Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

TOPOLOGICAL QUANTUM FIELD THEORIES

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Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 2 de juliol de 2020

Abstract

Topological quantum field theories (TQFTs) are functors from the category of bordisms to the category of vector spaces that preserve their monoidal structure. Such functors arose in Physics but have proven to be useful in various fields of Mathematics. TQFTs give topological and geometric invariants of manifolds, and thus may help in understanding and classifying them.

In this work, however, we perform the reverse process: the completely known classification of 1- and 2-dimensional manifolds will serve as the ground that permits us comprehend TQFTs in these dimensions and determine their underlying structure. In particular, we give a detailed description of 1- and 2-dimensional TQFTs in terms of finite-dimensional vector spaces and commutative Frobenius algebras, respectively.

We conclude by trying to elucidate the relation between TQFTs and Physics. We discuss the common structural properties shared by Hilbert spaces and spacetimes, which motivate the connection of quantum theory with general relativity via TQFTs.

Resum

Les teories quàntiques de camps topològiques (TQFTs, de l'anglès) són functors de la categoria de bordismes a la categoria d'espais vectorials que preserven la seva estructura monoidal. Aquests functors aparegueren en la Física, però s'ha demostrat la seva utilitat en altres àrees de les Matemàtiques. Les TQFTs donen invariants topològics i geomètrics de varietats, i per tant poden ajudar a entendre-les i classificar-les.

En aquest treball, no obstant, procedim de manera inversa: la classificació completament entesa de varietats uni- i bidimensionals serviran de base per entendre les TQFTs en aquestes dimensions i determinar la seva estructura subjacent. En particular donarem una descripció detallada de les TQFTs en dimensió 1 i 2 en termes d'espais vectorials de dimensió finita i d'àlgebres de Frobenius commutatives, respectivament.

Conclourem tractant de dilucidar la relació entre TQFTs i la Física. Discutirem les propietats estructurals comunes entre els espais de Hilbert i els espai-temps, fet que motiva la connexió entre teoria quàntica i relativitat general mitjançant les TQFTs.

2020 Mathematics Subject Classification. 57R56, 18M05, 19D23.

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor Joana Cirici for her guidance, knowledge and patience.

I also wish to thank Joachim Kock for his macros, which provided me with a really nice pair of LaTeX pants.

Finally I want to give special thanks to Fran and Francesca for their support during the confinement, and to Manel for his during the deconfinement.

Contents

1 Introduction

You should never wear your best  when you go out to fight for freedom and truth.

— Henrik Ibsen, *An enemy of the people*

I grow old ... I grow old ...
I shall wear the bottoms of my  rolled.

Shall I part my hair behind? Do I dare to eat a peach?
I shall wear white flannel , and walk upon the beach.
I have heard the mermaids singing, each to each.

I do not think that they will sing to me.

— T. S. Eliot, *The Love Song of J. Alfred Prufrock*

Yeah, you should think about getting yourself a ! I feel all exposed and nasty!

— Donkey [in Puss in Boots' body], in *Shrek 3*

Topological quantum field theories were introduced by the theoretical physicist E. Witten in [Wit]. He coined this term to describe a type of quantum field theories[1] in which the expectation values of observables encode information about the topology of spacetime. Shortly after, M. Atiyah provided a rigorous axiomatization of Witten's topological quantum field theories (see [Ati]) in purely mathematical terms, inspired on Segal's axioms for conformal field theories, which can be found in [Seg]. According to his definition, a topological quantum field theory of dimension d is a rule that assigns a Λ-module to each oriented closed smooth d-dimensional manifold and a Λ-module homomorphism to each smooth $(d+1)$-dimensional manifold with boundary, such that some requirements are satisfied.[2]

Roughly, in Atiyah's axiomatization d-dimensional manifolds correspond to the space, which are mapped to Hilbert spaces (whose vectors are quantum states of a physical system), whereas $(d+1)$-dimensional manifolds correspond to spacetime, which are mapped to operators between Hilbert spaces (that represent processes from one state to another one).

Topological quantum field theories (or TQFTs for short) are most elegantly described using the language of Category Theory. This theory is essentially used to find patterns and similarities between different mathematical structures. A category consists in a collection of objects (for example, objects can be sets, vector spaces and other algebraic objects, but can also be of topological or geometric origin) and a collection of morphisms between these objects (which are usually applications preserving the structure of the objects, but can also be of different nature, as we will see when we introduce bordisms). The notion of morphism can be generalized to that of functor: applications between categories. In this setting, topological quantum field theories are functors from the category of bordisms to

[1] As Peskin and Schoeder explain in their book [PS], a quantum field theory is a theoretical model that "combines three of the major themes of modern physics: the quantum theory, the field concept, and the principle of relativity."

[2] Notice the discrepancies between Atiyah's formulation and the one we will give: our n-dimensional topological quantum field theories correspond to Atiyah's $(n-1)$-dimensional ones. We will also require the ring Λ to be a vector space $\Bbbk$.

the category of vector spaces.

The concept of bordims (also known as cobordisms) is due to L. Pontrjagin and R. Thom; as Dieudonné points out in [Die], Pontrjagin's article [Pon1] "may be considered the germ of the much more extensive theory of *cobordism* inaugurated by Thom in 1953 [Tho]." Bordism theory was originally studied "as a revival of Poincaré's unsuccessful 1895 attempts to define homology using only manifolds", and flourished during the 1950s and early 1960s, with its various applications in Topology, such as the Hirzebruch–Riemann–Roch theorem, the Atiyah–Singer index theorem and the development of K-theory.

Grosso modo, given two closed oriented smooth manifolds, say Σ_1 and Σ_2, a bordism from Σ_1 to Σ_2 is an oriented smooth manifold of one dimension higher with the disjoint union $\Sigma_1 \sqcup \Sigma_2$ as its boundary. Hence the category of bordisms can be constructed with closed oriented smooth manifolds as the objects and bordisms between such objects as the morphisms.

Both the category of bordisms and the category vector spaces have a further structure: they are symmetric monoidal categories. Broadly speaking, a symmetric monoidal structure is a law that allows to take products understood in a general abstract sense: we want it to be associative, commutative and to have a unit. For instance, in the category of bordisms such multiplication will be disjoint union (with the empty set as the unit), and in the category of vector spaces it will be the tensor product (with the underlying field as the unit).

The functor defining a TQFT preserves, by assumption, these monoidal structures. In this case, we talk about symmetric monoidal functors. These functors allow us to study topological invariants, i.e. topological properties that remain unchanged under homeomorphisms, as other symmetric monoidal functors have been previously used: an important construction in Algebraic Topology which precedes TQFTs is the functor of singular homology, which associates, to each topological space, a graded vector space. This types of construction have been proven to be extremely useful in classifying topological spaces: if two topological spaces differ in its image under such functors (say, they have different homologies), they must be nonhomeomorphic.

Whereas the interest of TQFTs is that they are a useful tool in understanding high-dimensional manifolds, in lower dimensions—1 and 2—a complete classification of manifolds is known, so we can use these known results in order to get used to the notion of TQFTs and to grasp its structure in low dimensions. As we will see, there is a one-to-one correspondence between 1-dimensional TQFTs and finite vector spaces. But the main focus of this paper will be 2-dimensional TQFTs and its correspondence with Frobenius algebras.

The origins of Frobenius algebras can be found in G. Frobenius' work [Fro], but its name—*Frobeniusean* algebras to be precise—was first given by C. Nesbitt and T. Nakayama in a series of papers during the 1930s (for instance [Nak, Nes]). A Frobenius algebra is a finite-dimensional vector space that is both an algebra and a coalgebra in a compatible way, specifically such that a certain law, known as Frobenius relation, holds. These algebras exhibiting duality properties have been used in many fields of Mathematics, such Number Theory, Algebraic Geometry and Combinatorics.

Frobenius algebras may be more easily understood visually, by translating the operations that define them into the palpable 2-bordisms. For instance the multiplication

$\mu : A \otimes A \longrightarrow A$ can be represented by the bordism par excellence: the *pair of pants*:

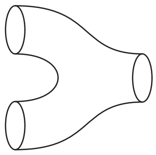


Figure 1.1: A pair of pants.

This work is organized in the following manner: In Section 2 we review the main concept on topological and smooth manifolds in order to introduce properly the bordisms; we construct the category of such bordisms and endow it with a monoidal structure. In Section 3 we present TQFTs and give a method to describe the 1- and 2-dimensional cases algebraically: we use presentations of categories—in a similar way as groups can be presented by a set of generators and relations—to understand the categories of bordisms in low dimensions. Then we will be able to determine 1-TQFTs' structure. In Section 4 we introduce the historical definition Frobenius algebras and deduce equivalent definitions that make its correspondence with 2-TQFT easier to prove. And finally, in the last section we explain in general terms the relation of the mathematical construction on TQFT and its physical motivation.

2 Bordisms

The theory of bordisms, also called cobordisms, establishes a fundamental equivalence relation on the class of compact manifolds by means of boundaries, in which two manifolds are related (*cobordant*) if their disjoint union is the boundary of a compact manifold one dimension higher. The foundations of bordism theory can be traced back to works of Pontrjagin [Pon2] and Thom [Tho].

In this section, we review the main definitions concerning topological and smooth manifolds. We also discuss orientations and boundaries in order to describe the category of bordisms and its main properties. In particular, we explain its symmetric monoidal structure. Main references for this section are [Tu] and [Koc].

2.1 Topological manifolds

Definition 2.1. A topological space M is **locally Euclidean** of dimension n if every point has a neighborhood U such that there is a homeomorphism ϕ from U to an open subset of $\mathbb{R}^n$. We call the pair (U, ϕ) a **chart**. A collection of charts that cover M is called **atlas**.

Definition 2.2. A **topological manifold** is a Hausdorff, second countable, locally Euclidean space. We say it has dimension n (and we call it n-manifold) if it is locally Euclidean of dimension n.

Remark. The empty set $\varnothing$ can be regarded as an n-manifold. 0-manifolds are defined to be disjoint unions of points.

The only compact connected manifold in dimension 1 is the circle, $\mathbb{S}^1$, and in dimension 2 the well known classification theorem of surfaces states that every compact connected 2-manifold is either

(1) a sphere, $\mathbb{S}^2$,

(2) a connected sum of g tori, $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$, or

(3) a connected sum of k projective planes, $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$.

A classification of 3-manifolds is not yet known. The are, though, some partial results: Poincaré conjectured in 1904 that every compact simply-connected 3-manifold is homeomorphic to $\mathbb{S}^3$. This remained unproven for almost a century, but in the meanwhile generalizations on higher dimensions have been shown: every compact n-manifold homotopy equivalent to $\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^n$. Smale proved it for $n > 4$ in 1961 (see [Sma]) and Freedman for $n = 4$ in 1982 ([Fre]). Finally Perelman proved Poincaré's original conjecture for $n = 3$ in 2003 ([Per]).

In dimension 4 manifolds cannot be classified. The idea to understand this is that every finitely presented group is the fundamental group of a 4-manifold. Finitely presented groups cannot be classified and therefore, as homeomorphic manifolds have isomorphic fundamental groups, 4-manifolds cannot either. However, Freedman classified simply connected 4-manifolds in the paper of 1982.

2.2 Smooth manifolds

We want to require topological manifolds to have a further structure, making them into geometric objects:

Definition 2.3. A $\boldsymbol{\mathcal{C}^\infty}$**-atlas** on a topological space M is an atlas $\mathcal{U} = \{(U_i, \phi_i)\}$ on M whose **transition maps**

$$\phi_{ij} = \phi_j \circ \phi_i^{-1}|_{\phi_i(U_i \cap U_j)} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

are $\mathcal{C}^\infty$.

We will say that a chart (U, ϕ) is **compatible with** $\boldsymbol{\mathcal{U}}$ if the union $\mathcal{U} \cup (U, \phi)$ is a $\mathcal{C}^\infty$-atlas. We will say that $\mathcal{U}$ is **maximal** if any compatible chart belongs to $\mathcal{U}$.

A **smooth manifold** is a topological manifold with a maximal $\mathcal{C}^\infty$-atlas.

One may ask if every topological manifold admits a smooth structure. This is the case in dimensions 1, 2 and 3. The first counterexample in higher dimensions was found by Kervaire in 1960; in [Ker] he constructs a 10-dimensional topological space not admitting smooth structures. Since then other counterexamples in dimension 4 have been found.

The notion of continuous map between topological spaces is similarly enriched to the geometric setting:

Definition 2.4. A continuous map $f : M \longrightarrow N$ between to smooth manifolds is said to be a **smooth map** if for all charts (U, ϕ) and (V, ψ) of M and N respectively, the composition

$$\psi \circ f \circ \phi^{-1}|_{\phi(U \cap f^{-1}(V))} : \phi(U \cap f^{-1}(V)) \longrightarrow \psi(V)$$

is $\mathcal{C}^\infty$.

Smooth manifolds and smooth maps form a category, denoted $\mathbf{Man}^\infty$. In this category isomorphisms are known as **diffeomorphisms**. Note that, for a smooth map to be a diffeomorphism, there has to be an inverse which is also smooth.

A natural question is whether a topological manifolds can be equipped with different geometric structures such that the resulting smooth manifolds are nondiffeomorphic. This is not possible in dimensions 1, 2 and 3. In 1956 Milnor found smooth manifolds homeomorphic but nondifferomorphic to the standanrd Euclidean 7-sphere (known as exotic spheres), and subsequently, in 1963 along with Kervaire, he showed that there are 28 nondiffeomorphic oriented 7-spheres (see [Mil1] and [KM]). A surprising result is that, while $\mathbb{R}^n$ with $n \neq 4$ has only a smooth structure (up to diffeomorphism), $\mathbb{R}^4$ has uncountably many. This was proven by Donaldson and Freedman in 1984 (their results are explained in [DK] and [FQ]).

One can attach to every point p of a smooth manifold M a tangent space T_pM. This is a real vector space that intuitively contains all the possible directions in which one can tangentially pass through p. This definition relies on a manifold's ability to be embedded into an ambient vector space. However, it is more convenient to define the notion of a tangent space depending only on the manifold.

Definition 2.5. For M a smooth manifold, denote by $\mathcal{F}(M)$ the set of **smooth functions on M**, given by functions $f : M \to \mathbb{R}$ such that the composition

$$f_i := f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}$$

is differentiable for any chart (U_i, φ_i).

The set $\mathcal{F}(M)$ has the structure of a real vector space with the point-wise operations:

$$(f+g)(p) := f(p) + g(p)\,,\ (\lambda f)(p) := \lambda \cdot f(p)$$

as well as a ring structure, with the multiplication

$$(fg)(p) := f(p) \cdot g(p).$$

Definition 2.6. The **tangent space T_pM of M at a point p** is the set of all *derivations* at p. These are the real linear maps $D : \mathcal{F}(M) \longrightarrow \mathbb{R}$ such that

$$D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g).$$

From now on all our manifolds will be smooth compact manifolds.

2.3 Orientations

We want to endow manifolds with orientation. First we define:

Definition 2.7. An **orientation** of the real vector space $\mathbb{R}^n$ is a choice of sign ($+$ or $-$) for every ordered basis, such that two ordered basis have the same sign if and only if the linear transformation from one to the other has positive determinant. In other words, an orientation is a choice of an ordered basis, which we assign to be positive (the sign of the other ordered basis are then uniquely determined).

Therefore a vector space $\mathbb{R}^n$ can only have 2 possible orientations.

Remark. The 0-dimensional vector space $\{0\}$ has only a basis (the empty set). Thus an orientation in this case is simply a choice of either $+$ or $-$.

Definition 2.8. We say that the **standard orientation of $\mathbb{R}^n$** is the one that assigns a $+$ sign to the ordered basis $((1,0,\dots,0),(0,1,\dots,0),(0,0,\dots,1))$. The **standard orientation of $\{0\}$** is $+$.

With the concept of tangent space T_pM of a manifold M at a point p we can define the **orientation of a manifold** to be a smooth choice of orientations of the tangent spaces. The smoothness condition is equivalent to asking that the differentials of the transition maps $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ should all preserve orientations.

Definition 2.9. A manifold M is said to be **orientable** if it admits an orientation.

Not every manifold is orientable. For example the real projetive plane $\mathbb{RP}^2$ and the Klein bottle $\mathbb{K}$ are nonorientable surfaces. All our manifolds will be hereafter orientable, so we will refer to them simply as manifolds. For example, in dimension 2, all the surfaces that we will consider are disjoint unions of the sphere, $\mathbb{S}^2$ (a surface of genus 0), and connected sums of g tori, $\#_{i=1}^g \mathbb{T}^2$ (surfaces of genus g).

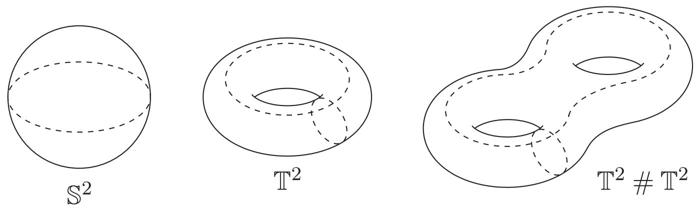


Figure 2.1: Orientable surfaces of genus 0, 1 and 2.

A connected manifold has 2 possible orientations and so a manifold with k connected components has 2^k possible orientations. The empty manifold $\varnothing$ has only one orientation.

Given an oriented manifold M, we denote $\overline{M}$ the same manifold with opposite orientation.

2.4 Manifolds with boundary

All the concepts described in the preceding subsections can be extended to manifolds with boundaries if we allow the charts $\phi : U \longrightarrow \mathbb{R}^n$ have as image open subsets of $\mathbb{H}^n = \{(x_1,\dots,x_n) \,|\, x_n \geq 0\}$, for example:

Definition 2.10. A **topological manifold with boundary** is a Hausdorff, second countable, locally homeomorphic to $\mathbb{H}^n$ space.

And similarly for the other definitions.

A point $p \in M$ is in the boundary if a chart maps it to a point in $\partial\mathbb{H}^n = \{(x_1,\dots,X_n) \,|\, x_n = 0\}$. The set of such points is denoted ∂M and is a submanifold without boundary of M with $\dim \partial M = \dim M - 1$.

Remark. A manifold without boundary has the empty space $\varnothing$ as boundary. In this case we say its a **closed** manifold.

1-dimensional connected manifolds with boundary are the segment $[0,1]$ and the sphere (empty boundary). In dimension 2, these are closed orientable manifolds with a finite number of open discs that have been removed. Hence surfaces with boundary are classified by its number of boundary components ($\mathbb{S}^1$) and its genus g (before removing the discs). For example:

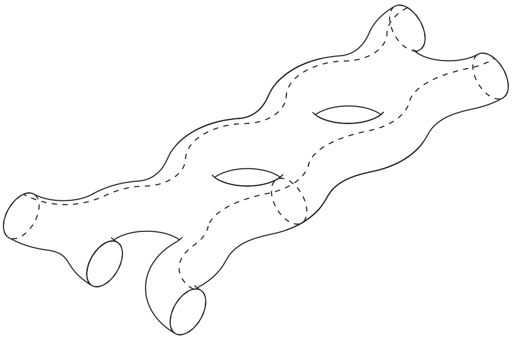


Figure 2.2: Connected manifold with 5 boundary components and genus 2.

Given an oriented manifold with boundary M, we may wonder how is the orientation of ∂M "with respect to M":

Definition 2.11. Let Σ be a closed submanifold of M of codimension 1. At a given point $p \in \Sigma$, if $(v_1, \ldots, v_{n-1})$ is a positive basis for $T_p\Sigma$, then a vector $w \in T_pM$ is **positive normal** if $(v_1, \ldots, v_{n-1}, w)$ is a positive basis for T_pM. Similarly, if $(v_1, \ldots, v_{n-1})$ is a negative basis for $T_p\Sigma$, then a vector $w \in T_pM$ is **positive normal** if $(v_1, \ldots, v_{n-1}, w)$ is a negative basis for T_pM.

Remark. The two given definitions of positive normal vector are equivalent. The second definition is redundant except in the case when there is no positive basis, that is to say when $\dim M = 1$, $\dim \Sigma = 0$ and the only basis of $T_p\Sigma$ (the empty set) is negative.

Let us give a visual example:

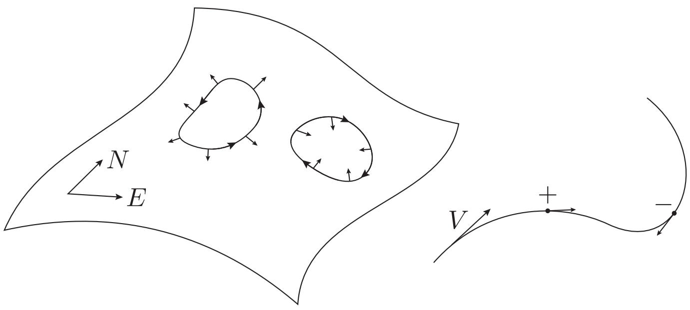


Figure 2.3: In the first manifold (with positive basis (N, E)) positive normal vectors point outwards for counterclockwise-oriented $\mathbb{S}^1$ and inwards for clockwise-oriented $\mathbb{S}^1$. In the second manifold (with positive basis (V)) positive normal vectors point in the same direction of V for positive-oriented points and in the opposite direction of V for negative-oriented points.

Now we can classify the connected components of the boundary of a manifold:

Definition 2.12. Let M be a manifold with boundary. We say that a connected component of ∂M is an **in-boundary** if one of its positive normal vector point inwards. Analogously, a connected component of ∂M is an **out-boundary** if one of its positive normal vector point outwards.

It can be proven that this definition does not depend on the choice of the positive normal vector. So we have that the boundary ∂M can be expressed as a disjoint union of its in-boundaries and out-boundaries.

2.5 Bordisms

A bordism (or cobordism) between two manifolds Σ_1 and Σ_2 is essentially a manifold of one higher dimension with boundary $\Sigma_1 \sqcup \Sigma_2$. We are interested in oriented bordisms:

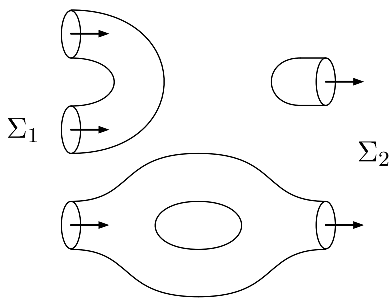
Definition 2.13. Let Σ_1 and Σ_2 be two closed oriented $(n-1)$-manifolds. An **(oriented) bordism** from Σ_1 to Σ_2 is an oriented n-manifold M, together with two smooth maps

$$\iota_{\text{in}} : \Sigma_1 \longrightarrow M \longleftarrow \Sigma_2 : \iota_{\text{out}}$$

such that ι_{in} [resp. ι_{out}] is an orientation-preserving diffeomorphism that maps Σ_1 [resp. Σ_2] onto the in-boundaries [resp. out-boundaries] of M.

For instance, if $n = 1$, bordisms are disjoint unions of segments (and loops) connecting two discrete sets of points (therefore if one has an even number of points, so must the other, and vice versa). If $n = 2$, bordisms are closed orientable surfaces between disjoint unions of $\mathbb{S}^1$: each connected component is determined by the number of its in-boundaries, the number of its out-boundaries and its genus.

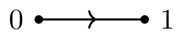
We will draw the bordisms with the in-boundaries to the left and the out-boundaries to the right, e.g., in dimension 2:



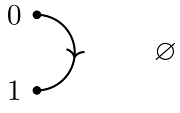
The arrows are the positive normal vectors on the boundary, and if we omit them it is to be understood that they point from left to right.

As an example, let us consider the unit interval $I = [0, 1]$ with standard orientation (i.e. from left to right). We can assign to each component of its boundary, 0 and 1, four possible orientations:

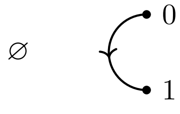
- If both 0 and 1 are positive, 0 is an in-boundary and 1 an out-boundary of I. Thus I is a bordism from 0 to 1:



- If 0 is positive and 1 negative, both are in-boundaries. Thus I is a bordism from $0 \cup 1$ to $\varnothing$:



- If 0 is negative and 1 positive, both are out-boundaries. Thus I is a bordism from $\varnothing$ to $0 \cup 1$:

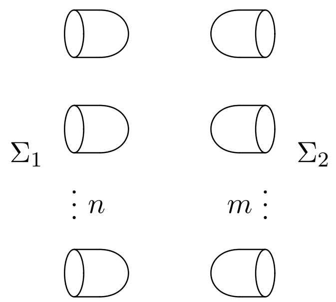


- If both 0 and 1 are negative, 0 is an out-boundary and 1 an in-boundary of I. Thus I is a bordism from 1 to 0:
$$1 \bullet\!\!\longleftarrow\!\!\bullet\ 0$$

It may be natural to ask if for any two given closed manifolds there exists a bordism connecting them:

Definition 2.14. Two closed oriented $(n-1)$-manifolds Σ_1 and Σ_2 are said to be **cobordant**[3] if there is a bordism $\Sigma_1 \longrightarrow M \longleftarrow \Sigma_2$.

With what we have seen above, it is straightforward to see that two 0-dimensional manifolds are cobordant if the sum of their signs, i.e. the number of positively oriented components minus the number of negatively oriented components, is equal. However any two 1-dimensional manifolds are cobordant: Suppose Σ_1 and Σ_2 are the disjoint unions of n and m circles respectively. Then we can always construct such bordisms:



Cobordancy is in fact an equivalence relation: cylinders give the reflexive property; gluing (see below), the transitive property; and the reflexive property is given by switching the ι_{in} and ι_{out} maps. Disjoint union makes the set of equivalences classes into an Abelian group, denoted as Ω^n, with n the dimension of the boundary manifolds. The above observations translate to $\Omega^0 = \mathbb{Z}$ and $\Omega^1 = 0$ respectively. As Thom points out in [Tho, Thm. IV.13], for $n < 8$,

$$\Omega^0 = \mathbb{Z}, \quad \Omega^1 = \Omega^2 = \Omega^3 = 0, \quad \Omega^4 = \mathbb{Z}, \quad \Omega^5 = \mathbb{Z}/2\mathbb{Z}, \quad \Omega^6 = \Omega^7 = 0.$$

2.6 The category of bordisms

We want to consider a category with closed manifolds as objects and oriented bordisms as morphisms connecting them. In order to construct such category we will first define an equivalence relation between two bordisms as follows:

Definition 2.15. Two bordisms M and N, both from Σ_1 to Σ_2, are said to be **equivalent** if there is a diffeomorphism $\varphi : M \longrightarrow N$ making the following diagram commute:

$$\begin{tikzcd} & M \arrow[dd, "\cong"', "\varphi"] & \\ \Sigma_1 \arrow[ur, tail] \arrow[dr, tail] & & \Sigma_2 \arrow[ul, tail] \arrow[dl, tail] \\ & N & \end{tikzcd}$$

[3] Etymologically two manifolds being *cobordant* would mean that they *collectively bound* something (i.e. their disjoint union is the boundary of something), as *"bord"* means edge, boundary, in French. This explains the alternative nomenclature *cobordism*.

It follows that this is an equivalence relation, and it defines equivalence classes of bordisms between two given manifolds. Our category will have these equivalence classes as morphisms.

In order to define the composition of morphisms, we will use the procedure of gluing two bordisms:

Definition 2.16. Given two bordisms M and N with a common boundary Σ (with morphisms $\iota_{\text{out}}^M : \Sigma \longrightarrow M$ and $\iota_{\text{in}}^N : \Sigma \longrightarrow N$), the bordism that results from **gluing $\boldsymbol{M}$ and $\boldsymbol{N}$ along $\boldsymbol{\Sigma}$** is

$$M \sqcup_\Sigma N := M \sqcup N/\sim,$$

where $\sim$ is the following equivalence relation: given two points $p \in M$, $q \in N$, $p \sim q$ if and only if there exists a point $x \in \Sigma$ such that $\iota_{\text{out}}^M(x) = p$ and $\iota_{\text{in}}^N(x) = q$.

The following result may be found in [Mil2, Thm. 1.4]:

Theorem 2.17. *Let M and N be two bordisms from Σ_1 to Σ_2 and from Σ_2 to Σ_3 respectively. There exists a smooth structure on $M \sqcup_{\Sigma_2} N$ such that each inclusion map $M \hookrightarrow M \sqcup_{\Sigma_2} N$, $N \hookrightarrow M \sqcup_{\Sigma_2} N$ is a diffeomorphism onto its image, and it is unique up to diffeomorphism fixing Σ_1, Σ_2 and Σ_3.*

The identity morphisms will be given by cylinders:

Definition 2.18. Given a closed oriented manifold Σ, we define the **cylinder** to be $C_\Sigma := \Sigma \times [0,1]$ oriented in such a way that $\Sigma \times \{0\}$ is its in-boundary and $\Sigma \times \{1\}$ its out-boundary. Hence, with the canonical maps

$$\iota_{\text{in}} : \Sigma \xrightarrow{\sim} \Sigma \times \{0\} \hookrightarrow C_\Sigma \hookleftarrow \Sigma \times \{1\} \xleftarrow{\sim} \Sigma : \iota_{\text{out}}$$

C_Σ is a bordism from Σ to itself.

It can be shown that gluing a bordism M with the cylinder of one of its boundary gives a bordism equivalent to M.

These results allow us to define the category of bordisms in the following manner:

Definition 2.19. The **category of bordisms** of dimension n, $\mathbf{Bord}_n$, is defined as follows:

(1) Its objects Σ are closed oriented $(n-1)$-manifolds,

(2) its morphisms $M : \Sigma_1 \longrightarrow \Sigma_2$ are equivalence classes of bordisms from Σ_1 to Σ_2,

(3) the identity morphisms $1_\Sigma : \Sigma \longrightarrow \Sigma$ are the equivalence classes of cylinders C_Σ, and

(4) the composition $N \circ M : \Sigma_1 \longrightarrow \Sigma_3$ of two morphisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Sigma_2 \longrightarrow \Sigma_3$ is the equivalence class of $M \sqcup_{\Sigma_2} N$.

From now on we will refer to equivalence classes of bordisms simply as bordisms.

2.7 Monoidal structure on the category of bordisms

We end this section by presenting the symmetric monoidal structure on the category of bordisms. A **monoidal structure** on a category is a tensor product operation, sending a pair of objects A and B to an object $A \otimes B$ in a functorial way and satisfying certain compatibility axioms that mimic the properties of the tensor product on the category of vector spaces. In particular, there is a **unit** element involved. Such a tensor product is **symmetric** if there is a natural isomorphism when exchanging the factors, called **symmetric braiding**. We refer to the Appendix for a precise definition. Here, we review some prototypical examples of symmetric monoidal categories which will provide sufficient intuition on symmetric monoidal structures.

- The category of sets **Set** with the disjoint union $\sqcup$ as tensor product and the empty set $\varnothing$ as the unit.
- The category of sets **Set** with the Cartesian product $\times$ as tensor product, any one-element set $\{\bullet\}$ as the unit, and the natural map σ that interchanges the two factors of $\times$ as the symmetric braiding.
- Similarly, the category of topological spaces, **Top**, (whose morphisms are continuous maps) with $\times$, $\{\bullet\}$ and σ.
- The category of vector spaces over a field $\Bbbk$, $\mathbf{Vect}_\Bbbk$, (whose morphisms are linear maps) together with the ordinary tensor product $\otimes$, the field $\Bbbk$ as the unit, and the natural map σ that interchanges the two factors of $\otimes$ as the symmetric braiding.
- The category of ($\mathbb{Z}$-)graded vector spaces over a field $\Bbbk$, $\mathbf{grVect}_\Bbbk$: Its objects are collections of $\Bbbk$-vector spaces $V = \{V_n\}_{n\in\mathbb{Z}}$, and its morphisms are collections of linear vector maps $f = \{f_n : V_n \longrightarrow W_n\}_{n\in\mathbb{Z}}$. The tensor product is defined by $V \otimes W = \{\bigoplus_{i+j=n} V_i \otimes W_j\}_{n\in\mathbb{Z}}$, the unit is $\{V_n\}_{n\in\mathbb{Z}}$ with $V_0 = \Bbbk$ and $V_n = \{0\}$ if $n \neq 0$, and the symmetric braiding κ (known as *Koszul braiding*) is given by $v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v$, where $|\cdot|$ denotes the degree ($|v| = n$ if $v \in V_n$).
- An important category used in Algebraic Topology, the category of *chain complexes* (or differential graded vector spaces), $\mathbf{Ch}_\Bbbk$: Its objects are pairs $(C_\bullet, \partial_\bullet)$ where $C_\bullet$ is a graded vector space and $\partial_\bullet = \{\partial_n : C_n \longrightarrow C_{n-1}\}_{n\in\mathbb{Z}}$ is a collection of linear maps (called *boundary operators*) such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. A morphism from $(C_\bullet, \partial_\bullet^C)$ to $(D_\bullet, \partial_\bullet^D)$ is a collection of linear maps $f_\bullet = \{f_n : C_n \longrightarrow D_n\}$ (called *chain map*) such that

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l, "\partial^C_{n-1}"'] \arrow[d, "f_{n-1}"] & C_n \arrow[l, "\partial^C_n"'] \arrow[d, "f_n"] & C_{n+1} \arrow[l, "\partial^C_{n+1}"'] \arrow[d, "f_{n+1}"] & \cdots \arrow[l, "\partial^C_{n+2}"'] \\
\cdots & D_{n-1} \arrow[l, "\partial^D_{n-1}"'] & D_n \arrow[l, "\partial^D_n"'] & D_{n+1} \arrow[l, "\partial^D_{n+1}"'] & \cdots \arrow[l, "\partial^D_{n+2}"']
\end{tikzcd}$$

  commutes. The tensor product $C_\bullet \otimes D_\bullet$ is defined as in the previous example and $\partial_\bullet^C \otimes \partial_\bullet^D = \partial_\bullet^{C\otimes D}$ is given by

$$\partial_{i+j}^{C\otimes D}(v \otimes w) = \partial_i^C(v) \otimes w + (-1)^i v \otimes \partial_j^D w.$$

The unit is $(C_\bullet, 0)$ with $C_0 = \Bbbk$ and $C_n = \{0\}$ if $n \neq 0$. The symmetric braiding is κ defined as before. (Note that κ is a chain map—it commutes with $\partial_\bullet^{C\otimes D}$—whereas σ is not.)

We now turn our attention to the category of bordisms. Given two closed oriented $(n-1)$-manifolds Σ and Π, the disjoint union $\Sigma \sqcup \Pi$ is a closed oriented $(n-1)$-manifolds as well. Similarly, given to bordisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Pi_1 \longrightarrow \Pi_2$, the disjoint union $M \sqcup N$ is a bordism from $\Sigma_1 \sqcup \Pi_1$ to $\Sigma_2 \sqcup \Pi_2$. By considering the empty manifold $\varnothing_n$ as a bordism from $\varnothing_{n-1}$ to itself, it follows that the tuple $(\mathbf{Bord}_n, \sqcup, \varnothing)$ is a monoidal category.

The diffeomorphism $\Sigma \sqcup \Pi \xrightarrow{\sim} \Pi \sqcup \Sigma$ induces a bordism $T_{\Sigma,\Pi}$ (known as **twist bordism**) from $\Sigma \sqcup \Pi$ to $\Pi \sqcup \Sigma$ via the cylinder morphism:

$$\Sigma \sqcup \Pi \xrightarrow{\sim} \Pi \sqcup \Sigma \hookrightarrow C_{\Pi \sqcup \Sigma} \hookleftarrow \Pi \sqcup \Sigma.$$

These bordisms are the components of the symmetric braiding, and indeed they satisfy the conditions in A.7. Thus, $(\mathbf{Bord}_n, \sqcup, \varnothing, T)$ is a symmetric monoidal category.

3 Topological quantum field theories

A topological quantum field theory is a functor from the category of bordisms to the category of vector spaces. Atiyah defined TQFTs in [Ati] by requiring some additional axioms. Those axioms can be summarized by stating that the functor must be symmetric monoidal, as we will see in our definition. In the next sections we confer TQFTs the structure of category, with natural transformations as morphisms. As a first application we determine the structure of one-dimensional topological quantum field theories and establish a one-to-one correspondence between these theories and finite-dimensional vector spaces.

3.1 Definition and properties

A symmetric monoidal functor is a functor F between categories that preserves the symmetric monoidal structures. In particular, there are natural isomorphisms

$$\Phi_{A,B} : F(A) \otimes_2 F(B) \longrightarrow F(A \otimes_1 B)$$

for all objects A, B of the domain category, as well as a natural isomorphism

$$\varphi : I_2 \longrightarrow F(I_1)$$

in the target category, comparing the unit objects I_1 and I_2 of the domain and target categories respectively. We refer to the Appendix for a precise definition. The idea of considering symmetric monoidal functors in order to study topological and geometric invariants is not new to topological quantum field theories. Before describing TQFTs, let us review some instances where such functors play important roles.

First, we examine a basic construction which assigns, in a symmetric monoidal fashion, a vector space to each set, thus obtaining a symmetric monoidal functor from the category of sets $(\mathbf{Set}, \times, \{\bullet\})$ to the category of vector spaces $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk)$. Let $F : \mathbf{Set} \longrightarrow \mathbf{Vect}_\Bbbk$

be defined by sending a set A to $\mathbb{k}[A]$, the $\mathbb{k}$-vector space spanned by A. A map $f : A \longrightarrow B$ is then sent to the unique linear map $\hat{f}$ defined by $\hat{f}(a) = f(a)$, $\forall a \in A$. We choose the components of Φ to be the isomorphism

$$\begin{aligned}\Phi_{A,B} : \mathbb{k}[A] \otimes \mathbb{k}[B] &\longrightarrow \mathbb{k}[A \times B]\\ \sum_i \lambda_i a_i \otimes \sum_i \mu_j b_j &\longmapsto \sum_{i,j} \lambda_i \mu_j (a_i, b_j)\end{aligned}$$

and φ to be the isomorphism $\varphi : \mathbb{k} \longrightarrow \mathbb{k}[\{\bullet\}]$, $\lambda \longmapsto \lambda\bullet$.

A fundamental construction that arises in Algebraic Topology is the *homology functor*. In a purely algebraic setting, the homology functor

$$H_\bullet(-, \mathbb{k}) : \mathbf{Ch}_\mathbb{k} \longrightarrow \mathbf{grVect}_\mathbb{k}$$

is given by

$$H_n(C_\bullet, \mathbb{k}) = \ker \partial_n / \operatorname{im} \partial_{n+1} \text{ and } H_n(f_\bullet, \mathbb{k})([v]) = [f(v)].$$

Here $[v]$ denotes the equivalence class of $v \in C_n$. The unit of $\mathbf{Ch}_\mathbb{k}$ is sent to the unit of $\mathbf{grVect}_\mathbb{k}$, and the Künneth theorem (see [Hat]) states that

$$H_n(C_\bullet \otimes D_\bullet, \mathbb{k}) \cong H_n(C_\bullet, \mathbb{k}) \otimes H_n(D_\bullet, \mathbb{k}).$$

This makes $H_\bullet(-, \mathbb{k})$ into a symmetric monoidal functor.

In the topological setting, there is the *functor of singular chain complexes.* This is the functor

$$S_\bullet : \mathbf{Top} \longrightarrow \mathbf{Ch}_\mathbb{k}$$

defined by sending a topological space to the direct sum of the $\mathbb{k}$-vector spaces generated by all the continuous maps σ_n from the n-simplex Δ^n to X:

$$S_n(X, \mathbb{k}) := \bigoplus_{\sigma_n} \mathbb{k}[\sigma_n].$$

This graded vector space is equipped with boundary operators ∂_n^X defined by

$$\partial_n^X([p_0, \dots, p_n]) := \bigoplus_{i=0}^{n} (-1)^i [p_0, \dots, \hat{p_i}, \dots, p_n],$$

where we represent σ by its vertices $[p_0, \dots, p_n]$.

On continuous maps, $S_\bullet(f, \mathbb{k})$ is defined by

$$S_n(f, \mathbb{k})(\sigma_n) = f \circ \sigma_n.$$

This commutes with $\partial_\bullet^X$ and so it is indeed a chain map.

The composition of the above two functors yields a symmetric monoidal functor, the *singular homology functor*

$$H_\bullet(-, \mathbb{k}) : \mathbf{Top} \longrightarrow \mathbf{grVect}_\mathbb{k}.$$

For more details on this topic you can refer to [NP, Hat].

The example of monoidal functors we are interested in are TQFTs, which are functors from $\mathbf{Bord}_n$ to $\mathbf{Vect}_\mathbb{k}$, and are related to homology functors in the sense that they give invariants of topological spaces (given two homeomorphic spaces their homologies will be isomorphic).

Definition 3.1. A **topological quantum field theory** (or **TQFT**) of dimension n is symmetric monoidal functor from $(\mathbf{Bord}_n, \sqcup, \varnothing, T)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$.

Essentially this means that if $\mathcal{Z}$ is a TQFT then

(1) $\mathcal{Z}(1_\Sigma) = 1_{\mathcal{Z}(\Sigma)}$ for any object Σ in $\mathbf{Bord}_n$,

(2) $\mathcal{Z}(N \circ M) = \mathcal{Z}(N) \circ \mathcal{Z}(M)$ for any bordisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Sigma_2 \longrightarrow \Sigma_3$,

(3) $\mathcal{Z}(\Sigma \sqcup \Pi) = \mathcal{Z}(\Sigma) \otimes \mathcal{Z}(\Pi)$ for any objects Σ and Π in $\mathbf{Bord}_n$,

(4) $\mathcal{Z}(M \sqcup N) = \mathcal{Z}(M) \otimes \mathcal{Z}(N)$ for any bordisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Pi_1 \longrightarrow \Pi_2$,

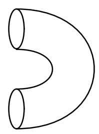
(5) $\mathcal{Z}(\varnothing) = \Bbbk$, and

(6) $\mathcal{Z}(T_{\Sigma,\Pi}) = \sigma_{\mathcal{Z}(\Sigma),\mathcal{Z}(\Pi)}$ for any objects Σ and Π in $\mathbf{Bord}_n$.

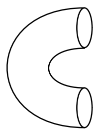
Note that, according to definitions A.9 and A.10, the equalities in (3), (4) and (5) are to be understood as isomorphisms.

Before studying TQFTs of dimension 1 and 2, we describe some general properties, so that we get used to the notion of TQFT. To prove the first proposition, we define a type of bordism:

Definition 3.2. Given an object Σ in $\mathbf{Bord}_n$, we define the **U-tubes** to be the bordisms

 $: \Sigma \sqcup \overline{\Sigma} \longrightarrow \varnothing$

and

 $: \varnothing \longrightarrow \overline{\Sigma} \sqcup \Sigma.$

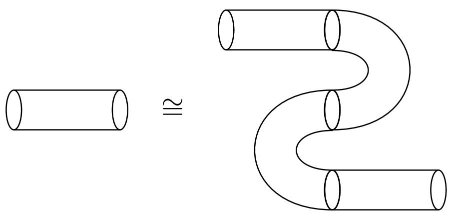
Remark. It may seem unnecessary to use $\overline{\Sigma}$ (i.e. Σ with the opposite orientation). However, it cannot be guaranteed the existence of a bordism $\Sigma \sqcup \Sigma \longrightarrow \varnothing$ (and vice versa). For instance, in $\mathbf{Bord}_1$, there is no bordism from $\{+,+\}$ to $\varnothing$. On the other hand, the two bordisms we give always exist—they can be constructed by reversing the orientation of one of the (out- or in-)boundaries of the cylinder C_Σ.

Proposition 3.3. *Let $\mathcal{Z} : \mathbf{Bord}_n \longrightarrow \mathbf{Vect}_{\Bbbk}$ be a TQFT. For any object Σ in $\mathbf{Bord}_n$, the vector space $\mathcal{Z}(\Sigma)$ is finite-dimensional.*

Proof. Let $U := \mathcal{Z}(\Sigma)$ and $V := \mathcal{Z}(\overline{\Sigma})$. The images of the U-tubes are

$\beta := \mathcal{Z}\big($  $\big) : U \otimes V \longrightarrow \Bbbk$ and $\gamma := \mathcal{Z}\big($  $\big) : \Bbbk \longrightarrow V \otimes U.$

The diffeomorphism



gives two bordisms of the same class, and by taking images, we have that

$$1_U = (\beta \otimes 1_U) \circ (1_U \otimes \gamma).$$

We can choose finitely many $v_i \in V, u_i \in U$ such that $\gamma(1) = \sum_i v_i \otimes u_i$ (every element of $U \otimes V$ can be expressed in this way). Using the above expression, and writing $\langle u|v\rangle$ for $\beta(u,v)$, we have that for any $u \in U$,

$$\begin{aligned} u &= (\beta \otimes 1_U)((1_U \otimes \gamma)(u \otimes 1)) \\ &= (\beta \otimes 1_U)\Big(u \otimes \sum_i v_i \otimes u_i\Big) = \sum_i \langle u|v_i\rangle \otimes u_i = \sum_i \langle u|v_i\rangle \cdot u_i. \end{aligned}$$

Thus, $\{u_i\}$ spans U and therefore U is finite-dimensional. Observe that this also gives a natural isomorphism between U and V^*, the dual vector space to V, namely, $u \longmapsto \langle u|-\rangle$. (See Subsection 4.3 where we discuss nondegenerate pairings.) In other words,

$$\mathcal{Z}(\overline{\Sigma}) = \mathcal{Z}(\Sigma)^*. \qquad \square$$

TQFTs produce invariants of manifolds in the following sense: a manifold M can be viewed as a bordism from $\varnothing$ to itself, so its image under a TQFT is a linear map $\Bbbk \longrightarrow \Bbbk$ that can be thought as a scalar. A simple example is the following:

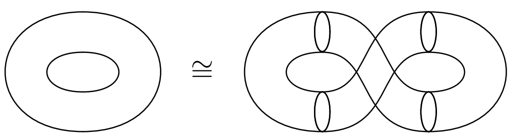
Proposition 3.4. *Let $\mathcal{Z}$ be a TQFT and Σ a closed manifold. Then*

$$\mathcal{Z}(\Sigma \times \mathbb{S}^1) = \dim(\mathcal{Z}(\Sigma)).$$

Proof. Using the notation of the previous proof, if $\{v_i\}$ is a basis of V and $\{u_j\}$ a basis of U then $\gamma(1) = \sum_{i,j} \lambda_{ij} v_i \otimes u_j$ for some coefficients λ_{ij}. As before, we can find that, for any vector $u \in U$, $u = \sum_{i,j} \lambda_{ij}\langle u|v_i\rangle \cdot u_j$. In particular $u_j = \sum_{i,k} \lambda_{ik}\langle u_j|v_i\rangle \cdot u_k$ for the elements of the basis, so

$$\sum_i \lambda_{ik}\langle u_j|v_i\rangle = \delta_{jk} = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k. \end{cases}$$

The diffeomorphism



gives the expression

$$\mathcal{Z}(\Sigma \times \mathbb{S}^1) = \beta \circ \sigma_{U,U} \circ \gamma.$$

Therefore

$$\mathcal{Z}(\Sigma \times \mathbb{S}^1)(1) = \beta\Big(\sigma_{U,U}\Big(\sum_{i,j} \lambda_{ij} v_i \otimes u_j\Big)\Big) = \sum_{i,j} \lambda_{ij}\langle u_j|v_i\rangle = \sum_j 1 = \dim(U). \qquad \square$$

We may wonder if $\mathcal{Z}(M \sqcup N) = \mathcal{Z}(N \circ M)$ for bordisms with only in- and out-boundary respectively, say $M : \Sigma \longrightarrow \varnothing$ and $N : \varnothing \longrightarrow \Pi$. Indeed, if $f = \mathcal{Z}(M) : \mathcal{Z}(\Sigma) \longrightarrow \Bbbk$ and $g = \mathcal{Z}(N) : \Bbbk \longrightarrow \mathcal{Z}(\Pi)$, then

$$\mathcal{Z}(M \sqcup N)(v \otimes 1) = f(v) \otimes g(1) = 1 \otimes f(v)g(1)$$

whereas

$$\mathcal{Z}(N \circ M)(v) = g(f(v)) = f(v)g(1),$$

so they are essentially the same.

3.2 The category of TQFTs

In Category Theory one can define the **functor category** between two given categories $\mathcal{C}$ and $\mathcal{D}$, $\mathbf{Fun}(\mathcal{C},\mathcal{D})$, to be the category whose objects are functors $F:\mathcal{C}\longrightarrow\mathcal{D}$ and whose morphisms are natural transformations between these functors. If the categories are monoidal and one only allows the objects to be monoidal functors and the morphisms to be monoidal transformations, the category is a **monoidal functor category**, $\mathbf{MonFun}(\mathcal{C},\mathcal{D})$. Similarly one can define the **symmetric monoidal functor category** between two symmetric monoidal categories, $\mathbf{SymMonFun}(\mathcal{C},\mathcal{D})$.

With this nomenclature, TQFTs form a symmetric monoidal functor category from $\mathbf{Bord}_n$ to $\mathbf{Vect}_{\Bbbk}$:

$$\mathbf{TQFT}_n^{\Bbbk}=\mathbf{SymMonFun}(\mathbf{Bord}_n,\mathbf{Vect}_{\Bbbk}).$$

3.3 Presentation of bordisms in low dimensions

Now we would like to determine the structure of $\mathbf{TQFT}_1^{\Bbbk}$ and $\mathbf{TQFT}_2^{\Bbbk}$ (see [CR] and [Koc] respectively). We can do so because a classification of manifolds of dimension 0, 1 and 2 are completely understood (but not in higher dimensions).

In order to describe $\mathbf{TQFT}_1^{\Bbbk}$ and $\mathbf{TQFT}_2^{\Bbbk}$ first we need to understand $\mathbf{Bord}_1$ and $\mathbf{Bord}_2$: similarly to Group Theory, one can give a **presentation of a category $\mathcal{C}$** (and particularly of a symmetric monoidal category) by a set of objects, generators and relations:

(1) The set of **objects**, O, is a set of objects such that every object in $\mathcal{C}$ is isomorphic to one in O,

(2) The set of **generators**, G, is a set of morphisms in $\mathcal{C}$ that generate via composition every morphism f in $\mathcal{C}$, i.e. $f=g_1\circ\cdots\circ g_n$ for some $g_i\in G$. If we talk about [symmetric] monoidal categories we allow G to generate all morphisms via composition and "tensoring":

$$f=(g_1^1\otimes\cdots\otimes g_1^{r_1})\circ\cdots\circ(g_n^1\otimes\cdots\otimes g_n^{r_n})$$

for some $g_i^j\in G$.

(3) The set of **relations**, R. A relation is the equality of two decompositions of a given morphism f in terms of the generators, i.e. equalities of the form

$$g_1\circ\cdots\circ g_n=h_1\circ\cdots\circ h_m.$$

In the symmetric monoidal setting:

$$(g_1^1\otimes\cdots\otimes g_1^{r_1})\circ\cdots\circ(g_n^1\otimes\cdots\otimes g_n^{r_n})=(h_1^1\otimes\cdots\otimes h_1^{r_1})\circ\cdots\circ(h_m^1\otimes\cdots\otimes h_m^{r_m}).$$

The set R is a set of relations such that every relation can be derived from the relations in R.

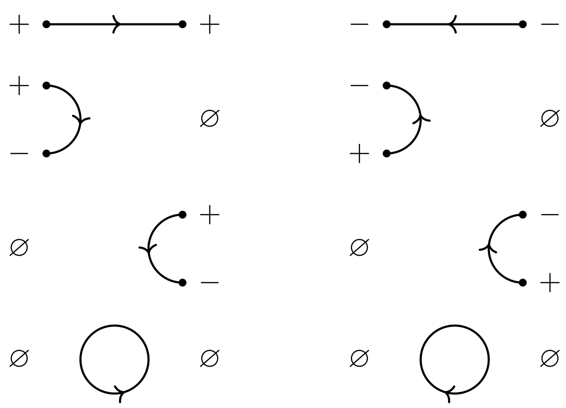
First let us find the presentation of $\mathbf{Bord}_1$:

(1) The set of objects consists of disjoint unions of positively oriented points, $\bullet_+$, and negatively oriented points, $\bullet_-$, e.g.

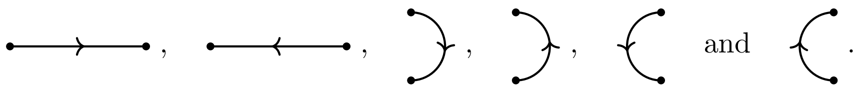
$$\varnothing, \quad \bullet_+, \quad \bullet_- \sqcup \bullet_+, \quad \bullet_+ \sqcup \bullet_+ \sqcup \bullet_- \sqcup \bullet_+ \ldots$$

The orientation is important as there is no preserving-diffeomorphism between $\bullet_+$ and $\bullet_-$, and the order of the disjoint unions is important as well, as there is no isomorphism (i.e. cylinder) from $\bullet_+ \sqcup \bullet_-$ to $\bullet_- \sqcup \bullet_+$.

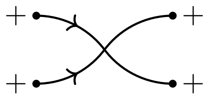
(2) All connected bordisms are isomorphic to one of these:



Thus, every connected bordism is generated by



Although generators will generate all bordisms via tensor products (i.e. disjoint unions) as well as compositions, this does not mean that we are done: there are disconnected bordisms that cannot be expressed as a disjoint union of bordisms, for example the twist:



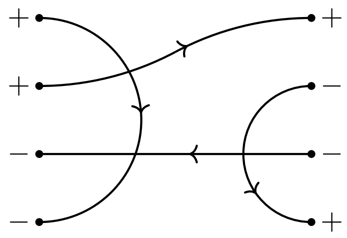
Similarly to how we defined the twist bordisms in subsection 2.7, given a permutation $\sigma \in \mathfrak{S}_n$, we can define the **permutation bordism** associated to σ to be the bordism induced by the diffeomorphism

$$\Sigma_1 \sqcup \cdots \sqcup \Sigma_n \xrightarrow{\sim} \Sigma_{\sigma(1)} \sqcup \cdots \sqcup \Sigma_{\sigma(n)},$$

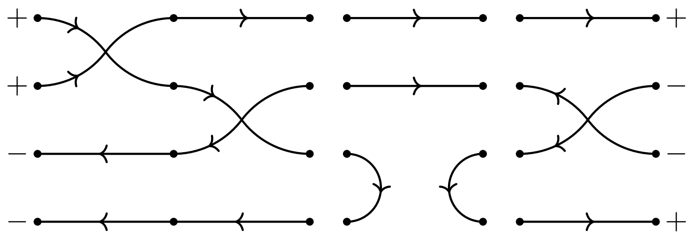
where in this case, Σ_i are oriented points. These bordisms can be generated by the twist bordisms $\Sigma_i \sqcup \Sigma_j \longrightarrow \Sigma_j \sqcup \Sigma_i$, just like every permutation can be generated by the transpositions. Now it is easy to see why the following lemma holds:

Lemma 3.5. *Every bordism can be expressed as the composition of a permutation bordism, a disjoint union of connected bordisms, and another permutation bordism.*

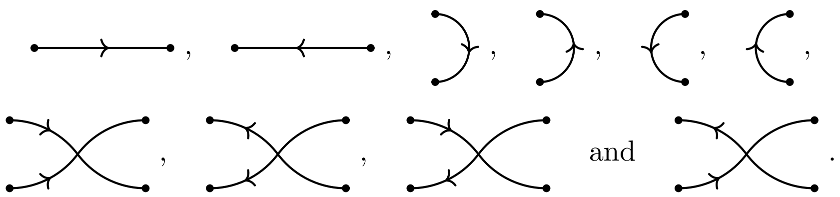
For example the following bordism



can be expressed as

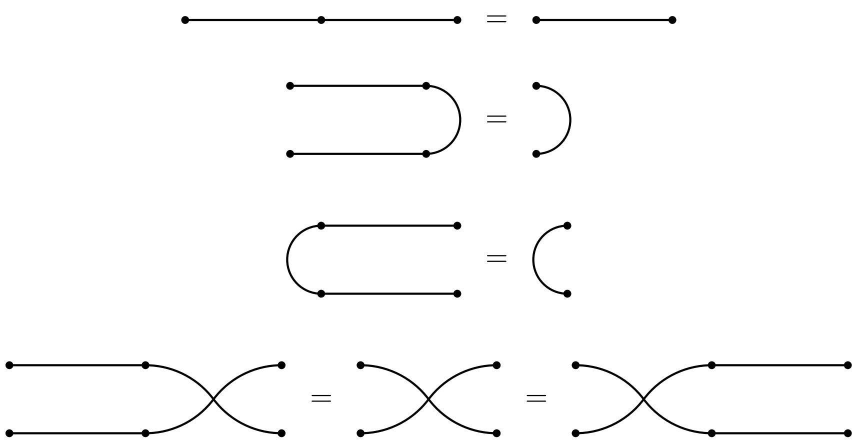


So in conclusion a nice set of generators is



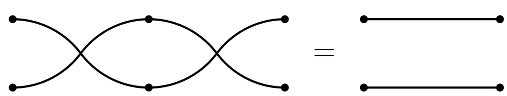
(3) The set of relations is the following:

(a) All the relations involving the identity morphisms:

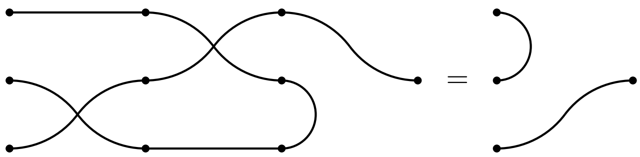


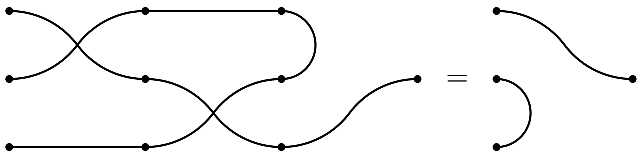
We have omitted the orientations, but one must take them into account, so in fact there are 14 equalities above.

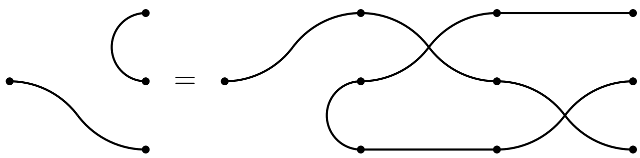
(b) A set of relations involving the twists. The first four ones refer to the fact that the twist is its own inverse:

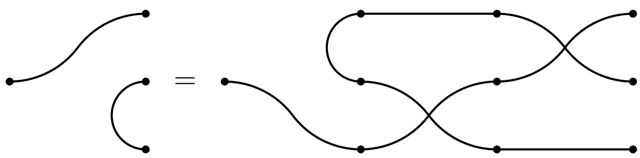


(c) The other ones refer to the naturality of the twists: for any pair of bordisms it is the same to apply the twist before their disjoint union as to apply it after the reversed disjoint union. We only need the relations involving the generators, and in fact, as every disjoint union of two bordisms can be understood as a composition of disjoint unions with the identity bordisms, it suffices to write the cases with a generator in disjoint union with an identity:







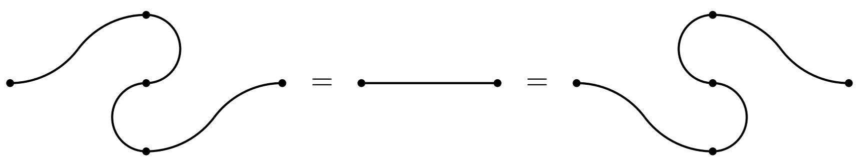


Relacions que involucren els bordismes de "twist":

Notice how, for example in the first relation, we expressed the twists , i.e. $(\bullet \sqcup \bullet) \sqcup \bullet \longrightarrow \bullet \sqcup (\bullet \sqcup \bullet)$, decomposed using the generators . (As we pointed out, every permutation bordism can be generated by these generators.)

(d) And finally some relations characteristic to $\mathbf{Bord}_1$ (since the other ones are due to the symmetric monoidal structure of $\mathbf{Bord}_1$):

i. The **snake relations**:



ii. The **commutativity** and the **cocommutativity relations**:

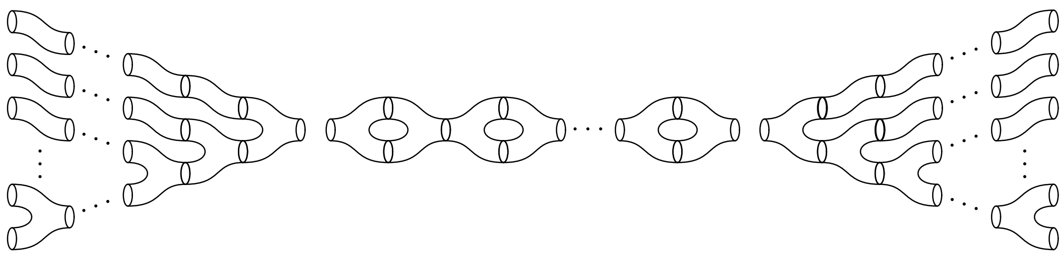
One can prove that this set of relations is complete (i.e. that every other possible relation can be derived from it), and obviously it is not by no means minimal.

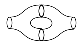
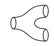
In a similar fashion we can find a presentation of $\mathbf{Bord}_2$:

(1) The set of objects of $\mathbf{Bord}_2$ consists of disjoint unions of circles, since every closed connected 1-manifold is diffeomorphic to $\mathbb{S}^1$. Observe that the disjoint unions we consider can be the disjoint unions of the same circle, say Σ, so we don't have to worry about orientations as we did before—there is an orientation-preserving diffeomorphism between a circle and the same circle with the opposite orientation.

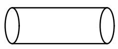
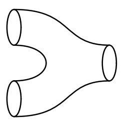
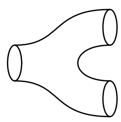
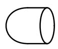
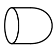
We will write $\mathbf{n}$ to denote the disjoint union of n copies of Σ (and $\mathbf{0} = \varnothing$). Our set of objects is $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$.

(2) To determine a set of generators first observe that, as we have seen, a connected bordism in $\mathbf{Bord}_2$ is uniquely determined by the number of in- and out-boundaries (say n and m resp.) and its genus (say g), so it can be expressed as the composition of a bordism $\mathbf{n} \longrightarrow \mathbf{1}$ with genus 0 (called **in-part**), a bordism $\mathbf{1} \longrightarrow \mathbf{1}$ of genus g (**topological part**) and a bordism $\mathbf{1} \longrightarrow \mathbf{m}$ of genus 0 (**out-part**):

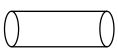
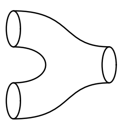
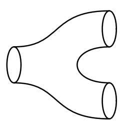
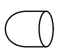
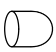
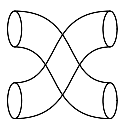


with $n-1$ copies of  in the in-part, g copies of  in the topological part and $m-1$ copies of  in the out-part. If $n = 0$ the in-part should be  instead, and if $m = 0$ the out-part should be .

Therefore we have that every connected bordism is generated by

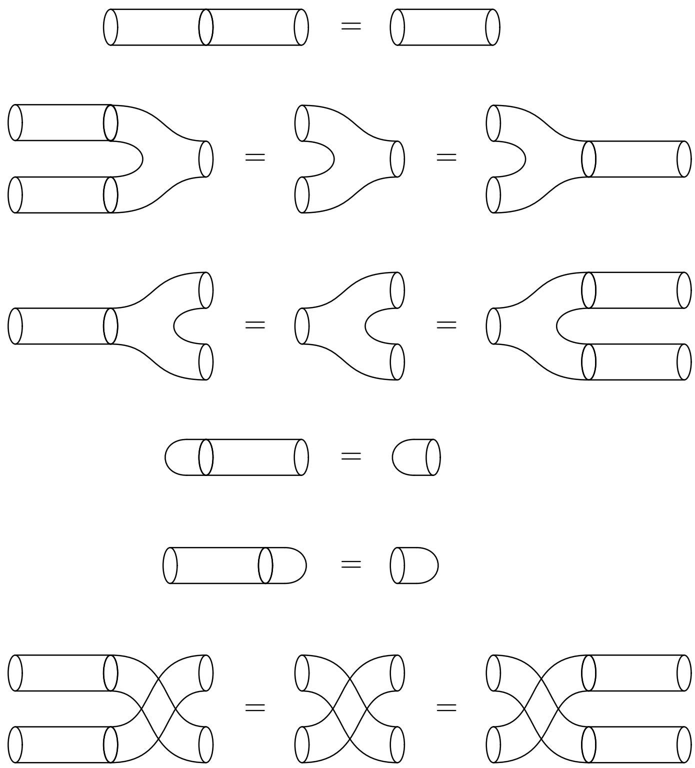
, , ,  and .

And finally using Lemma 3.5 as before, we have that the set of generators is

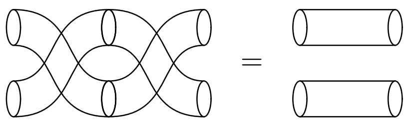
, , , ,  and .

(3) We now give the set of relations:

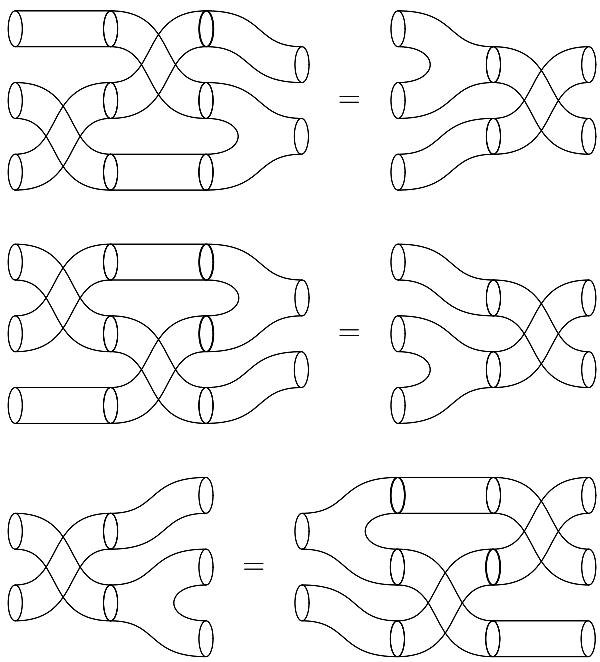
(a) Firstly, the ones relating to the identity morphisms:

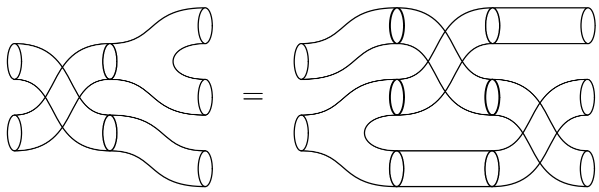


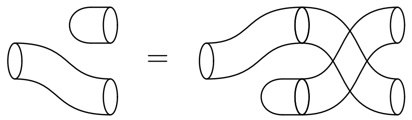
(b) The relation that expresses the fact that the twist is its own inverse:

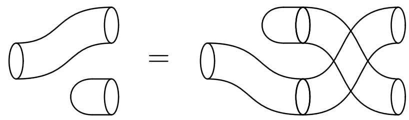


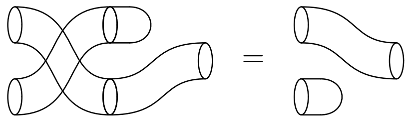
(c) The relations expressing the naturality of the twist:

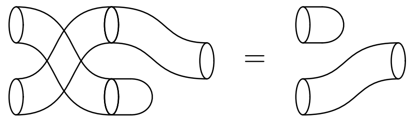


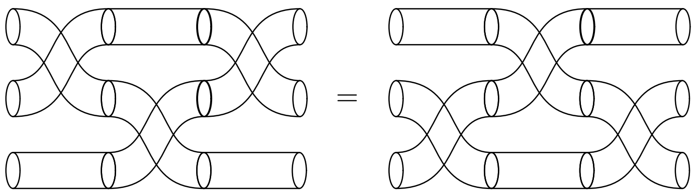






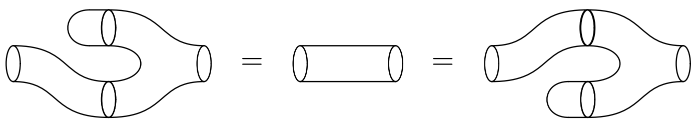


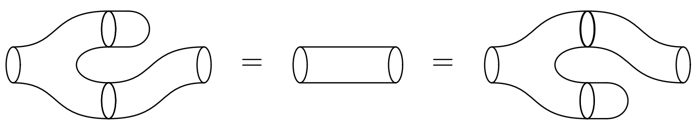




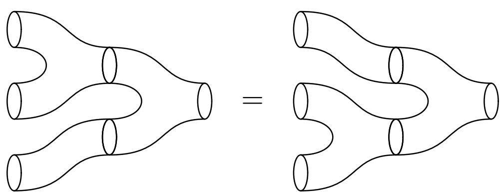
(d) And the relations characteristic of $\mathbf{Bord}_2$:

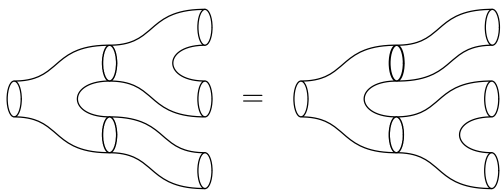
i. The **unit relations** and the **counit relations**:



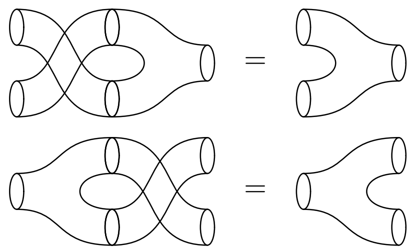


ii. The **associativity** and the **coassociativity relations**:

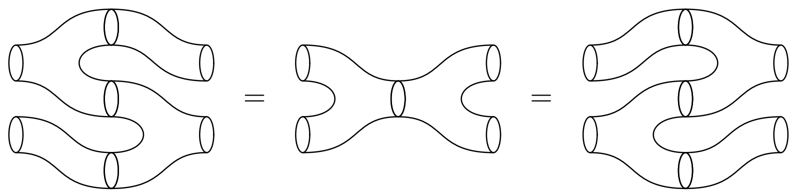




iii. The **commutativity** and the **cocommutativity relations**:



iv. And the **Frobenius relation**:[4]



To prove that all these relations hold, it is enough to see that in each case the bordisms have the same number of in- and out-boundaries and the same genus, 0. A proof of the completeness can be found in [Koc].

The point of having the presentation (O, G, R) of a [symmetric monoidal] category $\mathcal{C}$ is that, given another [symmetric monoidal] category $\mathcal{D}$, a [symmetric monoidal] functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is uniquely determined (up to [symmetric monoidal] isomorphism) by a choice of the images of the objects in O and the morphisms in G such that the relations of R still hold in $\mathcal{D}$, that is to say, if $f = g$ is a relation of R, then $F(f) = F(g)$ in $\mathcal{D}$.

3.4 Structure of 1-dimensional TQFTs

The presentation of $\mathbf{Bord}_1$ we have seen leads to a proof of the following statement, which establishes a one-to-one correspondence between 1-dimensional TQFTs and finite dimensional vector spaces in a functorial way.

Theorem 3.6. *There is a symmetric monoidal equivalence of categories*

$$\mathbf{TQFT}_1^{\Bbbk} \simeq \mathbf{FinVect}_{\Bbbk}^{\mathrm{iso}}$$

where $\mathbf{FinVect}_{\Bbbk}^{\mathrm{iso}}$ *is the category of finite-dimensional* $\Bbbk$*-vector spaces with invertible linear maps as morphisms (and with the tensor product* $\otimes$ *and the symmetric braiding* σ*).*

Proof. First, consider the symmetric monoidal functor

$$\begin{aligned} F : \mathbf{TQFT}_1^{\Bbbk} &\longrightarrow \mathbf{FinVect}_{\Bbbk}^{\mathrm{iso}} \\ [\mathcal{Z} : \mathbf{Bord}_1 \longrightarrow \mathbf{Vect}_{\Bbbk}] &\longmapsto V_{\mathcal{Z}} = \mathcal{Z}(\bullet_+) \end{aligned}$$

which is well-defined in virtue of Proposition 3.3.

Conversely, we contruct the symmetric monoidal functor

$$\begin{aligned} G : \mathbf{FinVect}_{\Bbbk}^{\mathrm{iso}} &\longrightarrow \mathbf{TQFT}_1^{\Bbbk} \\ V &\longmapsto [\mathcal{Z}_V : \mathbf{Bord}_1 \longrightarrow \mathbf{Vect}_{\Bbbk}] \end{aligned}$$

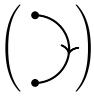
in the following manner:

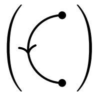
[4] The names of these relations will be understood later, when we introduce the Frobenius algebras.

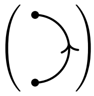
(1) For the objects in O, we define $\mathcal{Z}_V(\bullet_+) = V$ and $\mathcal{Z}_V(\bullet_-) = V^*$. The image on the other objects is defined by the "monoidality" of the functor, e.g.:

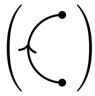
$$\mathcal{Z}_V(\bullet_+ \sqcup \bullet_-) = V \otimes V^*.$$

(2) For the generators in G, we define

$\mathcal{Z}_V$  $: V \otimes V^* \longrightarrow \Bbbk$, $\quad v \otimes \varphi \longmapsto \varphi(v)$

$\mathcal{Z}_V$  $: \Bbbk \longrightarrow V^* \otimes V$, $\quad \lambda \longmapsto \lambda \sum_i e_i^* \otimes e_i$

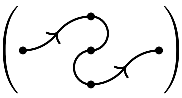
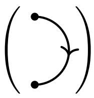
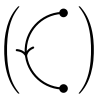
$\mathcal{Z}_V$  $: V^* \otimes V \longrightarrow \Bbbk$, $\quad \varphi \otimes v \longmapsto \varphi(v)$

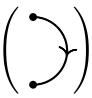
$\mathcal{Z}_V$  $: \Bbbk \longrightarrow V \otimes V^*$, $\quad \lambda \longmapsto \lambda \sum_i e_i \otimes e_i^*$

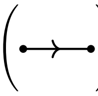
where $\{e_i\}$ is a basis of V and $\{e_i^*\}$ its dual basis. The images of the identity bordisms are fixed by the property (1) of functors (i.e. they are the identity maps), and the images of the twist bordisms are fixed by the symmetricity of the functor (i.e. they are defined by the symmetric braiding σ).

(3) We need to show that the relations in R still hold when taking images. The relations involving the identities and the twists are satisfied for the same reason in the previous paragraph.

The commutativity and cocommutativity relations are straightforward to prove. For one of the snake relations we have, for instance,

$\mathcal{Z}_V$  $(v) = \Big(\mathcal{Z}_V$  $\otimes 1_V\Big) \circ \Big(1_V \otimes \mathcal{Z}_V$  $\Big)(v \otimes 1) =$

$= \Big(\mathcal{Z}_V$  $\otimes 1_V\Big)\Big(\sum_i v \otimes e_i^* \otimes e_i\Big) = \sum_i e_i^*(v) \otimes e_i =$

$= \Big(\sum_i e_i^*(v)\Big) \cdot e_i = v = \mathcal{Z}_V$  $(v),$

and similarly for the other three.

We want to see that the compositions of these functors are equal (to be precise, naturally isomorphic) to the identities. On one hand we have

$$[\mathcal{Z} : \mathbf{Bord}_1 \longrightarrow \mathbf{Vect}_\Bbbk] \xmapsto{\;F\;} V_\mathcal{Z} = \mathcal{Z}(\bullet_+) \xmapsto{\;G\;} [\mathcal{Z}_{V_\mathcal{Z}} : \mathbf{Bord}_1 \longrightarrow \mathbf{Vect}_\Bbbk]$$

with $\mathcal{Z}_{V_\mathcal{Z}}(\bullet_+) = V_\mathcal{Z} = \mathcal{Z}(\bullet_+)$ and $\mathcal{Z}_{V_\mathcal{Z}}(\bullet_-) = V_\mathcal{Z}^* = \mathcal{Z}(\bullet_+)^* = \mathcal{Z}(\bullet_-)$. (Recall the last observation in the proof of 3.3.) And on the other hand,

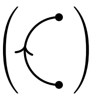
$$V \xmapsto{\;G\;} [\mathcal{Z}_V : \mathbf{Bord}_1 \longrightarrow \mathbf{Vect}_\Bbbk] \xmapsto{\;F\;} V_{\mathcal{Z}_V} = \mathcal{Z}_V(\bullet_+) = V.$$

Now we have to define how the functors F and G act on morphisms. Let $\alpha : \mathcal{Z} \Longrightarrow \mathcal{Y}$ be a symmetric monoidal natural transformation between two 1-TQFTs, $\mathcal{Z}$ and $\mathcal{Y}$, and denote $V_\mathcal{Z} = \mathcal{Z}(\bullet_+)$ and $V_\mathcal{Y} = \mathcal{Y}(\bullet_+)$ (and therefore $V_\mathcal{Z}^* = \mathcal{Z}(\bullet_-)$ and $V_\mathcal{Y}^* = \mathcal{Y}(\bullet_-)$). With this notation, α has in particular the two components $\alpha_{\bullet_+} : V_\mathcal{Z} \longrightarrow V_\mathcal{Y}$ and $\alpha_{\bullet_-} : V_\mathcal{Z}^* \longrightarrow V_\mathcal{Y}^*$. One can see that $\alpha_{\bullet_+}$ is in fact an isomorphism, with inverse $\tilde{\alpha}_{\bullet_+}$ defined as the composition

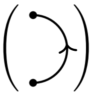
$$V_{\mathcal{Y}} \xrightarrow{\gamma_{\mathcal{Z}} \otimes 1_{V_{\mathcal{Y}}}} V_{\mathcal{Z}} \otimes V_{\mathcal{Z}}^* \otimes V_{\mathcal{Y}} \xrightarrow{1_{V_{\mathcal{Z}}} \otimes \alpha_{\bullet_-} \otimes 1_{V_{\mathcal{Y}}}} V_{\mathcal{Z}} \otimes V_{\mathcal{Y}}^* \otimes V_{\mathcal{Y}} \xrightarrow{1_{V_{\mathcal{Z}}} \otimes \beta_{\mathcal{Y}}} V_{\mathcal{Z}}$$

where the maps $\gamma_{\mathcal{Z}}$ and $\beta_{\mathcal{Y}}$ are

$$\gamma_{\mathcal{Z}} = \mathcal{Z}\left(\quad \right) : \Bbbk \longrightarrow V_{\mathcal{Z}} \otimes V_{\mathcal{Z}}^* \quad \text{and}$$



$$\beta_{\mathcal{Y}} = \mathcal{Y}\left(\quad \right) : V_{\mathcal{Y}}^* \otimes V_{\mathcal{Y}} \longrightarrow \Bbbk.$$



For instance, $\tilde{\alpha}_{\bullet_+} \circ \alpha_{\bullet_+} = 1_{V_{\mathcal{Z}}}$, since the following commutes:

$$\begin{tikzcd}
V_{\mathcal{Y}} \arrow[r, "\gamma_{\mathcal{Z}} \otimes 1_{V_{\mathcal{Y}}}"] & V_{\mathcal{Z}} \otimes V_{\mathcal{Z}}^* \otimes V_{\mathcal{Y}} \arrow[r, "1_{V_{\mathcal{Z}}} \otimes \alpha_{\bullet_-} \otimes 1_{V_{\mathcal{Y}}}"] & V_{\mathcal{Z}} \otimes V_{\mathcal{Y}}^* \otimes V_{\mathcal{Y}} \arrow[r, "1_{V_{\mathcal{Z}}} \otimes \beta_{\mathcal{Y}}"] & V_{\mathcal{Z}} \\
V_{\mathcal{Z}} \arrow[u, "\alpha_{\bullet_+}"] \arrow[r, "\gamma_{\mathcal{Z}} \otimes 1_{V_{\mathcal{Z}}}"] \arrow[rr, bend right, "1_{V_{\mathcal{Z}}}"'] & V_{\mathcal{Z}} \otimes V_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}} \arrow[u, "1_{V_{\mathcal{Z}}} \otimes 1_{V_{\mathcal{Z}}^*} \otimes \alpha_{\bullet_+}"'] \arrow[r, "1_{V_{\mathcal{Z}}} \otimes \beta_{\mathcal{Z}}"] \arrow[ur, "1_{V_{\mathcal{Z}}} \otimes \alpha_{\bullet_-} \otimes \alpha_{\bullet_+}"] & V_{\mathcal{Z}} \arrow[ur, "1_{V_{\mathcal{Z}}}"']
\end{tikzcd}$$

where the rightmost rhomboid is due to the naturality of α, and the curved identity is due to the snake relation and the functoriality of $\mathcal{Z}$. Therefore we can assign $F(\alpha) = \alpha_{\bullet_+}$.

Conversely, given an invertible linear map $f : V \longrightarrow W$, we define $G(f)$ to be the symmetric monoidal natural transformation $\alpha_f : \mathcal{Z}_V \Longrightarrow \mathcal{Z}_W$ with components $\alpha_{f_{\bullet_+}} = f$, $\alpha_{f_{\bullet_-}} = (f^{-1})^*$ (and the other ones defined by monoidality). $\square$

Some authors, cf. [CR], prefer to formulate this equivalence of categories using $\mathbf{DP}_{\Bbbk}$, the category of **dual pairs**. This is the category consisting of two $\Bbbk$-vector spaces V and W equipped with two linear maps $\gamma : \Bbbk \longrightarrow V \otimes W$ and $\beta : W \otimes V \longrightarrow \Bbbk$ such that

$$(1_V \otimes \beta) \circ (\gamma \otimes 1_V) = 1_V \quad \text{and} \quad (\beta \otimes 1_W) \circ (1_W \otimes \gamma) = 1_W.$$

This is equivalent to stating that V and W are duals of each other, but with this new category there is no symmetry break with the treatment of $\bullet_+$ and $\bullet_-$: A TQFT $\mathcal{Z}$ would be sent to the pair $(\mathcal{Z}(\bullet_+), \mathcal{Z}(\bullet_-))$.

On the other hand, a morphism $(V_1, W_1, \gamma_1, \beta_1) \longrightarrow (V_2, W_2, \gamma_2, \beta_2)$ in $\mathbf{DP}_{\Bbbk}$ consists of a pair $(f : V_1 \longrightarrow V_2, g : W_1 \longrightarrow W_2)$ such that

$$(f \otimes g) \circ \gamma_1 = \gamma_2 \quad \text{and} \quad \beta_1 = \beta_2 \circ (g \otimes f).$$

Therefore a natural transformation between TQFTs α would be mapped to the pair $(\alpha_{\bullet_+}, \alpha_{\bullet_-})$.

4 Frobenius algebras and structure of 2-dimensional TQFTs

The structure for two-dimensional topological quantum field theories is, of course, much more complex than in the one-dimensional case. Still, it admits a surprisingly transparent and intuitive description in terms of commutative Frobenius algebras. These algebras with special duality properties were originally considered in representation theory of finite groups. With important roles on Number Theory, Algebraic Geometry, and Combinatorics, they have been more recently used to study Hopf algebras, coding theory and cohomology rings of compact manifolds. Their key role in topological quantum field theory is one of the most recent expressions of this algebra structure, with applications to the study of knots and other low-dimensional manifold theories.

In this section, we review the correspondence between two-dimensional TQFTs and Frobenius algebras, by means of an equivalence of symmetric monoidal categories. This correspondance was first described by Dijkgraaf in his Ph.D. thesis [Dij], although we have mainly followed [Koc].

4.1 Preliminary concepts

Definition 4.1. A **$\Bbbk$-algebra** is a $\Bbbk$-vector space A equipped with two linear maps $\mu : A \otimes A \longrightarrow A$ and $\eta : \Bbbk \longrightarrow A$ (called **multiplication** and **unit** respectively) such that the following diagrams commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\mu\otimes 1_A"'] \arrow[dr, "1_A\otimes\mu"] & \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mu"] & \\
\Bbbk \otimes A \arrow[ur, "\eta\otimes 1_A"] \arrow[rr, equal] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mu"] & \\
A \otimes \Bbbk \arrow[ur, "1_A\otimes\eta"] \arrow[rr, equal] & & A
\end{tikzcd}$$

We will write $\mu(a \otimes b) = ab$ and $\eta(1) = 1$ when there is no confusion.

Observe how a $\Bbbk$-algebra (A, μ, η) is a ring $(A, +, \cdot)$ with $+$ the addition inherited from the $\Bbbk$-vector space structure and $\cdot$ defined

$$\begin{aligned} \cdot : A \times A &\longrightarrow A \\ (a, b) &\longmapsto \mu(a \otimes b). \end{aligned}$$

So we can talk about ideals of a $\Bbbk$-algebra, and just as a reminder:

Definition 4.2. Given a $\Bbbk$-algebra A, a $\Bbbk$-vector subspace $\mathfrak{a} \subset A$ is a **left ideal** [resp. **right ideal**] if $ax \in \mathfrak{a}$ [resp. $xa \in \mathfrak{a}$] for every $a \in A$ and $x \in \mathfrak{a}$.

4.2 First definition of Frobenius algebras

The first definition of Frobenius algebras can be found in [BN] and reads:

Definition 4.3. A **Frobenius $\Bbbk$-algebra** is a finite-dimensional $\Bbbk$-algebra A equipped with a linear map $\varepsilon : A \longrightarrow \Bbbk$ (called **Frobenius form**) whose nullspace

$$\mathrm{null}(\varepsilon) := \{a \in A | \varepsilon(a) = 0\}$$

contains no nontrivial left ideals.

Observe that, since every nontrivial left ideal contains a nontrivial principal left ideal (i.e. a ideal of the form $Ax = \{ax | a \in A\}$ with $x \in \mathrm{null}(\varepsilon)$),[5] the condition of the definition is equivalent to requiring the nullspace to contain no nontrivial principal left ideal; in other words:

$$\forall x \in A, \quad \varepsilon(Ax) = 0 \Longrightarrow x = 0.$$

With either the definition or this characterization we can give some examples of Frobenius algebras:

(1) First consider $\mathbb{R}$-algebra of complex numbers, $\mathbb{C}$, with μ being its usual multiplication and $1 \in \mathbb{R}$ as the unit.[6] Any nonzero linear map $\varepsilon : \mathbb{C} \longrightarrow \mathbb{R}$ is a Frobenius form: indeed, as $\mathrm{null}(\varepsilon) \neq \mathbb{C}$ because ε is nonzero, and due to the fact that every field has only two ideals –viz. $\{0\}$ and the field itself–, $(\mathbb{C}, \varepsilon)$ satisfies the definition of Frobenius algebra. In particular, we could have $\varepsilon(z) := \mathrm{Re}(z)$.

(2) Consider the set of $n \times n$ matrices over a field $\Bbbk$, $\mathrm{Mat}_{n\times n}(\Bbbk)$, which is a $\Bbbk$-algebra with the usual multiplication and the identity matrix as the unit. It is a Frobenius algebra with the trace as its form. Let us prove the contrapositive of the condition above: Let M be a nonzero matrix. To check that there exists a matrix N such that $\mathrm{Tr}(NM) \neq 0$, consider the natural basis of $\mathrm{Mat}_{n\times n}(\Bbbk)$ consisting of the matrices E_{ij} with $e_{ij} = 1$ as the only nonzero entry. We have

$$E_{ij}E_{kl} = \delta_{jk}E_{il}$$
$$\mathrm{Tr}(E_{ij}E_{kl}) = \delta_{jk}\delta_{il}.$$

If $M = \sum_{ij} \lambda_{ij}E_{ij}$ with $\lambda_{kl} \neq 0$, then

$$\mathrm{Tr}(E_{lk}M) = \sum_{ij} \lambda_{ij}\,\mathrm{Tr}(E_{lk}E_{ij}) = \sum_{ij} \lambda_{ij}\delta_{ki}\delta_{lj} = \lambda_{kl} \neq 0.$$

(3) As a last example let $\Bbbk[G]$ be the vector space spanned by a finite (multiplicative) group $G = \{g_0 = 1, g_1, \ldots, g_n\}$, whose elements are formal linear combinations with coefficients in $\Bbbk$. It is a $\Bbbk$-algebra with the multiplication inherited from the group structure and g_0 as the unit. Consider the linear map ε defined on the basis as follows:

$$\varepsilon(g_i) = \delta_{i0} = \begin{cases} 1 & \text{if} \quad g_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Like in the previous example, let $x = \sum_{i=0}^{n} a_i g_i$ with some $a_j \neq 0$. Then

$$\varepsilon(g_j^{-1}x) = \sum_{i=0}^{n} a_i \varepsilon(g_j^{-1}g_i) = \sum_{i=0}^{n} a_i\delta_{ji} = a_j \neq 0,$$

and thus $(\Bbbk[G], \varepsilon)$ is a Frobenius algebra.

We want to find equivalent definitions of the Frobenius algebras that set us closer to its correspondence to 2-TQFTs.

[5] For instance, given a nontrivial left ideal $\mathfrak{a} \subset \mathrm{null}(\varepsilon)$, choosing a nonzero element $x \in \mathfrak{a}$ gives $Ax \subset \mathfrak{a}$.

[6] With this common abuse of language we state that $\eta(1) = 1$, and hence $\eta(x) = x \ \forall x \in \mathbb{R}$.

4.3 Second definition of Frobenius algebras

Before introducing a second definition we review the notion of pairing and its dual counterpart.

Definition 4.4. A **pairing** between to $\Bbbk$-vector spaces V and W is a linear map $\beta : V \otimes W \longrightarrow \Bbbk$. We will write $\beta(v \otimes w) = \langle v|w\rangle$. Similarly, a **copairing** between V and W is a linear map $\gamma : \Bbbk \longrightarrow V \otimes W$.

Definition 4.5. A pairing $\beta : V \otimes W \longrightarrow \Bbbk$ is said to be **nondegenerate in its first variable** if it exists a copairing $\gamma_V : \Bbbk \longrightarrow W \otimes V$ such that the composition

$$V \xrightarrow{1_V \otimes \gamma_V} V \otimes W \otimes V \xrightarrow{\beta \otimes 1_V} V$$

is equal to the identity map 1_V. Analogously, β is said to be **nondegenerate in its second variable** if it exists a copairing $\gamma_W : \Bbbk \longrightarrow W \otimes V$ such that the composition

$$W \xrightarrow{\gamma_W \otimes 1_W} W \otimes V \otimes W \xrightarrow{1_W \otimes \beta} W$$

is equal to the identity map 1_W. If β is nondegenerate in both its components, it is simply said to be **nondegenerate**.

Lemma 4.6. *If $\beta : V \otimes W \longrightarrow \Bbbk$ is a nondegenerate pairing, then $\gamma_V = \gamma_W$.*

Proof. Consider the map λ defined to be

$$\Bbbk \xrightarrow{\gamma_W \otimes \gamma_V} W \otimes V \otimes W \otimes V \xrightarrow{1_W \otimes \beta \otimes 1_V} W \otimes V.$$

On one hand we can factor $\gamma_W \otimes \gamma_V$ as $(1_W \otimes 1_V \otimes \gamma_V) \circ \gamma_W$, so we have

$$\lambda = (1_W \otimes \beta \otimes 1_V) \circ (1_W \otimes 1_V \otimes \gamma_V) \circ \gamma_W = (1_W \otimes ((\beta \otimes 1_V) \circ (1_V \otimes \gamma_V))) \circ \gamma_W = \gamma_W$$

as $(\beta \otimes 1_V) \circ (1_V \otimes \gamma_V) = 1_V$ by nondegeneracy. And on the other hand we can factor $\gamma_W \otimes \gamma_V$ as $(\gamma_W \otimes 1_W \otimes 1_V) \circ \gamma_V$, so, similarly,

$$\begin{aligned}\lambda = (1_W \otimes \beta \otimes 1_V) \circ (\gamma_W \otimes 1_W \otimes 1_V) \circ \gamma_V \\ = (((1_W \otimes \beta) \circ (\gamma_W \otimes 1_W)) \otimes 1_V) \circ \gamma_V = \gamma_V. \quad \square\end{aligned}$$

In this case we will call the copairing simply γ.

Definition 4.7. Let A be a $\Bbbk$-algebra. A pairing $\beta : A \otimes A \longrightarrow \Bbbk$ is said to be **associative** if $\forall a, b, c \in A$, $\langle ab|c\rangle = \langle a|bc\rangle$, i.e. the following commutes:

$$\begin{tikzcd} & A \otimes A \otimes A \arrow[dl, "\mu \otimes 1_A"'] \arrow[dr, "1_A \otimes \mu"] & \\ A \otimes A \arrow[dr, "\beta"'] & & A \otimes A \arrow[dl, "\beta"] \\ & \Bbbk & \end{tikzcd}$$

Now we can give the second definition of Frobenius algebras:

Definition 4.8. A **Frobenius $\Bbbk$-algebra** is a finite-dimensional $\Bbbk$-algebra A equipped with a nondegenerate associative pairing $\beta : A \otimes A \longrightarrow \Bbbk$ (called **Frobenius pairing**).

We want to prove the equivalence of both definitions. First some lemmas:

Lemma 4.9. *Let V and W be two vector spaces and let $\beta : V \otimes W \longrightarrow \Bbbk$ be a pairing. The following are equivalent:*

(1) β is nondegenerate in W, and

(2) $\langle v|w\rangle = 0 \quad \forall v \in V \Longrightarrow w = 0$ and W is finite-dimensional.

Similarly, the following are equivalent:

(3) β is nondegenerate in V, and

(4) $\langle v|w\rangle = 0 \quad \forall w \in W \Longrightarrow v = 0$ and V is finite-dimensional.

Proof. We will show the first equivalence (the other one is analogous). To prove that (1) implies (2), suppose that β is nondegenerate with associated copairing γ. Suppose that $\gamma(1) = \sum_{i=0}^n v_i \otimes w_i$ for some vectors $v_i \in V$, $w_i \in W$. Then, by nondegeneracy, for every vector $w \in W$,

$$w \xmapsto{\gamma\otimes 1_W} \sum_{i=0}^n w_i \otimes v_i \otimes w \xmapsto{1_W\otimes\beta} \sum_{i=0}^n \langle v_i|w\rangle w_i = w.$$

This shows that $\{w_i\}$ spans W, which is therefore finite-dimensional. Also, if we suppose that $\langle v|w\rangle = 0$ for all $v \in V$, in particular $\langle v_i|w\rangle = 0$, so $w = 0$.

To prove the opposite implication, assume that W is finite-dimensional with a basis $\{w_1, \dots, w_n\}$. The first property of (2) guarantees that $\langle -|w_i\rangle$ are linearly independent (in V^*):

$$\sum_{i=0}^n \lambda_i \langle -|w_i\rangle = 0 \Longrightarrow \sum_{i=0}^n \lambda_i \langle v|w_i\rangle = 0 \quad \forall v \in V \Longrightarrow$$
$$\left\langle v \,\middle|\, \sum_{i=0}^n \lambda_i w_i \right\rangle \quad \forall v \in V \Longrightarrow \sum_{i=0}^n \lambda_i w_i = 0 \Longrightarrow \lambda_i = 0,$$

and thus there exist vectors $v_i, \dots, v_n$ such that $\langle v_i|w_j\rangle = \delta_{ij}$.

We define a copairing $\gamma : \Bbbk \longrightarrow W \otimes V$ by saying $\gamma(1) = \sum_{i=0}^n w_i \otimes v_i$. Now, given a vector $\sum_{j=0}^n \lambda_j w_j \in W$,

$$\sum_{j=0}^n \lambda_j w_j \xmapsto{\gamma\otimes 1_W} \sum_{i,j} w_i \otimes v_i \otimes \lambda_j w_j \xmapsto{1_W\otimes\beta} \sum_{i,j} \lambda_j \langle v_i|w_j\rangle w_i = \sum_{i=0}^n \lambda_i w_i.$$

So β is nondegenerate. $\square$

We are only interested when V and W are of the same finite dimension (in fact, we will use $V = W = A$). Notice that, in this case, the condition (2) says that the map $W \longrightarrow V^*$, $w \longmapsto \langle -|w\rangle$ is injective and therefore an isomorphism. Taking duals we get the isomorphism $V \longrightarrow W^*$, $v \longmapsto \langle v|-\rangle$, which corresponds to (4). We can state a weaker version of the previous lemma:

Lemma 4.10. *Let V and W be two vector spaces of the same finite dimension and let $\beta : V \otimes W \longrightarrow \Bbbk$ be a pairing. The following are equivalent:*

(1) β is nondegenerate,

(2) $\langle v|w\rangle = 0 \quad \forall v \in V \Longrightarrow w = 0$, and

(3) $\langle v|w\rangle = 0 \quad \forall w \in W \Longrightarrow v = 0$.

The next lemma will be useful:

Lemma 4.11. *Let A be a $\Bbbk$-algebra. There is a one-to-one correspondence between linear forms $A \longrightarrow \Bbbk$ and associative pairings $A \otimes A \longrightarrow \Bbbk$.*

Proof. Given a linear form $\varepsilon : A \longrightarrow \Bbbk$ we can construct a pairing

$$\begin{aligned} \beta_\varepsilon : A \otimes A &\longrightarrow \Bbbk \\ x \otimes y &\longmapsto \varepsilon(xy), \end{aligned}$$

which is obviously associative, and given an associative pairing $\beta : A \otimes A \longrightarrow \Bbbk$ we can construct a linear form

$$\begin{aligned} \varepsilon_\beta : A &\longrightarrow \Bbbk \\ x &\longmapsto \langle 1|x\rangle = \langle x|1\rangle. \end{aligned}$$

Observe that

$$\varepsilon \longmapsto \beta_\varepsilon \longmapsto \varepsilon_{\beta_\varepsilon} = \big[x \longmapsto \beta_\varepsilon(1 \otimes x) = \varepsilon(1x)\big] = \varepsilon$$

and that

$$\beta \longmapsto \varepsilon_\beta \longmapsto \beta_{\varepsilon_\beta} = \big[x \otimes y \longmapsto \varepsilon_\beta(xy) = \langle 1|xy\rangle = \langle x|y\rangle\big] = \beta.$$

□

Theorem 4.12. *Definitions 4.3 and 4.8 are equivalent.*

Proof. By Lemma 4.11, it is sufficient to see that null(ε) contains no nontrivial left ideals if and only if its corresponding pairing β is nondegenerate. Lemma 4.10 says that β is nondegenerate if and only if $\langle A|x\rangle = 0 \Longrightarrow x = 0$, or equivalently, if and only if $\varepsilon(Ax) = 0 \Longrightarrow x = 0$. As we observed above, this is the same as saying that null(ε) contains no nontrivial left ideals. □

Notice that in Definition 4.3 we could have defined Frobenius algebras in terms of right—instead of left—ideals.

Before proceeding to state a third definition, we can express the three previous examples in terms of pairings and copairings:

(1) According to the proof of Lemma 4.11, the pairing of $(\mathbb{C}, \mathrm{Re})$ is defined $\beta(z \otimes w) = \mathrm{Re}(zw)$. Its associated copairing is given by

$$\gamma(1) = 1 \otimes 1 - i \otimes i.$$

Let us check that β is nondegenerate in its first variable (and therefore simply nondegenerate) with γ:

$$a + bi \xmapsto{1_{\mathbb{C}} \otimes \gamma} (a+bi) \otimes 1 \otimes 1 - (a+bi) \otimes i \otimes i \xmapsto{\beta \otimes 1_{\mathbb{C}}} \mathrm{Re}(a+bi) \otimes 1 - \mathrm{Re}(ai - b) \otimes i = a + bi.$$

(2) In our second example, the pairing is $\beta(M \otimes N) = \mathrm{Tr}(MN)$. The copairing must be $\gamma(1) = \sum_{i,j} E_{ij} \otimes E_{ji}$, since

$$E_{ij} \xmapsto{1_{\mathrm{Mat}} \otimes \gamma} \sum_{k,l} E_{ij} \otimes E_{kl} \otimes E_{lk} \xmapsto{\beta \otimes 1_{\mathrm{Mat}}} \sum_{k,l} \mathrm{Tr}(E_{ij} E_{kl}) E_{lk} = \sum_{k,l} \delta_{jk} \delta_{il} E_{lk} = E_{ij}$$

for each element of the basis.

(3) Finally, in $\mathbb{k}[G]$ the pairing is

$$\beta(g_i \otimes g_j) = \begin{cases} 1 & \text{if} \quad g_i g_j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

and its associated copairing $\gamma(1) = \sum_{i=0}^{n} g_i^{-1} \otimes g_i$:

$$g_i \xmapsto{1_{\mathbb{k}[G]} \otimes \gamma} \sum_{j=0}^{n} g_i \otimes g_j^{-1} \otimes g_j \xmapsto{\beta \otimes 1_{\mathbb{k}[G]}} \sum_{j=0}^{n} \varepsilon(g_i g_j^{-1}) g_j = \sum_{j=0}^{n} \delta_{ij} g_j = g_i$$

for each element of G (which is the basis of $[G]$).

4.4 Third definition of Frobenius algebras

The definitions of Frobenius algebras we will give now resembles the most to the presentation of $\mathbf{TQFT}_2^{\mathbb{k}}$ we found. First we need some definitions:

Definition 4.13. A **$\mathbb{k}$-coalgebra** is a $\mathbb{k}$-vector space A equipped with two linear maps $\delta : A \longrightarrow A \otimes A$ and $\varepsilon : A \longrightarrow \mathbb{k}$ (called **comultiplication** and **counit** respectively) such that the following diagrams commute:

$$\begin{tikzcd} & A \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ A \otimes A \arrow[dr, "\delta \otimes 1_A"'] & & A \otimes A \arrow[dl, "1_A \otimes \delta"] \\ & A \otimes A \otimes A & \end{tikzcd}$$

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\varepsilon \otimes 1_A"] & \\ A \arrow[ur, "\delta"] \arrow[rr, equal] & & \Bbbk \otimes A \end{tikzcd} \qquad \begin{tikzcd} & A \otimes A \arrow[dr, "1_A \otimes \varepsilon"] & \\ A \arrow[ur, "\delta"] \arrow[rr, equal] & & A \otimes \Bbbk \end{tikzcd}$$

We can give some examples to illustrate this definition:

(1) Consider a 2-dimensional vector space with basis $\{c, s\}$. It has the structure of coalgebra with the following definitions for δ and ε:

$$\delta : \begin{cases} c \longmapsto c \otimes c - s \otimes s \\ s \longmapsto c \otimes s + s \otimes c, \end{cases}$$

$$\varepsilon : \begin{cases} c \longmapsto 1 \\ s \longmapsto 0. \end{cases}$$

This is known as the *trigonometric coalgebra*: If we let c and s be the linearly independent functions cos and sin respectively, there are the equivalences

$$\begin{aligned} \delta(\cos)(x \otimes y) &= \cos(x + y), \\ \delta(\mathrm{son})(x \otimes y) &= \sin(x + y), \\ \varepsilon(\cos) &= \cos(0) \quad \text{and} \\ \varepsilon(\sin) &= \sin(0). \end{aligned}$$

(2) Given an arbitrary set S and a field $\Bbbk$, $\Bbbk[S]$ is a $\Bbbk$-coalgebra with the comultiplication and the counit defined on the basis as

$$\delta : x \longmapsto x \otimes x$$
$$\varepsilon : x \longmapsto 1$$

for all $x \in S$.

(3) As the notion of a coalgebra is the dual of that of an algebra (i.e. it is defined by "reversing the arrows"), one could ask if given an algebra A, the dual, A^*, is a coalgebra. Whereas this is not true in general, it is in the finite-dimensional case:

We define the comultiplication and the counit as the transposes of the multiplication and unit respectively, i.e.

$$\begin{aligned} \delta : A^* &\longrightarrow (A \otimes A)^* \\ f &\longmapsto f \circ \mu \qquad \text{and} \\ \varepsilon : A^* &\longrightarrow \Bbbk \\ f &\longmapsto f(\eta(1)). \end{aligned}$$

Notice that δ is well-defined since there is always a canonical isomorphism between $(A \otimes A)^*$ and $A^* \otimes A^*$ if A is finite-dimensional (see [Rom, Thm. 14.7]). However given a coalgebra (A, δ, ε), its dual is always an algebra, since there is always a canonical monomorphism $A^* \otimes A^* \hookrightarrow (A \otimes A)^*$ that lets us define the multiplication μ in A^* as

$$\begin{aligned} \mu : A^* \otimes A^* &\hookrightarrow (A \otimes A)^* \xrightarrow{\ \delta^*\ } A \\ f \otimes g &\longmapsto [f \odot g : x \otimes y \longmapsto f(x)g(y)] \longmapsto (f \odot g) \circ \delta. \end{aligned}$$

The unit is defined as $\eta(1) = \varepsilon$.

Definition 4.14. We say that two linear maps $\mu : A \otimes A \longrightarrow A$ and $\delta : A \longrightarrow A \otimes A$ satisfy the **Frobenius relation** if the following diagram commutes:

$$\begin{tikzcd} & A \otimes A \otimes A \arrow[dr, "1_A \otimes \mu"] & \\ A \otimes A \arrow[ur, "\delta \otimes 1_A"] \arrow[r, "\mu"] \arrow[dr, "1_A \otimes \delta"'] & A \arrow[r, "\delta"] & A \otimes A \\ & A \otimes A \otimes A \arrow[ur, "\mu \otimes 1_A"'] & \end{tikzcd}$$

The third definition of Frobenius algebras is:

Definition 4.15. A **Frobenius $\Bbbk$-algebra** is a quintuple $(A, \mu, \eta, \delta, \varepsilon)$ such that

(1) (A, μ, η) is a $\Bbbk$-algebra,

(2) (A, δ, ε) is a $\Bbbk$-coalgebra, and

(3) the Frobenius relation holds for μ and δ.

To show that this definition is equivalent to the previous ones, firstly we assume (A, ε) to be a Frobenius algebra (according to Definition 4.3) and we want construct a comultiplication δ such that (A, δ, ε) is a $\Bbbk$-coalgebra. To do so we have at our disposal μ, η, β and γ as well as ε.

We define δ to be the composition

$$A \xrightarrow{\gamma \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \mu} A \otimes A.$$

To prove its coassociativity (i.e. the first property in the definition of $\Bbbk$-coalgebra), we will use the following:

Lemma 4.16. *The following diagram commutes*

$$\begin{tikzcd}
A\otimes A\otimes A\otimes A \arrow[r, "1_A\otimes 1_A\otimes\mu"] & A\otimes A\otimes A \arrow[d, "1_A\otimes\beta"] \\
A\otimes A \arrow[u, "\gamma\otimes 1_A\otimes 1_A"] \arrow[r, "\mu"] \arrow[d, "1_A\otimes 1_A\otimes\gamma"'] & A \\
A\otimes A\otimes A\otimes A \arrow[r, "\mu\otimes 1_A\otimes 1_A"'] & A\otimes A\otimes A \arrow[u, "\beta\otimes 1_A"']
\end{tikzcd}$$

Proof. All these kind of statements involving commutative diagrams will be proven by "diagram chasing." Let us prove the uppermost part of the diagram:

$$\begin{tikzcd}
A\otimes A\otimes A\otimes A \arrow[rr, "1_A\otimes 1_A\otimes\mu"] & & A\otimes A\otimes A \arrow[d, "1_A\otimes\beta"] \\
A\otimes A \arrow[u, "\gamma\otimes 1_A\otimes 1_A"] \arrow[r, "\mu", "(\mathbf{1})"'] & A \arrow[ur, "\gamma\otimes 1_A"] \arrow[r, "1_A", "(\mathbf{2})"'] \arrow[ur, phantom, shift right=4, "(\mathbf{1})"] & A
\end{tikzcd}$$

(**1**) are found by adding and removing identity maps, and (**2**) is a consequence of the nondegeneracy of β. ☐

Lemma 4.17. *The following diagram commutes*

$$\begin{tikzcd}
& A\otimes A\otimes A \arrow[dr, "1_A\otimes\mu"] & \\
A \arrow[ur, "\gamma\otimes 1_A"] \arrow[dr, "1_A\otimes\gamma"'] & & A\otimes A \\
& A\otimes A\otimes A \arrow[ur, "\mu\otimes 1_A"'] &
\end{tikzcd}$$

(i.e. we could have constructed δ in a "symmetrical" way.)

Proof. We use the notation $A^{\otimes n} = A \otimes \overset{n}{\cdots} \otimes A$ to declutter the diagrams.

$$
\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[ddl, "1_A \otimes 1_A \otimes 1_A \otimes \gamma", "(\mathbf{1})"'] \arrow[ddrr, "1_A \otimes \mu"] & & \\
 & & & A^{\otimes 4} \arrow[dr, "1_A \otimes \beta \otimes 1_A", "(\mathbf{1})"'] & \\
A \arrow[uurr, "\gamma \otimes 1_A"] \arrow[ddrr, "1_A \otimes \gamma"', "(\mathbf{3})"] & A^{\otimes 5} \arrow[urr, "1_A \otimes \mu \otimes 1_A \otimes 1_A", "(\mathbf{1})"'] \arrow[drr, "1_A \otimes 1_A \otimes \mu \otimes 1_A", "(\mathbf{2})"'] & & & A \otimes A \\
 & & & A^{\otimes 4} \arrow[ur, "1_A \otimes \beta \otimes 1_A", "(\mathbf{2})"'] & \\
 & & A \otimes A \otimes A \arrow[uul, "\gamma \otimes 1_A \otimes 1_A \otimes 1_A"', "(\mathbf{3})"] \arrow[uurr, "\mu \otimes 1_A"', "(\mathbf{4})"] & &
\end{tikzcd}
$$

Lemma 4.16 gives (**1**); the associativity of β gives (**2**); rearranging identities gives (**3**); and finally again using Lemma 4.16 we have (**4**). $\square$

Lemma 4.18. *The comultiplication δ we constructed is coassociative, i e. the following diagram commutes:*

$$
\begin{tikzcd}
 & A \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\
A \otimes A \arrow[dr, "\delta \otimes 1_A"'] & & A \otimes A \arrow[dl, "1_A \otimes \delta"] \\
 & A \otimes A \otimes A &
\end{tikzcd}
$$

Proof. Lemma 4.17 gives us two equivalent definitions of δ,

$$A \xrightarrow{\gamma \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \mu} A \otimes A \quad \text{and} \quad A \xrightarrow{1_A \otimes \gamma} A \otimes A \otimes A \xrightarrow{\mu \otimes 1_A} A \otimes A.$$

So we have

$$
\begin{tikzcd}
 & & A \arrow[dll, bend right, "\delta"'] \arrow[dl, "1_A\oplus\gamma"'] \arrow[d, "\gamma\otimes 1_A\otimes\gamma", "(\mathbf{1})"'] \arrow[dr, "\gamma\oplus 1_A", "(\mathbf{3})"'] \arrow[drr, bend left, "\delta"] & & \\
 & A^{\otimes 3} \arrow[dl, "\mu\oplus 1_A"'] & & A^{\otimes 3} \arrow[dr, "1_A\oplus\mu", "(\mathbf{3})"'] & \\
A\otimes A \arrow[dr, "\gamma\oplus 1_A\oplus 1_A"] \arrow[ddrr, bend right, "\delta\otimes 1_A"'] & & A^{\otimes 5} \arrow[dl, "1_A\oplus 1_A\oplus\mu\oplus 1_A"', "(\mathbf{1})"] \arrow[dr, "1_A\oplus\mu\oplus 1_A\oplus 1_A", "(\mathbf{2})"'] & & A\otimes A \arrow[dl, "1_A\oplus 1_A\oplus\gamma"', "(\mathbf{3})"] \arrow[ddll, bend left, "1_A\otimes\delta"] \\
 & A^{\otimes 4} \arrow[dr, "1_A\oplus\mu\oplus 1_A"] & & A^{\otimes 4} \arrow[dl, "1_A\oplus\mu\oplus 1_A"', "(\mathbf{2})"] & \\
 & & A\otimes A\otimes A & &
\end{tikzcd}
$$

where $(\mathbf{1})$ have been found rearranging identities, $(\mathbf{2})$ by associativity of μ and $(\mathbf{3})$ again rearranging identities. ☐

Now we shall prove the remaining property of $\Bbbk$-coalgebras.

Lemma 4.19. *The following diagrams commute:*

$$
\begin{tikzcd}
 & A \arrow[dr, "\varepsilon"] & \\
A\otimes A \arrow[ur, "\mu"] \arrow[rr, "\beta"'] & & \Bbbk
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A\otimes A \arrow[dr, "\beta"] & \\
A \arrow[ur, "1_A\otimes\eta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\eta\otimes 1_A"'] & & \Bbbk \\
 & A\otimes A \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

Proof. These diagrams are given in the proof of 4.11. ☐

Lemma 4.20. *The Frobenius form ε is a counit for the comultiplication δ, i.e.:*

$$
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\varepsilon\otimes 1_A"] & \\
A \arrow[ur, "\delta"] \arrow[rr, "1_A"] \arrow[dr, "\delta"'] & & A \\
 & A \otimes A \arrow[ur, "1_A\otimes\varepsilon"'] &
\end{tikzcd}
$$

(Observe how from now on, the diagrams will not distinguish A, $\Bbbk \otimes A$, $A \otimes \Bbbk$, etc.)

Proof. Let us proof the uppermost diagram:

$$
\begin{tikzcd}
 & & A \otimes A \arrow[ddrr, "\varepsilon\otimes 1_A"] & & \\
 & & A \otimes A \otimes A \arrow[u, "(\mathbf{1})"', "\mu\otimes 1_A"] \arrow[drr, "(\mathbf{2})", "\beta\otimes 1_A"'] & & \\
A \arrow[uurr, "\delta"] \arrow[urr, "(\mathbf{1})", "1_A\otimes\gamma"'] \arrow[rrrr, "1_A", "(\mathbf{3})"'] & & & & A
\end{tikzcd}
$$

($\mathbf{1}$) are the definition of δ, ($\mathbf{2}$) is given by Lemma 4.19, and ($\mathbf{3}$) by nondegeneracy of β. $\square$

Lemmas 4.18 and 4.20 state that A is a $\Bbbk$-coalgebra with δ and ε. Now we show that the Frobenius relation is satisfied.

Lemma 4.21. *μ and δ satisfy the Frobenius relation:*

$$
\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dr, "1_A\otimes\mu"] & \\
A \otimes A \arrow[ur, "\delta\otimes 1_A"] \arrow[r, "\mu"] \arrow[dr, "1_A\otimes\delta"'] & A \arrow[r, "\delta"] & A \otimes A \\
 & A \otimes A \otimes A \arrow[ur, "\mu\otimes 1_A"'] &
\end{tikzcd}
$$

Proof. Again, we will prove only the uppermost diagram.

$$\begin{tikzcd}
 & & A^{\otimes 3} \arrow[ddrr, "1_A\oplus\mu"] & & \\
 & A^{\otimes 4} \arrow[ur, "1_A\oplus\mu\oplus1_A"', "(\mathbf{1})"] \arrow[dr, "1_A\otimes1_A\otimes\mu", "(\mathbf{2})"'] & & & \\
 & & A^{\otimes 3} \arrow[drr, "1_A\otimes\mu", "(\mathbf{2})"'] & & \\
A\otimes A \arrow[uuurr, bend left, "\delta\otimes1_A"] \arrow[uur, "\gamma\oplus1_A\oplus1_A"', "(\mathbf{1})"] \arrow[rr, "\mu", "(\mathbf{3})"'] & & A \arrow[u, "\gamma\otimes1_A", "(\mathbf{3})"'] \arrow[rr, "\delta", "(\mathbf{4})"'] & & A\otimes A
\end{tikzcd}$$

The definition of δ gives $(\mathbf{1})$; the associativity of μ, $(\mathbf{2})$; a rearrangement of identities, $(\mathbf{3})$; and again the definition of δ, $(\mathbf{4})$. ☐

This lemma is analogous to 4.19:

Lemma 4.22. *The following diagrams commute:*

$$\begin{tikzcd}
 & A \arrow[dr, "\delta"] & \\
\Bbbk \arrow[ur, "\eta"] \arrow[rr, "\gamma"'] & & A\otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A\otimes A \arrow[dr, "1_A\otimes\varepsilon"] & \\
\Bbbk \arrow[ur, "\gamma"] \arrow[rr, "\eta"] \arrow[dr, "\gamma"'] & & A \\
 & A\otimes A \arrow[ur, "\varepsilon\otimes1_A"'] &
\end{tikzcd}$$

Proof. Firstly,

$$\begin{tikzcd}
 & A \arrow[dr, "1_A\oplus\gamma"', "(\mathbf{1})"] \arrow[ddrr, bend left, "\delta"] & & \\
 & & A\otimes A\otimes A \arrow[dr, "\mu\oplus1_A"', "(\mathbf{1})"] & \\
\Bbbk \arrow[uur, "\eta"] \arrow[r, "\gamma", "(\mathbf{2})"'] & A\otimes A \arrow[ur, "\eta\oplus1_A\oplus1_A"', "(\mathbf{2})"] \arrow[rr, "1_A\otimes1_A", "(\mathbf{3})"'] & & A\otimes A
\end{tikzcd}$$

where $(\mathbf{1})$ is by definition of δ, $(\mathbf{2})$ is found rearranging identities and $(\mathbf{3})$ is the property of units in an algebra. Then,

$$\begin{tikzcd}
 & A\otimes A \arrow[dr, "1_A\otimes\varepsilon"] & \\
\Bbbk \arrow[ur, "\gamma"] \arrow[r, "\eta", "(\mathbf{1})"'] & A \arrow[u, "(\mathbf{1})", "\delta"'] \arrow[r, "1_A", "(\mathbf{2})"'] & A
\end{tikzcd}$$

(**1**) is due to the property we just proved, and (**2**) is given by Lemma 4.20. The other part of the diagram is proven analogously. ☐

Now, at last, we have all the tools to proof the following two theorems, that state the equivalence of the three definitions of Frobenius algebras.

Theorem 4.23. *Given a Frobenius* $\Bbbk$*-algebra* (A, ε) *in the sense of 4.3, there exists a unique linear map* $\delta : A \longrightarrow A \otimes A$*, such that* A *is Frobenius* $\Bbbk$*-algebra* (A, ε) *in the sense of 4.15 with* δ *as comultiplication and* ε *as counit.*

Proof. With our construction of δ, and lemmas 4.18, 4.20 and 4.21, we only need to prove the unicity. Suppose there is another linear map ω satisfying the hypotheses of the theorem. We can construct the following diagram starting from the upper half of the Frobenius relation diagram (in this case the central square):

$$
\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[dr, "1_A \otimes \mu"] \arrow[ddrr, bend left, "1_A \otimes \beta", "\mathbf{(2)}"'] & & \\
 & A \otimes A \arrow[ur, "\omega \otimes 1_A"] \arrow[dr, "\mu"] & & A \otimes A \arrow[dr, "1_A \otimes \varepsilon"', "\mathbf{(1)}"] & \\
A \arrow[ur, "\mathbf{(1)}", "\eta \otimes 1_A"'] \arrow[rr, "1_A", "\mathbf{(3)}"'] & & A \arrow[ur, "\omega"] \arrow[rr, "1_A", "\mathbf{(5)}"'] & & A
\end{tikzcd}
$$

We extended the diagram adding the morphisms (**1**). Lemma 4.19 gives (**2**); (**3**) is due to the fact that η is the unit of the algebra; and (**4**) to the fact that ε is the counit of the coalgebra. Notice that the outer morphisms of the diagram state that $\omega \circ \eta$ is a copairing that makes β nondegenerate. But we saw that this copairing is unique; in particular $\omega \circ \eta = \gamma$. The left part of the previous diagram becomes

$$
\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[dr, "1_A \otimes \mu"] & \\
 & A \otimes A \arrow[ur, "\omega \otimes 1_A"] \arrow[dr, "\mu"] & & A \otimes A \\
A \arrow[uurr, bend left, "\gamma \otimes 1_A"] \arrow[ur, "\eta \otimes 1_A"] \arrow[rr, "1_A"'] & & A \arrow[ur, "\omega"] &
\end{tikzcd}
$$

Now the outer morphisms say that $(1_A \otimes \mu) \circ (\gamma \otimes 1_A) = \omega$, but this is precisely the definition of δ! ☐

Theorem 4.24. *Let* A *be a* $\Bbbk$*-vector space equipped with a multiplication* $\mu : A \otimes A \longrightarrow A$ *with unit* $\eta : \Bbbk \longrightarrow A$ *and a comultiplication* $\delta : A \longrightarrow A \otimes A$ *with counit* $\varepsilon : A \longrightarrow \Bbbk$*, and such that the Frobenius relation holds for* μ *and* δ*. Then*

(1) A is finite-dimensional,

(2) μ is associative, so (A, μ, η) is a $\Bbbk$-algebra,

(3) δ is coassociative, so (A, δ, ε) is a $\Bbbk$-coalgebra and

(4) ε is a Frobenius form, so (A, ε) is a Frobenius $\Bbbk$-algebra.

Proof. Let us define a pairing $\beta := \varepsilon \circ \mu$ and prove that it is nondegenerate with the copairing defined $\gamma := \delta \circ \eta$. To do so, we repeat the proof of the previous lemma, extending the Frobenius relation diagram with η and ε, etc.:

$$\begin{tikzcd}
& & A \otimes A \otimes A \arrow[dr, "1_A\otimes\mu"] \arrow[ddrr, bend left, "1_A\otimes\beta"] & & \\
& A \otimes A \arrow[ur, "\delta\otimes 1_A"] \arrow[dr, "\mu"] & & A \otimes A \arrow[dr, "1_A\otimes\varepsilon"] & \\
A \arrow[ur, "\eta\otimes 1_A"] \arrow[rr, "1_A"'] \arrow[uurr, bend left, "\gamma\otimes 1_A"] & & A \arrow[ur, "\delta"] \arrow[rr, "1_A"'] & & A
\end{tikzcd}$$

To prove the nondegeneracy of β on the other variable we should use the other counterpart of the Frobenius relation. By Lemma 4.9, A is therefore finite-dimensional.

Notice that the previous diagram also says that $\mu = (1_A \otimes \beta) \circ (\delta \otimes 1_A)$ and $\delta = (1_A \otimes \mu) \circ (\gamma \otimes 1_A)$. This allows us to prove the associativity of μ and the coassociativity of δ. For the associativity:

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[ddl, "1_A\otimes\mu"'] \arrow[d, "1_A\otimes\delta\otimes 1_A", "\mathbf{(1)}"'] \arrow[ddr, "\mu\otimes 1_A", "\mathbf{(3)}"'] & \\
& A^{\otimes 4} \arrow[dl, "1_A\otimes 1_A\otimes\beta"', "\mathbf{(1)}"] \arrow[dd, "\mu\otimes 1_A\otimes 1_A", "\mathbf{(2)}"'] & \\
A \otimes A \arrow[ddr, "\mu"'] & & A \otimes A \arrow[dl, "\delta\otimes 1_A", "\mathbf{(3)}"'] \arrow[ddl, "\mu", "\mathbf{(4)}"'] \\
& A^{\otimes 3} \arrow[d, "1_A\otimes\beta", "\mathbf{(2)}"'] & \\
& A &
\end{tikzcd}$$

(**1**) is due to the equality we mentioned; (**2**) to a rearrangement of identities; (**3**) to the Frobenius relation; and (**4**) to $\mu = (1_A \otimes \beta) \circ (\delta \otimes 1_A)$ as well. The proof of the coassociativity of δ is analogous.

Finally, as μ is associative, clearly $\beta := \varepsilon \circ \mu$ is associative as well, so (A, β) is a Frobenius $\Bbbk$-algebra. $\square$

Let us translate our three examples in this third definition:

(1) As $\{1, i\}$ is a basis of $\mathbb{C}$ as a $\mathbb{R}$-vector space, we may wonder if the trigonometric coalgebra structure is compatible with the Frobenius $\mathbb{R}$-algebra $(\mathbb{C}, \mathrm{Re})$. First observe that if we set $c = 1$ and $s = i$ the trigonometric counit coincides with Re. It is straightforward to see that the Frobenius relation holds for the multiplication $\mu(z \otimes w) = zw$ and trigonometric comultiplication; for example

$$\begin{aligned}
1 \otimes i &\xmapsto{\delta \otimes 1_{\mathbb{C}}} 1 \otimes 1 \otimes i - i \otimes i \otimes i \xmapsto{1_{\mathbb{C}} \otimes \mu} 1 \otimes i + i \otimes 1 \\
1 \otimes i &\xmapsto{\quad\quad\mu\quad\quad} i \xmapsto{\quad\quad\delta\quad\quad} 1 \otimes i + i \otimes 1 \\
1 \otimes i &\xmapsto{1_{\mathbb{C}} \otimes \delta} 1 \otimes 1 \otimes i + 1 \otimes i \otimes 1 \xmapsto{\mu \otimes 1_{\mathbb{C}}} 1 \otimes i + i \otimes 1
\end{aligned}$$

and similarly for the other three combinations of the basis vectors.

(2) To find the comultiplication compatible with $(\mathrm{Mat}_{n \times n}(\Bbbk), \mathrm{Tr})$, we use the construction above, i.e., for every element of the basis, δ must be given by the following composition:

$$E_{ij} \xmapsto{\gamma \otimes 1_{\mathrm{Mat}}} \sum_{k,l} E_{kl} \otimes E_{lk} \otimes E_{ij} \xmapsto{1_{\mathrm{Mat}} \otimes \mu} \sum_{k,l} E_{kl} \otimes E_{lk} E_{ij} = \sum_{k,l} E_{kl} \otimes \delta_{ki} E_{lj} = \sum_{l=1}^{n} E_{il} \otimes E_{lj}.$$

(3) The definition of the coalgebra $(\Bbbk[S], \delta, \varepsilon)$, with $S = G$, is not compatible the Frobenius structure of $\Bbbk[G]$ formulated above. However $(\Bbbk[S], \mu, \eta, \delta, \varepsilon)$ is a *bialgebra*, another kind of vector space that is both an algebra and a coalgebra, but satisfying other relations (which can be found in [DNR, §4.1]). Instead, the comultiplication associated with our Frobenius algebra must be given by

$$g_i \xmapsto{\gamma \otimes 1_{\Bbbk[G]}} \sum_{j=0}^{n} g_j^{-1} \otimes g_j \otimes g_i \xmapsto{1_{\Bbbk[G]} \otimes \mu} \sum_{j=0}^{n} g_j^{-1} \otimes g_j g_i.$$

4.5 Commutative Frobenius algebras

Recall the symmetric braiding σ that interchanges the factors of $\otimes$:

$$\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V.$$

We are interested when $V = W = A$ (and we will write $\sigma_A = \sigma_{A,A}$).

Definition 4.25. A $\Bbbk$-algebra (A, μ, η) [resp. $\Bbbk$-coalgebra (A, δ, ε)] is said to be **commutative** [resp. **cocommutative**] if the following diagram commutes:

$$
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mu"] & \\
A \otimes A \arrow[ur, "\sigma_A"] \arrow[rr, "\mu"'] & & A
\end{tikzcd}
\qquad
\left[\text{resp.} \quad
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\sigma_A"] & \\
A \arrow[ur, "\delta"] \arrow[rr, "\delta"'] & & A \otimes A
\end{tikzcd}
\right]
$$

Definition 4.26. A Frobenius $\Bbbk$-algebra $(A, \mu, \eta, \delta, \varepsilon)$ is said to be **commutative** if (A, μ, η) is a commutative $\Bbbk$-algebra.

We could have defined a Frobenius algebra to be commutative if (A, δ, ε) was a cocommutative coalgebra instead, as this proposition shows:

Proposition 4.27. *A Frobenius $\Bbbk$-algebra $(A, \mu, \eta, \delta, \varepsilon)$ is commutative if and only if (A, δ, ε) is a cocommutative $\Bbbk$-coalgebra.*

Proof. Suppose (A, μ, η) is commutative, and consider the composition

$$A \xrightarrow{\ \delta\ } A \otimes A \xrightarrow{\ \sigma_A\ } A \otimes A.$$

If we show that this is a comultiplication with ε as counit and that it satisfies the Frobenius relation with μ, by means of Theorem 4.24, $(A, \sigma_A \circ \delta, \varepsilon)$ will be a coalgebra, and therefore $\delta = \sigma_A \circ \delta$ due to unicity (see Theorem 4.23).

$$
\begin{tikzcd}
 & & A \otimes A \arrow[ddrr, "\varepsilon \otimes 1_A"] & & \\
 & A \otimes A \arrow[ur, "\sigma_A"] \arrow[drrr, "1_A \otimes \varepsilon", "(\mathbf{1})"'] & & & \\
A \arrow[ur, "\delta"] \arrow[rrrr, "1_A", "(\mathbf{2})"'] & & & & A
\end{tikzcd}
$$

The naturality of the symmetric braiding σ (recall the Definition A.7) gives (**1**); hence, as ε is the counit of δ we have (**2**). Similarly one can see that $(1_A \otimes \varepsilon) \circ (\sigma_A \circ \delta) = 1_A$.

Now we see the first equality of the Frobenius relation holds (the other is analogous):

$$
\begin{tikzcd}
 & & & A \otimes A \otimes A \arrow[dddrrr, "1_A \otimes \mu"] & & & \\
 & A \otimes A \otimes A \arrow[urr, "\sigma_A \otimes 1_A"] \arrow[drrr, bend right, "1_A \otimes \sigma_A"'] \arrow[drrr, phantom, "\mathbf{(1)}"] & & & & & \\
 & & & & A \otimes A \otimes A \arrow[ulll, bend right, "1_A \otimes \sigma_A"'] \arrow[ddll, "\sigma_A \otimes 1_A"', "\mathbf{(3)}"] \arrow[dddr, "\mu \otimes 1_A", "\mathbf{(2)}"'] & & \\
A \otimes A \arrow[uur, "\delta \otimes 1_A"] \arrow[ddr, "\sigma_A", "\mathbf{(4)}"'] \arrow[dddrrr, bend right, "\mu", "\mathbf{(6)}"'] & & & & & & A \otimes A \\
 & & A \otimes A \otimes A \arrow[drrr, "\mu \otimes 1_A", "\mathbf{(3)}"'] & & & & \\
 & A \otimes A \arrow[ur, "1_A \otimes \delta", "\mathbf{(4)}"'] \arrow[drr, "\mu", "\mathbf{(5)}"'] & & & & A \otimes A \arrow[uur, "\sigma_A", "\mathbf{(2)}"'] & \\
 & & & A \arrow[urr, "\delta", "\mathbf{(5)}"'] & & &
\end{tikzcd}
$$

The "loop" (**1**) can be added because $\sigma_A \circ \sigma_A = 1_A$; naturality of σ justifies (**2**); in (**3**) a symmetric braiding is added to the multiplication μ (it is commutative); (**4**) is another instance of naturality of σ; (**5**) is given by the Frobenius relation of μ and δ; and finally (**6**) is again due to the commutativity of μ.

The converse implication of the theorem is proven exactly the same (by "reversing the arrows"). □

In our examples, the only commutative Frobenius algebras are $\mathbb{C}$ and $\Bbbk[G]$ provided that G is Abelian.

4.6 The category of Frobenius algebras

We have to define the morphisms in order to construct the category of Frobenius algebras. First we describe the simpler categories of algebras and coalgebras.

Definition 4.28. A $\Bbbk$**-algebra homomorphism** $f : A_1 \longrightarrow A_2$ between two $\Bbbk$-algebras, (A_1, μ_1, η_1) and (A_2, μ_2, η_2), is a linear map such that the following diagrams commute:

$$
\begin{tikzcd}
A_1 \otimes A_1 \arrow[r, "\mu_1"] \arrow[d, "f \otimes f"'] & A_1 \arrow[d, "f"] \\
A_2 \otimes A_2 \arrow[r, "\mu_2"'] & A_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_1 \arrow[dd, "f"] \\
\Bbbk \arrow[ur, "\eta_1"] \arrow[dr, "\eta_2"'] & \\
 & A_2
\end{tikzcd}
$$

It is obvious that $\Bbbk$-algebras together with $\Bbbk$-algebra homomorphisms form a category, $\mathbf{Alg}_{\Bbbk}$, which is a symmetric monoidal category with $\otimes$.

Definition 4.29. A **$\Bbbk$-coalgebra homomorphism** $f : A_1 \longrightarrow A_2$ between two $\Bbbk$-coalgebras, $(A_1, \delta_1, \varepsilon_1)$ and $(A_2, \delta_2, \varepsilon_2)$, is a linear map such that the following diagrams commute:

$$\begin{tikzcd} A_1 \arrow[r, "\delta_1"] \arrow[d, "f"'] & A_1 \otimes A_1 \arrow[d, "f\otimes f"] \\ A_2 \arrow[r, "\delta_2"'] & A_2 \otimes A_2 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[dr, "\varepsilon_1"] \arrow[dd, "f"'] & \\ & \Bbbk \\ A_2 \arrow[ur, "\varepsilon_2"'] & \end{tikzcd}$$

Similarly we define the symmetric monoidal category $\mathbf{CoAlg}_{\Bbbk}$.

By combining these two notions we define Frobenius algebra homomorphisms:

Definition 4.30. A **Frobenius $\Bbbk$-algebra homomorphism** $f : A_1 \longrightarrow A_2$ between two Frobenius $\Bbbk$-algebras, $(A_1, \mu_1, \eta_1, \delta_1, \varepsilon_1)$ and $(A_2, \mu_2, \eta_2, \delta_2, \varepsilon_2)$, is a map that is both a $\Bbbk$-algebra homomorphism and a $\Bbbk$-coalgebra homomorphism.

This gives us the symmetric monoidal category of Frobenius $\Bbbk$-algebras, $\mathbf{FA}_{\Bbbk}$. If we require the objects to be commutative Frobenius algebras, we have the subcategory $\mathbf{cFA}_{\Bbbk}$.

4.7 Equivalence of categories

Lastly, we state and prove the equivalence between two-dimensional TQFTs and Frobenius algebras.

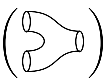
Theorem 4.31. *There is a symmetric monoidal equivalence*

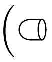
$$\mathbf{TQFT}_2^{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}.$$

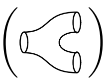
Proof. First, consider the symmetric monoidal functor

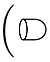
$$F : \mathbf{TQFT}_2^{\Bbbk} \longrightarrow \mathbf{cFA}_{\Bbbk}$$
$$[\mathcal{Z} : \mathbf{Bord}_2 \longrightarrow \mathbf{Vect}_{\Bbbk}] \longmapsto A_{\mathcal{Z}} = \mathcal{Z}(\mathbf{1})$$

(recall that $\mathbf{n}$ is the disjoint union of n circles). By monoidality of F we have that $F(\mathbf{n}) = A_{\mathcal{Z}}^{\otimes n}$. To endow $A_{\mathcal{Z}}$ with the structure of Frobenius algebra, we define

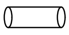
$\mu_{\mathcal{Z}} := \mathcal{Z}\Big($  $\Big) : A_{\mathcal{Z}} \otimes A_{\mathcal{Z}} \longrightarrow A_{\mathcal{Z}},$

$\eta_{\mathcal{Z}} := \mathcal{Z}\Big($  $\Big) : \Bbbk \longrightarrow A_{\mathcal{Z}},$

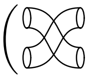
$\delta_{\mathcal{Z}} := \mathcal{Z}\Big($  $\Big) : A_{\mathcal{Z}} \longrightarrow A_{\mathcal{Z}} \otimes A_{\mathcal{Z}}$ and

$\varepsilon_{\mathcal{Z}} := \mathcal{Z}\Big($  $\Big) : A_{\mathcal{Z}} \longrightarrow \Bbbk.$

We also have that

$1_{A_{\mathcal{Z}}} = \mathcal{Z}$() $: A_{\mathcal{Z}} \longrightarrow A_{\mathcal{Z}}$

by functoriality of F, and since F is a symmetric monoidal functor,

$\sigma_{A_{\mathcal{Z}}} = \mathcal{Z}$() $: A_{\mathcal{Z}} \otimes A_{\mathcal{Z}} \longrightarrow A_{\mathcal{Z}} \otimes A_{\mathcal{Z}}.$

With these definitions, observe how the unit relations of $\mathbf{Bord}_2$ translate (via F) to $\mu_{\mathcal{Z}} \circ (\eta_{\mathcal{Z}} \otimes 1_{A_{\mathcal{Z}}}) = 1_{A_{\mathcal{Z}}} = \mu_{\mathcal{Z}} \circ (1_{A_{\mathcal{Z}}} \otimes \eta_{\mathcal{Z}})$ (i.e. $\eta_{\mathcal{Z}}$ is a unit for $\mu_{\mathcal{Z}}$); counit relations of $\mathbf{Bord}_2$ similarly translate to the fact that $\varepsilon_{\mathcal{Z}}$ is a counit for $\delta_{\mathcal{Z}}$; and the (until now unjustifiedly called) Frobenius relation of $\mathbf{Bord}_2$ translates to the Frobenius relation of 4.14. Therefore, by Theorem 4.24, $(A_{\mathcal{Z}}, \mu_{\mathcal{Z}}, \eta_{\mathcal{Z}}, \delta_{\mathcal{Z}}, \varepsilon_{\mathcal{Z}})$ is a Frobenius $\Bbbk$-algebra. Similarly, the commutativity relation of $\mathbf{Bord}_2$ translates to fact that $A_{\mathcal{Z}}$ is a commutative Frobenius $\Bbbk$-algebra. So in conclusion F is well-defined.

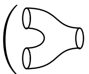
Conversely, we construct the symmetric monoidal functor

$$\begin{aligned} G : \mathbf{cFA}_{\Bbbk} &\longrightarrow \mathbf{TQFT}_2^{\Bbbk} \\ A &\longmapsto [\mathcal{Z}_A : \mathbf{Bord}_2 \longrightarrow \mathbf{Vect}_{\Bbbk}] \end{aligned}$$

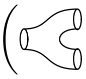
in the following manner (recall that we found a presentation (O, G, R) of $\mathbf{Bord}_2$, so $\mathcal{Z}_A$ will be determined by the images of the objects in O and the morphisms in G, provided that the relations of R still hold):

(1) For the objects in O, we define $\mathcal{Z}_A(\mathbf{1}) = A$, and thus $\mathcal{Z}_A(\mathbf{n}) = A^{\otimes n}$.

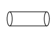
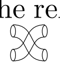
(2) For the generators in G, we define:

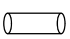
$\mathcal{Z}_A$() $:= \mu : A \otimes A \longrightarrow A,$

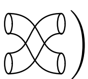
$\mathcal{Z}_A$() $:= \eta : \Bbbk \longrightarrow A,$

$\mathcal{Z}_A$() $:= \delta : A \longrightarrow A \otimes A$ and

$\mathcal{Z}_A$() $:= \varepsilon : A \longrightarrow \Bbbk.$

(3) We have to show that the relations in R are satisfied when taking images. The relations involving  and  must hold because G is a symmetric monoidal functor, that is,

$\mathcal{Z}_A$() $:= 1_A : A \longrightarrow A$

$\mathcal{Z}_A$() $:= \sigma_A : A \otimes A \longrightarrow A \otimes A.$

The other relations, when taking images, yield expressions satisfied by commutative Frobenius algebras that we saw in previous subsections.

Due to the way F and G are constructed, it is obvious that one is the inverse of the other.

Now we should define the functors F and G on morphisms. Let $\alpha : \mathcal{Z} \Longrightarrow \mathcal{Y}$ be a symmetric monoidal natural transformation between two 2-TQFTs, $\mathcal{Z}$ and $\mathcal{Y}$. In particular α has components $\alpha_{\mathbf{0}}$, $\alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{2}}$ making the following diagrams commute:

$$
\begin{tikzcd}
A_{\mathcal{Z}} \otimes A_{\mathcal{Z}} \arrow[r, "\mu_{\mathcal{Z}}"] \arrow[d, equal] & A_{\mathcal{Z}} \arrow[d, equal] \\
\mathcal{Z}(\mathbf{2}) \arrow[r, "\mathcal{Z}(\ )"] \arrow[d, "\alpha_{\mathbf{2}}"'] & \mathcal{Z}(\mathbf{1}) \arrow[d, "\alpha_{\mathbf{1}}"] \\
\mathcal{Y}(\mathbf{2}) \arrow[r, "\mathcal{Y}(\ )"'] \arrow[d, equal] & \mathcal{Y}(\mathbf{1}) \arrow[d, equal] \\
A_{\mathcal{Y}} \otimes A_{\mathcal{Y}} \arrow[r, "\mu_{\mathcal{Y}}"'] & A_{\mathcal{Y}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \arrow[r, "\eta_{\mathcal{Z}}"] \arrow[d, equal] & A_{\mathcal{Z}} \arrow[d, equal] \\
\mathcal{Z}(\mathbf{0}) \arrow[r, "\mathcal{Z}(\ )"] \arrow[d, "\alpha_{\mathbf{0}}"'] & \mathcal{Z}(\mathbf{1}) \arrow[d, "\alpha_{\mathbf{1}}"] \\
\mathcal{Y}(\mathbf{0}) \arrow[r, "\mathcal{Y}(\ )"'] \arrow[d, equal] & \mathcal{Y}(\mathbf{1}) \arrow[d, equal] \\
\Bbbk \arrow[r, "\eta_{\mathcal{Y}}"'] & A_{\mathcal{Y}}
\end{tikzcd}
$$

As α is monoidal, we have that $\alpha_{\mathbf{2}} = \alpha_{\mathbf{1}} \otimes \alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{0}} = 1_{\Bbbk}$, so the above diagrams are the ones in the definition of $\Bbbk$-algebra homomorphism. Similarly we see that $\alpha_{\mathbf{1}}$ is a $\Bbbk$-coalgebra homomorphism. Therefore $F(\alpha) = \alpha_{\mathbf{1}}$ is well-defined.

Conversely, given a Frobenius $\Bbbk$-algebra homomorphism $f : A \longrightarrow B$, the functor G can map it to the symmetric monoidal natural transformation $\alpha_f : \mathcal{Z}_A \Longrightarrow \mathcal{Z}_B$ with components $\alpha_{f_{\mathbf{n}}} = f^{\otimes n}$. It is obvious that F and G are inverse to each other in the morphism level as well. □

5 Digression on Physics

Dr. von Neumann, ich möchte gerne wissen, was ist dann eigentlich ein Hilbertscher Raum?[7]

— Question asked by Hilbert himself in a talk by von Neumann, 1929, Göttingen.

In this last section we try to shed some light on how the mathematical concept of topological quantum field theories was historically meant to reconcile Quantum Physics and General Relativity. This section is based on Baez's paper [Bae] and we refer to the original reference for further details.

The key to grasp the importance of TQFTs as proposed by Witten and Atiyah is to understand that both quantum processes and spacetime can be described categorically in a similar way. At first glance, general relativity and quantum theory "use different sorts of mathematics," says Baez: "one is based on objects such as manifolds, the other on objects such as Hilbert spaces."

Under the assumption that neither space nor spacetime have a fixed topology—an idea that general relativity put on the table—manifolds of dimension n are used to model spacetime, and submanifolds of dimension $(n-1)$, space at a given instant.[8] Thus a bordism can represent an evolution of the space along time. For instance the pair of pants would represent the collision of two different spaces that merge in a new one:

[7] Loosely translated as "Dr. von Neumann, I would like to know, what is actually a Hilbert space?," this quote can be found in [ML1].

[8] There are many models with n different from the usual 4, for example in string theory spacetime is 10-dimensional, while in M-theory it is 11-dimensional.

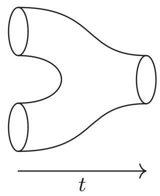


Figure 5.1: Spacetime as a bordism.

On the other hand, we want to establish a connection between quantum states and vector spaces. In fact, Quantum Mechanics is mathematically formulated by means of complex Hilbert spaces, a kind $\mathbb{C}$-vector spaces with further structure: the possible states of a quantum system are associated with the unit vectors of a Hilbert space. Moreover, the processes that occur between such sets of states are described by bounded linear maps (or more commonly *operators*). Therefore TQFTs, in this case, will be understood as functors from $\mathbf{Bord}_n$ to $\mathbf{Hilb}$, the category of the aforementioned vector spaces.

The resemblances between these two categories become more apparent when we consider the *Feynman diagrams*, which are used to visualize operators: intuitively these diagrams exhibit the analogy between Quantum Physics and Topology.

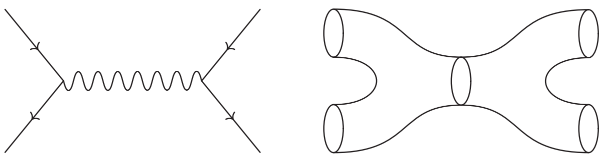


Figure 5.2: Feynman diagram understood as a 2-bordism.

Later in the 1970s, as it is said in [BS], "Penrose realized that generalizations of Feynman diagrams arise throughout quantum theory, and might even lead to revisions in our understanding of spacetime." Furthermore, in string theory Feynman diagrams are substituted by *worldsheets*, 2-dimensional bordisms that describe the embedding of a string in spacetime; and similarly in the loop quantum gravity (LQG) theory, et cetera.

5.1 The category of Hilbert spaces

Definition 5.1. A **Hilbert space** is a $\mathbb{C}$-vector space H equipped with a positive definite inner product whose induced norm makes H a complete metric space.

Let us unravel this definition: An inner product is a sesquilinear, conjugate-symmetric map $\langle -|-\rangle : H \times H \longrightarrow \mathbb{C}$; in other words, it satisfies

(1) $\langle \psi | a\phi + b\chi \rangle = a\langle \psi|\phi\rangle + b\langle \psi|\chi\rangle$ and

(2) $\langle \psi|\phi\rangle = \overline{\langle \phi|\psi\rangle}$

for all vectors $\psi, \phi, \chi \in H$ and all scalars $a, b \in \mathbb{C}$ (where $\bar{a}$ denotes the complex conjugate of a). These two properties imply that $\langle a\psi + b\phi|\chi\rangle = \bar{a}\langle\psi|\chi\rangle + \bar{b}\langle\phi|\chi\rangle$. Antilinearity on the first variable and linearity on the second is the usual convention in Physics, but it is sometimes defined conversely.

The inner product is said to be positive definite if $\langle\psi|\psi\rangle \geq 0$ with the equality if and only if $\psi = 0$. The induced norm is $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. Now the definition should be clear. Refer to [Rud, Chapter 12] for more details.

These structures are the objects of the category of Hilbert spaces, denoted **Hilb**. As for its morphisms, one would expect a reasonable choice to be linear operators that preserve the inner product, i.e. *unitary operators*: linear maps $T : H_1 \longrightarrow H_2$ such that

$$\langle T(\psi)|T(\phi)\rangle_{H_2} = \langle\psi|\phi\rangle_{H_1}$$

for all $\psi, \phi \in H_1$. However, many of the operators that arise in Quantum Physics are not unitary.

We require the morphisms only to be bounded linear operators.[9] Since it can be shown that in finite dimensions every linear map is bounded, the category of finite-dimensional Hilbert spaces is equivalent to $\mathbf{FinVect}_{\mathbb{C}}$, and hence the inner product is superfluous. If we allow Hilbert spaces to have any dimension, possibly infinite, the category **Hilb** is not equivalent to $\mathbf{Vect}_{\mathbb{C}}$, but to the subcategory of Hilbertizable vector spaces—those that can be equipped with the topology of a certain Hilbert space. Bounded linear maps do not preserve inner products, though, only the topology they provide. The fact that we require the objects to be Hilbert spaces, and not just vector spaces with a specific kind of topology, is that inner products allow us to imbue **Hilb** with some new structure, namely it makes it a †-category.

5.2 †-categories

Given a category $\mathcal{C}$, we can define a new category by formally reversing the direction of the morphisms. This is known as the **opposite category of $\mathcal{C}$** and is denoted as $\mathcal{C}^{\text{op}}$. In other words,

(1) $\mathcal{C}^{\text{op}}$ has the same objects as $\mathcal{C}$,

(2) Each morphism $f : A \longrightarrow B$ in $\mathcal{C}^{\text{op}}$ is in one-to-one correspondence with a morphism $f : B \longrightarrow A$ in $\mathcal{C}$, and

(3) the composition in $g \circ f$ in $\mathcal{C}^{\text{op}}$ is defined to be the composition $f \circ g$ in $\mathcal{C}$.

This construction allows us to define †-categories:

Definition 5.2. A **†-category** (or dagger category) is a category $\mathcal{C}$ equipped with a functor

$$\dagger : \mathcal{C}^{\text{op}} \longrightarrow \mathcal{C}$$

which

(1) is the identity on objects and

(2) is an involution, i.e. $\dagger \circ \dagger = 1_{\mathcal{C}}$.[10]

The image $f^\dagger : B \longrightarrow A$ is said to be the **†-adjoint** of $f : A \longrightarrow B$.

[9] This means that there exists a constant $M \geq 0$ such that $\|T(\psi)\| \leq M\|\psi\|$ for every vector $\psi \in H_1$, and is equivalent to continuity for linear operators.

[10] This is a harmless abuse of notation; we actually mean that $\dagger \circ -^{\text{op}} \circ \dagger \circ -^{\text{op}} = 1_{\mathcal{C}}$.

The common operation in Quantum Physics that makes **Hilb** into a †-category is the *Hermitian conjugate* or *adjoint*: given a bounded linear operator $T : H_1 \longrightarrow H_2$ we define its Hermitian conjugate to be unique bounded linear operator $T^\dagger : H_2 \longrightarrow H_1$ satisfying

$$\langle T^\dagger(\phi)|\psi\rangle_{H_1} = \langle \phi|T(\psi)\rangle_{H_2}$$

for all $\psi \in H_1$, $\phi \in H_2$.[11]

Therefore the inner product serves to regard **Hilb** as a †-category, and in fact, conversely, one can recover the inner product solely form the †-structure of **Hilb**. By means of the canonical correspondence between elements of a Hilbert space H and linear maps $\mathbb{C} \longrightarrow H$,

$$\begin{aligned} \psi &\longmapsto [T_\psi : 1 \longmapsto \psi] \\ T(1) &\longleftarrow\!\shortmid\; T \end{aligned}$$

the inner product of H can be expressed in terms of morphisms $\mathbb{C} \longrightarrow H$:

$$\begin{aligned} &T_\psi^\dagger \circ T_\phi : \mathbb{C} \longrightarrow \mathbb{C} \\ &\qquad 1 \longmapsto T_\psi^\dagger(\phi) = \langle 1|T_\psi^\dagger(\phi)\rangle_{\mathbb{C}} = \langle (T_\psi^\dagger)^\dagger(1)|\phi\rangle_H = \langle T_\psi(1)|\phi\rangle_H = \langle \psi|\phi\rangle_H. \end{aligned}$$

Dirac's *bra-ket notation* is commonly used to denote such morphisms: $T_\psi^\dagger = \langle\psi|$ and $T_\phi = |\phi\rangle$.

The category of bordisms is a †-category as well: the definition of the †-adjoint morphisms is simpler in this case. Given a bordism $M : \Sigma_1 \longrightarrow \Sigma_2$ we set as its adjoint $M^\dagger := \overline{M} : \Sigma_2 \longrightarrow \Sigma_1$, that is, the bordism M with opposite orientation. Notice that, since the orientations of Σ_1 and Σ_2 remain unchanged, the in-boundary becomes the out-boundary and vice versa.

If bordisms represent spacetime, the † functor is to be understood as a time reversal operation: If $M : \Sigma_1 \longrightarrow \Sigma_2$ describes a process in time where the space Σ_1 is transformed into the space Σ_2, $\overline{M}$ is the reverse process from Σ_2 to Σ_1, switching past and future.

TQFTs can be enriched by requiring it to preserve the †-structure:

Definition 5.3. A **†-functor** (or dagger functor) is a functor between †-categories, $F : (\mathcal{C}, \dagger) \longrightarrow (\mathcal{D}, \ddagger)$, such that it commutes with the †-structures, i.e.

$$F \circ \dagger = \ddagger \circ F^{\mathrm{op}},$$

where $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is the functor induced by F.

TQFTs that satisfy this condition are called **unitary TQFTs**. In our case that would mean that, if $\mathcal{Z}$ is a TQFT,

$$\mathcal{Z}(\overline{M}) = \mathcal{Z}(M)^\dagger,$$

which is the *Hermitian axiom* in Atiyah's paper, [Ati]. This establishes an analogy between time reversal in general relativity and taking the adjoint of an operator between Hilbert spaces.

In Quantum Physics, operators that describe the time evolution of a quantum system are usually assumed to be unitary—operators T such that $\langle T(\psi)|T(\phi)\rangle = \langle\psi|\phi\rangle$, or equivalently, such that $T^\dagger = T^{-1}$—a common hypothesis known as *unitarity*. However, if one

[11]The proof of uniqueness can be found in [Rud, §12.9], and it is key to show that † is an involution: $\langle T^{\dagger\dagger}(\psi)|\phi\rangle = \overline{\langle \phi|T^{\dagger\dagger}(\psi)\rangle} = \overline{\langle T^\dagger(\phi)|\psi\rangle} = \overline{\langle \phi|T(\psi)\rangle} = \langle T(\psi)|\phi\rangle$, hence $T^{\dagger\dagger} = T$.

accepts the possibility of topology changes in space along time, then, as Baez points out, other operators should be considered: It can be shown that a bordism $M : \Sigma \longrightarrow \Sigma$ is unitary (i.e. $\overline{M} = M$) if it involves no topology changes on Σ—or in other words, if it is a cylinder. (The converse is true only for $n \leq 3$.) Therefore

> *absence of topology change implies unitary time evolution* [italics are Baez's]. This fact reinforces a point already well-known from quantum field theory on curved spacetime, namely that unitary time evolution is not a built-in feature of quantum theory but rather the consequence of specific assumptions about the nature of spacetime.

5.3 Quantum entanglement

It is time to talk about how spacetime and Hilbert spaces fit into the monoidality of their associated categories.

We saw that the tensor product in $\mathbf{Bord}_n$ is the disjoint union. This corresponds in our analogy with stating that a disjoint union of two spacetimes is to be understood simply as letting them evolve in parallel, being independent from one another.

In the other hand, the fact that the monoidality of **Hilb** comes from the usual tensor product of vector spaces (instead of the Cartesian product), opens the Pandora's box of many paradoxes in Quantum Physics, in particular *quantum entanglement*. Intuitively, the state of a joint system, a system consisting on two separate parts, is uniquely determined by the state of each part. However, one of the most lurid discoveries in the twentieth century—discussed by Einstein, Podolsky and Rosen in their famous paper [EPR] and by Schrödinger shorty after—is that, whereas that is true in Classical Physics, in the quantum scenario there exist *entangled states*: systems that cannot be described by the sum (or, more rightly, the product) of its constituent parts, but by the *superposition* of such. Put in other words, tensor product best describes quantum systems, while in the classical context Cartesian product is enough.

This is another evidence for the similarity between **Hilb** and $\mathbf{Bord}_n$. To name another one, consider the Wootters–Zurek argument (see [WZ]), which states that no quantum system can be cloned. If joint system could be expressed with Cartesian product, there would be the canonical diagonal map $\Delta : H \longrightarrow H \times H$ that duplicated information. However, as **Hilb** is monoidal with the tensor product, the possibility of cloning would imply the existence of an operator $H \longrightarrow H \otimes H$; but it can be proven that there is no canonical way to define such map.

Appendix

A Monoidal Categories

In this appendix we collect some basic notions and properties on Category Theory, with special attention to the theory of symmetric monoidal structures. All the concepts and results that appear in this section can be found in [ML2].

A.1 Basic Definitions in Category Theory

Definition A.1. A **category** $\mathcal{C}$ consists of

(1) a collection of **objects** $\mathrm{Ob}(\mathcal{C})$ and

(2) a collection of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for every $A, B \in \mathrm{Ob}(\mathcal{C})$ (if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we write $f : A \longrightarrow B$)

equipped with

(1) a morphism $1_A : A \longrightarrow A$ for any object $A \in \mathrm{Ob}(\mathcal{C})$ (called **identity morphism**) and

(2) a morphism $g \circ f : A \longrightarrow C$ for any pair of morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ (called **composition morphism**),

such that the following laws hold:

(1) **Unit law:** $f \circ 1_A = f = 1_B \circ f$ for any morphism $f : A \longrightarrow B$.

(2) **Associative law:** $(h \circ g) \circ f = h \circ (g \circ f)$ for any triple of morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$.

We will often write $A \in \mathcal{C}$ instead of $A \in \mathrm{Ob}(\mathcal{C})$.

An **isomorphism** is a morphism $f : A \longrightarrow B$ with an inverse, i.e. a morphism $g : B \longrightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Definition A.2. Given two categories $\mathcal{C}, \mathcal{D}$, a **functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of

(1) an application $F : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$ and

(2) an application $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$,

such that

(1) $F(1_A) = 1_{F(A)}$ for any object $A \in \mathcal{C}$, and

(2) $F(g \circ f) = F(g) \circ F(f)$ for any pair of applications $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in $\mathcal{C}$.

Given a category $\mathcal{C}$ we can define the **identity functor** $1_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ with $1_{\mathcal{C}}(A) = A$ and $1_{\mathcal{C}}(f) = f$, and given two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ we can define the **composition functor** $G \circ F : \mathcal{C} \longrightarrow \mathcal{E}$ with $G \circ F(A) = G(F(A))$ and $G \circ F(f) = G(F(f))$. One can easily prove that the unit and associative law hold.

Definition A.3. Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a **natural transformation** $\alpha : F \Longrightarrow G$ consists of

(1) a morphism $\alpha_A : F(A) \longrightarrow G(A)$ in $\mathcal{D}$ for every object $A \in \mathcal{C}$ (called **component** of α at A),

such that

(1) for every morphism $f : A \longrightarrow B$ in $\mathcal{C}$ we have the following commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

Given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ we can define the **identity natural transformation** $1_F : F \Longrightarrow F$ with components $(1_F)_A = 1_{F(A)}$, and given two natural transformations $\alpha : F \Longrightarrow G$ and $\beta : G \Longrightarrow H$ we can define the **composition natural transformation** $\beta \circ \alpha : F \Longrightarrow H$ with components $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$. One can easily check that the corresponding commutative diagrams commute, and, again, that the unit and associative laws hold.

Definition A.4. Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a **natural isomorphism** $\alpha : F \Longrightarrow G$ is a natural transformation with an inverse, i.e. a natural transformation $\beta : G \Longrightarrow F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$. That is to say, according to the last paragraph, that every component of α is an isomorphism.

Definition A.5. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an **equivalence** if it has a weak inverse, i.e. a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that there exist two natural isomorphisms $\alpha : G \circ F \Longrightarrow 1_{\mathcal{C}}$ and $\beta : F \circ G \Longrightarrow 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to indicate that there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$.

A.2 Monoidal Categories

In order to define monoidal categories we need to set the notion of Cartesian product of categories as follows: given two categories $\mathcal{C}$ and $\mathcal{D}$,

(1) $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$,

(2) $\mathrm{Hom}_{\mathcal{C} \times \mathcal{D}}(A \times B, A' \times B') = \mathrm{Hom}_{\mathcal{C}}(A, A') \times \mathrm{Hom}_{\mathcal{D}}(B, B')$ for any $A, A' \in \mathcal{C}$, $B, B' \in \mathcal{D}$,

(3) composition is done component-wise, i.e. $(f,g) \circ (f',g') = (f \circ g, f' \circ g')$, and

(4) identity morphisms are defined component-wise, i.e. $1_{(f,g)} = (1_f, 1_g)$.

We define the empty product category, $\mathbf{1}$, to be the category with only an object and only its identity morphism.

Thus we can write $(\mathcal{C} \times \mathcal{D}) \times \mathcal{E} = \mathcal{C} \times \mathcal{D} \times \mathcal{E} = \mathcal{C} \times (\mathcal{D} \times \mathcal{E})$ and $\mathcal{C} \times \mathbf{1} = \mathcal{C} = \mathbf{1} \times \mathcal{C}$.

Definition A.6. A **monoidal category** is a category $\mathcal{M}$ equipped with

(1) a functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ called the **tensor product** (we write $\otimes(A,B) = A \otimes B$ and $\otimes(f,g) = f \otimes g$),

(2) a functor $I : \mathbf{1} \longrightarrow \mathcal{M}$ called **unit** (we write the image of the only object in $\mathbf{1}$ simply as $I \in \mathrm{Ob}(\mathcal{M})$),

(3) a natural isomorphism, α, called **associator**

$$\begin{tikzcd}
 & \mathcal{M} \times \mathcal{M} \times \mathcal{M} \arrow[dl, "\otimes \times 1_{\mathcal{M}}"'] \arrow[dr, "1_{\mathcal{M}} \times \otimes"] & \\
\mathcal{M} \times \mathcal{M} \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\xRightarrow{\alpha}"] & & \mathcal{M} \times \mathcal{M} \arrow[dl, "\otimes"] \\
 & \mathcal{M} &
\end{tikzcd}$$

(i.e. with components $\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$)

(4) and two natural isomorphisms, λ and ρ, called **unit isomorphisms**

$$\begin{tikzcd}
 & \mathcal{M} \times \mathcal{M} \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\lambda"] & \\
\mathbf{1} \times \mathcal{M} \arrow[ur, "I \times 1_{\mathcal{M}}"] \arrow[rr, equal] & {} & \mathcal{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{M} \times \mathcal{M} \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\rho"] & \\
\mathcal{M} \times \mathbf{1} \arrow[ur, "1_{\mathcal{M}} \times I"] \arrow[rr, equal] & {} & \mathcal{M}
\end{tikzcd}$$

(i.e. with components $\lambda_A : I \otimes A \longrightarrow A$ and $\rho_A : A \otimes I \longrightarrow A$ respectively),

such that the following diagrams commute for any $A, B, C, D \in \mathcal{M}$:

(1) The **pentagon diagram**:

$$
\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes 1_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B \otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}
$$

(2) and the **triangle diagram**:

$$
\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
 & A \otimes B &
\end{tikzcd}
$$

These two diagrams are known as **coherence conditions** and are to ensure that all diagrams involving α, λ and ρ commute.

Now we want to define monoidal categories with commutative tensor products. To do so first we define the **twist functor** $T : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D} \times \mathcal{C}$ as follows:

(1) $T(A \times B) = B \times A$ for any objects A, B in $\mathcal{C}$ and $\mathcal{D}$ respectively,

(2) $T(f \times g) = g \times f$ for any morphisms f, g in $\mathcal{C}$ and $\mathcal{D}$ respectively.

Definition A.7. A **symmetric monoidal category** is a monoidal category $\mathcal{M}$ equipped with a natural isomorphism, β, called **symmetric braiding**

$$
\begin{tikzcd}
\mathcal{M} \times \mathcal{M} \arrow[rr, "\otimes"] \arrow[dr, "T"'] \arrow[rr, phantom, shift right=6ex, "\Downarrow\beta"] & & \mathcal{M} \\
 & \mathcal{M} \times \mathcal{M} \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

(i.e. with components $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$), such that

(1) it satisfies the **symmetric condition**: β is its own inverse, i.e. its components satisfy $\beta_{B,A} \circ \beta_{A,B} = 1_{A \otimes B}$,

(2) and makes the **hexagon diagram** commute for any objects $A, B, C \in \mathcal{M}$:

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\beta_{A,B}\otimes 1_C"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[ur, "1_B \otimes \beta_{A,C}"'] &
\end{tikzcd}$$

Definition A.8. A [symmetric] monoidal category is said to be **strict** if $\alpha_{A,B,C}$, λ_A and ρ_A are identity morphisms, i.e. if $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ and $I \otimes A = A = A \otimes I$.

In the end of this section we will present a theorem stating that [symmetric] monoidal categories and strict [symmetric] monoidal categories are essentially the same.

A.3 Monoidal Functors

Definition A.9. A **monoidal functor** between two monoidal categories $(\mathcal{M}_1, \otimes_1, I_1)$ and $(\mathcal{M}_2, \otimes_2, I_2)$ is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ equipped with

(1) a natural isomorphism, Φ,

$$\begin{tikzcd}
& \mathcal{M}_1 \times \mathcal{M}_1 \arrow[dl, "F\times F"'] \arrow[dr, "\otimes_1"] & \\
\mathcal{M}_2 \times \mathcal{M}_2 \arrow[dr, "\otimes_2"'] \arrow[rr, phantom, "\overset{\Phi}{\Longrightarrow}"] & & \mathcal{M}_1 \arrow[dl, "F"] \\
& \mathcal{M}_2 &
\end{tikzcd}$$

(i.e. with components $\Phi_{A,B} : F(A) \otimes_2 F(B) \longrightarrow F(A \otimes_1 B)$)

(2) and an isomorphism $\varphi : I_2 \longrightarrow F(I_1)$ in $\mathcal{M}_2$

such that

(1) the following diagram commutes for any objects $A, B, C \in \mathcal{M}_1$,

$$
\begin{tikzcd}
& F(A \otimes_1 B) \otimes_2 F(C) \arrow[r, "\Phi_{A\otimes_1 B,C}"] & F((A \otimes_1 B) \otimes_1 C) \arrow[dr, "F(\alpha_{A,B,C})"] & \\
(F(A) \otimes_2 F(B)) \otimes_2 F(C) \arrow[ur, "\Phi_{A,B}\otimes_2 1_{F(C)}"] \arrow[dr, "\alpha_{F(A),F(B),F(C)}"'] & & & F(A \otimes_1 (B \otimes_1 C)) \\
& F(A) \otimes_2 (F(B) \otimes_2 F(C)) \arrow[r, "1_{F(A)}\otimes_2 \Phi_{B,C}"'] & F(A) \otimes_2 F(B \otimes_1 C) \arrow[ur, "\Phi_{A,B\otimes_1 C}"'] &
\end{tikzcd}
$$

(2) and the following diagrams commute for any object $A \in \mathcal{M}_1$.

$$
\begin{tikzcd}
I_2 \otimes_2 F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "\varphi\otimes_2 F(A)"'] & F(A) \\
F(I_1) \otimes_2 F(A) \arrow[r, "\Phi_{I_1,A}"'] & F(I_1 \otimes_1 A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \otimes_2 I_2 \arrow[r, "\rho_{F(A)}"] \arrow[d, "F(A)\otimes_2\varphi"'] & F(A) \\
F(A) \otimes_2 F(I_1) \arrow[r, "\Phi_{A,I_1}"'] & F(A \otimes_1 I_1) \arrow[u, "F(\rho_A)"']
\end{tikzcd}
$$

We can define a symmetric monoidal functor in a similar fashion:

Definition A.10. A **symmetric monoidal functor** is a monoidal functor, F, between two symmetric monoidal categories $(\mathcal{M}_1, \otimes_1, I_1)$ and $(\mathcal{M}_2, \otimes_2, I_2)$ that makes the following diagram commute for any $A, B \in \mathcal{M}_1$:

$$
\begin{tikzcd}
& F(B) \otimes_2 F(A) \arrow[dr, "\Phi_{B,A}"] & \\
F(A) \otimes_2 F(B) \arrow[ur, "\beta_{F(A),F(B)}"] \arrow[dr, "\Phi_{A,B}"'] & & F(B \otimes_1 A) \\
& F(A \otimes_1 B) \arrow[ur, "F(\beta_{A,B})"'] &
\end{tikzcd}
$$

A.4 Monoidal Natural Transformations

Finally we can extend the notion of monoidality to natural transformations:

Definition A.11. A **monoidal natural transformation** between two functors $(F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2, \Phi, \varphi)$ and $(G : \mathcal{M}_1 \longrightarrow \mathcal{M}_2, \Gamma, \gamma)$ is a natural transformation $\mu : F \Longrightarrow G$ that makes the two following diagrams commute for any objects $A, B \in \mathcal{M}_1$:

$$\begin{tikzcd} F(A) \otimes_2 F(B) \arrow[r, "\Phi_{A,B}"] \arrow[d, "\mu_A \otimes_2 \mu_B"'] & F(A \otimes_1 B) \arrow[d, "\mu_{A \otimes_1 B}"] \\ G(A) \otimes_2 G(B) \arrow[r, "\Gamma_{A,B}"'] & G(A \otimes_1 B) \end{tikzcd} \qquad \begin{tikzcd} I_2 \arrow[r, "\varphi"] \arrow[d, equal] & F(I_1) \arrow[d, "\mu_{I_1}"] \\ I_2 \arrow[r, "\gamma"'] & G(I_1) \end{tikzcd}$$

Definition A.12. A **symmetric monoidal natural transformation** is a monoidal natural transformation between two symmetric monoidal functors.

The equivalence between [symmetric] monoidal categories and strict [symmetric] monoidal we talked about is to be understood in the sense of the following definition.

Definition A.13. A [symmetric] monoidal functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ between [symmetric] monoidal categories is a **[symmetric] monoidal equivalence** if there is a [symmetric] monoidal functor $G : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ such that there exist two [symmetric] monoidal natural isomorphisms $\mu_1 : G \circ F \Longrightarrow 1_{\mathcal{M}_1}$ and $\mu_2 : F \circ G \Longrightarrow 1_{\mathcal{M}_2}$.

The following result can be found in [ML2, Chapter XI, §3].

Theorem A.14 (Mac Lane's Theorem)**.** *Given a monoidal category $\mathcal{M}$, there exists a strict monoidal category $\tilde{\mathcal{M}}$ for which there is a monoidal equivalence $F : \mathcal{M} \longrightarrow \tilde{\mathcal{M}}$. Similarly, given a symmetric monoidal category $\mathcal{M}$, there exists a strict symmetric monoidal category $\tilde{\mathcal{M}}$ for which there is a symmetric monoidal equivalence $F : \mathcal{M} \longrightarrow \tilde{\mathcal{M}}$.*

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