Introduction to Quantum Game Theory: The relevance of betting

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Abstract: The goal of this paper is to study, on an invented quantum game, the relevance that betting different amounts has on the outcome of the game. Two different scenarios are laid out to find the best strategy for each player: when both players know everyting, and when one of them has lack of information. Finally, the same game is thought from the viewpoint of a gambling house, where this game is presented in front of a crowd of players, who may or may not be professionals.

I. INTRODUCTION

Game theory has been studied by many scientists during the last century. It is common to describe and understand complex concepts of this field using simple games. In many articles, such as [2] and [3], some traditional well-known games have been studied, like the prisoner's dilemma among others.

Based on the work of John von Neumann, the mathematician David A. Meyer wrote paper [1], where a variation of the traditional penny flipover game was introduced, involving a "quantum penny".

The traditional penny flipover game consists on a penny placed head up and then three consecutive turns (player 1, then player 2, then player 1) flipping the penny over or not, without being able to see what the other player has done before. At the end of the game, player 1 wins if and only if the penny is still head up. Doing an easy probabilistic calculation, it can be seen that the probability that player 1 wins is exactly 1/2.

Founded on the well-known television series Star Trek, Meyer posed a game between Q (a player with quantum strategies) and Captain Picard (P, a player with traditional classic moves). In this paper, Meyer showed that Q (player 1), playing an appropriate quantum strategy, is always able to increase his expected payoff, in relation to the case that both players can only play classically.

Afterwards, other scientists wrote different variations on Meyer's paper. For instance, whereas in [1] both players act on a single coin, in [2] the game consists on a pair of entangled coins so that there is a single state space for the pair of coins. However, in spite of all the variations where the advantage of the quantum player was demonstrated, the obligation to pay more in order to be able to make a better (quantum) movement has not been studied enough, and it is not clear if, in that case, the game is still favourable for the quantum player.

Inspired by the work of Meyer, it is known that if Q knows everything about the game, then Q can always win. For this reason, the goal of this paper is to design a game such that trying not to get a clear victory for Q is possible, and it will be done keeping the relevance of betting in mind.

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II. QUANTUM GAME

An orthonormal basis $\mathcal{B} = \{|1\rangle, |2\rangle\}$ can be defined, where $|1\rangle$ and $|2\rangle$ represents the winning state of player 1 (who is a quantum player) and player 2 (who is a classic player), respectively. So any state can be represented as

$$|\psi\rangle = \sum_{i=1}^{2} a_i |i\rangle, \tag{1}$$

such that $|a_1|^2 + |a_2|^2 = 1$.

The two-player game presented consists in an alternate sequence of turns in which it is compulsory to put a certain amount of money on the table, which allows the player to do a specific movement over the penny. Starting with the penny in a quantum superposition fair (equiprobable) state

$$|\psi\rangle_0 = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \tag{2}$$

in the first turn, player 1, after putting P_Q (quantum payment) on the table, is allowed to change the state of the penny with a matrix rotation $\mathcal{R}(\theta)$ such that

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \theta \in [-\pi, \pi]. \tag{3}$$

This matrix can represent any rotation on the xy-plane in an anticlockwise way and satisfies that $\mathcal{R}^2(\theta) = \mathcal{I}$ (where $\mathcal{I}$ represents the identity matrix), so $\mathcal{R}^{-1}(\theta) = \mathcal{R}(\theta)$. The way to flip a quantum penny classically is applying the matrix Pauli σ_x over the state, which coincides with $\sigma_x = \mathcal{R}(\theta = \pi/2)$.

To further understand the theory, an aside can be made at this point to prove easily what Meyer postulated in [1]. The game starts with the penny placed completely head up, that is in the winning state of player 1, so $|\psi'\rangle_0 = |1\rangle$. Then, three consecutive movements have to be applied over the state $|\psi'\rangle_0$ in order to play the game. These movements can either be $\mathcal{R}(\theta_2) \circ \mathcal{I} \circ \mathcal{R}(\theta_1)$ if player 2 does not flip over the penny, or $\mathcal{R}(\theta_2) \circ \sigma_x \circ \mathcal{R}(\theta_1)$ if player 2 does so. Meyer showed that Q (player 1) can always win. That is because, in the first case, Q can choose $\theta_2 = \theta_1$, since $\mathcal{R}(\theta_1) \circ \mathcal{I} \circ \mathcal{R}(\theta_1) = \mathcal{I}\ \forall \theta_1$; and in the second case, Q can choose $\theta_2 = \pi/2 - \theta_1$, which satisfies that $\mathcal{R}(\pi/2-\theta_1) \circ \sigma_x \circ \mathcal{R}(\theta_1) = \mathcal{R}(\theta = 0)$, which, in turn, fulfills that $\mathcal{R}(\theta = 0)|\psi'\rangle_0 = |\psi'\rangle_0$. Thus, if Q does not know the action taken by the second player, Q can easily secure his victory playing with some θ_1 such that $\pi/2 - \theta_1 = \theta_1$, that is $\theta_1 = \pi/4$. This means that Q will play both turns with the matrix $\mathcal{R}(\theta = \pi/4)$, which is also called as the Hadamard matrix. Hence, it has just been showed that whether or not player 2 flips the penny, player 1 is guaranteed a win.

Continuing with the game that was being explained, in the second turn, player 2 can either pay the classic payment $P_C < P_Q$ with probability p, in order to reverse (flip over) classically the state of the penny (option A), or pay P_I (identity payment) with probability $1-p$, in order to stop the game and measure the final quantum state (option B). Finally, only in the case that player 2 has paid P_C (option A), there is a third turn, where player 1 has to pay P_I mandatorily to stop the game and measure the final state.

At the end of the game, as the penny can be in a superposition of states of $\mathcal{B}$, there is a measure of the final quantum state in order to decide, according to the probability of winning of each player, who is the winner of the game and can take all the money bet on the table. However, as developed below, this is equivalent to thinking, in terms of the expected value of the earnings, that all the money bet on the table will be shared out among both players, proportionally to the probability of winning of each one of the final state.

Following the rules of the game, player 1 will turn the initial state $|\psi\rangle_0$ into

$$|\psi\rangle_1 = \mathcal{R}(\theta)|\psi\rangle_0 = \frac{1}{\sqrt{2}}[(\sin\theta + \cos\theta)|1\rangle + (\sin\theta - \cos\theta)|2\rangle]. \tag{4}$$

Afterwards, the final state can either be

$$|\psi\rangle_f^A = \sigma_x|\psi\rangle_1 = \frac{1}{\sqrt{2}}[(\sin\theta - \cos\theta)|1\rangle + (\sin\theta + \cos\theta)|2\rangle] \tag{5}$$

with probability p (in option A), or $|\psi\rangle_f^B = |\psi\rangle_1$ with probability $1-p$ (in option B).

The earnings G_i of player $i = 1, 2$ can be defined as $G_i = pG_i^A + (1-p)G_i^B$, where $G_i^j, j = A, B$, represents, respectively, the earnings of player i in each option A (with probability p) or B (with probability $1-p$). Each of these G_i^j is calculated as follows: $G_i^j = \text{Prob}_j\{|i\rangle\} M_T^j - M_B^{i,j}$, where $\text{Prob}_j\{|i\rangle\}$ is the modulus squared of the coefficient a_i of the final state in choice j, M_T^j is all the money put on the table at the end of option j, and $M_B^{i,j}$ is only the money bet by player i in case j.

Therefore, the earnings of player 1 can be calculated as follows:

$$G_1 = (1-p)\left[\frac{1}{2}(1+\sin(2\theta))(P_Q + P_I) - P_Q\right]$$
$$+p\left[\frac{1}{2}(1-\sin(2\theta))(P_Q + P_C + P_I) - (P_Q + P_I)\right]. \tag{6}$$

Just from the definitions of the earnings G_1 and G_2, it is clear to see that $G_1 + G_2 = 0$.

III. OPTIMAL STRATEGIES

A. Complete information

The purpose of this section is to determine which rotation player 1 has to do in order to win, that is $G_1 > G_2$ or, equivalently, $G_1 > 0$. This inequality depends on a value p_0 defined as

$$p_0 \equiv \frac{P_Q + P_I}{2(P_Q + P_I) + P_C}, \tag{7}$$

such that $0 < p_0 \leq 1/2$, which is a restriction over the probability p.

(a) If $p < p_0$, then $G_1 > G_2$ if and only if

$$\sin(2\theta) > \Lambda = \frac{P_Q - P_I + p(2P_I - P_C)}{P_Q + P_I - p[2(P_Q + P_I) + P_C]}. \tag{8}$$

(b) If $p > p_0$, then $G_1 > G_2$ if and only if $\sin(2\theta) < \Lambda$.

(c) If $p = p_0$, then $G_1 < G_2$ always.

It can be observed that, in case (a), Λ should satisfy $\Lambda < 1$, which is equivalent to $p < p_0^- < p_0$, where p_0^- is defined as

$$p_0^- \equiv \frac{P_I}{2P_I + P_Q}. \tag{9}$$

Similarly, in case (b), Λ should satisfy $\Lambda > -1$, which is equivalent to $p > p_0^+ > p_0$, where p_0^+ is defined as

$$p_0^+ \equiv \frac{P_Q}{P_Q + P_C}. \tag{10}$$

Another way to determine which rotation is the best option for player 1 is optimising G_1 with respect to the angle θ, that is $\frac{\partial G_1}{\partial \theta}$. It can be seen that this optimisation also depends on the probability p of player 2 compared to p_0, so a rotation that maximises G_1 independently of the probability of player 2 does not exist in general.

(a) If $p < p_0$, then

$$G_1^{\max} = G_1\left(\theta = \frac{\pi}{4}, \frac{-3\pi}{4}\right) = P_I - p(2P_I + P_Q). \quad (11)$$

It is observed that the same restriction as before is obtained, because $G_1^{\max} > 0 \iff p < p_0^- < p_0$.

(b) If $p > p_0$, then

$$G_1^{\max} = G_1\left(\theta = \frac{-\pi}{4}, \frac{3\pi}{4}\right) = -P_Q + p(P_Q + P_C). \quad (12)$$

It can be seen that also the same restriction as before is obtained, since $G_1^{\max} > 0 \iff p > p_0^+ > p_0$.

(c) If $p = p_0$, then $G_1 < G_2$ always.

In conclusion, if everything is known, then the quantum player can not always win, because he needs to know the probability player 2 plays with, in order to choose the suitable θ. Regarding the classic player, if he plays with $p = p_0$, his victory is guaranteed. So it is not a very fair game because player 2 can always win easily, reason why it is necessary to change a little bit the rules of the game so it can be played appropriately.

B. Incomplete information

Currently, the following situation is set out: player 2 does not know P_Q, therefore he is not able to choose p_0 because it depends on P_Q, so the winning strategy can not be chosen. However, if player 2 plays with some $p \neq p_0$ and player 1 discovers this probability p, then the first player always has a possible movement θ to win, as it has already been said when Λ was introduced the first time in Eq. (8).

The main goal of this section is to determine which is the optimal movement for player 2 in front of his lack of knowledge, that is finding his minimum loss without knowing p_0.

It can be shown that

$$G_2(p = p_0) = \frac{P_Q(P_Q + P_I) - P_C P_I}{2(P_Q + P_I) + P_C} > 0. \quad (13)$$

Since P_Q is unknown, it is possible to think $G_2(p = p_0)$ as a function of P_Q, in order to determine the minimum value of $G_2(p = p_0)$ with respect to P_Q. It can be seen that $\dfrac{\partial G_2(p = p_0)}{\partial P_Q} \neq 0\ \forall P_Q$, which means that the minimum is on one of the endpoints of the interval $P_Q \in (P_C, \infty)$. However, $G_2(p = p_0)$ is an increasing function for P_Q, so if G_2^* is defined as $G_2^* = \min\limits_{P_Q}\{G_2(p = p_0)\}$, then

$$G_2^* = G_2(p = p_0, P_Q = P_C) = \frac{P_C^2}{3P_C + 2P_I}, \quad (14)$$

which coincides with the minimum earnings guaranteed by player 2 with the strategy adopted. This is, in fact, nothing more than a minimax problem: the maximum $G_2(p = p_0)$ is requested and the minimum $P_Q = P_C$ is looked for.

Therefore, after doing this reasoning, player 2 can play with a probability

$$p_0^* = p_0(P_Q = P_C) = \frac{P_C + P_I}{3P_C + 2P_I} < p_0, \quad (15)$$

since $G_2(p = p_0^*) > G_2^*$.

Even so, due to the fact that both players are rationals and each one is aware of the rationality of the other, player 1 can foresee his opponent's action and analyse which is the best angle θ to take. Owing to the fact that $p_0^- < p_0^* < p_0$ and according to Eq. (11), if player 2 plays with $p = p_0^*$, then $G_1^{\max} < 0$, which means that player 1 has lost the game because $G_1 < 0$, but his loss will be the least possible one when he takes an angle $\theta \in \{\pi/4, -3\pi/4\}$, that is, when he is winning (or rather, losing) $G_1^{\max}$.

So the classic player wins again, but his victory is fairer than the one exposed in section III A, due to the fact that his earnings is not the highest possible one (is not the one described by Eq. (13)), or in other words, the quantum player's loss is not that great.

Finally, player 2, who wants to achieve the maximum profit and knows that player 1 is rational and will lose as little as possible, will win

$$\begin{aligned} G_2^{\max}(p = p_0^*) &= G_2\left(p = p_0^*, \theta \in \left\{\frac{\pi}{4}, \frac{-3\pi}{4}\right\}\right) \\ &= \frac{P_C^2 + (P_Q - P_C)(P_C + P_I)}{3P_C + 2P_I}. \end{aligned} \quad (16)$$

Lastly, it can be shown that $0 < G_2^* < G_2^{\max}(p = p_0^*) \leq G_2(p = p_0^*) < G_2(p = p_0)$.

IV. MULTIPLE PLAYERS

Another way that the described quantum game can be dealt with is thinking of it as a gambling house would do so, that is analysing everything that can take place and always keeping in mind that, for their benefit, the condition $\langle G_1^{\max}\rangle_p > 0$ must be met. In this case, player 1 is the bank and it does not necessarily have to know if player 2 is clever, professional and perfectly logical or not.

Bearing in mind what has been done so far, Fig. 1 represents the current situation:

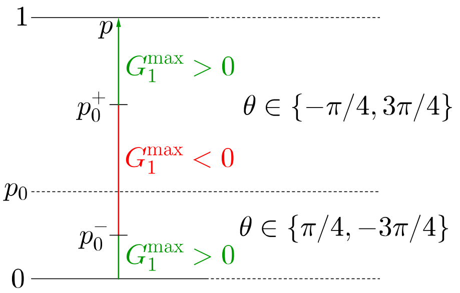

FIG. 1: Relationship between the sign of $G_1^{\max}$ and p_0, p_0^-, p_0^+.

The first way to think about the problem is to imagine playing it in front of a crowd of amateur players. Thinking of it as a bingo, in the sense that classic amateur players will choose randomly a probability p, the expected value of $G_1^{\max}$ can be calculated over all the probability $p \in [0, 1]$, and it results that

$$\langle G_1^{\max} \rangle_p = \frac{1}{2}\left[\frac{P_Q(P_C - P_Q) + (P_C + P_I)^2}{2(P_Q + P_I) + P_C}\right]. \tag{17}$$

Defining x and y as $x = \dfrac{P_C}{P_Q}$ (which satisfies $0 < x < 1$, because $P_C < P_Q$ is the only assumption made) and $y = \dfrac{P_I}{P_Q}$ ($0 \leq y$ is the only condition that can be said), the previous expected value can be rewritten as

$$\langle G_1^{\max} \rangle_p = \frac{1}{2}P_Q\left[\frac{x^2 + x - 1 + y^2 + 2yx}{2(1 + y) + x}\right]. \tag{18}$$

With these definitions, p_0 can also be rewritten as

$$p_0 = \frac{1 + y}{2(1 + y) + x}. \tag{19}$$

It is observed that for $0 < x < 1$ and defining $x_0 = \dfrac{\sqrt{5} - 1}{2} = \Phi - 1$, the following conditions are satisfied:

(i) $x^2 + x - 1 < 0 \iff 0 < x < x_0$.

(ii) $x^2 + x - 1 > 0 \iff x_0 < x < 1$.

In order to determine when $\langle G_1^{\max} \rangle_p > 0$, it is necessary to differentiate cases of $x = \dfrac{P_C}{P_Q}$ as a function of x_0.

(a) If x is supposed to satisfy $x_0 < x < 1$, then

$$\langle G_1^{\max} \rangle_p > \frac{1}{2}P_Q\frac{y^2 + 2yx}{2(1 + y) + x} \geq 0. \tag{20}$$

Thus, in this case, $\langle G_1^{\max} \rangle_p > 0$ always.

(b) If x is supposed to satisfy $0 < x < x_0$, then

$$\langle G_1^{\max} \rangle_p \geq 0 \iff y^2 + 2xy + (x^2 + x - 1) \geq 0, \tag{21}$$

that is

$$\langle G_1^{\max} \rangle_p \geq 0 \iff y \geq y_0(x) = \sqrt{1 - x} - x. \tag{22}$$

In addition, it can be shown that, in this case, $y_0(x) < 1$, as depicted in Fig. 2.

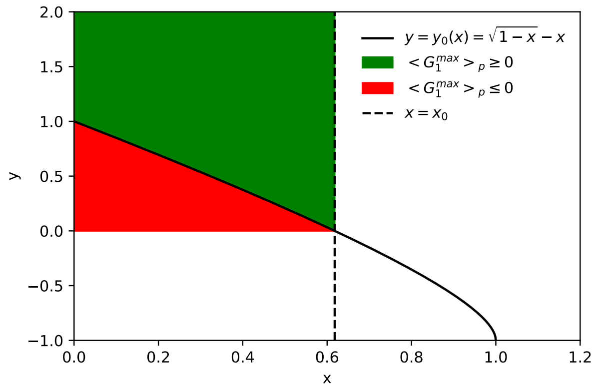

FIG. 2: Relationship between y and x with the expected value.

(c) If $x = x_0$, then $\langle G_1^{\max} \rangle_p = \dfrac{1}{2}P_Q\dfrac{y^2 + 2yx_0}{2(1 + y) + x_0} \geq 0$, and the inequality holds if and only if $y = 0$.

So, from the point of view of a casino, they will lay down that the relation between P_Q and P_C is such that $x_0 P_Q \leq P_C$, since in these cases $\langle G_1^{\max} \rangle_p > 0$ is always satisfied, and in this way their victory can be more easily secured.

Besides, it is also reasonable to think that players will be a little bit more clever and they will not play with a random $p \in [0, 1]$, but they will play with some probability such that it belongs to an interval of length δ around p_0, that is $[p_0 - \delta/2, p_0 + \delta/2]$. So thinking that the probability p follows a uniform distribution $\omega(p) = 1/\delta$, it can be shown that

$$\langle G_1^{\max} \rangle_p = \frac{[P_C P_I - P_Q(P_Q + P_I)]}{2(P_Q + P_I) + P_C} + \frac{\delta}{8}[2(P_Q + P_I) + P_C], \tag{23}$$

and it is easy to see that

$$\langle G_1^{\max} \rangle_p > 0 \iff \delta > \frac{8[P_Q(P_Q + P_I) - P_C P_I]}{[2(P_Q + P_I) + P_C]^2}. \tag{24}$$

Moreover, it is observed that, at least,

$$0 < \frac{8[P_Q(P_Q + P_I) - P_C P_I]}{[2(P_Q + P_I) + P_C]^2} < 2. \tag{25}$$

Furthermore, it is also logical to think that p follows a distribution like $\omega(p) = \dfrac{1}{(p-p_0)^2+\delta^2}$, instead of a uniform distribution. Now, in this case,

$$\langle G_1^{\max}\rangle_p = \frac{1}{\delta}\arctan\left(\frac{1}{2}\right)\left\{\frac{2[P_C P_I - P_Q(P_Q+P_I)]}{2(P_Q+P_I)+P_C}\right\} + \frac{1}{2}\ln\left(\frac{5}{4}\right)[2(P_Q+P_I)+P_C]. \tag{26}$$

Finally, similarly to what has been done before,

$$\langle G_1^{\max}\rangle_p > 0 \Leftrightarrow \delta > \frac{4\arctan\left(\frac{1}{2}\right)}{\ln\left(\frac{5}{4}\right)}\left\{\frac{P_Q(P_Q+P_I)-P_C P_I}{[2(P_Q+P_I)+P_C]^2}\right\}. \tag{27}$$

It can be noticed that this restriction over δ is very similar to the one obtained before in Eq. (24) in the uniform distribution of length δ, since $\dfrac{4\arctan\left(\frac{1}{2}\right)}{\ln\left(\frac{5}{4}\right)} \simeq 8.3112 \approx 8$.

V. CONCLUSIONS

The main goal of this paper was to study the relevance of betting in a quantum game. Knowing that a game is favourable for the quantum player, the question is if the obligation to pay more for doing a quantum movement than a classic one changes the fate of this game, making the classic player the most favourable winner.

The game explained in section II is a variation of the game between Q and Captain Picard exposed by Meyer in [1]. The main difference introduced here is that players have to pay to be able to do a quantum or classic movement, and also the fact that the second player has to choose with some probability p which option (A or B) wants to take. This distinction is what makes this plot twist and changes the favourable winner.

As seen in section III A, when all the information is known, a move which allows player 1 to always win does not exist (unlike in Meyer's game, where the first player could win with the Hadamard matrix), whereas the classic player can choose his probability as $p = p_0$ and his victory will be always assured.

In order to complicate the game for the second player, that is preventing him from using his winning strategy with $p = p_0$, some information is hidden. With this lack of information, player 2 looks for the maximum possible earnings with the details given. As seen in section III B, in spite of this lack of information, player 2 will always again win by following this strategy.

So, in both cases, the classic player, who has more restricted movements, with an optimal strategy is the favourable winner.

Finally, assuming that the quantum player role is taken by the bank, the last question is to determine if it would be a suitable game in a gambling house. If everybody was professional and knew the winning strategy, as it has been said before, the game would make no sense for the bank. However, taking into account that this game will be played in front of a crowd of players who might or might not be professionals, there is always an option for the bank to get $\langle G_1^{\max}\rangle_p > 0$.

As shown in section IV, choosing P_Q and P_C such that $x_0 P_Q \leq P_C$, the condition $\langle G_1^{\max}\rangle_p > 0$ is satisfied when all players act like monkeys with no strategy and choose randomly their probabilities $p \in [0, 1]$. Additionally, if the game is played in front of more experienced players, following their own rational strategies, the bank can choose values of P_Q, P_C and P_I such that Eq. (24) and Eq. (27) are fulfilled, depending on the level of skill of the players, or in other words, on how little δ is.

So in any case, there is always a way such that this game can be introduced in a gambling house, since there is always a strategy to obtain $\langle G_1^{\max}\rangle_p > 0$.

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