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Stable cores in information graph games

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Abstract: In an information graph situation, some agents that are connected by an undirected graph can share with no cost some information or technology that can also be obtained from a source. If an agent is not connected to an informed player, this agent pays a unitary cost to obtain this technology. A coalitional cost game can be defined from this situation, and the core of this game is known to be non- empty. We prove that the core of an information graph game is a von Neumann-Morgenstern stable set if and only if the graph is cycle- complete, or equivalently if the information graph game is concave. When the graph is not cycle-complete, whether there always exists a stable set is an open question. In this regard, we show that if the information graph consists of a ring that contains the source, then a stable set always exists and it is the core of a related information graph situation where one edge has been deleted..

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Keywords: Coalitional game, information graph, core, stable set.

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1 Introduction

In an *information graph game*, there is a finite set of agents that need to make use of information or some technology. Some of them have this information, we may think they obtain it from a source, and can share it with those agents connected to them by a relationship which can be represented by an undirected graph. Agents that are not connected with an informed agent have to pay a fixed cost of one unit to obtain this information. The cost of a coalition of agents is the minimum cost necessary so that all its agents achieve the information.

These games are known to have a non-empty core since they are a particular case of minimum cost spanning tree (*mcst*) games, whose balancedness[1] is known from Bird (1976). A mcst game is defined similarly from a finite set of nodes together with a source node, connected through a complete graph. The non-negative weights on the edges of the graph represent the cost of connecting the two nodes of the edge. When the weights only take the values 0 and 1, the mcst game is said to be *elementary*, and it is an information graph game.

Kuipers (1993) shows that each extreme core allocation of an information graph game is a marginal worth vector. Moreover, a concave information graph game can be associated with any information graph game and then the set of extreme core allocations of the latter coincides with the set of marginal worth vectors of the former.

When a (cost) game is balanced, we usually focus on the core when looking for fair cost allocations. Some single-valued solutions that lie in the core, for mcst games and hence also for information graph games, are studied in Bird (1976) and Granot and Huberman (1981).

Core allocations are undominated by any other allocation. However, we claim that, in information graph games, some out-of-core allocations should not be disregarded since they may well represent an acceptable standard

[1] A game is balanced when its core is nonempty.

of behavior. Take, for instance, an information graph situation with three agents, where two of them have the information and the third one is connected to both of them. The associated coalitional game has only one core allocation in which each agent pays zero. However, the uninformed agent needs the cooperation of at least one of the informed agents to obtain the information at zero cost. Hence any of these two agents could require the uninformed agent a side-payment $0 < \delta \leq 1$ for the information. These allocations[2] do not belong to the core and are not dominated by the unique core allocation.

This fact does not take place when the core of the game is a von Neumann-Morgenstern stable set. When the core is a stable set, it satisfies external stability, that requires that each allocation outside the core is dominated by some core allocation. The above example shows that the core of an information graph game may not be a stable set.

Some literature has studied the stability of the core of a coalitional game and the existence of stable sets. Characterization of those coalitional games with a stable core is still an open question although a stronger notion of the stable core is characterized in Jain and Vohra (2010). Stable sets have been found on several classes of games, such as assignment games (Núñez and Rafels, 2013), linear production games (Rosenmüller and Shitovitz, 2000, 2010), pillage games (MacKenzie et al., 2015), patent licensing games (Hirai and Watanabe, 2018), matching problems (Herings et al., 2017), tournaments (Brandt, 2011), voting games (Talamàs, 2018), and exchange economies (Graziano et al., 2015, 2017). Non-cooperative foundations of stable sets can be found in Anesi (2010); Diermeier and Fong (2012). More farsighted notions of stable sets have been related to the core in Einy (1996); Bhattacharya and Brosi (2011); Ray and Vohra (2015).

In this paper, we characterize those information graph games with a stable core. First, we notice that several information graph problems may lead

[2] $(\delta, 0, -\delta)$ or $(0, \delta, -\delta)$, assuming the informed players are 1 and 2, and the uninformed player is 3.

to the same cost game and we define one representative graph, the *saturated graph*, as the one that contains all those edges between two nodes that are connected to the source. We show that, given an information graph situation, there is only one saturated information graph that defines the same information cost game. We also show a bijection between saturated information graph games and elementary *mcst* games with no irrelevant links.

Then, in Theorem 3.1, we state that the core of an information graph game is a stable set if and only if its saturated graph is cycle-complete.

It is well known (Shapley, 1971) that a concave game has a stable core, although the reverse implication does not hold in general. Trudeau (2012) shows an equivalence between cycle-complete elementary mcst games with no irrelevant links and concave elementary mcst games. Hence, our result can be restated, saying that an information graph game has a stable core if and only if it is concave.

Roughly speaking, a graph is cycle-complete if each two nodes in a connected cycle are also connected. Hence, some inequalities between the cost of the edges of the graph determine the property of cycle-completeness. This fact somehow resembles the characterization of core stability in assignment games due to Solymosi and Raghavan (2001). Assignment games are another class of combinatorial optimization games introduced by Shapley and Shubik (1971) and defined by a weighted bipartite graph. The set of agents is partitioned in a finite set of buyers and a finite set of sellers, and the weight of each buyer-seller edge is the value this pair of agents can attain if they trade. Each agent can take part in only one trade, and the worth of a coalition of agents is the maximum value that can be attained by matching buyers to sellers. The assignment game has a non-empty core. Moreover, Solymosi and Raghavan (2001) prove that this core is a stable set if and only if the valuation matrix is dominant diagonal, that meaning that the value an agent attains with his/her optimally matched partner is the most he/she would attain with any other partner.

When the core of an assignment game is not a stable set, Núñez and

Rafels (2013) show how to enlarge the core with some non-core allocations to obtain a stable set. For information graph games, and consequentely also for *mcst* games, it is an open question whether this can also be done, that is to say, whether stable sets always exist. For the three-player information graph game mentioned in this introduction, we show that the sets of allocations $\{(-\delta, 0, -\delta) \mid 0 \leq \delta \leq 1\}$ and $\{(0, \delta, -\delta) \mid 0 \leq \delta \leq 1\}$ are stable sets. This fact somehow confirms our previous remark that these payoff vectors, although outside the core, could be expected to rise as a result of a negotiation process.

In the second part of the paper, we show how to find stable sets for some particular information graphs that consist of a ring that includes the source. We obtain stable sets that coincide with the core of other information graph games that are obtained either by deleting one node or by deleting one edge. Because of that, these stable sets represent standards of behavior with a clear interpretation.

We organize the paper as follows. In Section 2, we introduce notations and definitions, and we also analyze those information graphs that define the same cost game. In Section 3, we characterize the stability of the core of information graph games in terms of a graph property. In Section 4, we find some stable sets for information graph games consisting of a ring that can share the information because the ring includes the source. These stable sets coincide with the core of a related information game where one edge has been deleted. When the informed ring does not include the source, by deleting one edge we obtain a subset of imputations that is externally stable but not internally stable and hence it is not a stable set. In Section 5, we present some concluding remarks.

2 Information graph games

In an *information graph situation* there is a finite set of agents $N = \{1, \ldots, n\}$ and some of them have a particular information that we may assume have obtained from a source 0. There is also an undirected graph $G = (N \cup \{0\}, E)$, called the *information graph*, such that agents i and j can communicate and share the information at cost zero if and only if $\{i, j\} \in E$. For simplicity, we denote the undirected graph as the set of edges E. Moreover, we write ij instead of $\{i, j\}$ when referring to an edge in E.

Given an information graph situation E and $i, j \in N \cup \{0\}$, a *path* between nodes i and j is a sequence of different edges

$$\left\{i^0 i^1, i^1 i^2, \ldots, i^{K-1} i^K\right\} \subseteq E$$

such that $i^0 = i$ and $i^K = j$. When $i = j$, this path is called a *cycle*. Two nodes are connected in E if there is a path between i and j. This relation splits $N \cup \{0\}$ into connected components. An uninformed agent in a connected component of E that does not contain the source can obtain the information from the source, or from any informed player, at a fixed cost, say 1.

From an information graph situation E, we derive a coalitional cost game, the *information graph game* (N, C). Given $S \subseteq N$, we denote as $C(S)$ the minimum cost of making information available to all agents in coalition S, without the cooperation of agents outside S. Moreover, $C(\emptyset) = 0$.

An information graph situation E is *cycle-complete* if for each cycle and a pair of nodes i, j in this cycle, it holds $ij \in E$. That is, if two nodes are connected through two different paths, then they are also (directly) connected.

Moreover, different information graph situations may induce the same information graph game. Indeed, notice that if two agents i and j are informed, that is $0i \in E$ and $0j \in E$, then whether $ij \in E$ or not is irrelevant and does not affect the cost of the coalitions that contain these two agents. In this case, we say that ij is an *irrelevant edge.*

Definition 2.1 *An information graph situation E is* saturated *if whenever $0i \in E$ and $0j \in E$ for some $i, j \in N$, then $ij \in E$.*

Among all information graph situations that define the same cost game, there is only one that is saturated and we can then choose this one as a representative of the class.

Proposition 2.1 *For each information graph situation, there exists a unique saturated information graph situation that defines the same information cost game.*

Proof. Given an information graph situation E, let us define the saturated information graph situation E' given by

$$E' = E \cup \{ij \notin E \mid 0i \in E \text{ and } 0j \in E\}\,.$$

It is obvious that E' defines the same cost game (N, C') as E. To prove uniqueness, let us assume there is another saturated information graph situation E'' that defines a cost game (N, C'') such that $C(S) = C''(S)$ for all $S \subseteq N$. This implies that, for all $k \in N$, $0k \in E'$ if and only if $0k \in E''$. Moreover, since both graphs differ, we may assume there exists $ij \in E'' \setminus E'$. Since $ij \notin E'$ and E' is saturated, we can assume without loss of generality that $0i \notin E'$, which implies $0i \notin E''$. Now, if $0j \in E''$, then also $0j \in E'$ and we get $C'(\{i,j\}) = 1 \neq 0 = C''(\{i,j\})$. Similarly, if $0j \notin E''$, then also $0j \notin E'$ and $C'(\{i,j\}) = 2 \neq 1 = C''(\{i,j\})$. This contradicts $C' = C''$. ■

From the above remarks, it also follows that the correspondence between information graph situations and elementary *mcst* problems that define the same cost game is not one-to-one. But it becomes one-to-one if we restrict to saturated information graph situations and elementary *mcst* problems with no irrelevant arcs:[3] for each saturated information graph situation, there exists a unique elementary *mcst* problem with no irrelevant arcs such that their associated cost games coincide (and vice versa).

Hence, since we study the stability of the core, which is a property that relies on the coalitional cost function, we may assume without loss of generality that the information graph is saturated.

[3] In a *mcst* problem (N, c), ij is an irrelevant arc if $c_{ij} > \max\{c_{i0}, c_{0j}\}$. Hence, when the *mcst* problem is elementary, ij is an irrelevant arc if $c_{ij} = 1$ and $c_{i0} = c_{0j} = 0$. In the corresponding information graph, ij would be an irrelevant edge.

Let E be an information graph situation. An *imputation* in E is a cost allocation $x \in \mathbb{R}^N$ satisfying $x(N) = \sum_{i \in N} x_i = C(N)$ and $x_i \leq C(\{i\})$ for all $i \in N$, where x_i represents the cost allocated to agent $i \in N$ so that all the agents together cover the cost of making information avaliable to everybody with a minimum cost, and no agent alone pays more than the cost of getting the information by itself. Let $\mathcal{I}(E)$ denote the set of all imputations in E. When E is clear, we write $\mathcal{I}$ instead of $\mathcal{I}(E)$.

Given two imputations $x, y \in \mathcal{I}$, we say x *dominates* y via coalition $S \subseteq N$, and write $x \, dom_S \, y$, if $x_i < y_i$ for all $i \in S$ and $x(S) \geq C(S)$.

The *core* of an information graph situation E is the set of undominated imputations, and it is denoted as $\mathcal{C}(E)$. Namely,

$$\mathcal{C}(E) = \{x \in \mathcal{I} : x(S) \leq C(S) \text{ for all } S \subset N\} .$$

Notice that if $x \in \mathcal{C}(E)$, then $x_i \geq C(N) - C(N \setminus \{i\})$, for all $i \in N$.

A set of imputations $\mathcal{S} \subseteq \mathcal{I}$ is *internally stable* if any two imputations in $\mathcal{S}$ do not dominate one another. By its definition the core of any (cost) game is internally stable. A subset of imputations $\mathcal{S}$ is a *(von Neumann-Morgenstern) stable set* if in addition to being internally stable it is also *externally stable*, that is to say, any imputation outside $\mathcal{S}$ is dominated by some imputation in $\mathcal{S}$.

The core of an information graph game may not be a stable set, as the example in the Introduction shows.

Let E be an information graph situation and $\mathscr{P} = \{P_0, P_1, \ldots, P_K\}$ be the partition of $N \cup \{0\}$ into connected components, so that $0 \in P_0$. Notice that the case $K = 0$ is possible.

Given an information graph situation E and $i \in P_k \in \mathscr{P}$, by removing agent i, $P_k \setminus \{i\}$ is divided into one or more maximal connected components. Let $\mathscr{P}^i = \{P_0^i, P_1^i, \ldots\}$ denote these components, so that $0 \in P_0^i$ if $k = 0$. From this, it is straightforward to check that the marginal contribution of agent $i \in N$ to the grand coalition is

$$\overline{m}_i^C = C(N) - C(N \setminus \{i\}) = 1 - \left|\mathscr{P}^i\right| .$$

It is well-known that no allocation in the core assigns agent $i \in N$ less than $\overline{m}_i^C$.

3 Characterization of core stability

Our main result (Theorem 3.1) states that a saturated information graph game is core-stable if and only if its information graph is cycle complete. The reader will easily check that the three-player example in the Introduction, which has a non-stable core, is not cycle-complete. Also notice that, if we modify this example and take E with $N = \{1, 2, 3\}$ and $E = \{01, 02, 03, 13, 23\}$, the cost game is $C(S) = 0$ for all $S \subseteq N$ and then $\mathcal{C}(E) = \mathcal{I} = \{(0, 0, 0)\}$ is a stable set, although the graph is still not cycle-complete. However, the associated saturated graph is cycle-complete.

The example depicted in the information graph of Figure 1 shows, when the information graph is not cycle-complete, how to find a non-core imputation that cannot be dominated by a core imputation.

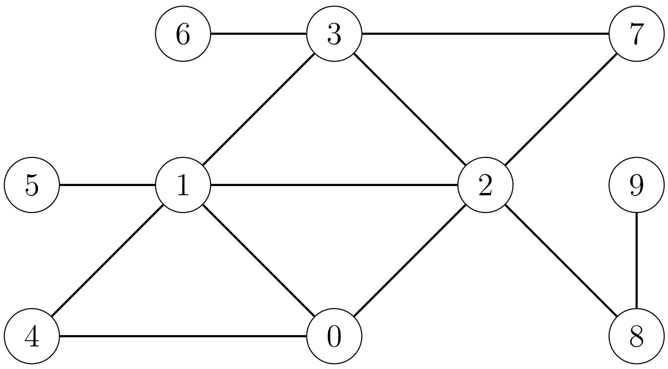


Figure 1: An information graph situation.

This information graph situation is not cycle-complete because $03 \notin E$ and, however, there exist more than one path that connect node 3 to the source.[4] Node 3 and her follower node 6 can then exploit this so that they pay zero in any core allocation. To see why, let $y \in \mathcal{C}(E)$ and assume $y_3 + y_6 > 0$.

[4]There are nine of them as for example $\{31, 14, 40\}$, $\{31, 10\}$, $\{37, 72, 20\}$, or $\{32, 20\}$.

Then, since $y(N) = 0$, we have that either $y(\{1, 4, 5\}) < 0$ or $y(\{2, 7, 8, 9\}) < 0$. Assume w.l.o.g. $y(\{2, 7, 8, 9\}) < 0$. Then, $y(\{1, 3, 4, 5, 6\}) > 0$ which is a contradiction because $C(\{1, 3, 4, 5, 6\}) = 0$ and $y \in \mathcal{C}(E)$.

We now define $x \notin \mathcal{C}(E)$ that will not be dominated by any core imputation. Let $A = \{1, 2, 6, 7\}$ be the set of nodes that have zero-cost to node 3. We define imputation x by assigning to these nodes their minimum payoffs in the core, which are $\overline{m}_1^C = -1$, $\overline{m}_2^C = -1$, $\overline{m}_6^C = 0$, and $\overline{m}_7^C = 0$. This assignment ensures that no core allocation can dominate x via a coalition that contains any of these nodes. The idea is to make node 3 pay a positive amount, since it is surrounded by nodes that cannot be tented by an allocation inside the core. First, we make the nodes that are hanging on A to pay so that no connected group pays more than 1. In particular, node 5 pays 1 so that it compensates $\overline{m}_1^C = -1$, and nodes 8 and 9 pay together 1 in order to compensate $\overline{m}_2^C = -1$. Additionally, we move out of the core by making node 3 to pay some $\delta \in (0, 1]$. Finally, we make one of the agents that are adjacent to agent 3 in one of the zero-cost paths to the source (i.e. either agent 1 or 2) to compensate this extra δ. For example, $x_1 = \overline{m}_1^C - \delta$. The rest of nodes pay zero.

We have then $x = (-1 - \delta, -1, \delta, 0, 1, 0, 0, x_8, x_9)$ where $x_8 + x_9 = 1$ and $x_8, x_9 \geq 0$. Notice that $x \notin \mathcal{C}(E)$ because coalition $T = \{2, 3, 8, 9\}$ objects it. Moreover, no core allocation can dominate x because nodes in A are paying their minimal core payoff (or less, case of agent 1) and the others either cannot connect to the source at cost zero without them (nodes 5, 8 and 9) or cannot find it profitable to reach node 3 (node 4).

We can generalize this idea to all cycle-complete saturated information graph games.

Theorem 3.1 *Let E be an information graph situation and (N, C) the related information graph game. The following statements are equivalent:*

1. *E has a stable core,*

2. *The associated saturated graph is cycle-complete, and*

3. (N, C) is concave.

Proof. It follows from Theorem 2 in Trudeau (2012) that a cycle-complete elementary *mcst* game is concave. This proves 2 $\Rightarrow$ 3. Moreover, from (Shapley, 1971) it is well known that any concave game has a stable core, which proves 3 $\Rightarrow$ 1. Hence it only remains to prove 1 $\Rightarrow$ 2. To this end, let E be an information graph situation, that we assume to be saturated, and assume E is not cycle-complete. To see that the core of E is not a stable set, we need to find an imputation $y \in \mathcal{I}$ such that no core allocation dominates y. We assume that either $\mathscr{P} = \{P_0\}$ with $P_0 = N \cup \{0\}$ or $\mathscr{P} = \{P_0, P_1\}$ with $P_0 = \{0\}$, so that all the agents are in the same connected component; otherwise, we can evaluate each connected component independently. Since E is not cycle-complete, there exist $\alpha, \beta \in N \cup \{0\}$ such that $\alpha\beta \notin E$ and a cycle $f = \{\alpha^0\alpha^1, \ldots, \alpha^{L-1}\alpha^L\}$, containing α and β, such that $\alpha^{k-1}\alpha^k \in E$ for all $k = 1, \ldots, L$. In particular, let $\alpha = \alpha^0 = \alpha^L$ and $\beta = \alpha^K$ with $1 < K < L$. We assume w.l.o.g. $\alpha \in N$. Let $A^\alpha = \{i \in N \cup \{0\} \mid i\alpha \in E\}$ be the set of nodes connected to agent α. Notice that $\alpha^1, \alpha^{L-1} \in A^\alpha$. Moreover, since the graph is saturated, we can assume w.l.o.g. $0 \notin A^\alpha$. To see why, notice that in case α and β were both agents connected to the source, then $\alpha\beta$ would be an irrelevant arc. Now, we have three cases:

1. If $\beta = 0$, then $\mathscr{P} = \{P_0\}$ and $C(N) = 0$. We define $y \in \mathbb{R}^N$ as follows:

   - $y_\alpha = \delta \in (0, 1]$,
   - $y_{\alpha^1} = \overline{m}^C_{\alpha^1} - \delta$,
   - $y_a = \overline{m}^C_a$ for all $a \in A^\alpha \setminus \{\alpha^1\}$,
   - $y_i = \frac{1}{|P|}$ for all $i \in N$ such that there exists $a \in A^\alpha$ and $i \in P \in \mathscr{P}^a \setminus \{P^a_0\}$, and
   - $y_i = 0$ otherwise.

   It is straightforward to check that y is an imputation. Moreover, y does not belong to the core, because $y(T) = \delta > 0 = C(T)$ where

$$T = \left\{\alpha^{K+1}, \ldots, \alpha^L\right\} \cup \left\{i \in P : P \in \mathscr{P}^{\alpha^{L-1}} \setminus \left\{P_0^{\alpha^{L-1}}\right\}\right\}.$$

We proceed by a contradiction argument. Assume $x \in \mathcal{C}(E)$ dominates y through coalition $S \subset N$. Hence, $x(S) = C(S)$ and $x_i < y_i$ for all $i \in S$. Since no core allocation can assign an agent i strictly less than $\overline{m}_i^C$, we deduce $A^\alpha \cap S = \emptyset$. Since S can be partitioned into one or more connected components, each of them should satisfy the required conditions, and so we assume S is a connected component. This implies $C(S) = 0$. Hence, $x(S) = 0$. Since $0\alpha \notin E$, we deduce $C(\{\alpha\}) = 1$ and hence $S \neq \{\alpha\}$. Moreover, S cannot contain α because S is a connected component and would then contain a zero-cost path between agent α and the source, which is not possible since $A^\alpha \cap S = \emptyset$. Also from $A^\alpha \cap S = \emptyset$, we deduce S cannot contain agents in $P \in P^a \setminus \{P_0^a\}$ for some $a \in A^\alpha$, because then S would not be connected. Hence, $y_i = 0$ for all $i \in S$ and $x_i < y_i = 0$ for all $i \in S$ contradicts $x(S) = 0$.

2. If $\beta \in N$ and $\mathscr{P} = \{P_0\}$, we can assume w.l.o.g. that either $0 \in \left\{\alpha^{K+1}, \ldots, \alpha^{L-1}\right\}$ or there exists a zero-cost path between the source and some agent in $\left\{\alpha^{K+1}, \ldots, \alpha^{L-1}\right\}$. We can then define y as in the previous case and prove, as before, that no core allocation dominates y.

3. If $\beta \in N$ and $\mathscr{P} = \{P_0, P_1\}$, we define y as before but with $y_\beta = 1$ instead of zero. This is still an imputation and, moreover, it does not belong to the core because $y(T \cup \{\beta\}) = 1 + \delta > 1 = C(T \cup \{\beta\})$, where T is defined as in case 1. The rest of the proof is similar to the previous cases. In particular, we can assume y is dominated by $x \in \mathcal{C}(E)$ via a coalition S that is connected and hence $x(S) = C(S) = 1$. Again, since in a core allocation no agent can be assigned a cost strictly below his/her marginal contribution, $A^\alpha \cap S = \emptyset$. From this, it is clear that no admissible S pays more than 1 under y, i.e. $y(S) \leq 1$. Hence, $x(S) \geq y(S)$ which contradicts $x_i < y_i$ for all $i \in S$.

■

As a consequence of the above theorem, the only concave information graph games are those which saturated information graph is cycle-complete. This parallels a result in Trudeau (2012) (see Theorem 2 and Lemma A.1) that shows that the only concave elementary *mcst* games with no irrelevant arcs are those with a cycle-complete graph.

Moreover, we have shown that, for information graph games, concavity is not only sufficient for the stability of the core but it is also a necessary condition.[5]

Finally, the characterization of concavity by means of cycle-completeness cannot be extended to more general *mcst* games. A concave *mcst* problem is not always cycle-complete, as next example shows:[6] Let $N = \{1, 2, 3\}$ and c be defined as $c_{01} = 3$, $c_{02} = 5$, $c_{03} = 5 + a$, $c_{12} = 2$, $c_{13} = 1$, and $c_{23} = 4$. This *mcst* problem is concave if $0 \leq a \leq 2$ and cycle-complete if $a = 0$.

It is shown in Kuipers (1993) that, with any information graph situation E, we can associate another information graph situation $\overline{E}$ that is cycle-complete and has the same core, $\mathcal{C}(E) = \mathcal{C}(\overline{E})$. To this end, we simply define the information graph situation

$$\overline{E} = E \cup \{ij \notin E \mid i \text{ and } j \text{ are nodes in a cycle of } E\}.$$

Notice that $\overline{E}$ will have a stable core, although this set may not be stable for E.

4 Stable sets for informed rings

When the core of an information graph game is not a stable set, we may ask whether we can enlarge the core to grant external stability, without losing the internal stability, and hence get a stable set. To this end, rings are the simplest structures that may fail to be cycle-complete. We first analyze the three-player example discussed in Section 1.

Example 4.1 *Let* $N = \{1, 2, 3\}$ *and* $E = \{01, 13, 23, 20\}$.

[5]It is well known that there exist non-concave cost games with a stable core.

[6]This example was suggested by C. Trudeau in a personal communication.

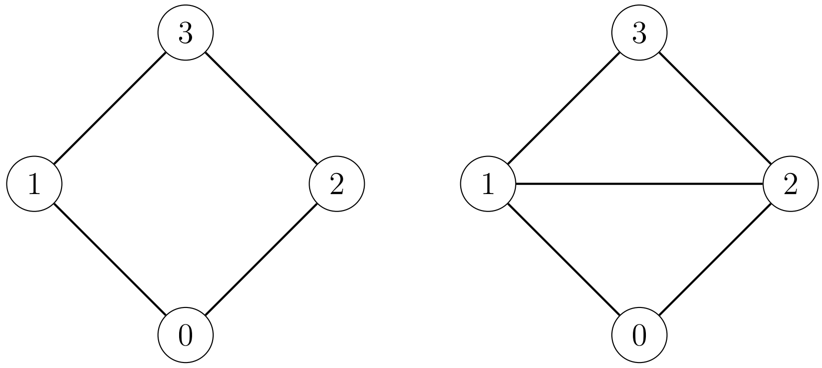


Figure 2: The information graph of Example 4.1 and its saturated form.

The only core allocation is $y = (0, 0, 0)$. Following the proof of Theorem 3.1, we know that, for any $\delta \in (0, 1]$, both $(-\delta, 0, \delta)$ and $(0, -\delta, \delta)$ are not core allocations but y does not dominate any of them via any coalition. In fact, it is not difficult to see that two stable sets for this problem are defined as follows: $\mathcal{A} = \{(-\delta, 0, \delta) : \delta \in [0, 1]\}$ and $\mathcal{B} = \{(0, -\delta, \delta) : \delta \in [0, 1]\}$. Notice these sets represent one standard of behavior in which agent 3 pays some positive amount either to agent 1 or agent 2, in reward for sharing the information.

Notice also that the above stable sets correspond with the core of a related situation where one edge has been deleted from the information graph. The stable set $\mathcal{A}$ is the core of the information graph situation where edge 23 has been deleted and the stable set $\mathcal{B}$ is the core of the information graph situation where edge 13 has been deleted.

4.1 Source as a node inside the ring

In this subsection, we generalize the situation of Example 4.1. We assume that the information graph is given by a ring topology that includes all the agents and the source, that is, there is a unique cycle that contains all the nodes. Without loss of generality, we can consider

$$E = \{01, 12, 23, \ldots, (n-1)n, n0, 1n\}.$$

Notice that the imputation set of the related information graph game is

$$\mathcal{I}(E) = \left\{x \in \mathbb{R}^N \mid x(N) = 0,\ x_1 \leq 0, x_n \leq 0, x_i \leq 1 \text{ for all } i \in N\right\}.$$

It is straightforward to check that the core of the corresponding information graph game (N, C) reduces to only one element, $\mathcal{C}(E) = \{(0, 0, \ldots, 0)\}$. This is because the marginal contribution of each agent is zero and hence $x_i \geq 0$ for all $i \in N$, while $x(N) = C(N) = 0$.

Clearly, if there are more than two agents in the cycle, the core of this information graph situation is not a stable set, since the graph is not cycle-complete. Notice that, as in Example 4.1, the unique core allocation does not reward agents 1 and n for providing the information.

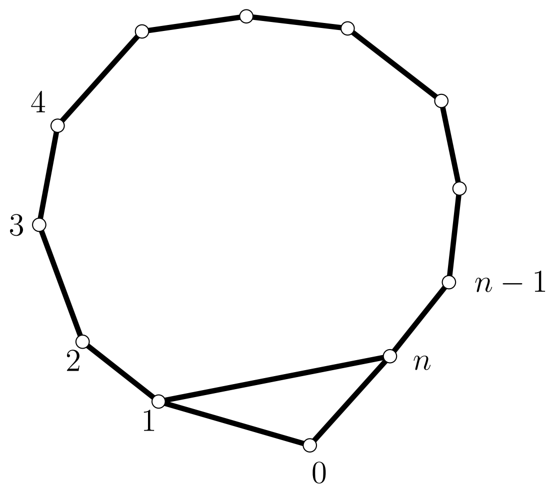


Figure 3: A saturated informed ring containing the source.

Recall that the core of an information graph game that is a ring topology is determined by the core constraints of those coalitions S that are intervals, that is, either

$$S = [i, j] = \{k \in N \mid i \leq k \leq j\}$$

if $i \leq j$, or

$$S = [i, j] = \{k \in N \mid k \leq j \text{ or } i \leq k\}$$

if $i > j$. Hence, given this ring topology (N, E), the core is

$$\mathcal{C}(E) = \left\{ x \in \mathbb{R}^N \mid x(N) = 0, x([i,j]) \leq C([i,j]) \text{ for all } i, j \in N \right\}.$$

The next proposition describes the core of the information graph game when one edge containing a node that is adjacent to the source is deleted.

Proposition 4.1 *In a ring topology (N, E) of informed agents given by $E = \{01, 12, \ldots, (n-1)n, n0, 1n\}$, $\mathcal{C}(E \setminus \{12\}) =$*

$$\{(0, \alpha_3, \alpha_4 - \alpha_3, \ldots, \alpha_n - \alpha_{n-1}, -\alpha_n) \mid \alpha_3, \ldots, \alpha_n \in [0,1]\}$$

and $\mathcal{C}(E \setminus \{(n-1)n\}) =$

$$\{(-\alpha_1, \alpha_1 - \alpha_2, \ldots, \alpha_{n-3} - \alpha_{n-2}, \alpha_{n-2}, 0) \mid \alpha_1, \ldots, \alpha_{n-2} \in [0,1]\}.$$

Proof. Let us focus on $\mathcal{C}(E \setminus \{(n-1)n\})$ since the proof for $\mathcal{C}(E \setminus \{12\})$ is analogous. Notice that an element x in $\mathcal{C}(E \setminus \{n-1n\})$ is defined by $x_n = 0$, since $\overline{m}_n = 0 \leq x_n \leq C(\{n\}) = 0$, and $x(N \setminus \{n\}) = 0$ together with the constraints

$$x([1,s]) \leq 0, \quad \text{for all} \quad 1 \leq s \leq n-1 \tag{1}$$

$$x([r,s]) \leq 1, \quad \text{for all} \quad 1 < r \leq s \leq n-1. \tag{2}$$

Notice that, for $s = 1$ in (1), we have $x_1 \leq 0$. Moreover, from $r = 2$ and $s = n-1$ in (2), we have $x_1 \geq -1$, so we may write $x_1 = -\alpha_1$ for some $\alpha_1 \in [0,1]$. Moreover, from $x_1 + x_2 \leq 0$, there exists $\alpha_2 \geq 0$ such that $x_2 + \alpha_2 = -x_1 = \alpha_1$. Also, since $x([3, n-1]) \leq 1$, we have $x_1 + x_2 \geq -1$ and hence $\alpha_2 = -x_1 - x_2 \leq 1$. Recursively, assume that for some $x \in \mathcal{C}(E \setminus \{n-1n\})$, there exist $\alpha_1 \ldots, \alpha_k \in [0,1]$, for some $2 < k < n-2$ such that $x_1 = -\alpha_1$ and $x_i = \alpha_{i-1} - \alpha_i$ for all $2 \leq i \leq k$. From the core constraint $x\left([1, k+1]\right) \leq 0$ we know there exists $\alpha_{k+1} \geq 0$ such that $x_{k+1} = -\alpha_{k+1} - x\left([1,k]\right) = \alpha_k - \alpha_{k+1}$. Moreover, since $x\left([k+2, n-1]\right) \leq 1$, we have that $\alpha_{k+1} \leq 1$. Finally, if there exist $\alpha_1, \ldots, \alpha_{n-2} \in [0,1]$ such that $x_1 = -\alpha_1$ and $x_i = \alpha_{i-1} - \alpha_i$ for all $1 < i < n-1$, the efficiency requires $x_{n-1} = \alpha_{n-2}$. ■

The main result in this section states that the two sets described in the above proposition are stable sets of the ring topology of informed agents. Hence, we find two stable sets. In one of them, all agents in the ring, except for agent 1, get the information from agent n. Each agent pays an amount to her successor in the ring to get the information, and receives a payment from her predecessor in the ring in exchange for the information. Conversely, in the second stable set, agent 1 spreads the information and hence each agent in the ring, except for agent n, pays an amount to the predecessor and receives a payment from the successor.

Theorem 4.1 *In a ring topology (N, E) of informed agents, the sets*

$$\mathcal{S}_i = \mathcal{C}(E \setminus \{ij\})$$

where $i, j \in N$ are such that $0i, ij \in E$, are stable sets.

Proof. Assume $E = \{01, 12, \ldots, (n-1)n, n0\}$. We prove the stability of the set $\mathcal{S}_n$. The proof for $\mathcal{S}_1$ is analogous. To prove internal stability, notice first the cost games related to the two information graphs (N, E) and $(N, E \setminus \{(n-1)n\})$ have the same imputation set. Moreover, the cost of an interval coalition in both games coincides, except if this coalition contains both agents $n-1$ and n and does not contain agent 1, since this edge has cost 0 in E but cost 1 in $E \setminus \{(n-1)n\}$. From Theorem 3, $\mathcal{C}(E \setminus \{(n-1)n\})$ is internally stable since it is cycle-complete. Take now $x, y \in \mathcal{S}_n$ and assume $x\, dom_S y$ for some $S \subset N$. Coalition S cannot contain agent n since $x_n = y_n = 0$. But $x\, dom_S y$ via a coalition that does not contain $\{n-1, n\}$ contradicts the internal stability of $\mathcal{C}(E \setminus \{(n-1)n\})$.

To prove external stability of $\mathcal{S}_n$, we must show that for all $y \in \mathcal{I}(E) \setminus \mathcal{S}_n$ there exists $x \in \mathcal{S}_n$ that dominates y via some coalition $S \subset N$. But recall that $\mathcal{S}_n = \mathcal{C}(E \setminus \{(n-1)n\})$ and $\mathcal{I}(E) = \mathcal{I}(E \setminus \{(n-1)n\})$. Hence, since $E \setminus \{(n-1)n\}$ is cycle-complete, Theorem 3.1 guarantees that $\mathcal{C}(E \setminus \{(n-1)n\})$ is externally stable in $\mathcal{I}(E \setminus \{(n-1)n\}) = \mathcal{I}(E)$. This means that any $y \in \mathcal{I}(E) \setminus \mathcal{C}(E \setminus \{(n-1)n\})$ is dominated by some $x \in \mathcal{C}(E \setminus \{(n-1)n\})$

via some coalition S which we may assume with no loss of generality that is connected in $E \setminus \{n-1n\}$, since otherwise x would dominate y via some connected coalition of S. Then, S is also connected in E and hence it has the same cost in both information graphs, which implies that also $x \, dom_S y$ in E. ■

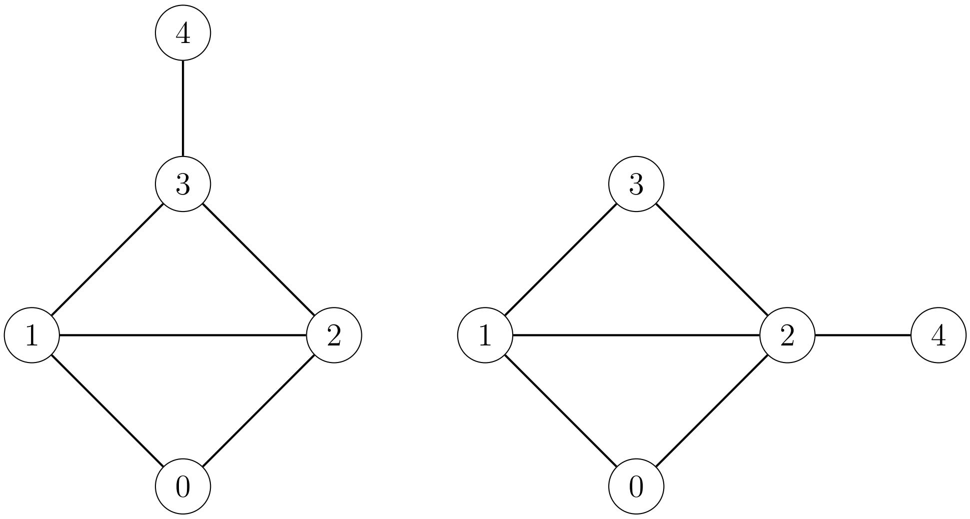
The stable sets obtained in the previous theorem can also been understood as the core of a subgame when one of the two agents connected to the source leaves the game paying zero and the remaining agents allocate the null total cost according to a core allocation of the subgame. More precisely, $x \in \mathcal{C}\left(E \setminus \{(n-1)n\}\right)$ is equivalent to assume that agent n leaves the network at a null cost ($x_n = 0$) and the remaining agents share the null connection cost according to a core element of the information subgraph $(N \setminus \{n\}, E^{-n})$, where $E^{-n} = \{(i,j) \in E \mid i \neq n, j \neq n\}$.

Viewed in this way, these stable sets resemble those obtained in Núñez and Rafels (2013) for the assignment games, which consist of the union of the core of the game with the core of some particular subagmes.

As opposed, if we delete a different edge from the information graph, that is, an edge that does not involve any agent adjacent to the source, then we do not obtain a stable set. That is to say, the set $\mathcal{C}(E \setminus \{ij\})$ where $i \neq 1$ and $j \neq n$, is not stable, since it is not internally stable. Notice first that both information graph games, E and $E \setminus \{ij\}$, have the same imputation set. Moreover, two elements in $\mathcal{C}(E \setminus \{ij\})$ cannot dominate one another through a coalition S not containing agents i and j, since core elements are undominated. But one such element can dominate another via a coalition S with $\{i,j\} \subseteq S$. This is the same difficulty we will find later on when we analyze informed rings that do not contain the source.

Before moving to the situation where the source does not belong to the ring, one may ask what happens when the information graph consists of something more than a ring. We cannot say anything for the general case of several connected rings, but the next example illustrates that if there is a single ring with some edges getting out of some nodes of the ring, similar stable sets can be obtained.

Example 4.2 *Consider the following two information graph situations with four players.*



For the first information graph game of Example 4.2, the core is

$$\mathcal{C}(E) = \{(0, 0, -\alpha, \alpha) \mid 0 \leq \alpha \leq 1\}$$

and two stable sets are

$$\mathcal{S}_1 = \{(0, -\delta, \delta - \alpha, \alpha) \mid \alpha, \delta \in [0, 1]\}$$

and

$$\mathcal{S}_2 = \{(-\delta, 0, \delta - \alpha, \alpha) \mid \alpha, \delta \in [0, 1]\}.$$

Each of these stable sets represents a standard of behavior in which agent 3 transfers some payoff either to agent 1 or 2 in order to have access to the information. At the same time, agent 3 receives a transfer from agent 4 in exchange for the information.

In the second game of this example, the core is

$$\mathcal{C}(E) = \{(0, -\alpha, 0, \alpha) \mid \alpha \in [0, 1]\}$$

and a stable set is

$$\mathcal{S}_3 = \{(0, -\delta - \alpha, \delta, \alpha) \mid \alpha, \delta \in [0, 1]\}.$$

This stable set represents the standard of behavior in which both agents 3 and 4 transfer some payoff to agent 2 to have access to the information. Instead, the set $\mathcal{S}_4 = \{(-\delta, -\alpha, \delta, \alpha) \mid \alpha, \delta \in [0, 1]\}$, that represents the standard of behavior in which agent 3 makes a transfer to agent 1 to have access to the information, does not lead to a stable set because it violates internal stability. For example, $(-\gamma, -\gamma, \gamma, \gamma)$ dominates any $(-\delta, -\alpha, \delta, \alpha)$ with $\delta > \gamma > \alpha$ via coalition $\{2, 3\}$.

The stable sets we obtain in Example 4.2 above also correspond to the cores of the information graph games that arise when removing an edge in the cycle. The stable set $\mathcal{S}_1$ for the left information graph corresponds with the core of the game that arises when removing edge 13 and the stable set $\mathcal{S}_2$ corresponds with the core of the game that arises when removing edge 23. The stable set $\mathcal{S}_3$ for the information graph on the right corresponds with the core of the game that arises when removing edge 13. However, set S_4 that corresponds with the core of the game that arises when removing edge 23 is not a stable set, since it is not internally stable.

An important difference of the information graph E on the left with respect to the one E' on the right (and also with the informed rings studied in this section) is that agent 2 does not have a fixed core payoff. Notice that $-1 = \overline{m}_2^{C'} < C'(\{2\}) = 0$, while $\overline{m}_2^{C} = C(\{2\}) = 0$ and also $\overline{m}_1^{C} = C(\{1\}) = 0$ and $\overline{m}_1^{C'} = C(\{1\}) = 0$.

4.2 Source as a node connected to the ring

We now focus on those ring networks E where the source does not belong to the ring but one of the agents, say agent 1, is connected to the source. That is, $E = \{01, 12, 23, 34 \ldots, (n-1)n, n1\}$.

In this case, the core of the corresponding information graph game contains more than one point. Next proposition precisely describes this core.

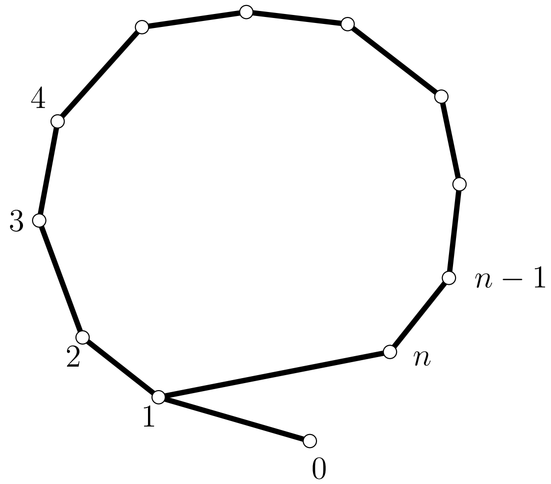


Figure 4: An informed ring not containing the source.

Proposition 4.2 *In a ring topology* (N, E) *of informed agents given by* $E = \{01, 12, 23, \ldots, (n-1)n, n1\}$*, the core of the corresponding cost game is* $\mathcal{C}(E) =$

$$\left\{(-\alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_{n-1}) \in \mathbb{R}^N \mid 1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq 0\right\}.$$

Proof. If $x \in \mathcal{C}(E)$, then $-1 \leq x_1 \leq 0$ and $0 \leq x_i \leq 1$ for all $2 \leq i \leq n$. This is because $C(\{1\}) = 0$, $\overline{m}_1 = -1$ and $C(\{i\}) = 1$ and $\overline{m}_i = 0$ for all $2 \leq i \leq n$. The remaining core constraints are $x([i,j]) \leq 0$ if $1 \in [i,j]$ and $x([i,j]) \leq 1$ otherwise. Then, from $x_1 \leq 0$ we know there exists $\alpha_1 \geq 0$ such that $x_1 + \alpha_1 = 0$ and hence $x_1 = -\alpha_1$. From $x_1 + x_2 \geq 0$ we deduce there exists $\alpha_2 \geq 0$ such that $x_1 + x_2 + \alpha_2 = 0$ and hence $x_2 = \alpha_1 - \alpha_2$. By repetedly applying this argument we get that $x_i = \alpha_{i-1} - \alpha_i$ for all $2 \leq i \leq n-1$ and by efficiency $x_n = \alpha_{n-1}$, with $\alpha_i \geq 0$ for all $1 \leq i \leq n-1$. From $-1 \leq x_1$ we obtain $\alpha_1 \leq 1$ and from $0 \leq x_i$ we get $\alpha_{i-1} \geq \alpha_i$ for all $2 \leq i \leq n-1$. It is now straightforward to check that for $x = (-\alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_{n-1})$ with $1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1}$ all core constraints are satisfied. ■

Notice that in a core allocation of an informed ring topology with the source outside the ring, all agents but the one connected to the source pay a non-negative amount to obtain the information. Agent $i > 1$ pays α_{i-1} to her predecessor in the path to the source, and receives α_i from the agent that follows. No payment can exceed the unitary cost of the information and the net payment for each agent is non-negative. Agent 1, that is connected to the source, receives a non-negative payment.

When $n > 3$, the core described in the above proposition is not a stable set, since the graph is not cycle-complete.

Given a ring topology as defined above, we may consider the information graph situation obtained by deleting one edge, take for instance $E \setminus \{n1\}$. It is straightforward to see that the core of this subgraph situation is $\mathcal{C}(E \setminus \{n1\}) =$

$$\left\{(-\alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_{n-1}) \in \mathbb{R}^N \mid \alpha_i \in [0,1] \text{ for all } i \in N\right\} \tag{3}$$

which means that, inside this set, agent 1 always receives a non-negative payment while agent n always pays a non-negative amount. Each intermediate agent (agents from 2 to $n-1$) receives some amount from the agent that follows in the graph and pays something to the one that precedes her in the path to the source. The balance for each intermediate agent may be positive or negative. Clearly the imputation set of both information graph situations, E and $E \setminus \{n1\}$ is the same, and $\mathcal{C}(E) \subseteq \mathcal{C}(E \setminus \{n1\})$.

Since $E \setminus \{n1\}$ is cycle-complete, $\mathcal{C}(E \setminus \{n1\})$ is a stable set for $E \setminus \{n1\}$ but not necessarily for E because the cost functions differ, the coalition $S = \{1, n\}$ has cost zero in (N, E) but cost one in $(N, E \setminus \{n1\})$. This situation is illustrated in the next example.

Example 4.3 *Let $E = \{01, 12, 23, 34, 45, 51\}$ be an informed ring not containing the source. Take the imputations $y = (-0.3, 0.1, -0.1, -0.2, 0.5)$ and $x = (-0.4, 0.2, -0.1, -0.1, 0.4)$. Notice that both imputations belong to $\mathcal{C}(E \setminus \{51\})$, since the first one is defined by taking $\alpha_1 = 0.3, \alpha_2 = 0.2, \alpha_3 = 0.3$ and $\alpha_4 = 0.5$ in (3), while the second corresponds to $\alpha_1 = 0.4, \alpha_2 = 0.2, \alpha_3 = 0.3$ and $\alpha_4 = 0.4$. Moreover, $x\,\mathrm{dom}\,y$ via coalition $S = \{1, 5\}$, which implies $\mathcal{C}(E \setminus \{51\})$ is not internally stable.*

One may think of restricting the set $\mathcal{C}(E \setminus \{51\})$ by imposing $\alpha_1 \geq \alpha_4$ to avoid internal domination via coalition $\{1, 5\}$, but the subset that results is still not internally stable. To see that, take the imputations

$$y' = (-0.7, -0.3, 0.7, -0.1, 0.4) \text{ and } x' = (-0.75, -0.1, 0.65, -0.15, 0.35).$$

Notice that $y', x' \in \mathcal{C}(E \setminus \{51\})$, since y' is defined by $\alpha_1 = 0.7, \alpha_2 = 1, \alpha_3 = 0.3, \alpha_4 = 0.4$ and x' corresponds to $\alpha_1 = 0.75, \alpha_2 = 0.85, \alpha_3 = 0.2, \alpha_4 = 0.35$. Moreover $x\, dom y$ via coalition $S' = \{1, 3, 4, 5\}$.

Nevertheless, the set $\mathcal{C}(E \setminus \{n1\})$ satisfies a weaker stability property. It is externally stable. The same result is obtained if we delete any other edge in the ring.

Proposition 4.3 *In a ring topology (N, E) of informed agents, the sets*

$$\mathcal{S}_{ij} = \mathcal{C}(E \setminus \{ij\})$$

where $i, j \in N$ and $ij \in E$, are externally stable.

Proof. Assume $E = \{01, 12, \ldots, (n-1)n, n1\}$ and fix $i, j \in N$ with $ij \in E$. Notice that $\mathcal{I}(E) = \mathcal{I}(E \setminus \{ij\})$. Hence, since $E \setminus \{ij\}$ is cycle-complete, Theorem 3.1 guarantees that $\mathcal{C}(E \setminus \{ij\})$ is externally stable in $\mathcal{I}(E \setminus \{ij\}) = \mathcal{I}(E)$. This means that any $y \in \mathcal{I}(E) \setminus \mathcal{C}(E \setminus \{ij\})$ is dominated by some $x \in \mathcal{C}(E \setminus \{ij\})$ via some coalition S which we may assume with no loss of generality that is connected in $E \setminus \{ij\}$. Then, S is also connected in E and hence it has the same cost in both information graphs, which implies that also $x\, dom_S y$ in E. ■

External stability of $\mathcal{S}_{ij} = \mathcal{C}(E \setminus \{ij\})$ means that whenever the negotiation on how to allocate the cost of sharing the information in an informed ring not containing the source leads to some proposal outside this set, there will be a coalition of agents that will object and propose an allocation in $\mathcal{S}_{ij}$. However, this allocation may not be final, since it can be dominated by another allocation, even by an allocation in $\mathcal{S}_{ij}$, as internal stability is not satisfied.

5 Concluding remarks

This paper shows a characterization of the stability of the core of information graph games and also, when the graph is has a ring structure that contains the source of information, provides a stable set for this game that coincides with the core of a related information graph where one edge has been deleted.

This fact resembles the situation of assignment games where some stable sets are obtained as the union of the cores of some subgames (Núñez and Rafels, 2013) and also of patent licensing games in which some stable sets coincide with the core of some suitable defined reduced game (Hirai and Watanabe, 2018).

Of course, many examples can be provided of stable sets (for instance of some three-player games) which are not a convex set and hence will not coincide with the core of any coalitional game. When a stable set corresponds with the core of another coalitional game, being it a subgame or a reduced game, it is more clear the rationale that is behind its standard of behavior.

It remains open whether stable sets always exist for information graph games and, if this is the case, whether there is always a stable set consisting of the core of some related information graph game after removing some nodes or edges in incomplete cycles.

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