Hydrodynamical limit of non-minimally coupled scalar fields

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Abstract: In this paper we study how the addition of a non-minimal derivative coupling modifies the energy-momentum tensor of conformally invariant scalar fields in a flat Robertson-Walker spacetime. We argue that the resulting energy-momentum tensor is not that of a perfect fluid with pressure proportional to density. This suggests that the departure from the cosmological fluid behaviour is a general feature of non-minimally coupled massless scalar fields.

I. INTRODUCTION

In the standard cosmology, the isotropy and homogeneity of the universe imposed by the cosmological principle require the metric tensor $g_{\mu\nu}$ to be of the Robertson-Walker (RW) form. From the Einstein field equations one then obtains that the energy-momentum tensor $T_{\mu\nu}$ must be that of a perfect fluid:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu},$$

where U_μ is the velocity of the fluid in the comoving frame, p its pressure and ρ its density.

In [1], P.D. Mannheim and D. Kazanas study a conformally coupled real massless scalar field φ in a RW geometry. The action of this system reads

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[\left(\frac{1}{8\pi G} - \frac{\varphi^2}{6}\right)R + g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right], \tag{1}$$

where $g = det(g_{\mu\nu})$, R is the Ricci scalar and G is the gravitational constant. Then, they explicitly derive a perfect fluid form for the energy-momentum tensor of the field, which the conformal invariance of the theory forces to be traceless. This determines the equation of state of a radiative fluid

$$p = \frac{\rho}{3}.$$

In this work, we shall restrict our attention to a spatially-flat RW spacetime

$$ds^2 = -dt^2 + a(t)^2\left(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right),$$

and consider the addition of a non-minimal derivative coupling to the action (1). As shown in [2], the only way to do so without introducing third order derivatives in the resulting field equations is to consider the non-minimal derivative coupling to be of the form

$$\frac{1}{M^2}G^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi,$$

where $G^{\mu\nu}$ is the Einstein tensor and M is a mass parameter. The addition of such a term modifies the scalar field equation, yields a new expression for the energy-momentum tensor and is found to break the conformal invariance of (1). Nevertheless, we assume a perfect fluid behaviour of the Hubble parameter H and treat the non-minimal coupling as a perturbation by setting $H^2M^{-2} \ll 1$. On this framework, we find a first-order perturbative solution to the non-minimally coupled scalar field equation, for which we find a valid regime of approximation. Under this regime, we use the incoherent averaging of [1] to show that even at first-order in the perturbation parameter, the pressure of the effective fluid is not proportional to the density.

II. A REVIEW ON P.D. MANNHEIM AND D. KAZANAS WORK

It is convenient for our purposes to give further insight into Mannheim and Kazanas work [1]. As we said, their starting point is action (1), a variation of which with respect to φ and the metric tensor gives the following scalar and gravitational field equations (we use the notation $\nabla_\mu\varphi = \varphi_\mu$ and $\nabla_\nu\nabla_\mu\varphi = \varphi_{\mu\nu}$).

$$g^{\mu\nu}\varphi_{\mu\nu} + \frac{R}{6}\varphi = 0, \tag{2}$$

$$\begin{aligned}\left(\frac{\varphi^2}{6} - M_P^2\right)G_{\mu\nu} = {}& \varphi_\mu\varphi_\nu - \frac{1}{2}g_{\mu\nu}\varphi_\alpha\varphi^\alpha \\ & - \frac{1}{6}(\varphi^2)_{\mu\nu} + \frac{1}{6}g_{\mu\nu}(\varphi^2)_\alpha{}^\alpha,\end{aligned} \tag{3}$$

where $M_P = (8\pi G)^{-1/2}$ is the reduced Planck mass. Rewriting the RHS of (3) and considering the curvature-dependent term $\frac{\varphi^2}{6}G_{\mu\nu}$ as part of the source, one can define the curvature-dependent energy-momentum tensor

$$\begin{aligned}T_{\mu\nu} = {}& \frac{2}{3}\varphi_\mu\varphi_\nu - \frac{1}{3}\varphi\varphi_{\mu\nu} \\ & - \frac{1}{6}g_{\mu\nu}\varphi_\alpha\varphi^\alpha + \frac{1}{3}g_{\mu\nu}\varphi\varphi_\alpha{}^\alpha - \frac{1}{6}G_{\mu\nu}\varphi^2,\end{aligned} \tag{4}$$

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that satisfies the usual Einstein field equations

$$G_{\mu\nu} = -M_P^{-2} T_{\mu\nu} \tag{5}$$

and thus is covariantly conserved: $\nabla^\mu T_{\mu\nu} = 0$.

The equation of motion (2) is invariant under conformal transformations with conformal weight $s = -2$ in the following sense. Consider a conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{6}$$

then the transformed version of (2) satisfies [3]

$$\left(\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu + \frac{\tilde{R}}{6}\right)\Omega^{-2}\varphi = \Omega^{-3}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{R}{6}\right)\varphi = 0.$$

An ansatz of a solution of the form

$$\varphi = \frac{u(\eta)g(r,\theta,\phi)}{a(t)}, \tag{7}$$

where η is the conformal time, leads to :

$$\frac{u''}{u} = \frac{1}{g(r,\theta,\phi)}\gamma^{-1/2}\partial_i\left[\gamma^{1/2}\gamma^{ij}\partial_j g(r,\theta,\phi)\right] = -\omega^2, \tag{8}$$

where $\gamma_{ij} = \frac{g_{ij}}{a(t)^2}$ is the Euclidean 3-space metric, γ its determinant and where a separation constant ω^2 is introduced. Here and henceforth, derivatives with respect to η are denoted by a prime and derivatives with respect to t are denoted by a dot. Therefore, u is harmonic in conformal time and one finds that

$$g(r,\theta,\phi) = j_l(\omega r) Re\, Y_l^m(\theta,\phi),$$

where j_l and Y_l^m are respectively the spherical Bessel functions and the spherical harmonics, for $l = 0, 1, \ldots$ and $m = -l, -l+1, \ldots, l$. Thus, a complete set of solutions to (2) is given by

$$\varphi = \frac{N_l^m}{a(t)}\begin{pmatrix}\cos\omega\eta \\ \sin\omega\eta\end{pmatrix} j_l(\omega r) P_l^m(\cos\theta)\begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}, \tag{9}$$

where P_l^m are the associated Legendre polynomials and N_l^m is a normalisation constant given by

$$N_l^m = (-1)^m\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}.$$

To find the hydrodynamical limit of the energy-momentum tensor, an incoherent averaging over the different field propagation modes is performed: for fixed values of m and l, one calculates $T_{\mu\nu}$ for each of the 4 modes given in (9) and then sums their contributions. This proves to be enough to eliminate the dependence on both η and ϕ. Then, summing over all the possible values of m using summation properties of the associated Legendre polynomials further eliminates the dependence on θ. Finally, summing over all the possible values of l using summation properties of the spherical Bessel functions ultimately removes the dependence on r. In this process, all off-diagonal terms are found to vanish, and what is left is a diagonal tensor with components

$$\rho = T^t{}_t = \frac{\omega^4}{2\pi^2 a^4},\;\; p = T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi = \frac{\omega^4}{6\pi^2 a^4}, \tag{10}$$

corresponding to a perfect radiative fluid.

III. THE NON-MINIMALLY COUPLED GRAVITATIONAL AND SCALAR FIELD EQUATIONS

The introduction of the non-minimal derivative coupling in (1) yields the following action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[\left(M_P^2 - \frac{\varphi^2}{6}\right)R + \left(g^{\mu\nu} - \frac{G^{\mu\nu}}{M^2}\right)\varphi_\mu\varphi_\nu\right]. \tag{11}$$

Again, the equation of motion for the scalar field and the gravitational field equations are obtained by varying the action (11) with respect to the field and the metric, respectively, and read (see [2] and [4])

$$\left(g^{\mu\nu} - \frac{G^{\mu\nu}}{M^2}\right)\varphi_{\mu\nu} + \frac{R}{6}\varphi = 0. \tag{12}$$

$$G_{\mu\nu} = -M_P^{-2}\left(T_{\mu\nu} + \Theta_{\mu\nu}\right). \tag{13}$$

where $T_{\mu\nu}$ is given by (4) and

$$\begin{aligned}\Theta_{\mu\nu} =& M^{-2}\Big\{-\frac{1}{2}\varphi_\mu\varphi_\nu R + 2\varphi_\alpha\varphi_{(\mu}R^\alpha{}_{\nu)} - \frac{1}{2}\varphi_\alpha\varphi^\alpha G_{\mu\nu} \\ &- \varphi^\alpha\varphi^\beta R_{\mu\alpha\nu\beta} - \varphi_{\alpha\mu}\varphi^\alpha{}_\nu + \varphi_{\mu\nu}\varphi_\alpha{}^\alpha \\ &+ \frac{1}{2}g_{\mu\nu}\left[\varphi_{\alpha\beta}\varphi^{\alpha\beta} - (\varphi_\alpha{}^\alpha)^2 - 2\varphi_\alpha\varphi_\beta R^{\alpha\beta}\right]\Big\}.\end{aligned} \tag{14}$$

Following a conformal transformation as (6), we have [3]

$$\tilde{G}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\varphi} = G^{\mu\nu}\nabla_\mu\nabla_\nu\tilde{\varphi} - G^{\mu\nu}C^\alpha_{\mu\nu}\nabla_\alpha\tilde{\varphi} + \cdots. \tag{15}$$

where $\tilde{\varphi} = \Omega^s\varphi$ and

$$C^\alpha_{\mu\nu} = 2\delta^\alpha_{(\mu}\nabla_{\nu)}\ln\Omega - g_{\mu\nu}g^{\alpha\beta}\nabla_\beta\ln\Omega.$$

The omitted terms in (15) involve only the conformal factor Ω, the field, the metric and their covariant derivatives. This is also the case for

$$\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\varphi} + \frac{\tilde{R}}{6}\tilde{\varphi}.$$

We then see that the term $G^{\mu\nu}C^\alpha_{\mu\nu}\nabla_\alpha\tilde{\varphi}$ cannot be cancelled unless $s = 0$. But in this case one would lose the conformal invariance of (2), that requires $s = -2$ (in the

4-dimensional case). We conclude that conformal invariance is broken by the addition of the non-minimal coupling, which means that the traceless energy-momentum tensor condition required for a radiative fluid is not fulfilled. Therefore, we already see that if a perfect fluid form was to be recovered from the non-minimally coupled energy-momentum tensor, it would certainly not be radiation.

Now, equation (12) can be unpacked as follows

$$a^{-3}\gamma^{-1/2}\Big\{-\gamma^{-1/2}\partial_t\left[a^3\left(1-\frac{3H^2}{M^2}\right)\partial_t\varphi\right] +a\left(1-\frac{2\dot{H}+3H^2}{M^2}\right)\partial_i(\gamma^{1/2}\gamma^{ij}\partial_j\varphi)\Big\} = (\dot{H}+2H^2)\,\varphi, \tag{16}$$

where $H=\frac{\dot{a}}{a}$ is the Hubble expansion parameter in cosmic time t. In terms of conformal time, the Hubble parameter is $\mathcal{H}=aH$. Equation (16) is separable and we set a solution of the form

$$\varphi(\eta,r,\theta,\phi)=f(\eta)g(r,\theta,\phi).$$

After the introduction of a separation constant $-\omega^2$ as in (8), we find that the functions $f(\eta)$ and $g(r,\theta,\phi)$ are given by

$$f''+2\mathcal{H}\,\frac{1-\frac{3\mathcal{H}'}{a^2M^2}}{1-\frac{3\mathcal{H}^2}{a^2M^2}}\,f'+ +\frac{\mathcal{H}'+\mathcal{H}^2+\omega^2\left[1-\frac{2\mathcal{H}'+\mathcal{H}^2}{a^2M^2}\right]}{1-\frac{3\mathcal{H}^2}{a^2M^2}}\,f=0. \tag{17}$$

$$\partial_i[\gamma^{1/2}\gamma^{ij}\partial_j g(r,\theta,\phi)]+\omega^2\gamma^{1/2}g(r,\theta,\phi)=0. \tag{18}$$

Equation (18) is the same as in (8). Therefore, the space-dependent part of the solution is not modified by the addition of the non-minimal derivative coupling, as expected by the fact that $G_{\mu\nu}$ is maximally symmetric in 3-space. We can then completely recover the analysis of [1] resulting in the same expression for $g(r,\theta,\phi)$.

IV. PERTURBATIVE SOLUTION IN A RADIATON-DOMINATED BACKGROUND

As we anticipated, we argue that the non-minimal derivative coupling is subdominant, as it would be at late times. To test whether $p\propto\rho$, we assume the following behaviour of the Hubble parameter:

$$H^2=\frac{a_0^nH_0^2}{a^n}\Longrightarrow\mathcal{H}^2=\frac{a_0^{n'}\mathcal{H}_0^2}{a^{n'}}, \tag{19}$$

where $n>0$ and hence $n'=n-2$ and we set the initial condition $a(\eta_0)=a(t_0)=a_0=1$ so that $\mathcal{H}(\eta_0)=\mathcal{H}_0$ and $H(t_0)=H_0$, for initial times η_0 and t_0. We then consider

$$\epsilon=H_0^2M^{-2},\qquad \epsilon\ll 1.$$

and seek a first-order perturbative solution to the non-minimally coupled scalar field equation (12) of the form

$$\varphi=\varphi_0+\epsilon\varphi_1=(f_0+\epsilon f_1)g(r,\theta,\phi),$$

where $f_0+\epsilon f_1$ is a first-order perturbative solution to (17). To this purpose, it is convenient to introduce, as in (7), the following rescaling for f

$$f=\frac{u}{a}=\frac{u_0+\epsilon u_1}{a}.$$

The perturbative treatment of equation (17) then yields the following equations for u_0 and u_1, at first order in ϵ

$$u_0''+w^2u_0=0, \tag{20}$$

$$u_1''+\omega^2u_1=-\frac{\mathcal{H}^2}{H_0^2a^2}\omega^2(2+n')u_0+3\frac{\mathcal{H}^4}{H_0^2a^2}\left(1+\frac{3n'}{2}\right)u_0 -6\frac{\mathcal{H}^3}{H_0^2a^2}\left(1+\frac{n'}{2}\right)u_0', \tag{21}$$

where we have also made use of the relation

$$\mathcal{H}'=-\frac{n'}{2}\mathcal{H}^2,$$

following from (19).

Equation (20) together with equation (18) imply that the unperturbed term of our perturbative solution is precisely that of Mannheim and Kazanas for (2), namely

$$\varphi_0=\frac{N_l^m}{a(\eta)}\begin{pmatrix}\cos\omega\eta\\ \sin\omega\eta\end{pmatrix}j_l(\omega r)P_l^m(cos\,\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}. \tag{22}$$

Of course, we had no reason to expect otherwise as (2) is precisely recovered from (12) by setting $\epsilon=0$.

As for the perturbed term u_1, we see that each of the two modes of u_0 will yield a different mode for u_1, which we denote by u_{11} and u_{12}. Let us denote by $q_1(\eta)$ and $q_2(\eta)$ the RHS of equation (21) for $u_0=\cos\omega\eta$ and $u_0=\sin\omega\eta$, respectively. Then, a general solution to (21) is given, for $i=1,2$, by

$$\begin{aligned} u_{1i}(\eta)=&\ C_{1i}\cos\omega\eta+C_{2i}\sin\omega\eta+\\ &+\omega sin\omega\eta\int_{\eta_0}^{\eta}q_i(s)\cos\omega s ds-\\ &-\omega\cos\omega\eta\int_{\eta_0}^{\eta}q_i(s)\sin\omega s ds, \end{aligned} \tag{23}$$

where the values of the constants are fixed by the fact that for $\eta\to+\infty$ we expect to recover the unperturbed solution. Thus,

$$\begin{aligned} C_{1i}&=\omega\int_{\eta_0}^{+\infty}q_i(s)\sin\omega s ds,\\ C_{2i}&=-\omega\int_{\eta_0}^{+\infty}q_i(s)\cos\omega s ds. \end{aligned} \tag{24}$$

V. TESTING THE COSMOLOGICAL ASSUMPTION

We now find a valid approximation for our perturbative solution that will allow us to study the effect of the non-minimal coupling on the energy-momentum tensor.

Making use of the Einstein field equations (13) together with the expression for ρ in (10), we have that

$$H^2 \propto G\omega^4 \left(\frac{a_0}{a}\right)^4 + \mathcal{O}(\epsilon),$$

In order to avoid quantum gravity regimes, we consider $G^{-1} \gg \omega^2$. If we now set $t = t_0$, we have

$$\frac{H_0^2}{\omega^2} \propto \omega^2 G \ll 1 \quad \Longrightarrow \quad \omega \gg H_0.$$

But H is a decreasing function of t so the previous relation actually holds for every t:

$$\omega \gg H. \tag{25}$$

We can use this to compare the three terms of $h_i(\eta)$:

$$\begin{aligned}
-\frac{\mathcal{H}^2}{H_0^2 a^2}\omega^2(2+n')\omega^2 u_0 &\propto \frac{\mathcal{H}^2}{a^2}\omega^2 \propto H^2\omega^2 \\
3\frac{\mathcal{H}^4}{H_0^2 a^2}\left(1+\frac{3n'}{2}\right)u_0 &\propto \frac{\mathcal{H}^4}{a^2} \quad \propto a^2 H^4 \\
-6\frac{\mathcal{H}^3}{H_0^2 a^2}\left(1+\frac{n'}{2}\right)u_0' &\propto \frac{\mathcal{H}^3}{a^2}\omega \quad \propto aH^3\omega
\end{aligned} \tag{26}$$

At first sight, we already realise that in account of condition (25) the first term in (26) dominates over the others for t sufficiently close to t_0 so that $a(t) \approx a(t_0) = 1$. For increasing times, the other two terms will become larger due to their dependence on the scale factor suggesting that this approximation might fail eventually. However, since we are taking G^{-1} to be various orders of magnitude larger than ω^2, we actually expect the regime of validity of the approximation to extend even to times well into the future.

Therefore, we restrict our attention to the regime of validity of our approximation, and assume that the differential equation (21) for u_1 can be well approximated by

$$u_1'' + \omega^2 u_1 \approx -\frac{\mathcal{H}^2}{H_0^2 a^2}\omega^2(2+m)u_0. \tag{27}$$

At this point, we can use (25) yet again in order to simplify the calculations of the integrals in (23) and (24). Indeed, from the fact that the frequency of oscillation is large compared to the Hubble parameter follows that $H^2 = \frac{\mathcal{H}^2}{a^2}$ is approximately constant over a period T of oscillation. Therefore, in each period, $\frac{\mathcal{H}^2}{a^2}$ comes out of the integral and what is left are the familiar integrals

$$\int_{\eta_j}^{\eta_j+T} \cos^2\omega s ds = \int_{\eta_j}^{\eta_j+T} \sin^2\omega s ds = \frac{1}{2},$$
$$\int_{\eta_j}^{\eta_j+T} \cos\omega s \sin\omega s ds = 0.$$

Breaking the whole integration interval into intervals of width T and following this procedure on each of them gives the following approximation for the required integrals

$$\begin{aligned}
\int_{\eta_0}^{\eta} \frac{\mathcal{H}^2}{a^2}\cos^2\omega s ds \approx \int_{\eta_0}^{\eta} \frac{\mathcal{H}^2}{a^2}\sin^2\omega s ds &\approx \frac{1}{2}\int_{\eta_0}^{\eta}\frac{\mathcal{H}^2}{a^2}ds, \\
\int_{\eta_0}^{\eta} \frac{\mathcal{H}^2}{a^2}\cos\omega s\sin\omega s ds &\approx 0.
\end{aligned} \tag{28}$$

After these considerations, we find the values of the constants to be

$$C_{11} = C_{22} = 0 \quad ; \quad C_{21} = -C_{12} = \frac{n}{n+2}\frac{\omega}{H_0}.$$

Substituting in (23) and performing the integrals as in (28) finally gives for the two modes of u_1

$$\begin{aligned}
u_{11} &= -\frac{n}{n+2}\frac{H}{aH_0^2}\omega\sin\omega\eta \\
u_{12} &= \frac{n}{n+2}\frac{H}{aH_0^2}\omega\cos\omega\eta
\end{aligned} \tag{29}$$

A. Incoherent averaging of the perturbed energy-momentum tensor

We now study the appearance of the energy-momentum tensor for our perturbative solution

$$\varphi(\eta, r, \theta, \phi) = \begin{pmatrix} \cos\omega\eta - \epsilon\,\frac{n}{n+2}\frac{H\omega}{aH_0^2}\sin\omega\eta \\ \sin\omega\eta + \epsilon\,\frac{n}{n+2}\frac{H\omega}{aH_0^2}\cos\omega\eta \end{pmatrix} \frac{g(r,\theta,\phi)}{a} \tag{30}$$

in the region of validity of approximation (27). If we denote it by $\tilde{T}_{\mu\nu}$, then from (13) in the perturbative setting we get

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \Theta_{\mu\nu}, \qquad \epsilon \ll 1, \tag{31}$$

Note that, since $\Theta_{\mu\nu}$ is already of first order in ϵ,

$$\Theta_{\mu\nu} = {}^0\Theta_{\mu\nu} + \mathcal{O}(\epsilon^2),$$

where ${}^0\Theta_{\mu\nu}$ is calculated solely from the unperturbed solution φ_0. Thus, we can further set

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + {}^0\Theta_{\mu\nu}, \qquad \epsilon \ll 1, \tag{32}$$

In the case of $T_{\mu\nu}$, however, we do need to consider the full peturbative solution. We proceed in a more general way and consider a solution like (30) but with arbitrary time-dependent modes v_1 and v_2, and we perform the incoherent averaging of [1], as explained before. By doing so, all off-diagonal terms are found to vanish and the dependence with respect to the space variables is removed in exactly the same way as in [1] for the unperturbed solution. Indeed, this is a consequence only of the particular space-dependence of the solution given by $g(r,\theta,\phi)$ and does not involve its time-dependent part all. As for the diagonal elements, we find

$$\begin{aligned}
T^t{}_t &= \frac{\omega^2}{4\pi^2 a^4}\left[(v_1^2+v_2^2)\omega^2 + v_1'^2 + v_2'^2\right] \\
T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi &= \frac{\omega^2}{12\pi^2 a^4}\big[-(v_1^2+v_2^2)\omega^2 \\
&\qquad + v_1'^2 + v_2'^2 - 2(v_1 v_1'' + v_2 v_2'')\big],
\end{aligned}$$

which, interestingly enough, is of the perfect fluid form. In passing, we also note that by putting $v_1 = \cos\omega\eta$ and $v_2 = \sin\omega\eta$ one recovers (10), as expected. Putting instead the two modes in (30) and keeping only the terms of leading order in ϵ $(= H_0^2 M^{-2})$, we find

$$\begin{aligned}
T^t{}_t &= \rho = \frac{\omega^4}{2\pi a^4}\left(1 - \frac{nH^2}{2M^2}\right) \\
T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi &= p = \frac{\omega^4}{2\pi a^4}\left(\frac{1}{3} - \frac{nH^2}{2M^2}\right).
\end{aligned} \tag{33}$$

Note how in the limit $M \to \infty$ we recover the radiation fluid (10). We also see that at first order in ϵ

$$w = w(t) = \frac{p}{\rho} = \frac{1}{3} - \frac{nH^2}{3M^2}, \tag{34}$$

so that $p \not\propto \rho$.

The incoherent averaging of ${}^0\Theta_{\mu\nu}$ is also found to reduce it to a diagonal form. For all the non-diagonal elements in (14) involving a product of two first covariant derivatives of the field the vanishing of the off-diagonal terms is obtained by the use of the same summation properties as in [1]. The only non-diagonal elements left are

$$\varphi_{0\,\alpha\mu}\varphi_0{}^\alpha{}_\nu \qquad ; \qquad \varphi_{0\,\mu\nu}\varphi_{0\,\alpha}{}^\alpha .$$

The off-diagonal terms in $\varphi_{0\,\mu\nu}\varphi_{0\,\alpha}{}^\alpha$ are found to vanish by the same properties after using that

$$\varphi_{0\,\alpha}{}^\alpha = -\frac{R}{6}\varphi_0 .$$

since φ_0 is a solution to (2). Finally, the vanishing of the off-diagonal elements in $\varphi_{0\,\alpha_\mu}\varphi_0{}^\alpha{}_\nu$ is obtained only after having derived further summation properties obtained simply by differentiation of the ones used in [1]. This is necessary in account of the new terms that appear when one considers a product of second covariant derivatives, as is the case.

The calculation of the diagonal terms in ${}^0\Theta_{\mu\nu}$, however, is found to be non-trivial as it requires further summation properties that cannot be obtained from the ones in [1]. Although this exact calculation is left for future work, we do realise that these elements will contain terms that scale like different powers of the scale factor. This means that the contribution of ${}^0\Theta_{\mu\nu}$ cannot cancel out the perturbative term in (34). Therefore, the perturbation in the energy-momentum tensor introduced by the non-minimal coupling breaks the linear dependence when $\epsilon = 0$ between p and ρ.

VI. CONCLUSIONS

In this paper, we have investigated whether the proportionality between p and ρ is a feature of the hydrodynamical limit of a non-minimal derivatively coupled theory of massless scalar fields. We have found that, even when treated perturbatively, the non-minimal derivative coupling not only rules out the possibility to recover a radiative fluid, but it breaks the perfect fluid with $p \propto \rho$ behaviour.

In future research, one would like to study whether a general perfect fluid form can be recovered in any regime for a non-minimal derivatively coupled massless scalar field. If this was the case, one would aim to calculate its equation of state and search for applications in realistic cosmological scenarios.

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