's famous "I am not provable" formula needs the concept of provability from the metamathematics of a formalized system), and it would become the foundation of the field of ordinal analysis.

These successes lead proof theory to further developments, in particular to find proof systems for mathematical theories. For example, many continued Gentzen's and Hilbert's aim of a proof theory of analysis. Furthermore many proof systems have been developed for theories of order, lattice theory, and elementary geometry, thanks to some recent developments in the sequent calculi of geometrical theories[4].

Yet one simple theory that lacked a syntactic characterization for a long time is the theory of the calculus of relation. This theory has a history as long as logic itself, given that it was first developed by De Morgan and Peirce, and then in the 1940s Tarski developed it to what is now known as the theory of relation algebras, and found many results of great mathematical importance. For example, one famous result is that relation algebras can express all first order logic containing no more than 3 variables, and that this suffices to express PA. Thus relation algebras can be seen as an alternative formalization of almost all mathematics.

Yet not much is known about the syntactical properties of such theory, given that the only calculus in the literature[5] is provably not cut-free. Thus the aim of this thesis will be to fill this gap in the literature, by finding a cut-free sequent calculus for the calculus of relations.

To do so in the first chapter a broad introduction to the proof theory of sequent calculi will be given. Then in the second chapter we will explain how to find sequent calculi rules from any geometric theory, while preserving cut-elimination. The third chapter presents how to obtain a geometric first-order theory out of any first order theory. Finally the fourth chapter will apply the results from the previous chapters to the calculus of relations.

More specifically for this chapter, we will give a broad introduction to the calculus of sequents, and present some of the earliest results related to it. Given the availability of resources on this subject[6][4], we will not give many details about the materials presented. We will only try to give intuitions about the important features of sequent calculi.

We will first give an introduction and the definition of a sequent calculus for propositional logic. Then we will present some of the biggest benefits of sequent calculi, that is, their ease of use when searching for a proof, and which characteristics the calculi have to maintain to preserve this ease of search. Next we present some results (inversion, weakening, contraction) that are indispensable for proving later results about sequent calculi. Lastly, we will expand our propositional calculus to a first-order logic one, which we will be using for the rest of the thesis.

1.2 Introduction to sequent calculus

We fix an alphabet of propositional atoms $\{P, Q, R, ...\}$, countably infinite, furthermore we will have the logical constant $\perp$. Then we define the propositional formulas

$\{A, B, C, ...\}$ inductively from the connectives $\vee, \wedge, \rightarrow$, and $\perp$.

We will call $\Gamma \Rightarrow \Delta$ a sequent, where Γ and Δ are finite multisets of propositional logic formulas. The standard interpretation of a sequent will be $\bigwedge \Gamma \rightarrow \bigvee \Delta$, as we want to express that from everything in Γ follows at least something in Δ. We will call Γ the antecedent, and Δ the consequent. Then clearly if we have an empty Γ, the sequent $\Rightarrow \Delta$ expresses that $\bigvee \Delta$ is a tautology. While if Δ is empty, then we have $\Gamma \rightarrow \perp$, so the antecedent Γ is inconsistent. As an example to get the intuition behind sequent calculus, consider how we might prove the following tautology: $\Rightarrow (((p \wedge q) \rightarrow w) \wedge p) \rightarrow (q \rightarrow w)$.

We start by looking at the meaning of the most external connective ($\rightarrow$). So if we assume $((p \wedge q) \rightarrow w) \wedge p$, then we should be able to derive $q \rightarrow w$. For the moment, we write this as $((p \wedge q) \rightarrow w) \wedge p \Rightarrow q \rightarrow w$. Then we notice that at this level we can deal with other 2 connectives (the $\wedge$ and the $\rightarrow$). So we start by trying to get rid of $\wedge$: if we are assuming $((p \wedge q) \rightarrow w) \wedge p$, then then it's the same as assuming $(p \wedge q) \rightarrow w$ and p, thus we write $(p \wedge q) \rightarrow w, p \Rightarrow q \rightarrow w$. Then we want to get rid of the $\rightarrow$ in $q \rightarrow w$: we are assuming some formulas and we want to prove $q \rightarrow w$, so it is the same as assuming those formulas and q, to prove w. Thus $(p \wedge q) \rightarrow w, p, q \Rightarrow w$. We have another $\rightarrow$ to get rid of, but this time is on our assumptions. So we use the equivalence $A \rightarrow B \equiv \neg A \vee B$. So we get $\neg(p \wedge q) \vee w, p, q \Rightarrow w$. Now we have the connective $\vee$ in our assumptions, that means we could reason by cases: we want $w, p, q \Rightarrow w$ and $\neg(p \wedge q), p, q \Rightarrow w$. The first case now has no more connectives, and we notice that we have a w in both the antecedent and consequent, so is trivially true. What about $\neg(p \wedge q), p, q \Rightarrow w$? We have to get rid of the negations, so we can notice, given our semantic interpretation, that $\bigwedge \Gamma \wedge \neg A \rightarrow \bigvee \Delta$ is equivalent to $\bigwedge \Gamma \rightarrow (\neg A \rightarrow \bigvee \Delta)$, and that is $\bigwedge \Gamma \rightarrow A \vee \bigvee \Delta$. So we have $p, q \Rightarrow p \wedge q, w$ for the second case. The last connective is a $\wedge$ in the conclusion, so if we want to prove $p \wedge q$, that is the same as proving first p and then q, so this case is further split into $p, q \Rightarrow p, w$ and $p, q \Rightarrow q, w$. In both cases we can see that the have reached some tautologies.

The idea behind sequent calculus is that we want to imitate the kind of reasoning behind this example: we want to start with a sequent, and simplify the connectives in the formulas in Γ and Δ until all the connectives are eliminated and there are only atomic formulas on each side of the sequent. So we want our proofs to look like a rooted tree graph, with the root of the tree being the formula to be proven. Then each step of the tree simplifies some connective, and sometimes branches the tree. At the end we want to have only tautologies as the leaves of the tree. So we need to give our inference rules, two for each connective, based whether they are on the left or the right of the sequent. So for the $\wedge$ we have:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\,\mathrm{R}\wedge$$

for the right and

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\,\mathrm{L}\wedge$$

for the left.

Given a rule, we call the formula whose connective is currently being eliminated the principal formula (in the above case, $A \wedge B$). Furthermore we call the formula above the inference line the premise, while the formula below it the conclusion.

For right $\vee$:

$$\frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\,\mathrm{R}\vee$$

and left $\vee$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\,\mathrm{L}\vee$$

for right $\rightarrow$

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}\,\mathrm{R}\rightarrow$$

and left $\rightarrow$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}\ \mathrm{L}{\rightarrow}$$

Then we need some axioms, our search for a proof will be successful only if at the end of our proof tree search we find something that is obviously true. So our first axiom represents the idea that a sequent is provable if one of the atoms on the right correspond to an atom to the left:

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ \mathrm{Ax}$$

with P an atom.

Then we need a way to treat negation. We will use the equivalence $\neg A \equiv A \rightarrow \bot$, and the rule (a sort of *ex falso quodlibet*) for $\bot$:

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot$$

By inspecting all our rules, we can notice an important feature of our sequent calculus, called the subformula property: in any proof-tree of a sequent $\Gamma \Rightarrow \Delta$, the formulas appearing above the inference line will be subformulas of the formulas in Γ and Δ. This is an important observation that makes us conclude that the empty sequent $\Rightarrow$ is not derivable.

Furthermore all of our rules have an important property (easily seen with truth-tables): whenever the premise of a rule is valid, then also the conclusion is valid. This observation allows to immediately prove the soundness of the calculus: by induction, the axioms we have given are clearly valid, and the rules preserve validity, so clearly if we can prove a sequent with this calculus, then this sequent is valid.

On the other hand we can also see that completeness holds: if a sequent $\Gamma \Rightarrow \Delta$ is valid, then we will find a proof with our calculus. If we have a valid sequent, then it's validity will be preserved upward. At the end our leaves can be axioms or a general sequent $\Gamma' \Rightarrow \Delta'$ made by atoms (when the proof fails). Yet this last case never happens, since the validity is preserved to the leaves, and having a general sequent $\Gamma' \Rightarrow \Delta'$ on some leaf means that we could put all the atoms in Γ' to true and all the atoms in Δ' to false, and thus have a not valid sequent. So we will never find a regular sequent on the top level, and so our leaves will only be axioms. So we will always have a proof.

Finally, if a sequent $\Gamma \Rightarrow \Delta$ is derivable in our sequent calculus, we write $\vdash_n \Gamma \Rightarrow \Delta$, where n is the depth (the length of the longest branch) of the proof tree of $\Gamma \Rightarrow \Delta$. This depth will be often used in induction proofs, like in the paragraph above.

This concludes our definition of the classical propositional logic sequent calculus, which we will call G3cp[6]. The reason for the "3" is that this version of the calculus was not historically the first one to be developed. We will give some motivations about why this calculus is adopted in the next sections.

1.3 Desideratum: cut-free

In Gentzen's original calculus, called LK, we had a rule of the following form:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

This rule can be seen as representing some sort of transitivity of the sequent: it states that when we can conclude A, and this A can also be in the antecedent for another derivation, then the A can be "cut", and the derivations joined.

We can notice that this rules appears to be very different from the rules of G3cp: in our bottom-up proof construction, we could always know how to proceed using the rules, which can be applied in a unique way. Yet if the cut rule is available then we face a problem: how do we guess the formula A? (we notice that that formula doesn't appear in the below sequent $\Gamma, \Gamma'\Delta, \Delta'$). This creates some problems when it comes to automated theorem searches, as we no longer have a simple algorithm as for the rules without cut.

Also another problem is that with a cut rule we are introducing a totally new formula A in our proof tree, and this breaks the subformula property.

This motivates the importance of cut-elimination theorems. In case of G3cp+Cut, we could do this easily given that we have the completeness and soundness theorems.

Theorem 1.3.1. *If G3cp+Cut* $\vdash \Gamma \Rightarrow \Delta$ *then G3cp* $\vdash \Gamma \Rightarrow \Delta$.

Proof. if G3cp+Cut $\vdash \Gamma \Rightarrow \Delta$ then by soundness we know $\Gamma \Rightarrow \Delta$ is valid (since the cut rule is also sound), then by completeness G3cp $\vdash \Gamma \Rightarrow \Delta$.

□

Yet this proof does not give an algorithm that transforms an instance of a proof containing cut, to a cut-free proof. This was first done by Gentzen in his "Main Theorem" (Hauptsatz) about LK[7].

1.4 Desideratum: shared contexts

Another important property that our calculus contains is that, going upwards in the proof tree search, it is always uniquely determined how to apply a rule. Consider these two possible rules for the R$\wedge$ connective

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}\,\mathrm{R}\wedge$$

and

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,\mathrm{R}\wedge$$

The first type of rule we call context independent, the second context sharing. They can be seen to be equivalent (i.e. simulate the work of one using the other) in the following way. If we have $\Gamma \Rightarrow A \wedge B$, then by contraction we have $\Gamma, \Gamma \Rightarrow A \wedge B$: so we can use the context independent rule to get $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$; if we have $\Gamma, \Delta \Rightarrow A \wedge B$, then by the context sharing we have $\Gamma, \Delta \Rightarrow A$ and $\Gamma, \Delta \Rightarrow B$, then by weakening we have $\Gamma \Rightarrow A$ and $\Delta \Rightarrow B$). However we can easily see that these two type of rules are not equivalent when it comes to proof-search algorithms. We want the rules of the sequent calculus to be always used in a determined way, but with the context independent rules, we would have to consider a lot of different possibilities when it comes to splitting the context between the two antecedents of the premises. With context sharing this doesn't happen: a proof search is uniquely determined by the principal formula of the sequent.

With this we conclude that the calculus G3cp is a good candidate for a sequent calculus that has the subformula property and that has effective proof-search algorithms.

1.5 Some initial theorems: inversion, weakening, contraction

Now we can proceed to explain some of the first results related to sequent calculi. Given the ample availability of textbooks related to these results[6][8], our exposition will be rather short.

First we want to talk about how, when we inspect the rules of G3cp, we can notice that not only the direction from premises to conclusions is valid, but also the opposite. That is, we can prove that if we assume that the conclusion is derivable, also the premise is derivable. Furthermore this result preserves proof depth. So, for example, for the rule R$\rightarrow$, we have that:

Theorem 1.5.1. *If* $\vdash_n \Gamma \Rightarrow A \rightarrow B, \Delta$ *then* $\vdash_n \Gamma, A \Rightarrow B, \Delta$

And we have a result of this shape, which we will call inversion principle, for each rule of G3cp. As in many other cases, all of these theorems can be proved by induction.

Another important result is weakening:

Theorem 1.5.2. *If* $\vdash_n \Gamma \Rightarrow \Delta$ *then* $\vdash_n \Gamma, A \Rightarrow \Delta$ *and* $\vdash_n \Gamma \Rightarrow A, \Delta$

This result basically states that if we have proven a sequent, then we can weaken the antecedent and/or the consequent, and we still have a valid sequent.

Lastly we present contraction:

Theorem 1.5.3. *If* $\vdash_n \Gamma, A, A \Rightarrow \Delta$ *then* $\vdash_n \Gamma, A \Rightarrow \Delta$. *Also if* $\vdash_n \Gamma \Rightarrow \Delta, A, A$ *then* $\vdash_n \Gamma \Rightarrow \Delta, A$

This theorem states that when dealing with the multisets in a sequent, we can disregard repetitions.

An important thing to note is that in all these results, the height of the proof $\vdash_n$ is preserved. This is important because it means that we can use these results as useful lemmas while proving by induction on the depth of a proof. Indeed in many textbooks[6][8] these results are used as important lemmas while proving cut-elimination results.

1.6 The intuitionistic variant and the classical logic variant

Now we extend our sequent calculus to first-order logic. We need a new first-order language, with or without identity, and without function symbols except constants. A term t of our language is either a variable x (we allow countably infinite many of them) or a constant a. Then the formulas are defined inductively from the atoms $p_i^n(t_1, ...t_n)$, with $i \geq 1$ and $n \geq 0$, and the propositional constant $\perp$, and using the connectives and the quantifiers (existential, $\exists$, and universal, $\forall$).

The substitution of a variable x with a term t in a term s (or in a formula A, or in a multiset Γ) is written as $s[t/x]$(or $A[t/x]$, or $\Gamma[t/x]$). To indicate simultaneous substitution of a list of variables $x_1, ..., x_n$ with a list $t_1, ..., t_n$, we write $[\bar{t}/\bar{x}]$.

Now to complete the definitions of our first-order calculus: we will use the same rules of G3cp plus some rules for the existential quantifiers:

$$\frac{\Gamma \Rightarrow \exists x A, A[t/x], \Delta}{\Gamma \Rightarrow \exists x A, \Delta}\,\mathrm{R}\exists$$

$$\frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta}\,\mathrm{L}\exists$$

and universal quantifier:

$$\frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x A, \Delta}\,\mathrm{R}\forall$$

$$\frac{\Gamma, A[t/x], \forall x A \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}\,\mathrm{L}\forall$$

We also need that in the rules R∀ and L∃ the variable y does not occur free in the conclusion. We will call y the eigenvariable of R∀ and L∃.

We will call this calculus G3c.

With the quantifier rules at our disposal we can obtain the intuitionistic variant rather easily. We will call this intuitionistic variant of our calculus G3im. The propositional part of it is equal to G3cp, except for the rules of implication[9]:

$$\frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\,\mathrm{L}\!\to$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma, A \to B \Rightarrow \Delta}\,\mathrm{R}\!\to$$

The L→ rule has a repetition of the principal in the left premise. Furthermore the succedent is only A. The right rule is similar in this. The existential quantifier will be the same as in the classical calculus, while the universal quantifiers will be similar to the implication:

$$\frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x A, \Delta} \text{ R}\forall$$

$$\frac{\Gamma, A[t/x], \forall x A \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \text{ L}\forall$$

With y an eigenvariable in R∀. These modifications might appear weird but are necessary to make the weakening possible. We will not deal much with the intuitionistic variant in this thesis, so we leave a more thorough description to [4].

Going back to G3c, an important lemma about substitutions of terms for variables in a sequent is:

Lemma 1.6.1. *If* $\vdash_n \Gamma \Rightarrow \Delta$ *then* $\vdash_n \Gamma[t/x] \Rightarrow \Delta[t/x]$*, where* t *is freely substitutable for* x *in* Δ, Γ

which basically states that term renaming doesn't impact our proof depth (again, an important aspect for proofs by induction). Furthermore the previous result can be strengthened even more, to substitute for a term instead that for a variable:

Lemma 1.6.2. *If* $\vdash_n \Gamma \Rightarrow \Delta$ *then* $\vdash_n \Gamma[t/u] \Rightarrow \Delta[t/x]$*, where the term* t *is freely substitutable for the term* u *in* Δ, Γ

With these results at our disposal we can prove the inversion, weakening, and contraction results also for G3c[8][6].

Finally a historically important result that we will only state is the cut elimination theorem for G3c:

Theorem 1.6.3. *If G3c+Cut* $\vdash \Gamma \Rightarrow \Delta$ *then G3c* $\vdash \Gamma \Rightarrow \Delta$*.*

For a proof, the reader is redirected to [6].

With these definitions and results fixed for sequent calculi about logics, now we are ready for the next chapter, where we will give a look at how we could devise a sequent calculus for a mathematical theory.

Chapter 2

Extending the calculus

2.1 Extending the calculus

In the previous chapter we have seen how sequent calculus is an effective method for finding proofs in first-order logic. Now we might ask ourselves: what about finding proofs of a mathematical theory? Indeed first-order logic has a long history when it comes to formalization of mathematics, but to do so it needs some axioms relative to the mathematical theory that it wants to be speaking about. So it is spontaneous to ask how the axioms of a theory can be added to the G3c sequent calculus, so that it can derive the valid formulas of the theory.

One first idea might be to add the axioms of the theory as end sequents: if for each axiom A of the theory we have a sequent rule of the form $\Rightarrow A$ from which the derivation can start, then our calculus should still be complete. Let's see an example of this.

We think about the theory of equivalence relations, and add two rules for a relation R that is reflexive and Euclidean:

$$\frac{}{\Rightarrow xRx}\text{ Ref}$$

$$\frac{}{xRz, yRz \Rightarrow xRy}\text{ Euc}$$

Now form these two rules we can derive that R is symmetric ($yRx \Rightarrow xRy$) by the following derivation:

$$\dfrac{\dfrac{}{\Rightarrow xRx}\text{Ref} \qquad \dfrac{}{xRx, yRx \Rightarrow xRy}\text{Euc}}{yRx \Rightarrow xRy}\text{Cut}$$

Yet this proof uses a Cut rule. And clearly it wouldn't be possible to derive $yRx \Rightarrow xRy$ without it. So if we want to preserve the cut-free property of G3c we have to add the axioms of a theory in a different way.

Historically there have been different approaches to extend the rules of a sequent calculus.

For example, Gentzen[10] tried to add some mathematical basic sequents, that is substitution instances of sequents of the form $P_1, ..., P_i \Rightarrow Q_1, ..., Q_j$, with P and Q atomic formulas. Then one can prove that the cut rule can be eliminated also when used with these basic sequents.

Another way of adding axioms, found in Gentzen's consistency proof of elementary arithmetic[7], is to put axioms as an antecedent Γ, such that to prove a theorem T our sequent calculus would always prove results of the form $\Gamma \Rightarrow T$. In this way the sequent calculus derivations are basically the same as the G3c system, and so we still have cut elimination. Yet we have to force such Γ set of mathematical axioms even in derivations of logical rules, which might appear not natural.

Yet more recently[4] a fourth way of adding rules to the G3c sequent calculus was developed, this will be the topic that we will explore in the rest of this chapter. The aim of this approach will be to find some "good" rule only for a particular type of mathematical theories, called geometric theories. Then in the next chapter we will see how this approach can be extended to all theories.

2.2 Geometric axioms and rules

We start by defining geometric implications:

Definition 2.2.1. *Geometric implication: a geometric implication (sometimes called special coherent implication) is a first-order sentence that is the universal closure of an implication $C \to D$, where C is a conjunction of atoms and D is a disjunction of existentially quantified conjunctions of atoms.*

That is geometric implications are formulas of the general form:

$$\forall \bar{x}(P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m)$$

With all the $\bar{P}$ being atoms and each M_j being a conjunction of atoms $\bar{Q}$. Furthermore we assume we rename the variables in $\bar{P}$s so that no $\bar{y}$ appears free in $\bar{P}$. Also we notice that in the geometric implication we can have an empty antecedent, by swapping the conjunction with a single $\top$, or an empty consequent, by swapping the disjunction with a single $\bot$. So for example the axiom for a reflexive relation R becomes

$$\forall x \top \to xRx$$

Finally a theory axiomatized by geometric implications will be called a geometric theory.

We can find many examples of geometric theories[11], for example Robinson's arithmetic. Another example are all algebraic theories, like group theory and ring theory, and all essentially algebraic theories, like category theory, the theory of fields, the theory of local rings, lattice theory, projective geometry.

Furthermore it is interesting to notice that sometimes a theory that doesn't immediately appear as geometric can be easily transformed into one. For example, Robinson arithmetic is a finitely axiomatized theory of first-order logic with identity. It's axioms are[12]:

1. $\neg s(a) = 0$
2. $s(a) = s(b) \to a = b$
3. $\neg a = 0 \to \exists y\, s(y) = a$

4. $a + 0 = a$

5. $a + s(b) = s(a + b)$

6. $a \cdot 0 = 0$

7. $a \cdot s(b) = a \cdot b + a$

We can notice that the third axiom is not in a geometric form, yet we can use the equivalent $a = 0 \vee \exists y\, s(y) = a$ instead, and thus obtain a geometric theory.

So it would be useful to temporarily focus on theories axiomatized by such formulas, as geometric implications have a shape that looks promising for being transformed in a sequent calculi rule. Given a geometric implication G, such rule, called *Grule* will be:

$$\frac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta \qquad ... \qquad \bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta}{\bar{P}, \Gamma \Rightarrow \Delta}\; Grule$$

With the condition that the eigenvariables $\bar{z}_i$ must not be free in $\bar{P}, \Gamma, \Delta$. With this condition, and by inspecting the rule, one can notice that we took care in a rather simple way of the existential quantifiers.

Another aspect that might be puzzling at first is the presence of the repetition of the $\bar{P}$ in the premises. This is a trick that is necessary to be able to still have the contraction lemma.

Suppose we have a derivation of $A, A, \Gamma \Rightarrow \Delta$, eith the last rule used being a *Grule*. Then we have 3 different cases for the origin of that repeated A. First, both instances of A are in Γ; second, one is principal; third, both of them are. In the first case we would handle the contraction by a usual induction. But in the second case we would have to use a trick used first to handle the contraction for the $L \rightarrow$ rule in G3i[4], that is, putting a repeated principal formula in the premises. Also the third case poses a problem, this time specific to the shape of geometric implications. The solution we will use is to add an ad hoc condition to have contraction, called the closure condition:

Definition 2.2.2. *Closure condition: a sequent calculus has the closure condition if, for every rule of the type*

$$\frac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_{m-2}, P, P, \Gamma \Rightarrow \Delta \quad ... \quad \bar{Q}_n(\bar{z}_n/\bar{y}_n), P_1, ...P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, ...P_{m-2}, P, \Gamma \Rightarrow \Delta}\, Grule$$

it has a rule

$$\frac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_{m-2}, P, \Gamma \Rightarrow \Delta \quad ... \quad \bar{Q}_n(\bar{z}_n/\bar{y}_n), P_1, ...P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, ...P_{m-2}, P, \Gamma \Rightarrow \Delta}\, Grule$$

Now clearly the number of rules to be added to a sequent calculus satisfying this condition is bounded. So adding this fix is not problematic.

Now we have all the intuitions to understand the *Grule*. To further cement the idea that this rule is able to convey the same expressiveness of the axiom it is representing, we give the following theorem.

Theorem 2.2.1. *A geometric axiom is derivable from the corresponding geometric rule. Also, a geometric rule is derivable from the corresponding geometric axiom plus the use of contraction, cut and inversion.*

Proof. First we want to derive the sequent $\Rightarrow \forall \bar{x}(P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Rightarrow M_1(\bar{z}_1/\bar{y}_1), ..., M_m(\bar{z}_n/\bar{y}_n)}{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Rightarrow \exists \bar{y}_1 M_1, ..., \exists \bar{y}_m M_m}\, R\exists \quad ... \quad \dfrac{\bar{Q}_m(\bar{z}_1/\bar{y}_1), \bar{P}, \Rightarrow M_1(\bar{z}_1/\bar{y}_1), ..., M_m(\bar{z}_n/\bar{y}_n)}{\bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Rightarrow \exists \bar{y}_1 M_1, ..., \exists \bar{y}_m M_m}\, R\exists}{\bar{P} \Rightarrow \exists \bar{y}_1 M_1, ..., \exists \bar{y}_m M_m}\, Grule}{\bar{P} \Rightarrow \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m}\, \text{repeated } R\vee}{P_1 \wedge ... \wedge P_n \Rightarrow \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m}\, \text{repeated } R\wedge}{\Rightarrow P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m}\, R\to}{\Rightarrow \forall \bar{x}(P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m)}\, R\forall$$

For the second part of the theorem we want to check we can derive the *Grule*:

$$\dfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta \qquad \ldots \qquad \bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta}{\bar{P}, \Gamma \Rightarrow \Delta}\, Grule$$

Given that we also can use the geometric implication $\Rightarrow \forall \bar{x}(P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m)$ as an axiom to the derivation. This can be checked by the derivation:

.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \forall \bar{x}(P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m)}{\Rightarrow P_1 \wedge ... \wedge P_n \to \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m}\, invR\forall}{P_1 \wedge ... \wedge P_n \Rightarrow \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m}\, invR\to}{\bar{P} \Rightarrow \exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m}\, invL\wedge \quad \dfrac{\dfrac{\dfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta}{M_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta}\, L\wedge}{\exists \bar{y}_1 M_1, \bar{P}, \Gamma \Rightarrow \Delta}\, L\exists \quad \ldots \quad \dfrac{\dfrac{\bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta}{M_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta}\, L\wedge}{\exists \bar{y}_m M_m, \bar{P}, \Gamma \Rightarrow \Delta}\, L\exists}{\exists \bar{y}_1 M_1 \vee ... \vee \exists \bar{y}_m M_m, \bar{P}, \Gamma \Rightarrow \Delta}\, Grule}{\dfrac{\bar{P}, \bar{P}, \Gamma \Rightarrow \Delta}{\bar{P}, \Gamma \Rightarrow \Delta}\, Contraction}\, Cut$$

□

If we analyze the last derivation we can see that the *Grule* is actually hiding a cut. This indeed creates a small problems for the desired subformula property. Yet as we will see later we can still obtain a weaker version of the subformula property, namely the subterm property. Another important remark is that it is convenient to always assume that in a derivation the sets of free and bound variables are disjoint, so we don't have to face problems with the substituted variables y_i.

Now that we have the definitions and rationale for the geometric rules, we can define the sequent calculus that is formed by a geometric theory.

Definition 2.2.3. *Given a Geometric theory T, G3cT is defined as the sequent calculus obtained from G3c plus the Grules obtained from the axioms of T, plus the rules that are necessary to satisfy the closure condition.*

In the next section we will see how the sequent calculus thus defined holds the desiderata (namely, cut elimination) that we explained in the previous chapter. But as for the proof of G3c, before having the cut-elimination result we will have to prove that the inversion, weakening, substitution and contraction lemmas also hold for G3cT.

2.3 Inversion, substitution, weakening, contraction

We first prove a new version of the substitution lemma.

Lemma 2.3.1. *Substitution: if G3cT $\vdash_n \Gamma \Rightarrow \Delta$ then G3cT $\vdash_n \Gamma(t/x) \Rightarrow \Delta(t/x)$, with x a free variable in Γ and Δ, t a term free for x in Γ and Δ, and not containing any of the variables of the geometric rules in the derivation.*

Proof. Proof. By induction on the height of the derivation. The logical part of the proof is already done. So we need to consider only the inductive step with the last rule of the derivation being a *Grule*. So we have a $\bar{P}, \Gamma' \Rightarrow \Delta$ Derived by the premises $\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma' \Rightarrow \Delta, ..., \bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma' \Rightarrow \Delta$. We know that the term t is free for x in these premises. So by induction hypothesis we get derivations of $\bar{Q}_1(\bar{z}_1/\bar{y}_1)(t/x), \bar{P}(t/x)$ $,\Gamma'(t/x) \Rightarrow \Delta(t/x), ..., \bar{Q}_m(\bar{z}_m/\bar{y}_m)(t/x), \bar{P}(t/x),$ $\Gamma'(t/x) \Rightarrow \Delta(t/x)$. We are assuming that in a derivation the sets of free and bound variables are disjoint, so we have $x \neq y_i$. The z_i are not free so $x \neq z_i$. Also t does not contain any of the y_i. So we can swap the substitutions $(\bar{z}_1/\bar{y}_1)$ and (t/x): $\bar{Q}_1(t/x)(\bar{z}_1/\bar{y}_1), \bar{P}(t/x)$ $,\Gamma'(t/x) \Rightarrow \Delta(t/x), ..., \bar{Q}_m(t/x)(\bar{z}_m/\bar{y}_m), \bar{P}(t/x),$ $\Gamma'(t/x) \Rightarrow \Delta(t/x)$. Then we apply the geometric rule scheme to get $\bar{P}(t/x), \Gamma'(t/x) \Rightarrow \Delta(t/x)$. So we are done.

□

By using this lemma we have the useful remark that in a derivation in G3cT, the collections of proper variables of any two geometric rules can be considered as disjoint.

Now we can prove weakening.

Lemma 2.3.2. *Weakening: If G3cT $\vdash_n \Gamma \Rightarrow \Delta$ then G3cT $\vdash_n \Gamma, A \Rightarrow \Delta$ and G3cT $\vdash_n \Gamma \Rightarrow A, \Delta$*

Proof. The proof is by induction, with the base case and almost all of the induction step done with the classical result. We only consider the case where the last rule that has been used is a geometric rule. We can use the induction hypothesis almost immediately, except that it could be the case that A contains some variables used in the geometric rule. At this point adding the A would cause problems with the condition on eigenvariables not being free in $\bar{P}, \Gamma, \Delta$. But we have the substitution lemma, so we can apply it to prevent any variable clash. So we are done

□

By a quick observation, we can prove the inversion lemma.

Lemma 2.3.3. *Inversion: if* $G3cT \vdash_n \bar{P}, \Gamma \Rightarrow \Delta$ *then* $G3cT \vdash_n \bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta, ...\bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta$

Proof. We simply notice that by a quick weakening we are done. □

Finally we prove contraction.

Lemma 2.3.4. *Contraction: If* $G3cT \vdash_n \Gamma, A, A \Rightarrow \Delta$ *then* $G3cT \vdash_n \Gamma, A \Rightarrow \Delta$*. Also if* $G3cT \vdash_n \Gamma \Rightarrow \Delta, A, A$ *then* $G3cT \vdash_n \Gamma \Rightarrow \Delta, A$

Proof. We prove by induction. The classical cases are already proved, so we only need to consider the inductive step with the last rule being used a geometric rule. We consider both left and right contraction. We have to consider different cases. We could have that both occurrences of A are not principal in the rule. In this case we can apply directly the induction hypothesis and then the rule and be done. The other case is when only one occurrence of A is principal. In this case A is one of the atoms P, so we handle this by repeating the principal formulas P in the geometric rule. The third and final case is when both occurrences of A are principal. This is the case why we introduced the closure condition, so we are done.

□

Now that we have these lemmas we are ready to prove the cut elimination theorem for the sequent calculi of geometric theories.

2.4 Cut elimination

Theorem 2.4.1. *If G3cT+Cut* $\vdash \Gamma \Rightarrow \Delta$ *then G3cT* $\vdash \Gamma \Rightarrow \Delta$.

Proof. This proof will have the same structure as usual proofs of cut elimination[6]. We assume we have a rule of the form

$$\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

used somewhere in our sequent calculus derivation, and we show by induction how to eliminate it. The induction is on the length of A, with an auxiliary induction on the sums of the derivations heights of the two premises of the cut rule (h_l for left height, h_r for right height). For $h_l + h_l = 0$ the theorem follows the usual proof of cut elimination. So we only consider the cases where the height of the cut is $h_l + h_l > 0$, and that arise from the addition of the geometric rule. So we have 2 sub-cases, depending on the length of A.

• Length of A is 0. Then we have two further cases whether we consider the left premise of the cut or the right one. Considering the left, then A is some atom P (other than in $\bar{P}$). So we have a derivation of the form:

$$\dfrac{\dfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta, P \qquad ... \qquad \bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta, P}{\bar{P}, \Gamma \Rightarrow \Delta, P}\,Grule \qquad \Gamma', P \Rightarrow \Delta'}{\bar{P}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

In which case the first thing we notice is that we can always use the substitution lemma, so we can consider the variables z_i as fresh. So we can push the cut up in the derivation, without problems (this will be the general strategy also in the rest of the proof, we want to push the cut one step up in the derivation, so that we can apply the IH):

.

$$\cfrac{\cfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta, P \quad \Gamma', P \Rightarrow \Delta'}{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut \quad ... \quad \cfrac{\bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta, P \quad \Gamma', P \Rightarrow \Delta'}{\bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut}{\bar{P}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Grule$$

Now we consider P being in the right premise. In this case P could be either not principal, so we proceed like above and we are done, or principal (Say P_1 in $\bar{P}$), in which case we have:

$$\cfrac{\Gamma \Rightarrow \Delta, P_1 \quad \cfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_n, \Gamma' \Rightarrow \Delta \quad ... \quad \bar{Q}_m(\bar{z}_m/\bar{y}_m), P_1, ...P_n, \Gamma' \Rightarrow \Delta}{P_1, ...P_n, \Gamma' \Rightarrow \Delta'}\,Grule}{P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

So we have to look at the left premise now, to know how to eliminate/push up the cut. if $\Gamma$and Δ have one atom in common, then we have this situation also in $P_2, ...P_n, \Gamma, \Gamma'\Delta, \Delta'$, so we are done. If $\perp$ is in Γ, or $\top$ is in Δ, then the same happens in the conclusion of the cut, so also in this case we are done immediately. If P_1 is in Γ, then the conclusion is of the form $P_1, P_2, ...P_n, \Gamma'', \Gamma'\Delta, \Delta'$, and is obtained from $P_1, ...P_n, \Gamma' \Rightarrow \Delta'$ by weakening.

So we only have to take care of the cases, for each rule, where the left premise is from some rule, and without P_1 being principal. To do these we use the usual lemmas, and are straightforward. As examples we give the ones for $L\exists$, $L \rightarrow$, and $Grule$.

For $L\exists$ we have a derivation of the form: .

$$\cfrac{\cfrac{\Gamma, A[t/x] \Rightarrow \Delta, P_1}{\Gamma, \exists x A \Rightarrow \Delta, P_1}\,\text{L}\exists \quad \cfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_n, \Gamma' \Rightarrow \Delta' \quad ... \quad \bar{Q}_m(\bar{z}_m/\bar{y}_m), P_1, ...P_n, \Gamma' \Rightarrow \Delta'}{P_1, ...P_n, \Gamma' \Rightarrow \Delta'}\,Grule}{\exists x A, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

This is dealt first by noticing we can substitute fresh variables in our geometric rule to not have eigenvariables problems, then with:

$$\dfrac{\dfrac{\Gamma, A[t/x] \Rightarrow \Delta, P_1 \qquad P_1, ...P_n, \Gamma' \Rightarrow \Delta'}{A[t/x], P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}}{\exists x A, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{L}\exists$$

For $L \rightarrow$ we have a derivation of the form:

.

$$\dfrac{\dfrac{\Gamma \Rightarrow A, \Delta, P_1 \qquad \Gamma, B \Rightarrow \Delta, P_1}{\Gamma, A \rightarrow B \Rightarrow \Delta, P_1}\,\text{L}\rightarrow \qquad \dfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_n, \Gamma' \Rightarrow \Delta' \quad ... \quad \bar{Q}_m(\bar{z}_m/\bar{y}_m), P_1, ...P_n, \Gamma' \Rightarrow \Delta'}{P_1, ...P_n, \Gamma' \Rightarrow \Delta'}\,Grule}{A \rightarrow B, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

But we deal with this by:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, P_1 \qquad P_1, ...P_n, \Gamma' \Rightarrow A, \Delta'}{P_2, ...P_n, \Gamma, \Gamma' \Rightarrow A, \Delta, \Delta'}\,\text{Cut} \qquad \dfrac{\Gamma, B \Rightarrow \Delta, P_1 \qquad P_1, ...P_n, \Gamma' \Rightarrow \Delta'}{P_2, ...P_n, \Gamma, \Gamma', B \Rightarrow \Delta, \Delta'}\,\text{Cut}}{A \rightarrow B, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{L}\rightarrow$$

The case of the *Grule* is totally novel, although very standard. We have a derivation of the form:

.

$$\dfrac{\dfrac{\bar{Q}'_1(\bar{z}'_1/\bar{y}'_1), P'_1, ...P'_n, \Gamma \Rightarrow \Delta, P_1 \quad ... \quad \bar{Q}'_m(\bar{z}'_m/\bar{y}'_m), P'_1, ...P'_n, \Gamma \Rightarrow \Delta, P_1}{P'_1, ...P'_n, \Gamma \Rightarrow \Delta, P_1}\,Grule \qquad \dfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_n, \Gamma' \Rightarrow \Delta' \quad ... \quad \bar{Q}_m(\bar{z}_m/\bar{y}_m), P_1, ...P_n, \Gamma' \Rightarrow \Delta'}{P_1, ...P_n, \Gamma' \Rightarrow \Delta'}\,Grule}{P'_1, ...P'_n, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

After pondering how we don't have variable clashes by using the substitution lemma, we can transform this derivation into:

.

$$\dfrac{\bar{Q}'_1(\bar{z}'_1/\bar{y}'_1), P'_1, ...P'_n, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad ... \quad \bar{Q}'_m(\bar{z}'_m/\bar{y}'_m), P'_1, ...P'_n, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{P'_1, ...P'_n, P_2, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Grule}$$

With all the m premises derived from a cut on $P_1, ...P_n, \Gamma' \Rightarrow \Delta'$ (which we get from a *Grule* we have) and a $\bar{Q}'_x(\bar{z}'_1/\bar{y}'_1), P'_1, ...P'_n, \Gamma \Rightarrow \Delta, P_1$. Then by IH we are done.

• Now we have the cases where the cut is applied to a formula A that is not an atom. We again have to distinguish the cases where we are looking at the right or left premise.

For the left, almost all the cases are done by the classical cut-elimination proof, we only need to focus on the case where the left rule is derived from a *Grule*:

$$\cfrac{\cfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), \bar{P}, \Gamma \Rightarrow \Delta, P \qquad ... \qquad \bar{Q}_m(\bar{z}_m/\bar{y}_m), \bar{P}, \Gamma \Rightarrow \Delta, P}{\bar{P}, \Gamma \Rightarrow \Delta, P}\,\textit{Grule} \qquad \Gamma', P \Rightarrow \Delta'}{\bar{P}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

But we already dealt with this case, the fact that A is a formula doesn't matter much.

For the right, the fact that A is a formula makes it so that it must be from Γ' (not principal). So we have the case:

$$\cfrac{\Gamma \Rightarrow \Delta, A \qquad \cfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_n, A, \Gamma' \Rightarrow \Delta' \qquad ... \qquad \bar{Q}_m(\bar{z}_m/\bar{y}_m), P_1, ...P_n, A, \Gamma' \Rightarrow \Delta'}{P_1, ...P_n, A, \Gamma' \Rightarrow \Delta'}\,\textit{Grule}}{P_1, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

Which is dealt with:

$$\cfrac{\bar{Q}_1(\bar{z}_1/\bar{y}_1), P_1, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \qquad ... \qquad \bar{Q}_m(\bar{z}_m/\bar{y}_m), P_1, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{P_1, ...P_n, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\textit{Grule}$$

With each of the m premises obtained with a cut from $\Gamma \Rightarrow \Delta, A$ and $\bar{Q}_x(\bar{z}_x/\bar{y}_x), P_1, ...P_n, A, \Gamma' \Rightarrow \Delta'$. So we are done.

$\square$

This result is useful because it gives us directly the weak subformula property

Theorem 2.4.2. *If $G3cT \vdash \Gamma \Rightarrow \Delta$, then in the derivation appear only subformulas of Γ, Δ, or atomic formulas.*

Which is a stepping stone in proving the subterm property for the rules of a calculus. This in turn is of great interest because with the subterm property we get an immediate solution to the derivability problem of a sequent.

This on ultimate analysis depends on the atomic formulas coming from the geometric rule, as it might be that they are introduced in such a way that we cannot track them, making a calculus undecidable. Yet this strategy might still be useful, as it pinpoints the origin of the undecidability, and it can be used to give some subset of a theory which could be decidable (we will see a simple example of this in chapter 4).

Meanwhile in the next section we are going to show an important consequence of having a calculus for geometric theories.

2.5 Barr's theorem

In this section we present a general and interesting application of the ability to get a sequent calculus out of any geometric theory.

First we notice without proof that the results of this chapter could have been very well presented with the intuitionistic counterpart of G3c, that is, G3im (the interested reader is redirected to [11] for more information). Indeed it can be shown that the results for the cut-elimination theorem hold for both systems, without many differences in the proofs presented above. So we call the intuitionistic variant enhanced with geometric rules G3imT(where the "m" stands for multisuccedent). We only remind that the main difference between the two calculi is that the principal formula of rule $L \rightarrow$ is repeated in the antecedent of the left premise. Also in the rule $R \rightarrow$ and $R\forall$, there is no Δ in the consequent of the premise. As we expect, these differences make so that the formulas provable with G3cT and G3imT are different. Yet geometric theories present us with an interesting observation:

Theorem 2.5.1. *Barr's theorem: Let T be a geometric theory, and let A be a geometric formula. Then if $G3cT \vdash\Rightarrow A$, then $G3imT \vdash\Rightarrow A$.*

Proof. A is of the form $\forall x(B \to C)$. This tells us that in a G3cT proof of such a formula, we first have a $R\forall$ step/steps, then a $R \to$ step, and then any combination of rules to derive C and B: for which we will only use the $\wedge, \vee, \exists$ and geometric rules. This is because A is geometric, so B and C do not contain any $\to$ or $\forall$. We also notice that a geometric rule has the same consequent in the premises and conclusions, so they will preserve the sequents that have a single consequent. So we are sure that in the G3cT derivation, the $R\forall$ and $R \to$ derivations are applied to a single consequent. So the intuitionistic rules are applicable. So G3imT $\vdash\Rightarrow A$.

□

This is an interesting result originally coming from a different field. Indeed the study of geometric theories has influenced many areas of logic and mathematics. And one such example is in topos theory, where Barr's theorem is a classical result.

Thus we end the chapter by noticing how this classical theorem becomes almost immediately provable through the methods of proof theory, by a simple analysis of the rules used by a calculus. This testifies for the strength of this approach.

In the next chapter we will try to extend the results just presented, to theories that are originally non-geometric.

Chapter 3

Non-geometric theories

3.1 Introduction

The aim of this chapter will be to prove how to obtain a sequent calculus from non-geometric theories. In the last chapter we obtained a procedure to obtain sequent calculi rules for geometric theories. Now we will give a method to get a geometric conservative extension out of every first-order theory, so that we can use the result from the past chapter to get a sequent calculus out of any theory.

This method can be considered a modification of Skolem's "normal form" theorem from 1920[13][14]. Skolem showed that for any first-order sentence is possible to construct a $\forall\exists$ sentence that is satisfiable iff the initial sentence is satisfiable. This was done by the familiar procedure of extending the original language with new predicate symbols.

Although this result is very similar to what we need, in Skolem's paper geometric axioms are not treated. As reported in[13] the first to publish a similar result for geometric axioms was Antonius[15], that showed in 1975 that given any first order theory, one can always translate it to a geometric theory provided that we can add sufficiently many new relational symbols. Yet this result is due to a completeness theorem proved through category-theoretic models, so in this chapter we will follow the results of Dyckhoff and Negri[13], that extended the results of Antonius and relate it to the work of Skolem.

So the aim of this chapter will be to augment Skolem's proof by showing that given a first-order sentence A in a language L we can construct an extension of L' of L and a finite set of geometric formulas that have the same consequences as A (that is a geometric conservative extension). To do so we will first give some starting definitions about formulas, theories, and extensions.

3.2 Starting definitions

To begin it is useful to recall the definition of geometric formula from the past chapter. Yet it will be more useful if we distinguish a geometric formal and its component part, that is a positive formula.

Definition 3.2.1. *Positive formula: a positive formula is a first-order logic formula composed only of atoms, conjunction, disjunction, and existential quantification.*

Starting from this definition we give a more useful one:

Definition 3.2.2. *Special positive formula: a special positive formula is a disjunction of existentially quantified conjunctions of atoms.*

Given these two definitions it is immediate to notice that we can use the distribution rules over the conjunctions and disjunction, to obtain a special positive formula out of every positive formula.

Now we can give a more general definition of geometric implication:

Definition 3.2.3. *Geometric implication: a geometric implication is a universal closure of an implication $A \to B$, where each A, B, are positive formulas.*

this more general definition will help to make clarity in later proofs, now we give the definition we where using in chapter 2 (we will later prove that these 2 definitions are almost equivalent).

Definition 3.2.4. *Special geometric implication: a special geometric implication is the universal closure of an implication $A \to B$, where A is a conjunction of atoms and B is a special positive formula.*

This is the definition we where used to, and we can prove the following:

Theorem 3.2.1. *Every geometric implication is equivalent to a conjunction of special geometric implications.*

Proof. We pick a geometric implication, it will have the form $A \to B$. We can transform B into a special positive formula. Then for A we use the equivalence $(\forall x(P \vee Q) \to S) \equiv ((\forall xP \to S) \wedge (\forall xQ \to S))$ to eliminate the disjunctions. And $\forall x(\exists yP \to Q) \equiv \forall xy(P \to Q)$ to eliminate the existential quantifiers. So we are done. $\square$

This is useful because of the following definition:

Definition 3.2.5. *Geometric theory: a theory is geometric if it is axiomatized by geometric implications.*

So without loss of generality we can consider a geometric theory to be axiomatized by special geometric implications.

Now we give some (more famous) definitions.

Definition 3.2.6. *Conservative extension: a theory T in a language L is extended by a theory T' in a language L' if L' is extending L and if for every formula $\phi \in L$, we have that $T \vdash \phi$ iff $T' \vdash \phi$.*

Definition 3.2.7. *L-equivalence: two formulae (or theories) are said to be L-equivalent when they prove the same L-formulae.*

Given these definitions we can present our principal result as: any first-order L-theory T is L-equivalent to a geometric theory.

To finish we will quickly recall the standard notions of definitional extension of a theory. The idea is that we can add new predicate symbols to the language, by also adding new axioms to the theory to specify their meaning.

Definition 3.2.8. *Definitional equivalence: Given L a language and P a new predicate symbol not in the language, a definitional equivalence are the sentences $\forall x P(x) \to M(x)$ and $\forall x M(x) \to P(x)$, with M an L-formula.*

this next definition will be the basic "step" necessary to build a definitional extension.

Definition 3.2.9. *Immediate definitional extension: Given a theory T in the language L, and immediate definitional extension of T is given by the addition of a new predicate symbol P to L and a definitional equivalence of P to T as an axiom.*

We notice that after an immediate definitional extension of a theory T by a predicate P that has definitional equivalence with an original predicate M, we might modify the original axioms in T by replacing the M with the P.

Definition 3.2.10. *Definitional extension: Given a theory T in the language L, a definitional extension of T is the multiple application of an immediate definitional extension (we allow the predicate symbols added early to be modified in later steps)*

We can modify these definitions to something weaker but more useful to us, that is, semidefinitional extensions. That is the same idea expressed in the previous definitions, but with only one direction of the implication of the definitional equivalence. But before doing that the following well known definition will be useful:

Definition 3.2.11. *Positive/Negative occurrence in a formula. Given a formula F:*

1. *F is a positive occurrence of F*

2. *If $A \vee B$ is a positive(negative) occurrence in F, then both A and B are positive (negative) occurrences in F*

3. *If $A \wedge B$ is a positive(negative) occurrence in F, then both A and B are positive (negative) occurrences in F*

4. *if $A \rightarrow B$ is a positive(negative) occurrence in F, then both A is a negative(positive) occurrence and B is a positive (negative) occurrence in F.*

With this we can define the following:

Definition 3.2.12. *Positive semidefinitional extension: Given a theory T and a formula $\forall x P(x) \rightarrow M(x)$, we obtain a positive semidefinitional extension of T by replacing any positive occurrence of $M(x)$ by $P(x)$ in the axioms of T, and by adding $\forall x P(x) \rightarrow M(x)$ to the axioms.*

Definition 3.2.13. *Negative semidefinitional extension: Given a theory T and a formula $\forall x M(x) \rightarrow P(x)$, we obtain a negative semidefinitional extension of T by replacing any negative occurrence of $M(x)$ by $P(x)$ in the axioms of T, and by adding $\forall x M(x) \rightarrow P(x)$ to the axioms.*

Definition 3.2.14. *Immediate semidefinitional extension: Given a theory T in the language L, and immediate semidefinitional extension of T is given by the addition of a new predicate symbol P to L and the extension of T by either a negative or positive semidefinitional extension.*

Definition 3.2.15. *Semidefinitional extension. Given a theory T in the language L, a semidefinitional extension of T is the multiple application of non-interacting immediate semidefinitional extension or immediate definitional extensions.*

In this last definitions we are including immediate definitional steps as well for the cases where we want to abbreviate both a positive and negative M. Yet we cannot do this with two semidefinitional extensions because after the first our P would no longer be fresh.

One immediate result that follows immediately from the definitions and is important to notice is

Lemma 3.2.2. *Every definitional extension is a semidefinitional extension.*

Now we are ready to present the result of Skolem.

3.3 Skolem's result

We start with an important definition of Skolem extension.

Definition 3.3.1. *Skolem extension: Given a language L and a theory T in it, a theory T' in a language L' extending L is a Skolem extension when*

1) every theorem of T is a theorem of T'

2) there is a substitution of L-formulae for predicate symbols in $L' \setminus L$ s.t. every theorem of T' is a theorem of T.

With this we prove the following useful theorem.

Theorem 3.3.1. *1) Every semidefinitional extension is a Skolem extension*

2) Every Skolem extension is a conservative extension

3) If T' is a Skolem extension of T, then they are both satisfiable in the same domains, with L being interpreted the same way.

Proof. 1) By definition we can pick an immediate semidefinitional extension. It will be either positive or negative.

If it is positive then we have an axiom $\forall x P(x) \to M(x)$ added to T and a new predicate P added to L. Then pick an axiom A of T with some positive occurrences of $M(x)$, these will be replaced by $P(x)$ in the new axiom A' of T'. Since these occurrences are positive with the new axiom $\forall x P(x) \to M(x)$ we can infer that $A' \to A$. So A is a theorem of T'. This reasoning works similarly for the negative immediate semidefinitional extension. So the first part of the definition of Skolem extension is done.

The second part is also immediate because swapping any occurrence of $P(x)$ with $M(x)$ will get us back to the original theorems or with the tautology $\forall x M(x) \to M(x)$. So we are done.

2) This is immediate because by the second part of the definition of Skolem extension we have a substitution that preserves L-formulae.

3)We have a T' satisfiable in D, we want to prove that T is satisfiable in D as well, but then by the first point of the definition of Skolem extension we are immediately done. Conversely we have a theory T in a domain D with an interpretation I, and we want it's Skolem extension T' to be satisfied in D. We can do this because by the second point of the definition of Skolem extension we have a substitution σ that substitutes predicate symbols in $L' \setminus L$ with L-formulae s.t. every theorem of T' is a theorem of T. Then we can define a new interpretation I' s.t $I'(A) = I(A^\sigma)$ if A is a predicate symbol in $L' \setminus L$, otherwise we set it equal to I. Then we have that everything in T is satisfied in D. So we are done. $\square$

Now we will give a quick example of how to employ Skolem extensions, before extending it to geometric theories.

3.4 Example

Suppose we have the following formula:

$$\forall x \exists y \forall z U(x, y, z)$$

Then we might want to get rid of the last quantifier, by substituting $\forall z U(x, y, z)$ with an single predicate $R(x, y)$, so to obtain the formula $\forall x \exists y R(x, y)$. Intuitively we want a formula expressing that $\forall x, y(R(x, y) \equiv \forall z U(x, y, z))$. To do so we use the definitional equivalence definition from the past sections, that gives us the formulae:

$$\forall x, y\ (R(x, y) \to \forall z U(x, y, z))$$

and

$$\forall x, y\ (\forall z U(x, y, z) \to R(x, y))$$

which become:

$$\forall x, y, z\,(R(x, y) \to U(x, y, z))$$

and

$$\forall x, y\,\exists z\,(U(x, y, z) \to R(x, y))$$

From here it's clear that with this formulae and $\forall x \exists y R(x, y)$ we can prove the initial formula $\forall x \exists y \forall z U(x, y, z)$. More than that, only the formula

$$\forall x, y, z\,(R(x, y) \to U(x, y, z))$$

and $\forall x \exists y R(x, y)$ suffice to prove $\forall x \exists y \forall z U(x, y, z)$, which is the result we desire given that we want geometric axioms for our theories. So now we will argue more formally how to obtain such result.

3.5 An extension of Skolem's extensions

Our first aim will be to consider a generic $\forall\exists$ formula and prove that it can be replaced by special geometric implications. So we consider a generic formula $\forall x \exists y M_1(x, y) \vee \ldots \vee M_n(x, y)$, with each M_j being a conjunction of literals. So each M_j will have the generic form $P_{j1}(x, y) \vee \ldots \vee P_{jr}(x, y) \vee \neg Q_{j1}(x, y) \vee \ldots \vee \neg Qjs(x, y)$. Then we replace each of the s formulas $\neg Q$ with a new predicate N, so to obtain a new formula $\forall x \exists y M'_1(x, y) \vee \ldots \vee M'_n(x, y)$, with each M' being a conjunction of only positive atoms. After distribution of the quantifier this formula becomes $\forall x(\exists\, y M'_1(x, y) \vee \ldots \vee \exists y\, M'_n(x, y))$, which is equivalent to

$$\forall x\ \top \to (\exists\, y M'_1(x, y) \vee \ldots \vee \exists y\, M'_n(x, y))$$

that is, a special geometric implication.

Likewise we have to add an axiom for each literal Q that was substituted. These axioms will be of the form $\forall x, y\, N_{ji}(x, y) \to \neg Q_{ji}(x, y)$, which is

$$\forall x, y\ (N_{ji}(x,y) \wedge Q_{ji}(x,y)) \to \bot$$

again a special geometric implication.

Given these observations we are ready to prove the following

Theorem 3.5.1. *Every $\forall\exists$ theory has a Skolem extension axiomatized by special geometric implications*

Proof. We first need to notice that the formulae $\forall x, y N_{ji}(x,y) \to \neg Q_{ji}(x,y)$ together with $\forall x(\exists\, y M_1'(x,y) \vee ... \vee \exists y\ M_n'(x,y))$ imply the original formula $\forall x \exists y M_1(x,y) \vee ... \vee M_n(x,y)$.

Conversely we also need to show that anything that is provable from the formulae $\forall x, y\ N_{ji}(x,y) \to \neg Q_{ji}(x,y)$ together with $\forall x(\exists\, y M_1'(x,y) \vee ... \vee \exists y\ M_n'(x,y))$, and that doesn't use the new predicate symbols, is also provable from the original $\forall x \exists y M_1(x,y) \vee ... \vee M_n(x,y)$, but we get this from the previous result that every semidefinitional extension is a Skolem extension. So we are done. □

With this result in our hands we now only have to prove that we can replace any theory with a $\forall\exists$ theory. The intuition here is that if we have a formula of the form

$$\forall x \exists y \forall z \exists u\, U(x,y,z,u)$$

Then we can transform it to a $\forall\exists$ formula by using the "definitional" axiom $\forall x, y P(x,y) \to \forall z \exists u\, U(x,y,z,u)$, i.e. $\forall x, y, z\ P(x,y) \to \exists u\, U(x,y,z,u)$, and our old sentence becomes

$$\forall x \exists y\ P(x,y)$$

that is, a $\forall\exists$ formula.

So we have to prove that this works for a finite number of quantifier alternations. In this way we will have the following theorem.

Theorem 3.5.2. *Given A be any first-order sentence, we can construct a finite set special geometric implication that are axioms of a Skolem extension of the theory axiomatised by A.*

Proof. We pick a generic formula A. We can consider it as being of the form:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 .... \forall x_n \exists y_n \; B(x_1, x_2, ...x_n, y_1, y_2, y_n)$$

With B being in disjunctive normal form. Then we can start the process of replacing it with the formulae

$$\forall x_1 \exists y_1 \; P_1(x_1, y_1)$$

and

$$\forall x_1, x_2, y_1 \; (P_1(x_1, y_1) \to \exists y_2, \forall x_3, \exists y_3 ... \forall x_n \exists y_n \; B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$$

In this last formula the subformula $\exists y_2 ... \forall x_n \exists y_n \; B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$ is not yet of the form we want, so we consider another "definition" axiom

$$\forall x_1, x_2, y_1, y_2 \; (P_2(x_1, x_2, y_1, y_2) \to \forall x_3, \exists y_3 ... \forall x_n \exists y_n \; B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$$

And so our formula $\forall x_1, x_2, y_1 (P_1(x_1, y_1) \to \exists y_2, \forall x_3, \exists y_3 ... \forall x_n \exists y_n B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$ can be simplified to a special geometric implication:

$$\forall x_1, x_2, y_1 \; (P_1(x_1, y_1) \to \exists y_2 \; P_2(x_1, x_2, y_1, y_2))$$

But now the formula $\forall x_1, x_2, y_1, y_2 (P_2(x_1, x_2, y_1, y_2) \to \forall x_3, \exists y_3 ... \forall x_n \exists y_n B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$ is still not done yet, we need the addition of another formula of the form

$$\forall x_1, x_2, x_3, y_1, y_2\ (P_2(x_1, x_2, y_1, y_2) \to \exists y_3\ P_3(x_1, x_2, x_3, y_1, y_2, y_3))$$

and for that it will need a definition of the form :

$$\forall x_1, x_2, x_3, y_1, y_2, y_3(P_3(x_1, x_2, x_3, y_1, y_2, y_3) \to \forall x_4, \exists y_4 ... \forall x_n \exists y_n B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$$

which will lead us to another series of implications.

From this it follows that a series of axioms of the following form will be created:

$$\forall x_1 \exists y_1\ P_1(x_1, y_1)$$

$$\forall x_1, x_2, y_1\ (P_1(x_1, y_1) \to \exists y_2\ P_2(x_1, x_2, y_1, y_2))$$

$$\forall x_1, x_2, x_3, y_1, y_2\ (P_2(x_1, x_2, y_1, y_2) \to \exists y_3\ P_3(x_1, x_2, x_3, y_1, y_2, y_3))$$

...

$$\forall x_1, ...x_{i+1}, y_1, ...y_i\ (P_i(x_1, ...x_i, y_1, ..y_i) \to \exists y_{i+1}\ P_{i+1}(x_1, ...x_{i+1}, y_1 ... y_{i+1}))$$

until we reach the final

$$\forall x_1, ...x_n, y_1, ...y_{n-1}\ (P_{n-1}(x_1, ...x_{n-1}, y_1, ..y_{n-1}) \to \exists y_n\ B(x_1, x_2, ...x_n, y_1, y_2, ...y_n))$$

Now that we have a procedure to generate e finite list of $\forall\exists$ axioms, we can use the result of theorem 5.1, to obtain a Skolem extension axiomatized by special geometric implications. $\square$

With this result and with theorem 3.1 we get the immediate result that we want

Theorem 3.5.3. *For every first-order theory T, we can construct a theory T' that is a geometric conservative extension of T.*

Given this we can conclude that we are able to use the results from chapter 2 on any theory. That is, we can get a cut-free calculus out of any first order theory, by first transforming it into a geometric theory.

Chapter 4

Relation Algebra and Sequent Calculi

In this chapter we will see how to utilize the techniques that we learned in the previous chapters. In particular we will find a cut-free calculus for the theory of relation algebras.

4.1 Historical remarks

In 1864, after the pioneering work of George Boole [16], Augustus de Morgan writes the first paper on the subject of the logic of relations. De Morgan realized that the logic of that time was inadequate to express some types of arguments, even some trivial ones. As he famously noted, the logic of Aristotle was not able to deduce, from the fact that a horse is an animal, that the head of a horse is the head of an animal. In his efforts to expand the limits of traditional logic, he was the first to try to use the concept of relations, and to recognize its significance.

Later Charles Saunders Peirce would give a foundation to the theory of relations in 1883[17], by making precise all the fundamental concepts of the theory of relations and formulating its fundamental laws. In this work he would also discuss the operation of relational composition, and give one of the first intuitions about the need of a notion of quantifier distinct from the classical boolean connectives[18]. In doing this, he made clear that a part of the theory theory of relations can be presented as a calculus, formally similar to the calculus of classes developed by G. Boole, but with greater expressive power.

An extension of the work of Peirce on the calculus of relations was then accomplished by Ernst Schröder[19], and with this work the theory seemed to be interesting and rich enough to become a field of study per se. For example, the initial paper[20][21] by Leopold Löwenheim (where a version of the downward Löwenheim-Skolem theorem was first proved) was written based on the calculus of relations. Furthermore Whitehead and Russel, in the Principia Mathematica, included the theory of relations as one of the core parts of symbolic logic. They would write [22]:

> "The subject of symbolic logic is formed by three parts: the calculus of propositions, the calculus of classes, and the calculus of relations"

Yet the calculus of relations soon experienced a setback, when Alwin Korselt proved the existence of some formulae in which four deep nested quantifiers had no equivalent in the calculus of relations[23].

This led to a loss of interest in the field until the work of Tarski. In 1941 he published the paper "On the calculus of relations"[24] in which he formulated the calculus as a finitely axiomatized FOL theory. In this paper Tarski notes also how this system might be better expressed by using an equational formulation, that is by developing a calculus that only deals with relations and doesn't need to delve into FOL. This created the field of relation algebra. Throughout the years various results succeeded in showing the expressive power of such system, with one of the most famous results due to Tarki[25], establishing that a relation algebra can express any FOL formula with at maximum three variables. Then it is provable that this fragment of FOL is sufficient to express Peano arithmetic, and this entails that relation algebras can be used to effectively algebrize almost all of mathematics, without the cost that comes with quantifiers and connectives from FOL. Because of these reasons we choose relation algebras as an interesting theory to be explored with a sequent calculus.

4.1.1 Relation algebra

We start by giving a quick presentation of the current formalization of a relation algebra. We do this just to give an understanding of the calculus under the most current mathematical framework, as later we will construct our sequent calculus starting from the original formalization of Tarski, given in FOL.

A Relation Algebra(RA) is a structure $\mathbb{A} = \langle A, +, -, \odot, \smile, \mathring{1} \rangle$ of type $\langle 2, 1, 2, 1, 0 \rangle$. The intuitive meaning of A is that of a set of binary relations. $+$ is the absolute or Boolean addition of relations, that is if xRy is "x loves y" and xPy is "x hates y", then $xR + Py$ is "x loves or hates y". $-$ is the complementation, so if R is the relation of "loving", then R^- is the relations of "not loving". $\odot$ is the relative or Peircean multiplication, so if R is the relation of "behaving kindly" and P is the relation of "being a friend of" then $R \odot P$ is "behaving kindly with a friend of". $\smile$ the formation of converses, so if R is "loving" then $R^{\smile}$ is "being loved". And $\mathring{1}$ the identity element or Peircean unit, with the meaning of "is the same as". (for a more intuitive meaning, we will present the FOL description of the Relation operation symbols after presenting the conditions of a RA). We notice that RA does not use individual variables $x, y, ...$ to specify the relations, the algebra has as elements only the binary relations.

There are more operations in the classical calculus of relations, but they can be all described in terms of these fundamental operations. So, given $R, P \in A$ we have that:

- The absolute or Boolean multiplication $R \cdot P$ is $(R^- + P^-)^-$ (if R is "loving" and P is "hating", then $R \cdot P$ is "loving and hating").
- The relative or Peircean addition $R \oplus P$ is $(R^- \odot P^-)^-$ (if R is the relation of "behaving kindly" and P is the relation of "being a friend of" then $R \oplus P$ is the relation "behaving kindly with everyone except the friends of")
- The relation of inclusion $R \leq P$ is $R + P = P$

- The Boolean unit 1 is $\mathring{1} + \mathring{1}^-$, and has standard meaning of a relation that always holds.

- The Boolean zero 0 is 1^- or $(\mathring{1}+\mathring{1}^-)^-$, and has standard meaning of a relation that never holds.

- The diversity element or Peircean zero $\mathring{0}$ is $\mathring{1}^-$, and has standard meaning of a relation that holds between two elements only if the two elements are different.

Then we have some conditions, that can be seen as axioms schemata, that have to be satisfied. The first 3 conditions express the fact that our addition and complementation operators are the same as in a Boolean algebra:

$$R + P \mathring{=} P + R \tag{A1}$$

$$R + (P + S) \mathring{=} (P + R) + S \tag{A2}$$

$$(R^- + P)^- + (R^- + P^-)^- \mathring{=} R \tag{A3}$$

Them the second group of conditions say that an interpretation of the Peircean multiplication is a semigroup operation and the interpretation of the identity is an identity element with respect to the Peircean multiplication:

$$R \odot (P \odot S) \mathring{=} (P \odot R) \odot S \tag{A4}$$

$$R \odot \mathring{1} \mathring{=} R \tag{A5}$$

The third group of conditions say that an interpretation of the converse operator is an involution with respect to the semigroup operation:

$$R^{\smile\smile} \mathrel{\mathring{=}} R \tag{A6}$$

$$(R \odot P)^{\smile} \mathrel{\mathring{=}} P^{\smile} \odot R^{\smile} \tag{A7}$$

And the fourth group says that the interpretations of the Peircean multiplication and converse operators distribute across addition:

$$(P + R) \odot S \mathrel{\mathring{=}} (P \odot S) + (R \odot S) \tag{A8}$$

$$(R + P)^{\smile} \mathrel{\mathring{=}} R^{\smile} + P^{\smile} \tag{A9}$$

The final condition is due to Tarski, it is a simplification of a set of equivalence results of De Morgan, put into a single equational form:

$$(R^{\smile} \odot (R \odot P)^{-}) + P^{-} \mathrel{\mathring{=}} P^{-} \tag{A10}$$

with the original de Morgan equivalences being:

$$R \odot P \leq S^{-}\ iff\ R^{\smile} \odot S \leq P^{-}\ iff\ S \odot P^{\smile} \leq R^{-}$$

With the standard interpretation of $R \leq P$ iff $R + P = P$.

Yet this formalization of relation algebra emerged only in a second moment, with the original article by Tarski first describing the operations between relations as axioms of a FOL theory, and then deducing the rules that we have just seen as theorems of this theory. We will focus on the following FOL description, as it is easier to derive a sequent calculi from a FOL theory.

We augment our FOL language with new relation variables, $R, S, T, \ldots$. Then from the symbols for the relational constants and operations described earlier ($+, -, \odot, \smile, \cdot, \oplus, \mathring{1}, \mathring{0}, 1, 0$) and a special relation equality symbol $=$, we augment the relational variables: we combine relation constants and relation variables with a single unary relation operation symbol, or we join two of them with a binary operation symbol, and so on. So we form compounds such as 1^-, R^-, $R^- \odot P$, $R^- \odot P + 1^-$, etc. Then we also add any expression of the form xRy with x, y individual variables and R a relation variable. Then the axioms of the theory of relation are the following

$$\forall x \forall y \; x1y \tag{T1}$$

$$\forall x \forall y \; \neg x0y \tag{T2}$$

$$\forall x \; x\mathring{1}x \tag{T3}$$

$$\forall x \forall y \forall z \; (xRy \wedge y\mathring{1}z) \rightarrow xRz \tag{T4}$$

$$\forall x \forall y \; x\mathring{0}y \iff \neg x\mathring{1}y \tag{T5}$$

$$\forall x \forall y \; xR^-y \iff \neg xRy \tag{T6}$$

$$\forall x \forall y \; xR^{\smile}y \iff yRx \tag{T7}$$

$$\forall x \forall y \; xR + Py \iff xRy \vee xPy \tag{T8}$$

$$\forall x \forall y\, xR \cdot Py \iff xRy \wedge xPy \tag{T9}$$

$$\forall x \forall y\, xR \oplus Py \iff \forall z (xRz \vee zPy) \tag{T10}$$

$$\forall x \forall y\, xR \odot Py \iff \exists z (xRz \wedge zPy) \tag{T11}$$

$$R \mathrel{\mathring{=}} P \iff \forall x \forall y (xRy \iff xPy) \tag{T12}$$

All of these axioms can be turned into sequent calculi rules with the procedure defined previously, by converting them first into geometric axioms.

4.1.2 Geometric axioms for RA

The axioms of Tarski can easily be converted to geometric axioms, then in the next section we will use these geometric axioms to get to a sequent calculus for relation algebras.

The axiom T1 is easily converted into the geometric counterpart using *verum ad quodlibet*:

$$\forall x \forall y\, \top \to x1y \tag{GT1}$$

T2 is similarly simple:

$$\forall x \forall y\, x0y \to \bot \tag{GT2}$$

T3 is the same as T1:

$$\forall x\, \top \to x\mathring{1}x \tag{GT3}$$

T4 is already geometric:

$$\forall x \forall y \forall z\, (xRy \wedge y\mathring{1}z) \to xRz \tag{GT4}$$

With T5 ($\forall x \forall y\, x\mathring{0}y \iff \neg x\mathring{1}y$)we start to be more careful. Given the biconditional we can split it in into two axioms T5.1 ($\forall x \forall y\, x\mathring{0}y \to \neg x\mathring{1}y$)and T5.2 ($\forall x \forall y\, \neg x\mathring{1}y \to x\mathring{0}y$). T5.1 can be transformed using logical equivalences into $\forall x \forall y\, \neg(x\mathring{0}y \wedge x\mathring{1}y)$, which is geometric:

$$\forall x \forall y\, (x\mathring{0}y \wedge x\mathring{1}y) \to \bot \tag{GT5.1}$$

T5.2 is easily transformed into $\forall x \forall y\, x\mathring{1} \vee x\mathring{0}y$, which is:

$$\forall x \forall y\, \top \to y\, x\mathring{1}y \vee x\mathring{0}y \tag{GT5.2}$$

For T6 ($\forall x \forall y\, xR^{-}y \iff \neg xRy$) we follow the exact same reasoning:

$$\forall x \forall y\, (xR^{-}y \wedge xRy) \to \bot \tag{GT6.1}$$

$$\forall x \forall y\, \top \to\ xRy \vee xR^{-}y \tag{GT6.2}$$

for T7 ($\forall x \forall y\, xR^{\smile}y \iff yRx$), we split it into two geometric axioms:

$$\forall x \forall y\, xR^{\smile}y \to yRx \tag{GT7.1}$$

$$\forall x \forall y\, yRx \rightarrow xR^{\smile}y \tag{GT7.2}$$

For T8 ($\forall x \forall y\, xR+Py \iff xRy \vee xPy$) we split it as usual into T8.1 ($\forall x \forall y\, xR+Py \rightarrow xRy \vee xPy$), which is geometric, and T8.2 ($\forall x \forall y\, (xRy \vee xPy) \rightarrow xR+Py$). For T8.2 we can use the equivalence $(A \vee B) \rightarrow C \iff (A \rightarrow C) \wedge (B \rightarrow C)$ to get T8.2.1 $\forall x \forall y\, xRy \rightarrow xR+Py$) and T8.2.2 $\forall x \forall y\, xPy \rightarrow xR+Py$). This leads us to the geometric axioms:

$$\forall x \forall y\, xR+Py \rightarrow xRy \vee xPy \tag{GT8.1}$$

$$\forall x \forall y\, xRy \rightarrow xR+Py \tag{GT8.2.1}$$

$$\forall x \forall y\, xPy \rightarrow xR+Py \tag{GT8.2.2}$$

For T9 we are immediately done:

$$\forall x \forall y\, xR\cdot Py \rightarrow xRy \wedge xPy \tag{GT9.1}$$

$$\forall x \forall y\, xRy \wedge xPy \rightarrow xR\cdot Py \tag{GT9.2}$$

For T10 ($\forall x \forall y\, xR \oplus Py \iff \forall z(xRz \vee zPy)$)we have to be more careful. We first split it into T10.1: $\forall x \forall y\, xR \oplus Py \rightarrow \forall z(xRz \vee zPy)$, which is geometric after moving out the quantifier, and T10.2: $\forall x \forall y\, \forall z(xRz \vee zPy) \rightarrow xR \oplus Py$. After some logical equivalences, we get T10.2: $\forall x \forall y \exists z\, (\neg xRz \wedge \neg zPy) \vee xR \oplus Py$, which we can make geometric if we can get rid of the negations. So we have T10.2.1 : $\forall x \forall y \exists z\, (xR'z \wedge zP'y) \vee xR \oplus Py$ and T10.2.2: $\forall x \forall z\, xR'z \rightarrow \neg xRz$, which is

$\forall x \forall z\, xR'z \wedge xRz \to \bot$, and T10.2.3: $\forall x \forall z\, xP'z \wedge xPz \to \bot$. This gives us the geometric axioms:

$$\forall x \forall y \forall z\, xR \oplus Py \to (xRz \vee zPy) \tag{GT10.1}$$

$$\forall x \forall y\, \top \to \exists z((xR'z \wedge zP'y) \vee xR \oplus Py) \tag{GT10.2.1}$$

$$\forall x \forall z\, xR'z \wedge xRz \to \bot \tag{GT10.2.2}$$

$$\forall x \forall z\, xP'z \wedge xPz \to \bot \tag{GT10.2.3}$$

For T11 ($\forall x \forall y xR \odot Py \iff \exists z(xRz \wedge zPy)$) we have it split into T11.1 $\forall x \forall y xR \odot Py \to \exists z(xRz \wedge zPy)$ and T11.2 $\forall x \forall y \forall z\, (xRz \wedge zPy) \to xR \odot Py$, which are geometric:

$$\forall x \forall y\, xR \odot Py \to \exists z(xRz \wedge zPy) \tag{GT11.1}$$

$$\forall x \forall y \forall z\, (xRz \wedge zPy) \to xR \odot Py \tag{GT11.2}$$

For T12 ($R \circeq P \iff \forall x \forall y(xRy \iff xPy)$) we have to first split it into T12.1: $R \circeq P \to \forall x \forall y(xRy \iff xPy)$ and T12.2: $(\forall x \forall y(xRy \iff xPy)) \to R \circeq P$. Then T12.1 is equivalent to : $\forall x \forall y\, R \circeq P \to ((\neg xRy \wedge \neg xPy) \vee (xRy \wedge xPy))$, which we can make geometric by introducing new relations: T12.1.1 is $\forall x \forall y\, R \circeq P \to ((xR'y \wedge xP'y) \vee (xRy \wedge xPy))$, T12.1.2 is $\forall x \forall y\, xR'y \to \neg xRy$ and T12.1.3 is $\forall x \forall y\, xP'y \to \neg xPy$, but these last two axioms we already have (up to renaming). For T12.2 we can rewrite it as: $\exists x \exists y\, \neg(xRy \iff xPy) \vee R \circeq P$ which is $\exists x \exists y\, \neg((\neg xRy \vee xPy) \wedge (\neg xPy \vee xRy)) \vee R \circeq P$, which is $\exists x \exists y\, ((xRy \wedge \neg xPy) \vee (xPy \wedge \neg xRy)) \vee R \circeq P$, for which we use the renaming trick again, and it becomes T12.2.1: $\exists x \exists y((xRy \wedge xP'y) \vee (xPy \wedge xR'y)) \vee R \mathring{=} P$ and T12.2.2:$\forall x \forall y\, xR'y \rightarrow \neg xRy$ and T12.2.3: $\forall x \forall y\, xP'y \rightarrow \neg xPy$, but again these last two axioms we already have. So for this last axioms we have:

$$\forall x \forall y\; R \mathring{=} P \rightarrow ((xR'y \wedge xP'y) \vee (xRy \wedge xPy)) \tag{GT12.1.1}$$

$$\top \rightarrow \exists x \exists y\; ((xRy \wedge xP'y) \vee (xPy \wedge xR'y)) \vee R \mathring{=} P \tag{GT12.2.1}$$

4.1.3 A sequent calculus for RA

Now that we have the geometric version of the axioms we are ready to present a sequent calculus for relation algebras. If we transform GT1 to a rule, we would get the rule that introduces the "1" relation, and we would get:

$$\frac{\top, x1y, \Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}\ 1$$

Now to deal with this rule and the following ones that contain a $\top$ symbol, we would need to introduce the rule

$$\frac{\top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \top$$

But to avoid unnecessary complications we will take out the $\top$ symbol from all the rules. So our rule for the introduction of the 1 is:

$$\frac{x1y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ 1$$

The GT2 axiom will corresponds to the impossibility of the "0" relation:

$$\frac{}{x0y, \Gamma \Rightarrow \Delta}\ 0$$

GT3 is the axiom that introduces the $\mathring{1}$ relation:

$$\frac{x\mathring{1}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{I}\mathring{1}$$

GT4 is a sort of identity rule for the $\mathring{1}$, that states:

$$\frac{xRy, y\mathring{1}z, xRz\Gamma \Rightarrow \Delta}{xRy, y\mathring{1}z, \Gamma \Rightarrow \Delta}\ \mathrm{E}\mathring{1}$$

GT5.1 and GT5.2 specify the behaviour of $\mathring{0}$, with one being an axiomatic rule and another being an introduction:

$$\frac{}{x\mathring{1}y, x\mathring{0}y, \Gamma \Rightarrow \Delta}\ \mathrm{Ax}\mathring{0}$$

$$\frac{x\mathring{1}y, \Gamma \Rightarrow \Delta \qquad x\mathring{0}y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{I}\mathring{0}$$

GT6.1 and GT6.2 deal with R^{-}, similarly to the previous rules:

$$\frac{}{xR^{-}y, xRy, \Gamma \Rightarrow \Delta}\ \mathrm{Ax}R^{-}$$

$$\frac{xRy, \Gamma \Rightarrow \Delta \qquad xR^{-}y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{I}R^{-}$$

GT7.1 and GT7.2 specify $R^{\smile}$, saying that you can always get the converse of a relation:

$$\frac{yRx, xR^{\smile}y, \Gamma \Rightarrow \Delta}{yRx, \Gamma \Rightarrow \Delta}\ \mathrm{I1}R^{\smile}$$

$$\frac{xR^{\smile}y, yRx, \Gamma \Rightarrow \Delta}{xR^{\smile}y, \Gamma \Rightarrow \Delta}\ \mathrm{I2}R^{\smile}$$

GT8 specifies how $R+P$ acts, with the first rule GT8.1 "eliminating" the $+$, and GT8.2.1 and GT8.2.2 being equal up to renaming, and "introducing" the $+$:

$$\frac{xR+Py, xRy, \Gamma \Rightarrow \Delta \qquad xR+Py, xPy, \Gamma \Rightarrow \Delta}{xR+Py, \Gamma \Rightarrow \Delta}\ \mathrm{E}+$$

$$\frac{xRy, xR+Py, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ \mathrm{I}+$$

GT9.1 similarly eliminates the $\cdot$, while GT9.2 introduces it:

$$\frac{xR\cdot Py, xRy, xPy, \Gamma \Rightarrow \Delta}{xR\cdot Py, \Gamma \Rightarrow \Delta}\ \mathrm{E}\cdot$$

$$\frac{xRy, xPy, xR\cdot Py, \Gamma \Rightarrow \Delta}{xRy, xPy, \Gamma \Rightarrow \Delta}\ \mathrm{I}\cdot$$

Given the meaning of $+$ and $\cdot$, we notice how their rules look a bit similar to the rules for $\wedge$ and $\vee$.

GT10.1 deals with how to eliminate $R\oplus P$, while GT10.2.1 is a sort of introduction. GT10.2.2 and GT10.2.3 are the same up to renaming, and are used as axiomatic rules:

$$\frac{xR\oplus Py, xRz, \Gamma \Rightarrow \Delta \qquad xR\oplus Py, zPy, \Gamma \Rightarrow \Delta}{xR\oplus Py, \Gamma \Rightarrow \Delta}\ \mathrm{E}\oplus$$

$$\frac{xR'z, zP'y, \Gamma \Rightarrow \Delta \qquad xR\oplus Py, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{I}\oplus$$

$$\frac{}{xRy, xR'y, \Gamma \Rightarrow \Delta}\ \mathrm{Ax}R'$$

GT11.1 eliminates the $\odot$ while GT11.2 introduces it:

$$\frac{xR\odot Py, xRz, zPy, \Gamma \Rightarrow \Delta}{xR\odot Py, \Gamma \Rightarrow \Delta}\ \mathrm{E}\odot$$

(and z is not free in the conclusions)

$$\frac{xRz, zPy, xR \odot Py, \Gamma \Rightarrow \Delta}{xRz, zPy, \Gamma \Rightarrow \Delta}\ \mathrm{I}\odot$$

GT12 is important because it tells us how to deal with equations between relations. GT12.1.1 is a sort of elimination while GT12.2.1 is a sort of introduction.

$$\frac{R\mathring{=}P, xR'y, xP'y, \Gamma \Rightarrow \Delta \qquad R\mathring{=}P, xRy, xPy\Gamma \Rightarrow \Delta}{R\mathring{=}P, \Gamma \Rightarrow \Delta}\ \mathrm{E}\mathring{=}$$

$$\frac{R\mathring{=}P, \Gamma \Rightarrow \Delta \qquad R\mathring{=}P, xRy, xP'y\Gamma \Rightarrow \Delta \qquad R\mathring{=}P, xR'y, xPy\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{E}\mathring{=}$$

4.2 Example of use

To give some intuitions about the us of the calculus, suppose we want to prove the obvious equivalence $R^{--}\mathring{=}R$, we have:

$$\frac{\dfrac{}{R^{--}\mathring{=}R, \Gamma \Rightarrow R^{--}\mathring{=}R}\ \mathrm{Ax} \qquad xRy, xR^{--\prime}y \Rightarrow R^{--}\mathring{=}R \qquad xR^{--}y, xR'y \Rightarrow R^{--}\mathring{=}R}{\Rightarrow R^{--}\mathring{=}R}\ \mathrm{E}\mathring{=}$$

The first sequent is an axiom by a rule of FOL sequent calculi. For the rest of the sequents in the antecedent we notice that a formula of the type A' is tractable by only AxR'. So if we consider the middle sequent, we conclude that we have to find a way to transform the xRy into an $xR^{--}y$, so that we can use the last one in conjunction with the $xR^{--\prime}y$ to use the AxR' rule. So we prove the middle sequent by:

$$\frac{\dfrac{}{xRy, R^{--\prime}, xR^{-}y,\ \Rightarrow \Delta}\ \mathrm{Ax}R^{-} \qquad \dfrac{}{xRy, R^{--\prime}, xR^{--}y \Rightarrow \Delta}\ \mathrm{Ax}R'}{xRy, R^{--\prime} \Rightarrow R^{--}\mathring{=}R}\ \mathrm{I}R^{-}$$

Where the first sequent is an axiom by AxR^- and indeed the second is an axiom by AxR'. For the sequent on the right $R^{--}, xR'yR^{--} \mathrel{\mathring{=}} R$ the proof is similar.

We notice we have to use a bit of creativity in the proof, but we are still limited in what we can use. This limit comes from the rules of the sequent calculus: indeed one could say that we have obtained some rules of the form

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where one is free to chose what relations to chose for A, B. Yet in searching for a proof of $R \mathrel{\mathring{=}} P$ we will always be limited by the necessity to reconstruct R from P and vice-versa, so to use the AxR' rule. This makes the possible rules to use limited to the relation operations in R, P. And for the same reasons also the possible instantiations of A, B from our rules will be limited to the subterms of R, P.

4.3 Simplifying the calculus

We could argue that the calculus we obtained is overburdened by rules, so in this section we will quickly try to reduced the number of rules needed.

One of the first things to notice is that the addition of the $\mathring{=}$ symbol, while clearly necessary in a FOL domain to distinguish between different meanings of equality, in the calculus plays a very limited role. Indeed if the only sequents that are to be proved are of the form $\Rightarrow R \mathrel{\mathring{=}} P$, then we might as well omit any rule for the $\mathring{=}$ and instead accept that we are always proving equivalent sequents of the form $\Rightarrow \forall x \forall y\, ((xRy \to xPy) \land (xPy \to xRy))$. This means that for any R, P we would always have a proof starting like this:

$$\dfrac{\dfrac{xPy \Rightarrow xRy}{\Rightarrow xPy \to xRy}\to \qquad \dfrac{xRy \Rightarrow xPy}{\Rightarrow xRy \to xPy}\to}{\Rightarrow \forall x \forall y\, ((xRy \to xPy) \land (xPy \to xRy))}\forall, \land$$

This leads to observe that to prove $\Rightarrow R \mathrel{\mathring{=}} P$, what we want is to prove $R \Rightarrow P$ and $P \Rightarrow R$ (this can be seen informally as proving two subset inclusions). This is interesting because by inspecting our rules for the relations calculus we notice that all our rules are modifying the left part of a sequent. This leads to the conclusion that one possible strategy for proving a sequent of the form $R \Rightarrow P$ will be to use the relation calculus rules we derived to construct P from R, to obtain a sequent $P \Rightarrow P$ that we can close by using the Ax rule from the G3c calculus.

Furthermore we can simplify the calculus even more by noticing that the famous equivalences $R \cdot P \stackrel{\circ}{=} (R^- + P^-)^-$, $R \oplus P \stackrel{\circ}{=} (R^- \odot P^-)^-$, $1 \stackrel{\circ}{=} \mathring{1} + \mathring{1}^-$, $0 \stackrel{\circ}{=} (\mathring{1} + \mathring{1}^-)^-$, $\mathring{0} \stackrel{\circ}{=} \mathring{1}^-$, hold in our calculus. As an example we prove the first one, by first proving $xR \cdot Py \Rightarrow (R^- + P^-)^-$:

$$\dfrac{\dfrac{}{\dots, xRy, xPy, x(R^-+P^-)^-y \Rightarrow (R^-+P^-)^-}\,\text{Ax} \qquad \dfrac{\dots, xRy, xR^-y \Rightarrow (R^-+P^-)^- \qquad \dots, xPy, xP^-y \Rightarrow (R^-+P^-)^-}{\dots, xRy, xPy, x(R^-+P^-)y \Rightarrow (R^-+P^-)^-}\,\text{E}+}{\dfrac{\dots, xRy, xPy \Rightarrow (R^-+P^-)^-}{xR \cdot Py \Rightarrow (R^-+P^-)^-}\,\text{E}\cdot}\,\text{I}R^-$$

And then proving $(R^- + P^-)^- \Rightarrow xR \cdot Py$:

$$\dfrac{\dfrac{\dfrac{}{(R^-+P^-)^-, xR^-y, xR^-+P^-y \Rightarrow xR \cdot Py}\,\text{Ax}R-}{(R^-+P^-)^-, xR^-y \Rightarrow xR \cdot Py}\,\text{I}+ \qquad \dfrac{\dfrac{\dots, xR \cdot Py \Rightarrow xR \cdot Py}{(R^-+P^-)^-, xRy, xPy \Rightarrow xR \cdot Py}\,\text{I}\cdot \qquad \dfrac{\dots, (R^-+P^-)^-, (R^-+P^-) \Rightarrow xR \cdot Py}{(R^-+P^-)^-, xRy, xP^-y \Rightarrow xR \cdot Py}\,\text{I}+}{(R^-+P^-)^-, xRy \Rightarrow xR \cdot Py}\,\text{E}+}{(R^-+P^-)^- \Rightarrow xR \cdot Py}\,\text{I}R^-$$

This leads us to the conclusion that we can drop all the rules related to the $\oplus, \cdot, 1, 0, \mathring{0}$ operations. Now we can finally define our sequent calculus for relation algebras, which we will call SeRA. SeRA will be composed by all the rules from G3c (although only the Ax rule will be useful for proving equivalences between relations), and the following rules:

$$\dfrac{x\mathring{1}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{I}\mathring{1}$$

$$\frac{xRy, y\mathring{1}z, xRz, \Gamma \Rightarrow \Delta}{xRy, y\mathring{1}z, \Gamma \Rightarrow \Delta}\ \mathrm{E}\mathring{1}$$

$$\frac{}{xR^{-}y, xRy, \Gamma \Rightarrow \Delta}\ \mathrm{Ax}R^{-}$$

$$\frac{xRy, \Gamma \Rightarrow \Delta \qquad xR^{-}y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{I}R^{-}$$

$$\frac{yRx, xR^{\smile}y, \Gamma \Rightarrow \Delta}{yRx, \Gamma \Rightarrow \Delta}\ \mathrm{I1}R^{\smile}$$

$$\frac{xR^{\smile}y, yRx, \Gamma \Rightarrow \Delta}{xR^{\smile}y, \Gamma \Rightarrow \Delta}\ \mathrm{I2}R^{\smile}$$

$$\frac{xR+Py, xRy, \Gamma \Rightarrow \Delta \qquad xR+Py, xPy, \Gamma \Rightarrow \Delta}{xR+Py, \Gamma \Rightarrow \Delta}\ \mathrm{E}+$$

$$\frac{xRy, xR+Py, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ \mathrm{I}+$$

$$\frac{xR\odot Py, xRz, zPy, \Gamma \Rightarrow \Delta}{xR\odot Py, \Gamma \Rightarrow \Delta}\ \mathrm{E}\odot$$

(and z is not free in the conclusions)

$$\frac{xRz, zPy, xR\odot Py, \Gamma \Rightarrow \Delta}{xRz, zPy, \Gamma \Rightarrow \Delta}\ \mathrm{I}\odot$$

4.4 A comparison with the Maddux sequent calculus of relation algebras

Historically this is not the first sequent calculus developed for the calculus of relations[5]. In 1983 Roger Maddux developed a sequent calculus, which we will call MaRA, in which a sequent $P \Rightarrow R$ had the interpretation of $P \subseteq R$, and was characterized by the following rules:

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\,\mathrm{MAx}$$

$$\frac{}{x0y, \Gamma \Rightarrow \Delta}\,\mathrm{M0}$$

$$\frac{}{\Gamma \Rightarrow x1y, \Delta}\,\mathrm{M1}$$

$$\frac{}{\Gamma \Rightarrow x\mathring{1}y, \Delta}\,\mathrm{M}\mathring{1}$$

$$\frac{xRz, \Gamma \Rightarrow \Delta}{xRy, y\mathring{1}z, \Gamma \Rightarrow \Delta}\,\mathrm{Mid}$$

$$\frac{\Gamma \Rightarrow xRy, \Delta}{xR^{-}y, \Gamma \Rightarrow \Delta}\,\mathrm{ML\text{-}}$$

$$\frac{\Gamma, xRy \Rightarrow \Delta}{\Gamma \Rightarrow xR^{-}y, \Delta}\,\mathrm{MR\text{-}}$$

$$\frac{\Gamma, yRx \Rightarrow \Delta}{\Gamma, xR^{\smile}y \Rightarrow \Delta}\,\mathrm{ML}\smile$$

$$\frac{\Gamma \Rightarrow yRx\Delta}{\Gamma \Rightarrow xR^{\smile}y, \Delta}\,\mathrm{MR}\smile$$

$$\frac{xRy, \Gamma \Rightarrow \Delta \qquad xPy, \Gamma \Rightarrow \Delta}{xR + Py, \Gamma \Rightarrow \Delta}\,\mathrm{ML}+$$

$$\frac{\Gamma \Rightarrow xRy, xPy, \Delta}{\Gamma \Rightarrow xR + Py, \Delta}\,\mathrm{MR}+$$

$$\frac{xRy, xPy, \Gamma \Rightarrow \Delta}{xR \cdot Py, \Gamma \Rightarrow \Delta}\,\mathrm{ML}\cdot$$

$$\frac{\Gamma \Rightarrow xRy, \Delta \qquad \Gamma \Rightarrow xPy, \Delta}{\Gamma \Rightarrow xR \cdot Py, \Delta}\,\mathrm{MR}\cdot$$

$$\frac{xRz, zPy, \Gamma \Rightarrow \Delta}{xR \odot Py, \Gamma \Rightarrow \Delta}\,\mathrm{ML}\odot$$

with z not appearing in the conclusion of the ML$\odot$

$$\frac{\Gamma \Rightarrow xRz, \Delta \qquad \Gamma \Rightarrow zPy, \Delta}{\Gamma \Rightarrow xR \odot Py, \Delta}\,\mathrm{MR}\odot$$

plus the rules for cut elimination, weakening and contraction.

Maddux used this calculus to successfully find a new proof for the older results of Tarski that a FOL sentence with no more than three variables can be translated into an equation in the calculus of relations. And furthermore Maddux was also able to find an algebraic semantics for this calculus.

We first prove that our calculus and MaRA are equivalent if we allow the cut rule in SeRA. To do so we employ a proof strategy based on showing how the rules of a calculus are derivable in another one.

Definition 4.4.1. *We say that a calculus A is able to simulate a calculus B when any rule R of B is admissible in A, that is, when the conclusion of R is derivable in A, if the premises of A are derivable in A.*

Theorem 4.4.1. *SeRA can simulate MaRA. Furthermore MaRA can simulate the non-logical part of SeRA.*

Proof. We prove that we can simulate all the rules of MaRA using SeRA+Cut.

MAx is the same as in our calculus. M0 as well.

M$\mathring{1}$ is (from now on we will omit the Γ and Δ for clarity):

$$\dfrac{\overline{x\mathring{1}x \Rightarrow x\mathring{1}x}\,\mathrm{Ax}}{\Rightarrow x\mathring{1}x}\,\mathrm{I}\mathring{1}$$

M1 is similar. Mid is immediate with our rule E$\mathring{1}$.

For ML- we have to use a cut:

$$\dfrac{\Rightarrow xRy \qquad \dfrac{}{xR^{-}y, xRy \Rightarrow}\text{AxR-}}{xR^{-}y \Rightarrow}\text{cut}$$

with the first premise being a premise in ML- and the second premise being our AxR-.

For MR-:

$$\dfrac{\dfrac{}{xR^{-}y \Rightarrow xR^{-}y}\text{Ax} \qquad xRy \Rightarrow xR^{-}y}{\Rightarrow xR^{-}y}\text{R-}$$

with the first being an axiom and the second being a premise in MR-.

For ML$\smile$ is the same as in our calculus. for MR$\smile$:

$$\dfrac{\Rightarrow yRx \qquad \dfrac{}{xR^{-}y, yRx \Rightarrow xR^{\smile}y}\text{AxR-}}{\Rightarrow xR^{\smile}y}\text{cut}$$

With the first being a premise from Maddux and the second our axiom AxR-.

For ML+ is the same as in our calculus. For MR+:

$$\dfrac{\dfrac{\Rightarrow xRy, xPy}{\Rightarrow xRy \vee xPy}\text{R}\vee \qquad \dfrac{\dfrac{\dfrac{}{xRy, xR+Py \Rightarrow xR+Py}\text{Ax}}{xRy \Rightarrow xR+Py}\text{I+} \qquad \dfrac{\dfrac{}{xPy, xR+Py \Rightarrow xR+Py}\text{Ax}}{xPy \Rightarrow xR+Py}\text{I+}}{xRy \vee xPy \Rightarrow xR+Py}\text{L}\vee}{\Rightarrow xR+Py}\text{cut}$$

With the first premise being one of the premises of Maddux.

For ML$\cdot$ is the same as in our calculus, for MR$\cdot$:

$$\dfrac{\dfrac{\Rightarrow xRy \qquad \Rightarrow xPy}{\Rightarrow xRy \wedge xPy}\text{R}\wedge \qquad \dfrac{\dfrac{\dfrac{}{xRy, xPy, xR\cdot Py \Rightarrow xR\cdot Py}\text{Ax}}{xRy, xPy \Rightarrow xR\cdot Py}\text{I}\cdot}{xRy \wedge xPy \Rightarrow xR\cdot Py}\text{L}\wedge}{\Rightarrow xR\cdot Py}\text{cut}$$

For ML$\odot$ is easy, for MR$\odot$ is similar to above:

$$
\dfrac{\dfrac{\Rightarrow xRy \qquad \Rightarrow xPy}{\Rightarrow xRy \wedge xPy}\,\mathrm{R}\wedge \qquad \dfrac{\dfrac{\dfrac{}{xRy, xPy, xR \odot Py \Rightarrow xR \odot Py}\,\mathrm{Ax}}{xRy, xPy \Rightarrow xR \odot Py}\,\mathrm{I}\odot}{xRy \wedge xPy \Rightarrow xR \odot Py}\,\mathrm{L}\wedge}{\Rightarrow xR \odot Py}\,\mathrm{cut}
$$

This concludes the first part of the proof. Now we prove that MaRA can simulate the non-logical part of SeRA.

To simulate I$\mathring{1}$ we use a cut:

$$
\dfrac{\dfrac{}{\Gamma \Rightarrow \Delta, x\mathring{1}x}\,\mathrm{M}\mathring{1} \qquad \Gamma, x\mathring{1}x \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathrm{cut}
$$

with the second premise of the cut being a premise of I$\mathring{1}$.

For E$\mathring{1}$ is the same as in MaRA.

For AxR- (we again omit the Γ and Δ):

$$
\dfrac{\dfrac{}{xRy \Rightarrow xRy}\,\mathrm{Ax}}{xR^{-}y, xRy \Rightarrow}\,\mathrm{ML}\text{-}
$$

For IR-:

$$
\dfrac{\dfrac{xRy \Rightarrow}{\Rightarrow xR^{-}y}\,\mathrm{MR}\text{-} \qquad xR^{-}y \Rightarrow}{\Rightarrow}\,\mathrm{cut}
$$

With all the premises being our premises in IR-

For I1R$\smile$:

$$
\dfrac{\dfrac{\dfrac{}{yRx \Rightarrow yRx}\,\mathrm{Ax}}{yRx \Rightarrow xR^{\smile}y}\,\mathrm{MR}\smile \qquad \dfrac{yRx \Rightarrow}{xR^{\smile}y \Rightarrow}\,\mathrm{ML}\smile}{yRx \Rightarrow}\,\mathrm{cut}
$$

For I2R$\smile$ is the same.

For E+ is the same as in the calculus of Maddux. For I+:

$$\dfrac{\dfrac{\dfrac{}{xRy \Rightarrow xRy, xPy}\,\mathrm{Ax}}{xRy \Rightarrow xR+Py}\,\mathrm{MR}+ \qquad xRy, xR+Py \Rightarrow}{xRy \Rightarrow}\,\mathrm{cut}$$

With the right premise of the cut being a premise in I+.

E$\odot$ is the same as in our calculus, while I$\odot$:

$$\dfrac{\dfrac{\dfrac{}{xRz, zPy \Rightarrow zPy}\,\mathrm{Ax} \qquad \dfrac{}{xRz, zPy \Rightarrow xRz}\,\mathrm{Ax}}{xRz, zPy \Rightarrow xR\odot Py}\,\mathrm{MR}\odot \qquad xRz, zPy, xR\odot Py \Rightarrow}{xRz, zPy \Rightarrow}\,\mathrm{cut}$$

With this we conclude the proof.

$\square$

With this theorem we can claim that SeRA has the same expressiveness of the other main calculus in the literature of relation algebra. But we can go a step further given that our calculus is cut-free. Indeed Maddux proved that MaRA has some shortcomings in terms of its cut-elimination rule. That is, there exist some particular equivalences that can only be proved by the MaRA calculus through a derivation that used a cut rule.

For example, one such derivation is $x(R\odot 1)\odot 1y \Rightarrow xR\odot 1y$. Yet clearly in our calculus this result is obtained by a cut-free derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{..., xRz', z'1z, z1y, z'1y, xR\odot 1y \Rightarrow xR\odot 1y}\,\mathrm{Ax}}{..., xRz', z'1z, z1y, z'1y \Rightarrow xR\odot 1y}\,\mathrm{I}\odot}{..., xRz', z'1z, z1y \Rightarrow xR\odot 1y}\,1}{..., x(R\odot 1)z, z1y \Rightarrow xR\odot 1y}\,\mathrm{E}\odot}{x(R\odot 1)\odot 1y \Rightarrow xR\odot 1y}\,\mathrm{E}\odot$$

Indeed we will always be able to deduce a cut-free proof in our sequent calculus. Thanks to our previous result that sequent calculi of geometric theories are cut-free we are able to immediately deduce:

Theorem 4.4.2. *If SeRA+Cut* $\vdash \Gamma \Rightarrow \Delta$ *then SeRA* $\vdash \Gamma \Rightarrow \Delta$

Yet in our proof of $x(R \odot 1) \odot 1y \Rightarrow xR \odot 1y$ we can notice that we are introducing a $z'1y$ at a certain point, that is quite "creative" in its use, as we are instantiating a relation between variables that didn't appear before in the proof. This makes our calculus a bit troubling to use if we want a totally deterministic calculation unfolding from the sequent to be proved.

One possible objection to this is that sequent calculi that are cut-free are easy to inspect to determine if there some decidable fragments of the theory. That is, by looking at the sequent calculi rules for the theory, we can sometimes make some observations that lead us to a proof of decidability. As an example of this, one easy result that we can obtain is that if in a sequent $A \Rightarrow B$ the only operations appearing are the symbols for $+$ and $\smile$, then there is an algorithm that can decide if the sequent $A \Rightarrow B$ holds or not.

Theorem 4.4.3. *Let A, B be two relations, formed by some relations $R_1, ...R_n$, and with the only symbols appearing in them being $+$ and $\smile$. Then $xAy \Rightarrow xBy$ is decidable.*

Proof. The first thing we notice is that in a derivation of $xAy \Rightarrow xBy$ with only $+$ and $\smile$ appearing we will have the subterm property. That is, in a proof of $xAy \Rightarrow xBy$, we will have only the variables x, y appear, and the relations appearing will be only among A, B and the sub-relations in them.

This is because there is no rule to introduce other variables except x, y (indeed we cannot use the only rule that introduces new variables, E$\odot$ because the only symbols appearing are $+$ and $\smile$). Likewise for the relations, we notice that given our assumptions the only way to close a branch of a sequent proof is through the Ax rule, that is through recreating a sequent $..., xBy \Rightarrow xBy$. Clearly that xBy will not have any new relations, and by inspecting our rules we can also be sure we don't need any new relations to construct it that is not already in B (this is because there isn't any rule that has in its conclusion a relation symbol that is totally eliminated in its premises).

Thanks to the subterm property we can get a decidable algorithm. If $xAy \Rightarrow xBy$

is provable, with a finite number of relations $R_1, ...R_n$ to instantiate with a finite number of rules, we are guaranteed to eventually reconstruct the relation xBy from xAy. If $xAy \Rightarrow xBy$ is not provable, then there si no such reconstruction, so we can stop when we start to instantiate relations we already have in a previous step. □

Even if such result might not appear that interesting, the point to note is that is possible to get such results by only inspecting the calculus. So the method presented of extending the sequent calculus with rules for a mathematical theory has the added benefit of providing new methods for proving results about that theory.

On the other hand, looking back at relations algebras, one might ask which other decidability results we could find. One easy answer is that also for sequents only containing $+$ and $-$ we have decidability, but this is because of results coming from the decidability of Boolean algebras. Another observation stemming from the same line of thought is that the full theory of RA was proved[23]to be able to express a fragment of FOL capable of expressing Peano arithmetic. Thus from the well know undecidability of Peano arithmetic we conclude the undecidability of RA, and consequently of our calculus.

4.5 Conclusions

In this thesis we wanted to present some of the current developments in proof theory, specifically how to create cut-free sequent calculi for a given theory. After a brief historical introduction about proof theory, in the first chapter we focused on explaining the basics of sequent calculi. We explained how to obtain some rules for propositional, intuitionistic, and first-order logic. Furthermore we gave some intuitions about what makes a "good" calculus and introduced the cut elimination rule.

The second chapter was devoted to obtain a technique capable of extending the sequent calculus for FOL to a sequent calculus for a theory axiomatized in FOL. Specifically we saw how to obtain rules from geometric axioms, and how these rules still preserve the cut-elimination theorem. At the end we gave an application of this technique, and presented an intuitive exposition of Barr's theorem.

The third chapter presented a brief digression about how to use Skolem extensions to get a geometric theory out of any theory. In this way the results from the second chapter are applicable to an even broader range of theories.

Finally in the fourth chapter we gave a concrete application of these methods for the well-known theory of the calculus of relations. We obtained some rules from the axioms originally used by Tarski, and then we simplified them in a more compact and usable form. At the end we found that our calculus, being cut-free, compares favourably with the other sequent calculus found in the literature.

We hope that in this work we have achieved a compact but thorough exposition of these themes. Likewise we think that the work presented here holds some importance at multiple levels. For example, the technique that allows us to obtain a cut-free sequent-calculi out of a geometrical theory allows us to derive a quick proof of Barr's theorem. This is to testify that the method is not a mathematical island, but can be connected with previous work from a different field. Even more we have a constructive proof that the sequent calculi obtained by the method are cut-free. By using the method we are able to obtain a sequent calculus for many theories, which might not have had a cut-free calculus before. In particular, we applied the method to the field of the calculus of relations, a powerful theory with a rich history, and many current applications in computer science[25]. And by doing so we were the first to obtain a cut-free calculus for this theory.

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