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# Continuous-time Optimal Pension Indexing in Pay-as-You-Go Systems

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**Abstract**: Ageing population and economic crisis have placed pay-as-you-go pension systems in need of mechanisms to ensure its financial stability. In this paper, we consider optimal indexing of pensions as an instrument to cope with the financial imbalances typically found in these systems. Using dynamic programming techniques in a stochastic continuous-time framework, we compute the optimal pension index and portfolio strategy that best target indexing and liquidity objectives determined by the government. A numerical example is provided to illustrate the results.

JEL Codes: H55.

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# 1 Introduction

Over the last two decades we have witnessed a far-reaching wave of reforms in Social Security Systems around the world, particularly in developed economies, due to the growing financial imbalances emerging into their systems. The nature of pay-as-you-go systems, where contributions are used to pay benefits, require them to be periodically adjusted. Otherwise, account deficits place pension systems in a situation of vulnerability that can lead to question the intergenerational solidarity in which are based on.

Financial imbalances can be caused by a variety of reasons. Recently, the main factors driving the reforms mainly include a severe economic crisis and an acute demographic pressure. On the one hand, the deterioration of the economic situation negatively impacts the short-term stability of the systems, lowering contributions (due to higher unemployment rates and reduction of salaries) while increasing benefit payments. On the other hand, the demographic context, with a decreasing ratio of workers to pensioners, raises benefit costs relative to contributions.

Generally, recent pension reforms tend to focus on the benefit side more than on the contribution side. Administrations consider contribution levels to be already in the high range of possible values, and increasing the tax burden is not the favoured option in the long term because of its unpopularity, the limited room for increases and the potential loss of competitiveness it may cause. Also, funding increasing benefits through debt is also a limited alternative if it must be consistent with a long-term fiscal sustainability.

Reforms adopt a wide range of forms, the simplest and most popular being the modification of the parameters of the system. Increasing the retirement age, changing the eligibility conditions to access an old-age pension (for example, increasing the necessary number of years of contributions for eligibility), tightening the minimum requirements for early retirement in order to prolong the working life (and raise the effective retirement age), or increasing the number of years used in the computation of the base pension are common parametric modifications. The popularity of parametric changes lies in their ease of implementation, and in the fact that they require less political consensus. On the contrary, structural changes, such as the transformation from a pay-as-you-go to a capitalization or funded system, are not possible without a wide political consensus.

More sophisticated reforms include the adoption of the so-called "automatic balance mechanisms" (Diamond, 2004; Vidal-Meliá et al., 2009). An automatic balance mechanism induces changes in a pension system based on the information provided by external demographic and economic factors. They take as input significant economic magnitudes such as growth rate of real wages, GDP variations or population life expectancy in order to stabilize the system. One of the main advantages of these mechanisms is that they liberate political institutions to decide over the system, which monitors and adjusts itself without external intervention.

Generally, these mechanisms are based on actuarially fair principles, and present a direct link between contributions and benefits at individual level. To restore the financial equilibrium of the system, the mechanisms tends to act on the benefit side rather than affecting the contribution rates. By triggering the balance mechanism, the system adjusts itself and re-establishes its equilibrium according to a solvency or sustainability indicator.

Although regulated automatic balance mechanisms tend to rely on equilibrium equations, the idea of controlling the pension system throught the modification of some selected variables has also been proposed in the past using more complex dynamic control techniques. For example, using optimal control techniques in a deterministic setting, Haberman and Zimbidis (2002) introduce the idea of a contingency fund for a pay-as-you-go pension system to absorb the short-term fluctuations in mortality or fertility patterns. The main utility of the fund is that it allows us to smooth the optimal path of the control variables, which in their model are the contribution rate and the age of eligibility for normal retirement.

The fund approach has also been considered in fully-funded pension systems relying on capitalization, as in Vigna and Haberman (2001) in a discrete time-setting, or in Devolder et al. (2003) in a continuous-time setting. Both models permits the fund to be invested in a risky and a riskless asset, following Merton (1975), while searching optimal investment policies for defined contribution plans. Similar approach in defined contribution plans is used in Josa-Fombellida and Rincón-Zapatero (2001) and Josa-Fombellida and Rincón-Zapatero (2004), which compute optimal contribution rates and investment strategies in a continuous-time stochastic framework while minimizing both the contribution rate risk and the solvency risk along the lines of Haberman and Sung (1994).

Although the fund method is employed in both capitalization and pay-as-you-go systems, the role of the fund is different in the two cases. In the case of capitalization, the goal is to accumulate savings to pay future obligations. Pay-as-you-go systems, in contrast, do not require accumulation of funds, so funds are collected to satisfy immediate payments to beneficiaries. Nevertheless, funds can also be collected in advance to provide a liquidity cushion in times of necessity, as it may happen in times of economic crisis where lowering contributions are insufficient to cover benefits, or to build up a large reserve in anticipation of an adverse demographic scenario.

The focus of governments to balance pension systems by controlling pension expenditure has led regulators to replace the usual indexing of pensions to the Consumer Price Index (CPI) for new methodologies. These new methods take into account economic and demographic variables to automatically establish a value of existing pensions so that expenditure matches available funds. For example, this is the case of the 2013 Spanish's pension reform, where each year the value of existing pensions is indexed according to the result of a formula meant to ensure a budget equilibrium (a comprehensive analysis of this aspect of the Spanish reform can be found in Roch et al., 2017).

This interest has led researchers to build mathematical models to optimally control the indexing of pensions. Godínez-Olivares, Boado-Penas, and Haberman (2016) include the indexing of pensions as a key variable for controlling the system in a discrete-time nonlinear dynamic programming approach, in order to guarantee a certain liquidity level. Also, in Godínez-Olivares, Boado-Penas, and Pantelous (2016), the method is extended to restore the long-term sustainability of the system.

In this paper we aim to expand the related literature on optimal pension indexing by extending its use as a control variable in a pay-as-you-go pension system in a stochastic continuous-time framework. According to our proposal, the aim of the government is to find an adequate level of pension indexing without compromising the financial equilibrium of the system, while maintaining fixed the level of contribution rates and other structural parameters.

Following the fund approach, the budget equilibrium is stabilized maintaining the pension fund close to a target level. Given the liquidity aspect of the pension fund in pay-as-you-go systems, opposite to the accumulation of funds to match actuarial liabilities, a stable value of the fund implies that income and expenditure must be closely related.

The main advantage of this approach is it that allows us to obtain closed form solutions for the optimal indexing paths. Our model contemplates the possibility of investing the fund assets both in a riskless and a risky asset, modelled by a continuous-time stochastic process. We further generalize the model allowing the fund to be invested in $n$ risky assets.

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3 finds the optimal indexing paths for different market structures. In Section 4, a numerical example is presented to illustrate the use of the model. Finally, Section 5 concludes the paper.

# 2 Mathematical model

In this section we define the general framework and the variables that conform the pension model. Later, in Section 3, we will consider particular cases of this general framework, restricting the universe of possible investment assets, in order to facilitate the interpretation and computation of the optimal indexing paths.

The most fundamental element of the model is the pension fund. It contains the assets hold by the sponsor (in this case, the government) necessary to satisfy present payments to the beneficiaries of the system. Its basic sources of income are dedicated tax revenues, generally in the form of Social Security contributions, and, to a lesser extent, cash-flows and capital gains generated by the invested reserves. Other sources of income are also accepted, as direct transfers from the state or federal budget to the fund originated from unrelated tax revenues.

Current assets of the fund are used to pay obligations, so benefit payments are drawn from the fund. The pension system defined in this model is of the defined benefit type, which means that benefits corresponding to each beneficiary are determined according a set of rules established in advance. Revenue collection and payments take place continuously, and decisions related to the system are made for a finite planning horizon, in the interval $[0, T]$.

Let $F(t)$ be the value of the assets in the fund at time $t$. The fund acts as a state variable of the model and follows a continuous-time stochastic process, with a continuous flow of money entering and leaving the fund. Income from contributions increase the value of the fund, and payments to the beneficiaries reduce its value.

At each moment in time, the remaining funds can be invested in a riskless asset and $m$ risky assets. The riskless asset, denoted by $S_0(t)$, $0 \leq t \leq \infty$, grows at a continuously compounded interest rate and therefore evolves according to

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$

being $r > 0$ the constant rate of interest. The risky assets are denoted by $S_1(t), \dots, S_m(t)$ and verify

$$dS_i(t) = \mu_i S_i(t)dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t) dW_j(t), \quad S_i(0) = s_i, \;\; i = 1, \dots, m. \tag{1}$$

Parameters $\mu_i$ and $\sigma_{ij}$ are assumed to be positive constants, and the interest rate $r$ is assumed to be strictly smaller than the mean rates of return of the risky assets, so $r \leq \mu_i$ for all $i = 1, \dots, m$. The vector $\mathbf{W}(t) = (W_1(t), \dots, W_m(t))^T$ is an $m$-dimensional standard Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the completion of the filtration $\sigma\{\mathbf{W}(s) \,|\, 0 \leq s \leq t\}$. We assume that the filtered probability space satisfies the usual 'technical' conditions of completeness and right-continuity. Bold font denotes vectors.

We denote by $\alpha_i(t)$ the fraction of the fund invested in each risky asset, $i = 1, \dots, m$, so that $\alpha_i(t)F(t)$ is the total amount invested in the risky asset $i$. The risky assets do not need to be particular assets, they can be broad categories or classes of risky assets. The fraction of the portfolio invested in the riskless asset at time $t$ is $1 - \sum_{i=1}^{m} \alpha_i(t)$.

The trading strategy $\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t))^T$ is a measurable process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, so that anticipation of future values of the random variables is not permitted. Short selling is allowed, as well as borrowing at the riskless rate of interest. Negative values of $\alpha_i(t)$ indicate short positions in the risky assets. If $1-\sum_{i=1}^{m} \alpha_i(t)$ takes a negative value, then the fund borrows at interest rate $r$ to support the long position in the risky assets.

As time goes on, the pension system funds expenditure from contributions and possibly from other sources of income. Total income to the system is given by a deterministic function $I(t)$, which may depend on factors such as population wages or contribution rate. Although some pension models take as a control variable the contribution rate of the system, recent reforms have not generally contemplated its modification, so we consider all income to be exogenously determined.

Total expenditure in the pension system is denoted by $G(t, g(t))$. Clearly, the main source of expenditure is the direct transfer to beneficiaries. In a pay-as-you-go system, benefits of each worker at the moment of retirement can be computed either using a flat rate principle or be earnings (salary) related. Later, benefits are indexed at a factor $g(t)$. In the model proposed, the index factor is a decision variable that controls the amount spent in benefits. The index process $g(t)$ is a measurable adapted process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that verifies

$$\int_0^t |g(s)|^2 ds < \infty \text{ a.s., for every } t < \infty.$$

Moreover, total expenditure should also include other costs required to run the pension system, such as those associated to cover employee, leasing and other administrative expenses.

Changes in the value of assets, contributions and payment of benefits provoke changes in the value of the fund. Therefore, the change in the fund level is given by

$$dF(t) = \left(\sum_{i=1}^{n} \alpha_i(t) \frac{dS_i(t)}{S_i(t)}\right) F(t) + \left(1 - \sum_{i=1}^{m} \alpha_i(t)\right) F(t) \frac{dS_0(t)}{S_0(t)} + (I(t) - G(t, g(t)))dt,$$

so that the fund satisfies

$$dF(t) = \left(rF(t) + \sum_{i=1}^{m} \alpha_i(t)(\mu_i - r)F(t) + I(t) - G(t, g(t))\right) dt + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i(t)\sigma_{ij} F(t) dW_j(t) \tag{2}$$

with initial condition $F(0) = F_0$.

Depending on the inflow and outflow of capital and the value of the investment portfolio, the value of the fund at every moment $t$ takes a positive or a negative sign. A negative value of $F(t)$ means an instant deficit that must be financed.

While controlling the evolution of the pension system, the government aims to index pensions cumulative close a target level $g_c$ during the planning horizon. The value of $g_c$ can be set to maintain the nominal value of pensions, can be related to an economic indicator like expected CPI (to maintain the real value of pensions) or can be related to any other value or indicator that may be considered desirable.

At the same time, the government aims that the level of the fund is close to a target level $F_c$. In a regular defined benefit pension plan, the value of $F_c$ corresponds to the actuarial liabilities of the plan. However, in a regular pay-as-you-go system, the sponsor uses current income to pay for current expenditure, so the value of the fund relates to liquidity objectives. In this sense, the target fund value could represent some months of future benefit payments, a percentage of the GDP or a percentage of total insured wages, for example. To simplify notation, and without loss of generality, in the following expressions we denote $g_c$ and $F_c$ as constant terms, but they can be functions of time as long as they are exogenously determined.

In these conditions, the sponsor does not want the fund to reach negative values, which would imply liquidity problems, but at the same time does not want to accumulate funds at the expense of a lower pension indexing. Thus, the sponsor aims to apply increasing values of the index factor, for a higher satisfaction of its pensioners, but not a a level that can provoke budget imbalances.

The quadratic loss function reflects the described equilibrium. Deviations of the two variables from their target levels from above or below are equally penalized.

Analytically, the problem is

$$\min_{\{g(t),\alpha_1(t),\ldots,\alpha_m(t)\}} E_{F_0}\left[\int_0^T e^{-\rho t}\left(\theta\left(\frac{g(t)-g_c}{g_c}\right)^2+(1-\theta)\left(\frac{F(t)-F_c}{F_c}\right)^2\right)dt\right] \quad (3)$$

subject to (2), where $E_{F_0}$ denotes the expectation conditional to an initial value of the fund $F_0$. The deviation of the variables $g(t)$ and $F(t)$ with respect to its corresponding target levels $g_c$ and $F_c$ is defined in relative terms, to compensate for the very different magnitude of the two variables.

The parameter $\theta$ ($0 < \theta \leq 1$) is a weight that allows the sponsor to ponder the importance of each objective. Therefore, a value of the parameter $\theta$ equal to 0.5 in expression (3) assigns the same importance to relative deviations of $g(t)$ and $F(t)$ with respect to its target levels. If the sponsor, for example, considers more important during the planning horizon to minimize the relative deviations of $g(t)$ with respect to its target level than those of $F(t)$, it can increase the value of $\theta$ from $\theta = 0.5$ up to $\theta = 1$.

The weighted relative deviations are discounted at a positive discount rate $\rho$. The higher the value of $\rho$, the more importance is given to the present. The rate $\rho$ is constant during the whole planning horizon.

# 3 Optimal pension control

Once the basic elements of the model are established, we aim to find a solution to the problem defined in (3) subject to (2). With the purpose of clarifying the necessary steps to find the solution, in this section we consider three particular cases of the problem, in increasing complexity. Starting with a market where only one riskless asset exists, we then extend the universe of available assets with one risky asset. Finally, we discuss the general case where an arbitrary number of risky assets is available for investing.

The following three subsections contain the discussion of the particular cases. We opt to maintain the notation of the value and auxiliary functions without distinction between cases. Being clear that the characterization of the functions is confined to the corresponding problem in each subsection, we avoid the use of subscripts or superscripts in the solutions to distinguish between cases, which hopefully clarifies the exposition of results.

## 3.1 No risky asset case

We start by considering the case where the market consists of a unique riskless asset. Despite its apparent simplicity, it describes a common situation, since pension reserves are frequently used to fund government expenditure due to regulatory impositions.

The differential equations that governs the dynamics of the fund is

$$F'(t) = rF(t) + I(t) - G(t, g(t)), \tag{4}$$

with initial condition $F(0) = F_0$, that corresponds to a particular case of expression (2).

According to this expression, the value of the fund is incremented by the income $I(t)$ and decremented by the expenditure $G(t, g(t))$. The remainder of the fund is invested in a riskless asset that grows at a constant rate $r$. Depending on the financial situation of the system, it may be necessary to borrow funds to cover current expenditure. We assume that borrowing and lending take place without restrictions at the same rate $r$.

In this setting, the sponsor of the fund controls the indexing of pensions in order to affect the total expenditure. On the one hand, the sponsor aims to set the index factor close the target level $g_c$, but on the other hand, it does not want to affect in excess the budget equilibrium.

We use dynamic programming techniques to solve the problem. Following Bellman (1957), the corresponding Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned} \rho V(t, F) - \frac{\partial V(t, F)}{\partial t} = \min_{g(t)} \Bigg\{ \theta \left( \frac{g(t) - g_c}{g_c} \right)^2 + (1 - \theta) \left( \frac{F - F_c}{F_c} \right)^2 + \Bigg( rF + I(t) \\ - G(t, g(t)) \Bigg) \frac{\partial V(t, F)}{\partial F} \Bigg\}, \end{aligned} \tag{5}$$

where $V(t, F)$ is the value function. The income function $I(t)$ and the expenditure function $G(t, g(t))$ are assumed to be continuous and twice differentiable functions. If $\phi(g(t); t, F)$ denotes the argument of the minimum function above, the first-order condition yields

$$\frac{\partial \phi(g(t); t, F)}{\partial g(t)} = \frac{2\theta(g(t) - g_c)}{g_c^2} - \frac{\partial G(t, g(t))}{\partial g(t)} \frac{\partial V(t, F)}{\partial F} = 0.$$

Let $g^*(t)$ be the value of the index factor that minimizes $\phi(g(t); t, F)$. Then, the optimal value $g^*(t)$ can be expressed in terms of the value function $V(t, F)$ and must satisfy

$$\frac{2\theta(g^*(t) - g_c)}{g_c^2} = \frac{\partial V(t, F)}{\partial F} \frac{\partial G(t, g(t))}{\partial g(t)} \Big|_{(t, g^*(t))}.$$

For the expenditure function $G(t, g(t))$, we assume the form

$$G(t, g(t)) = B(t)(1 + g(t))\delta, \tag{6}$$

where $B(t)$ denotes the expenditure related to payments to beneficiaries subject to the index factor $g(t)$ and $\delta$ is a factor incrementing the total expenditure due to administrative expenses ($\delta > 1$) and other costs or transfers. Here $B(t)$ is assumed to be a non-negative differentiable function and the parameter $\delta$ is assumed to be constant. Nevertheless, $\delta$ could be a variable function of time. For example, if it includes administrative expenses, $\delta$ could be a decreasing function due to technological advances or reforms that lead to a rationalization of the administration. Conversely, if it includes other transfers to the beneficiaries, $\delta$ could be an increasing function of time.

Note that $g(t)$ can be related to an instantaneous force of indexation $h(s)$ through the relation

$$(1 + g(t)) = e^{\int_0^t h(s)ds},$$

so that

$$h(t) = \frac{g'(t)}{1 + g(t)}.$$

Given the choice of the expenditure function, the optimal value of $g(t)$ is given by

$$g^*(t) = \frac{B(t)\delta g_c^2}{2\theta} \frac{\partial V(t, F)}{\partial F} + g_c. \tag{7}$$

In order to solve the HJB equation, the quadratic form of the functional suggests the use of quadratic guessing function for $V(t, F)$. So, we choose

$$V(t, F) = a(t)F^2 + b(t)F + c(t),$$

where the parameters $a(t)$, $b(t)$ and $c(t)$ are functions of time to be found.

By substituting the guessing function in (5) and matching coefficients we find that the parameter $a(t)$ must satisfy

$$a'(t) = \frac{1}{\theta}B(t)^2\delta^2 g_c^2 a(t)^2 + (\rho - 2r)a(t) - \frac{1 - \theta}{F_c^2}, \tag{8}$$

with final condition $a(T) = 0$. This differential equation can be identified as a Ricatti type equation that can be solved numerically.

Also, by matching coefficients in equation (5) the parameter $b(t)$ must satisfy

$$b'(t) + \left(r - \rho - \frac{1}{\theta}B(t)^2\delta^2 g_c^2 a(t)\right) b(t) = 2a(t)\left(B(t)\delta(1 + g_c) - I(t)\right) + \frac{2(1 - \theta)}{F_c},$$

with final condition $b(T) = 0$. Being a linear first-order differential equation, we can find an explicit solution for $b(t)$. Letting $p(t) = \frac{1}{\theta}B(t)^2\delta^2 g_c^2 a(t)$ and also letting $q(t) = 2a(t)\left(B(t)\delta(1+g_c) - I(t)\right) + \frac{2(1-\theta)}{F_c}$ we have that

$$b(t) = -\int_t^T q(s)e^{(r-\rho)(s-t)-\int_t^s p(u)du}ds. \tag{9}$$

With respect to the parameter $c(t)$, by matching coefficients it must satisfy the differential equation

$$c'(t) - \rho c(t) = \frac{1}{4\theta}B(t)^2\delta^2 g_c^2 b(t)^2 + B(t)\delta(1+g_c)b(t) - I(t)b(t) - (1-\theta),$$

with final condition $c(T) = 0$. Similar to the case of $b(t)$, the above differential equation can be identified as a linear first-order differential equation. If we let $\eta(t) = \frac{1}{4\theta}B(t)^2\delta^2 g_c^2 b(t)^2 + B(t)\delta g_c b(t) - I(t)b(t) - (1-\theta)$, the solution of the differential equation is

$$c(t) = -\int_t^T \eta(s)e^{-\rho(s-t)}ds.$$

Once we have found the three time parameters, we have an expression for the value function $V(t, F(t))$, so substituting $\dfrac{\partial V(t,F)}{\partial F}$ in (7) we obtain

$$g^*(t) = \frac{B(t)\delta g_c^2}{2\theta}\left(2a(t)F(t) + b(t)\right) + g_c. \tag{10}$$

Being $F(t)$ of deterministic nature, it is possible to find its expression by solving the differential equation (4) at the optimal $g^*(t)$. So,

$$\begin{aligned} F'(t) &= rF(t) + I(t) - G(t, g^*(t)) \\ &= rF(t) + I(t) - B(t)\left(1 + \frac{B(t)\delta g_c^2}{2\theta}\left(2a(t)F(t) + b(t)\right) + g_c\right)\delta, \end{aligned}$$

with $F(0) = F_0$, which can be identified as the linear first-order differential equation

$$F'(t) - \left(r - \frac{B(t)^2\delta^2}{\theta}a(t)g_c^2\right)F(t) = I(t) - \left(B(t)\delta(1+g_c) + \frac{B(t)^2\delta^2 g_c^2 b(t)}{2\theta}\right),$$

with solution

$$F(t) = F_0 e^{\int_0^t A(s)ds} + \int_0^t \xi(s)e^{\int_s^t A(u)du}ds,$$

where $A(t) = r - \frac{B(t)^2\delta^2}{\theta}a(t)g_c^2$ and $\xi(t) = I(t) - B(t)\delta(1+g_c) - \frac{B(t)^2\delta^2 g_c^2 b(t)}{2\theta}$.

An important observation from the results is that the value of $g(t)$ must reach the value $g_c$ at time $T$. This effect can be verified from expression (10), since $a(T)$ and $b(T)$ take the value 0. So, the model is useful in situations where the government has a clear indexing target to reach at the end of the planning horizon but accepts its temporary relaxation in order to cope with a foreseeable financial imbalance.

## 3.2 One risky asset case

Now we extend the universe of possible investments with the inclusion of one risky asset in the market, denoted by $S(t)$. Its dynamics is given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

with initial condition $S(0) = s_0$. This is a particular case of expression (1) corresponding to a unique risky asset.

The fund takes into account the results from investing both in the riskless and the risky asset, so its dynamics are given by

$$dF(t) = (rF(t) + \alpha(t)F(t)(\mu - r) + I(t) - G(t, g(t)))\,dt + \alpha(t)\sigma F(t)dW(t),$$

with initial condition $F(0) = F_0$. The fraction of the fund invested in the risky asset is $\alpha(t)$.

As in the previous case, we first set the Hamilton-Jacobi-Bellman equation, which is

$$\begin{aligned}\rho V(t,F) - \frac{\partial V(t,F)}{\partial t} = \min_{g(t),\alpha(t)} \Bigg\{ \theta \left( \frac{g(t) - g_c}{g_c} \right)^2 + (1-\theta) \left( \frac{F - F_c}{F_c} \right)^2 + \Bigg( rF + \alpha(t)F(\mu - r) \\ + I(t) - G(t, g(t)) \Bigg) \frac{\partial V(t,F)}{\partial F} + \frac{1}{2}\alpha(t)^2\sigma^2F^2\frac{\partial^2 V(t,F)}{\partial F^2} \Bigg\}\end{aligned} \tag{11}$$

Compared to (5), the above HJB equation incorporates an extra term to account for the stochastic nature of the fund, caused by the risky investment.

Let $\phi(g(t), \alpha(t); t, F)$ denote the argument of the minimum function above. Then, the first-order conditions are

$$\begin{cases} \dfrac{\partial \phi(g(t), \alpha(t); t, F)}{\partial g(t)} = 2\theta \left( \dfrac{g(t) - g_c}{g_c^2} \right) - \dfrac{\partial G(t, g(t))}{\partial g(t)} \dfrac{\partial V(t,F)}{\partial F} = 0, \\ \dfrac{\partial \phi(g(t), \alpha(t); t, F)}{\partial \alpha(t)} = (\mu - r)F\dfrac{\partial V(t,F)}{\partial F} + \alpha(t)\sigma^2F^2\dfrac{\partial^2 V(t,F)}{\partial F^2} = 0. \end{cases}$$

If $g^*(t)$ and $\alpha^*(t)$ are the value of the index factor and the fraction of fund invested in the risky asset that minimize $\phi(g(t), \alpha(t); t, F)$, the optimal values satisfy

$$\begin{cases} \dfrac{2\theta(g^*(t) - g_c)}{g_c^2} = \dfrac{\partial V(t,F)}{\partial F} \dfrac{\partial G(t, g(t))}{\partial g(t)} \Big|_{(t, g^*(t))}, \\ \alpha^*(t) = \dfrac{-(\mu - r)}{\sigma^2 F} \dfrac{\partial V(t,F)/\partial F}{\partial^2 V(t,F)/\partial F^2}. \end{cases}$$

For the choice of expenditure function $G(t, g(t)) = B(t)(1 + g(t))\delta$, the optimal value of $g(t)$ is given by

$$g^*(t) = \frac{B(t)\delta g_c^2}{2\theta} \frac{\partial V(t,F)}{\partial F} + g_c.$$

The guessing function for $V(t, F)$ is of quadratic form, so

$$V(t, F) = a(t)F^2 + b(t)F + c(t).$$

The time parameters $a(t)$, $b(t)$ and $c(t)$ are to be found by substituting the guessing function in (11) and matching coefficients. Then, the differential equation that $a(t)$ must satisfy is

$$a'(t) = \frac{1}{\theta} B(t)^2 \delta^2 g_c^2 a(t)^2 + (\rho - 2r + \epsilon)a(t) - \frac{1-\theta}{F_c^2},$$

with $a(T) = 0$ and where $\epsilon = \left(\frac{\mu - r}{\sigma}\right)^2$. Note that the difference between the above expression and (8) is the term $\epsilon$, which can be interpreted as the square of the market price of risk.

With respect to the function $b(t)$, it follows

$$b'(t) + \left(r - \rho - \epsilon - \frac{1}{\theta} B(t)^2 \delta^2 g_c^2 a(t)\right) b(t) = 2a(t)\left(B(t)\delta(1 + g_c) - I(t)\right) + \frac{2(1-\theta)}{F_c},$$

with $b(T) = 0$. Being a linear first-order differential equation, defining $p(t) = \frac{1}{\theta} B(t)^2 \delta^2 g_c^2 a(t)$ and $q(t) = 2a(t)\left(B(t)\delta(1 + g_c) - I(t)\right) + \frac{2(1-\theta)}{F_c}$ we have that

$$b(t) = -\int_t^T q(s) e^{(r - \rho - \epsilon)(s-t) - \int_t^s p(u)du} ds.$$

Comparing the value of $b(t)$ with expression (9), we find that the only difference comes from the inclusion of the term $\epsilon$ in the exponential function.

Finally, the function $c(t)$ must satisfy the differential equation

$$c'(t) - \rho c(t) = \frac{b(t)^2}{4} \left(\frac{B(t)^2}{\theta} \delta^2 g_c^2 + \frac{\epsilon}{a(t)}\right) + B(t)\delta(1 + g_c)b(t) - I(t)b(t) - (1 - \theta),$$

with final condition $c(T)=0$. Letting $\eta(t)=\frac{b(t)^2}{4}\left(\frac{B(t)^2}{\theta}\delta^2 g_c^2+\frac{\epsilon}{a(t)}\right)+B(t)\delta(1+g_c)b(t)-I(t)b(t)-(1-\theta)$, the solution of the differential equation is

$$c(t)=-\int_t^T \eta(s)e^{-\rho(s-t)}ds.$$

So, given a trajectory of $F(t)$, the optimal controls are given by

$$g^*(t)=\frac{B(t)\delta g_c^2}{2\theta}\left(2a(t)F(t)+b(t)\right)+g_c. \tag{12}$$

and

$$\alpha^*(t)=\frac{-\left(\mu-r\right)\left(2a(t)F(t)+b(t)\right)}{\sigma^2 F(t)2a(t)}. \tag{13}$$

## 3.3 Multiple risky assets case

Finally we solve problem (3), subject to the complete dynamics of the fund expressed in (2). For ease of computation we use vector notation for $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_m)^T$, representing the column vector of expected returns. Also, let $\Sigma=\sigma\sigma^T$, where $\sigma$ is the matrix $\sigma=(\sigma_{ij})$, and let $\mathbf{1}$ be the unit (column) vector of dimension $m$.

Proceeding as in the previous cases, the HJB equation for the problem with multiple risky assets is

$$\begin{aligned}\rho V(t,F)-\frac{\partial V(t,F)}{\partial t}=\min_{g(t),\boldsymbol{\alpha}(t)}\Bigg\{&\theta\left(\frac{g(t)-g_c}{g_c}\right)^2+(1-\theta)\left(\frac{F-F_c}{F_c}\right)^2+\\ &\left(rF+\boldsymbol{\alpha}^T(t)(\boldsymbol{\mu}-r\mathbf{1})F+I(t)-G(t,g(t))\right)\frac{\partial V(t,F)}{\partial F}+\frac{1}{2}\boldsymbol{\alpha}^T(t)\Sigma\boldsymbol{\alpha}(t)F^2\frac{\partial^2 V(t,g(t))}{\partial F^2}\Bigg\}.\end{aligned} \tag{14}$$

The first-order conditions are

$$\begin{cases}\dfrac{\partial\phi(g(t),\boldsymbol{\alpha}(t);t,F)}{\partial g(t)}=2\theta\left(\dfrac{g(t)-g_c}{g_c^2}\right)-\dfrac{\partial G(t,g(t))}{\partial g(t)}\dfrac{\partial V(t,F)}{\partial F}=0,\\ \dfrac{\partial\phi(g(t),\boldsymbol{\alpha}(t);t,F)}{\partial\boldsymbol{\alpha}(t)}=(\boldsymbol{\mu}-r\mathbf{1})F\dfrac{\partial V(t,F)}{\partial F}+\Sigma\boldsymbol{\alpha}(t)F^2\dfrac{\partial^2 V(t,F)}{\partial F^2}=0.\end{cases}$$

where $\phi(g(t),\boldsymbol{\alpha}(t);t,F)$ denotes the argument of the minimum function in the HJB equation.

Selecting the usual functional form of the expenditure function $G(t,g(t))=B(t)(1+$

$g(t))\delta$, we find the optimal value of $g(t)$ as

$$g^*(t) = \frac{B(t)\delta g_c^2}{2\theta}\frac{\partial V(t,F)}{\partial F} + g_c.$$

The optimal value of $\boldsymbol{\alpha}(t)$ can also be solved from the first order conditions, resulting in

$$\boldsymbol{\alpha}^*(t) = \frac{-1}{F}\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})\frac{\partial V(t,F)/\partial F}{\partial^2 V(t,F)/\partial F^2}.$$

As in the previous two cases, we choose the quadratic guessing function

$$V(t,F) = a(t)F^2 + b(t)F + c(t),$$

so that $\frac{\partial V(t,F)}{\partial F} = 2a(t)F + b(t)$, $\frac{\partial^2 V(t,F)}{\partial F^2} = 2a(t)$ and $\frac{\partial V(t,F)}{\partial t} = a'(t)F^2 + b'(t)F + c'(t)$. Substituting these expressions and the optimal controls in the HJB equation (14), by matching coefficients we obtain the set of differential equations relating $a(t)$, $b(t)$ and $c(t)$.

Starting with the $a(t)$ function, matching terms in $F^2$ we find that it must obey

$$a'(t) = \frac{1}{\theta}B(t)^2\delta^2 g_c^2 a(t)^2 + (\rho - 2r + \epsilon)\, a(t) - \frac{1-\theta}{F_c^2},$$

with final condition $a(T) = 0$, where $\epsilon = (\boldsymbol{\mu} - r\mathbf{1})^T\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})$. The differential equation is of Riccati type, and can be solved numerically.

Similarly, matching terms in $F$ in expression (14) we have that $b(t)$ follows the linear first-order ordinary differential equation

$$b'(t) + \left(r - \rho - \epsilon - \frac{B(t)^2}{\theta}\delta^2 g_c^2 a(t)\right) b(t) = 2a(t)\left(B(t)\delta(1+g_c) - I(t)\right) + \frac{2(1-\theta)}{F_c},$$

with final condition $b(T) = 0$. Defining the functions $p(t)$ and $q(t)$ by

$$p(t) = \frac{B(t)^2}{\theta}\delta^2 g_c^2 a(t)$$

and

$$q(t) = 2a(t)\left(B(t)\delta(1+g_c) - I(t)\right) + \frac{2(1-\theta)}{F_c},$$

the above linear differential equation has solution

$$b(t) = -\int_t^T q(s)e^{(r-\rho-\epsilon)(s-t) - \int_t^s p(u)du}ds.$$

The solution is similar to the case of one risky asset, only modified by the value of $\epsilon$, which now accounts for the market risk of the overall portfolio.

Finally, matching the rest of the terms in (14), we find that function $c(t)$ also follows a linear first-order differential equation, in this case

$$c'(t) - \rho c(t) = \eta(t),$$

with $c(T) = 0$, where

$$\eta(t) = \frac{b(t)^2}{4}\left(\frac{B(t)^2}{\theta}\delta^2 g_c^2 + \frac{\epsilon}{a(t)}\right) + B(t)\delta(1+g_c)b(t) - I(t)b(t) - (1-\theta).$$

The differential equation has the solution

$$c(t) = -\int_t^T \eta(s)e^{-\rho(s-t)}ds.$$

So, for a given trajectory of $F(t)$, the optimal controls are given by

$$g^*(t) = \frac{B(t)\delta g_c^2}{2\theta}(2a(t)F(t) + b(t)) + g_c.$$

and

$$\boldsymbol{\alpha}^*(t) = \frac{-\,(2a(t)F(t) + b(t))}{2a(t)F(t)}\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1}).$$

Comparing the solutions of the three cases, we see that the difference arises from the the market price of risk $\epsilon$. In the multivariate case, $\epsilon$ accommodates for the multivariate nature of the market, incorporating the diversification effect provided by a larger number of assets. In the case where there are no risky assets, the value of $\epsilon$ turns to be 0.

# 4 Numerical Example

In this section, we aim to illustrate the use of the model presented in the previous sections with a simple numerical example. We consider a hypothetical pension system to study the optimal behaviour of the fund, the index factor and the portfolio strategy, for different values of the parameters in the model.

## 4.1 Baseline case

A basic aspect of the model is the purpose to balance income and expenditure during the specified planning horizon. So, in order to obtain numerical results, we need to stipulate both elements.

A general specification of expenditure is already given in expression (6)

$$G(t, g(t)) = B(t)(1 + g(t))\delta,$$

but the pension roll $B(t)$ still needs to be determined. Although a more complex decomposition of $B(t)$ could be established, we consider the exponential function

$$B(t) = B_0 e^{(\kappa+\lambda)t},$$

where $B_0$ is the expenditure at the initial time, $\kappa$ defines the force of growth of the number of pensions in the system and $\lambda$ is the rate accounting for the difference between the mean pension of new pensions with respect to exiting pensions. For the sake of this example, we set $B_0 = 9{,}000$, $\kappa = 1\%$ and $\lambda = 1.5\%$.

Similarly, we consider an exponential function for the income of the pension system. In this case, we define

$$I(t) = I_0 e^{\beta t},$$

where $I_0$ denotes the initial income and $\beta$ accounts for the growth of income taking into account the variation of insured salaries (both in value and number), and the variation of transfers from the general budget to the Social Security budget, if they exist. We set $I_0 = 10{,}000$ and $\beta = 0.03$.

The structure of income and expenditure is completed defining the value of the parameter $\delta$, related to administrative costs and other transfers not included in $B(t)$. We consider these costs to account for an extra 10% of the pension expenditure, so we set $\delta = 1.1$. This structure reflects the general situation of pensions systems in many developed economies, where pension expenditure increases at a faster rate than income. Under this scenario, the value of $B(t)\delta$ will overcome the value of $I(t)$ between the third and four year after the initial time, and the financial imbalance will start to threaten the stability of the system.

Using the controls at its disposal, the government may choose to temporary relax the indexing of pensions to reduce the imbalance, or, on the contrary, decide to index the pensions close to a target level, possibly sacrificing budget equilibrium. Depending on the government's preference between the two objectives, pensions will be more or less generous.

In this example, the government aims to index the pensions at a rate of 1.5% for the next 10 years. So, the planning horizon of the problem is considered to be 10 years ($T = 10$), and the corresponding target factor is $g_c(t) = e^{0.015t} - 1$. Note that, according to the model, at the end of the time period considered, $T = 10$, $g(T)$ will converge to the target value $g(T) = e^{0.015T} - 1$, but its path along the 10 years will differ depending on the specification of the rest of the model.

The remaining funds can be invested in a riskless asset that yields a 1% annual return, or in a risky asset with $\mu = 6\%$ and $\sigma = 25\%$, also measured in annual terms. Although there only exists one risky asset in the market, from the results in Section 3, this is equivalent to a market with multiple risky assets providing the same market price of risk.

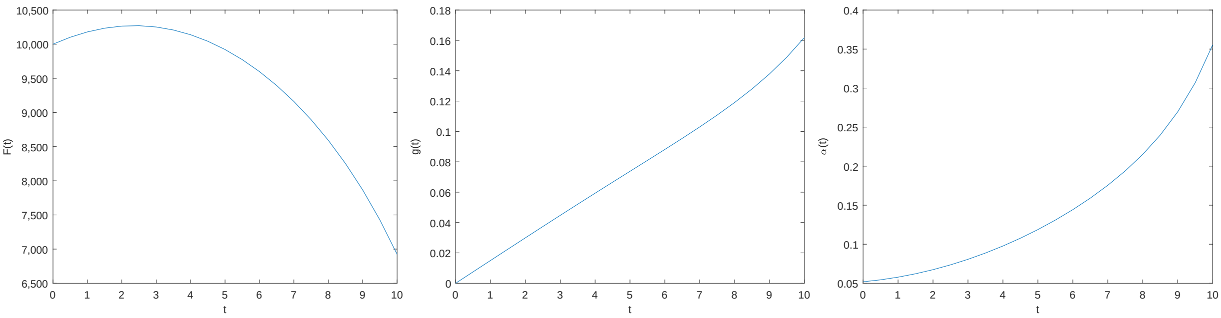
Let's assume that the government decides to assign the same weight to maintain an stable pension indexing than to maintain a stable budget equilibrium, choosing a value of $\theta = 0.5$. Also, parameter $\rho$ is set at $\rho = 2\%$. The initial value of the fund is considered to be equal to one year of present income, so $F_0 = 10{,}000$, and the target level for the fund is set equal to its initial value $F_c = 10{,}000$.

Now, for a given level of the fund, expressions (12) and (13) provide the optimal value of the control variables. For the sake of this illustration, we consider the trajectory of $F(t)$ given by its expected value, which can be computed as

$$E_{F_0}[F(t)] = F_0 e^{\int_0^t A(s)ds} + \int_0^t \xi(s) e^{\int_s^t A(u)du} ds,$$

where $A(t) = r - \epsilon - \frac{B(t)^2}{\theta}\delta^2 g_c^2 a(t)$ and $\xi(t) = I(t) - B(t)\delta(1 + g_c) - \frac{B(t)^2}{2\theta}\delta^2 g_c^2 b(t) - \frac{\epsilon}{2}\frac{b(t)}{a(t)}$. For details on this later result, see Arnold (1974, p. 139).

Once the model is specified, we run it. Figure 1 shows the optimal trajectories obtained for the value of the fund, $F(t)$, the optimal index factor $g(t)$, and the optimal fraction of the fund invested in the risky asset, $\alpha(t)$.




(a) $F(t)$ (b) $g(t)$ (c) $\alpha(t)$

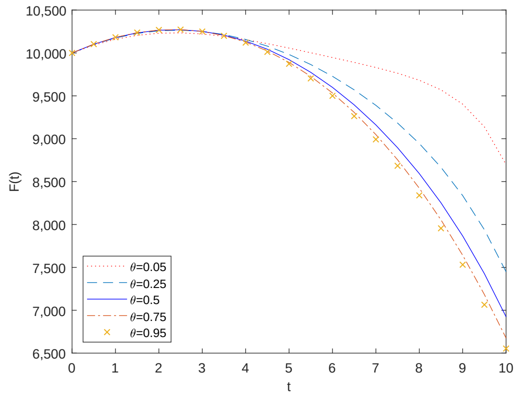
Figure 1: Sensibility analysis with respect to $F_c$.

According to the results, during the first years of application, when income is higher than expenditure, the value of the fund mildly increases, and the index factor can stay close to its target level. Once the financial situation begins to deteriorate, the index factor is set lower to its target level in order to reduce total expenditure, and the fraction of the fund invested in the risky asset increases to compensate for the reduction of fund value.

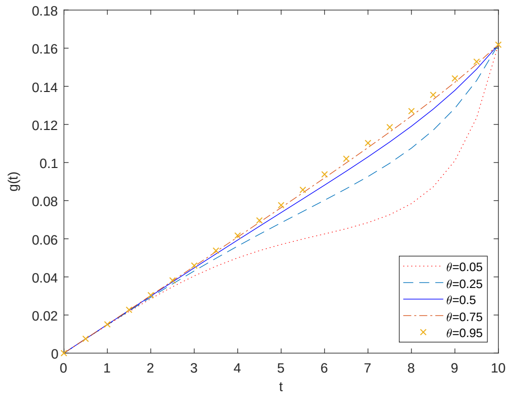
Finally, in the last years of application, expenditure overcomes income, so the value of the fund is reduced in order to keep with the indexation, which eventually must converge to its target level. The value of $\alpha(t)$ keeps increasing as the value of the fund is reduced in order to obtain capital gains that increase the value of the fund. Note that the last term of $\alpha(t)$ is computed taking its limit, since at the exact final moment its value is undetermined.

## 4.2 Sensitivity analysis

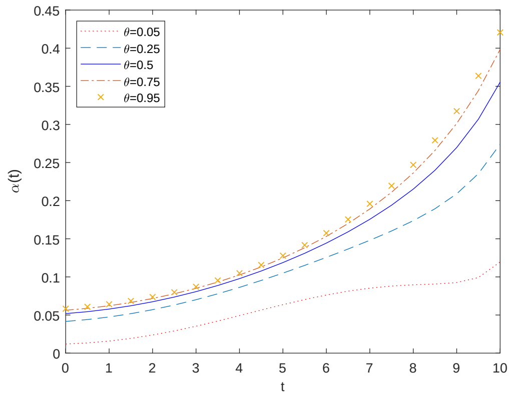
Taking as reference the baseline case, we can explore the effect of modifying some parameters in the optimal paths. First, the most fundamental parameter to study is $\theta$, which determines the relative importance of the target strategies. High values of $\theta$ (close to 1) sacrifice financial balance for a closer indexing to its target level, whereas low values of $\theta$ (close to 0) affect in the opposite direction. Figure 2 collects plots for five different values of $\theta$, being the middle one the baseline case.




(a) $F(t)$




(b) $g(t)$

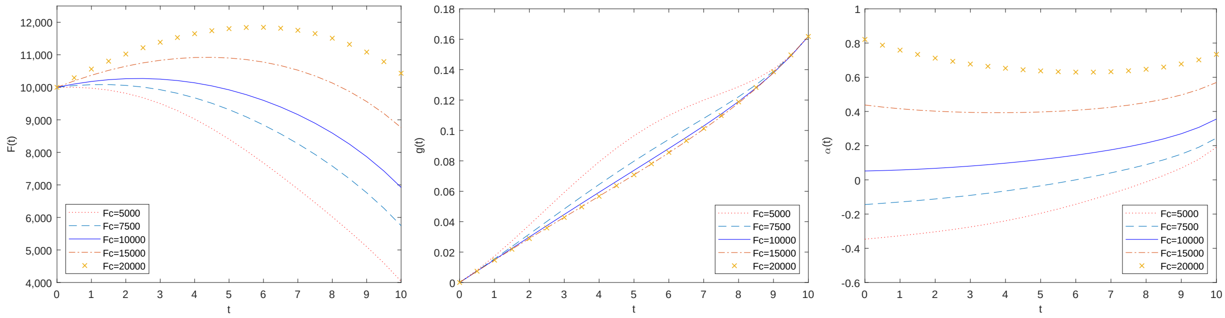



(c) $\alpha(t)$

Figure 2: Sensitivity analysis with respect to parameter $\theta$.

As expected, high values of $\theta$ result in lower fund values, since maintaining the indexing close to its target level increases total expenditure. The higher the $\theta$, the higher the indexation and therefore the lower the value of the fund. On the contrary, low values of $\theta$ imply a reduction of the indexation generosity, so the value of the fund gets close to the desired level $F_c$. The lower the value of $\theta$, the closer it gets to $F_c$.

Another important parameter to be chosen by the sponsor is the value of $F_c$. Figure 3 shows the evolution of the optimal paths when $F_c$ is modified.

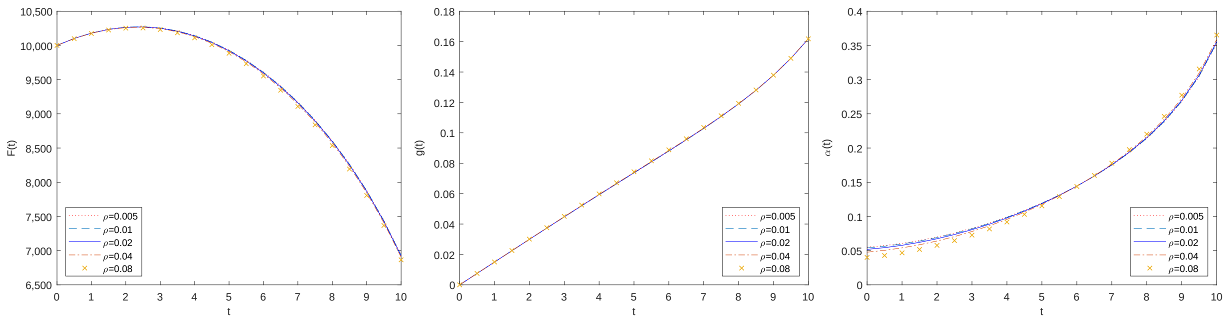



(a) $F(t)$ (b) $g(t)$ (c) $\alpha(t)$

Figure 3: Sensitivity analysis with respect to $F_c$.

Now, lower values of $F_c$ result in lower values of $F(T)$. Conversely, higher values of target level $F_c$ result in higher values of the fund at terminal time $T$. Therefore, lower values of $F_c$ permits a more generous indexing, as seen in plot (b). Note, however, that high values of $F_c$ produce small differences in the optimal values of $g(t)$. The explanation of this effect can be found in plot (c). In order to obtain a higher value of $F(T)$, the optimal strategy tries to compensate the cost of indexing investing in assets with higher mean return, meaning higher values of $\alpha(t)$.

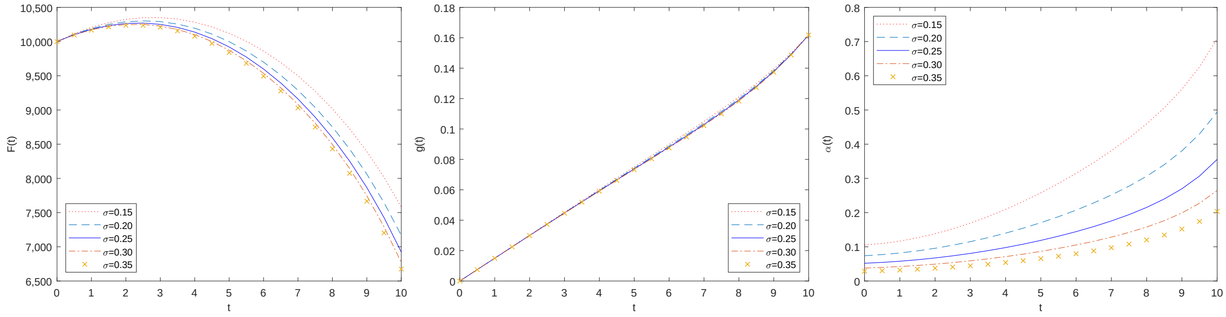
The last parameter to be chosen by the sponsor is $\rho$. Figure 4 collects the results for different values of the parameter. According to the plots, variations of the parameter have little effect on the results, at least for reasonable choices of $\rho$.




(a) $F(t)$ (b) $g(t)$ (c) $\alpha(t)$

Figure 4: Sensitivity analysis with respect to $\rho$.

The model also includes parameters exogenously determined as $\sigma$. As $\sigma$ increases, the market price of risk $\epsilon$ diminishes, and vice versa. The effect of altering the value of $\sigma$ is shown is Figure 5.

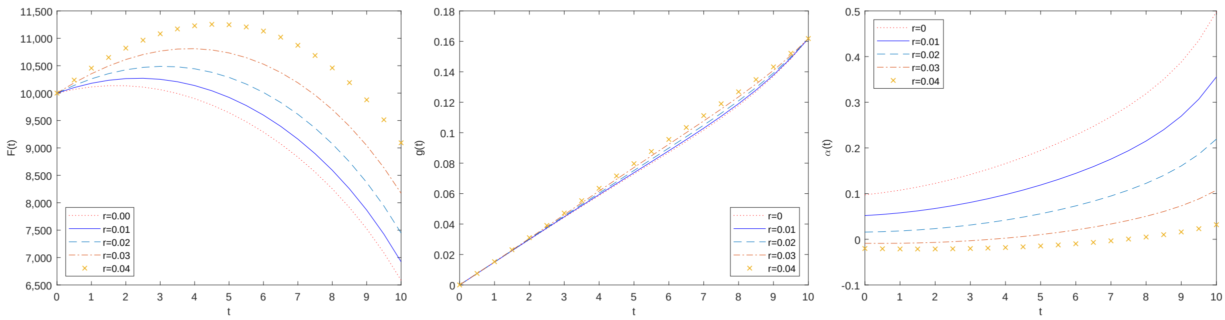



(a) $F(t)$ (b) $g(t)$ (c) $\alpha(t)$

Figure 5: Sensitivity analysis with respect to $\sigma$.

According to the results, modifications of the value of $\sigma$ barely affect the optimal trajectory of $g(t)$. However, it significantly affects the optimal value of $\alpha(t)$: the smaller the $\sigma$, the higher the fraction of the fund invested in the risky asset, which in turn leads to a higher value of $F(t)$. Note that similar changes in the value of the parameter $\mu$ would cause the opposite effect than those of $\sigma$ in the optimal paths, since as $\mu$ increases, so does the market price of risk.

Finally, we check the effect of modifying the interest rate $r$ in Figure 6.




(a) $F(t)$ (b) $g(t)$ (c) $\alpha(t)$

Figure 6: Sensitivity analysis with respect to $r$.

Increasing the value of $r$ relative to the baseline case increases the value of the fund, since it is invested with a higher return. At the same time, the higher value of the fund permits a higher indexing of pensions. Also, due to the higher riskless return, the market price of risk decreases, so the risky asset becomes less attractive and the fraction of the fund invested in the risky asset gets smaller.

# 5 Conclusions

General theory on pensions systems tend to formulate models that are in immediate equilibrium in terms of liquidity or solvency. However, results might not be socially acceptable when they imply a significant reduction of pensions levels, particularly in nominal terms. In these scenarios, governments might prefer to borrow funds, particularly in times when the cost of debt is small, and gradually delay the implementation of policies that might hurt the generosity of the pension system.

In this paper, we have considered the situation where the sponsor of a pay-as-you-go pension system (in this case, the government) must optimally decide, for a particular planning horizon, the indexing of pensions while taking into account its effect on the financial balance. Through the selection of an adequate pension indexing and investment portfolio, the sponsor can weigh the indexing and budget equilibrium objectives to find an optimally acceptable solution in social terms.

In the proposed model, income and expenditure functions are formulated in very wide terms, so that results in this paper constitute a general framework where different structures of pay-as-you-go systems can fit in. In general, sources of income are not limited to contributions, and costs are not limited to the pension roll. So, a general framework where all cash-flows are considered is necessary to make suitable decisions regarding the elements of the pension system.

Results on this paper have focused in situations where sponsors deal with liquidity constrains to index the pensions, but where eventually the indexing will reach its target value. It is left for further research the construction of models for alternative scenarios, in particular for those where solvency requirements have to be met. Also, results for different dynamics of the fund, or constraints on the control variables, are left for further research.

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