# Time consistent pension funding in a defined benefit pension plan with non-constant discounting

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**Abstract**

We consider the time consistent management of a defined benefit stochastic pension plan where the participants have different rates of time preference and the fund manager collects this heterogeneity when discounting the future. The main objective is to select the amortization rate and the investment strategy minimizing both the contribution rate risk and the solvency risk. The problem is formulated as a stochastic control problem with non-constant rate of discount and is solved analytically by means of the dynamic programming approach and the technical interest rate is selected in order to keep stable the fund evolution within prescribed targets. A numerical illustration shows a comparative of the stability of the fund assets and the rate of contribution for a convex combination of exponential functions as discount function and for the constant discount case.

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# 1 Introduction

The increase in life expectancy and the periods of financial crisis have stressed the importance of alternative pension plans in order to complement public protection at retirement. As a consequence, the study of the management of pension plans is an important subject in the economic field, and also in the financial field because the fund managers use the financial markets to invest the fund assets of the pension plan.

The analysis of the optimal management of pension plans from the dynamic optimization approach has appeared in the literature in discrete time, see Berkelaar and Kouwenberg (2003), Chang *et al.* (2003), Cox *et al.* (2013), Haberman and Sung (1994, 2005) and Haberman and Vigna (2002), and in continuous time, see Battocchio *et al* (2007), Cairns (2000), Deelstra *et al* (2003), Haberman *et al.* (2000), Menoncin and Vigna (2017) and Josa-Fombellida and Rincón-Zapatero (2001, 2004, 2008ab, 2010, 2018).

We focus our attention in plans of defined benefit type where the benefits are fixed in advance by the manager and contributions are designed to amortize the fund according to a previously chosen actuarial scheme. The pension plan manager can built an investment portfolio of the fund. The main aim of the plan manager is the minimization of both the solvency risk and the contribution rate risk. This objective is generally accepted since the papers Haberman and Sung (1994) and Josa-Fombellida and Rincón-Zapatero (2001). These risk concepts are defined as quadratic deviations of fund wealth and amortization rates with respect to liabilities and normal cost, respectively. The solvency risk is related with the security of the pension fund in attaining the comprised liabilities and the contribution risk with its stability, see Haberman *et al.* (2000). Note that we are considering as measure of the solvency risk the expected quadratic deviation of the fund with respect to the actuarial liability. In the literature, see for instance Devolder *et al.* (2012), we can also find other alternative approaches that consider the solvency risk as a probability (so the probability that the fund falls below a threshold is minimized), or as a risk measure like the Value at Risk or the expected shortfall.

In this paper we depart from Josa-Fombellida and Rincón-Zapatero (2004), where the authors study the optimal management of a defined benefit pension plan with stochastic benefits correlated with the financial market with the objective of minimizing the contribution and the solvency risks along an infinite horizon. We extend this setting by supposing that the discount function is not necessarily of the exponential type with the aim of capturing the diversity in the temporal preferences of the participants in the plan. Collective temporal decisions for agents with different rates of time preferences lead to time inconsistent aggregate temporal preferences (Jackson and Yariv (2015)). As a consequence, standard optimization techniques (Pontryagin Maximum Principle or Hamilton-Jacobi-Bellman (HJB) equation) fail in characterizing time consistent optimal policies. As preferences change with time, as long as the decision maker goes through the time horizon, they differ depending on the instant $t$ at which solutions are obtained, so a $t'$-agent, in general, will not find optimal the solutions computed by the $t$-agent, for any $t$ and $t'$ in the time horizon. Note that in the literature of models with general time preferences, by "non-constant" or "variable" discount rates we refer to the case where temporal preferences depend explicitly on the current position of the decision maker and not only con the calendar time and, as it will be seen below, this dependence will imply that preferences become time-inconsistent.

Karp (2007) introduced the analysis of dynamic optimization problems in continuous time setting with non-constant rate of discount, deriving in infinite time horizon a modified HJB equation. Later Marín-Solano and Navas (2009) extended the approach to the finite horizon case and study the application to a non-renewable resource problem with non-constant discount. The methodology for stochastic control problems with non-constant discount in a finite time horizon is developed in Marín-Solano and Navas (2010). The classical optimal consumption and portfolio problem by Merton (1971) with non-constant discount is detailed studied for logarithmic, power and exponential utility functions in this last paper.

Stochastic control problems with a non-constant rate of discount show an increasing and recent interest in the literature of economics, finance and insurance, and consequently in pension funding. In this sense, Liang *et al.* (2014) considers time-consistent strategies in a mean-variance optimization problem but in a defined benefit pension scheme starting from the model in Josa-Fombellida and Rincón-Zapatero (2008b) for strategies pre-commitment. Li *et al.* (2016) derives the time-consistent investment strategy under the mean-variance criterion for a defined contribution pension plan with stochastic salary and where the risky asset is a CEV process. Zhao *et al* (2016a) analyses a defined benefit pension plan model with non-constant discount where the aim is the minimization of the solvency and the contribution rate risks but in a finite horizon, and the manager invests the fund in a portfolio with one risky asset and one risk-free asset. The model follows Josa-Fombellida and Rincón-Zapatero (2001), where the benefit is constant. Zhao *et al.* (2016b) considers a consumption-investment problem for a member of a defined contribution pension plan with non-constant time preferences, with power and logarithmic utility functions and with the exponential discounting, the mixture of exponential discounting and the hyperbolic discounting.

The main findings of the paper are that the rate of discount intervenes in the time consistent strategies and in their associated fund evolution. Moreover, it is possible to select the technical rate of interest in order that the time consistent contribution does not depend on the parameters of the benefit process and it has the form of a spread method of funding, providing the stability of the plan at the long-term. We find that the speed of convergence of the fund to the actuarial liability is inversely related with the degree of impatience in the collective of participants in the plan.

The paper is organized as follows. Section 2 defines the elements of the pension scheme and describes the financial market where the fund operates. We consider the fund is invested in a portfolio with several risky assets and one riskless asset. The management of the defined benefit plan is formulated as a stochastic optimal control problem with non-constant discount where the objective is to minimize on a infinite horizon the contribution rate risk and the solvency risk. In Section 3 the time consistent strategies of contribution and investment, and the corresponding fund dynamics are obtained with dynamic programming techniques. Some properties of the time consistent solutions are found. A particular case where the technical rate of interest is selected to lead to a spread method of funding is analyzed. Section 4 serves as a numerical illustration of previous results based on real data. Finally, Section 5 establishes some conclusions. All proofs are developed in Appendix A.

## 2 The pension model

Consider a pension plan of aggregated type where, at every instant of time, active participants coexist with retired participants. The plan is of defined benefit type, that is to say, the benefits paid to the participants at the age of retirement are fixed in advance by the manager. The benefit is modeled by a stochastic process correlated with the financial market.

The main elements intervening in the pension plan are the following. We denote $F(t)$ the value of the fund assets at time $t$ and $C(t)$ the contribution rate made by the manager at time $t$ to the funding process in order to accrue the benefit at the moment of retirement. The risk-free market interest rate is the constant $r$. The technical rate of valuation $\delta$ is the constant used for the valuation of the liabilities. This valuation is made using the distribution function $M$ on $[a, d]$, that is, $100M(x)\%$ is the percentage of the value of the future benefits accumulated until age $x \in [a, d]$, where $a$ is the common age of entrance in the fund and $d$ is the common age of retirement.

$P(t)$ denotes the benefits promised to the participants at time $t$. Though it is related with the salary at the moment of retirement, we will not consider the salary process into the model, as in Josa-Fombellida and Rincón-Zapatero (2008a). $NC(t)$ is the normal cost at time $t$ for all participants. $NC$ is the ideal value of the contribution rate $C$. The ideal value of the fund $F$ at time $t$ is the actuarial liability at time $t$, denoted by $AL(t)$, that is, the total liabilities of the manager. The unfunded actuarial liability at time $t$ (equal to $AL(t) - F(t)$) is denoted by $UAL(t)$, and the supplementary cost at time $t$ (the difference $C(t) - NC(t)$) by $SC(t)$. We are considering that these actuarial functions are stochastic processes.

In this section we describe the financial market where the fund is invested, we built the actuarial functions assuming correlation with the financial market and we establish the optimization problem.

We consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on $\Omega$ and $\mathscr{F} =$

$\{\mathscr{F}_t\}_{t\geq 0}$ is a complete and right continuous filtration generated by the $(n+1)$-dimensional standard Brownian motion $(W_0, W_1, \ldots, W_n)$, that is to say, $\mathscr{F}_t = \sigma\{W_0(s), W_1(s), \ldots, W_n(s);\, 0 \leq s \leq t\}$.

### 2.1 The financial market

The pension scheme manager invests the fund in the financial market and for that she/he chooses a portfolio formed by $n$ risky assets $S^1, \ldots, S^n$, which are geometric Brownian motions correlated with the benefit process, and a riskless asset $S^0$, as proposed Merton (1971). The assets evolution is given by the equations:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{1}$$

$$dS^i(t) = S^i(t)\Big(b_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)\Big), \quad S^i(0) = s_i, \quad i = 1, \ldots, n. \tag{2}$$

We have denoted by $r > 0$ the short risk-free rate of interest, $b_i > 0$ the mean rate of return of the risky asset $S^i$, and $\sigma_{ij} > 0$ the uncertainty parameters. We assume that $b_i > r$, for each $i = 1, ..., n$, so the manager has incentives to invest with risk. The matrix $(\sigma_{ij})$ is denoted by $\sigma$ and the Sharpe ratio or market price of risk for this portfolio, $\sigma^{-1}(b - r\bar{1})$, by $\theta$, where $b = (b_1, \ldots, b_n)^\top$ and $\bar{1}$ is a (column) vector of 1's. We suppose that the symmetric matrix $\Sigma = \sigma\sigma^\top$ is positive definite.

### 2.2 The actuarial functions

The stochastic actuarial liability and the stochastic normal cost are defined as in Josa-Fombellida and Rincón-Zapatero (2004):

$$AL\,(t) = \int_a^d e^{-\delta(d-x)}\, M(x)\mathbb{E}\left(P(t+d-x)\,|\,\mathscr{F}_t\right)\, dx,$$

$$NC\,(t) = \int_a^d e^{-\delta(d-x)}\, M'(x)\mathbb{E}\left(P(t+d-x)\,|\,\mathscr{F}_t\right)\, dx.$$

for every $t \geq 0$, where $\mathbb{E}(\cdot|\mathscr{F}_t)$ denotes conditional expectation with respect to the filtration $\mathscr{F}_t$. The actuarial liability $AL$ is the total expected value of the promised benefits accumulated according to the distribution function $M$, and the normal cost $NC$ is the total expected value of the promised benefits accumulated according to the density function $M'$, both discounted at the constant rate $\delta$.

In order to get analytical tractability, we assume that the benefit $P$ is given by a geometric Brownian motion, as in Josa-Fombellida and Rincón-Zapatero (2004). Thus the expected benefit grows exponentially which is coherent because the benefit depends on the salary and the population pension plan size.

**Assumption 1** *The benefit $P$ satisfies the stochastic differential equation (SDE therefore)*

$$dP(t) = \mu P(t)\,dt + \eta P(t)\,\left(\sqrt{1-q^\top q}\,dW_0(t) + q^\top dW(t)\right), \quad t \geq 0,$$

*where $W = (W_1 \ldots, W_n)^\top$, $q = (q_1, \ldots, q_n)^\top$, with $q_i \in [-1, 1]$, and where $\mu \in \mathbb{R}$ is the instantaneous growth rate of the benefit and $\eta \in \mathbb{R}_+$ is the instantaneous volatility of the benefit. The initial condition $P(0) = P_0$ is a constant that represents the initial liabilities.*

We are supposing that there exists a correlation $q_i$ between the standard Brownian motion $B = \sqrt{1-q^\top q}\,W_0 + q^\top W$ and $W_i$, for $i = 1, \ldots, n$. Thus the financial market influences the evolution of liability $P$. We remark that the market is incomplete, because the stochastic benefit $P$ cannot be traded in the financial market and therefore the manager cannot hedge the inherent risk in the benefit. This particular form of an incomplete market is one of the few that can be managed because there only exists one risk source that cannot be spanned and all the variance and covariance parameters of this risk are assumed to be constant. Classical references of incomplete markets in this context are Duffie *et al.* (1997), Koo (1998) and Stoikov and Zariphopoulou (2005), which consider a consumption and portfolio choice in continuous time with stochastic income (or a stochastic factor) that cannot be replicated by trading the available securities. Note, however, that we have a complete market when benefit and risky assets are perfectly correlated, $q^\top q = \sum_{i=1}^n q_i^2 = 1$, and the corresponding model can be analytically solved also.

Under Assumption 1 the actuarial functions satisfy $AL\,(t) = \psi_{AL}P(t)$ and $NC\,(t) = \psi_{NC}P(t)$, where $\psi_{AL} = \int_a^d e^{(\mu-\delta)(d-x)}M(x)\,dx$ and $\psi_{NC} = \int_a^d e^{(\mu-\delta)(d-x)}M'(x)\,dx$, and they are linked by the identity

$$(\delta - \mu)AL(t) + NC(t) - P(t) = 0, \tag{3}$$

for every $t \geq 0$. See Proposition 2.1 in Josa-Fombellida and Rincón-Zapatero (2004). As a consequence, $AL$ and $NC$ are geometric Brownian motions also.

By Assumption 1 and using $AL = \psi_{AL} P$, the actuarial liability $AL$ satisfies the SDE

$$dAL(t) = \mu AL(t)\, dt + \eta AL(t) \sqrt{1 - q^\top q}\, dW_0(t) + \eta AL(t)\, q^\top dW(t), \tag{4}$$

with the initial condition $AL(0) = AL_0 = \psi_{AL} P_0$. Thus the benefit $P$ and actuarial liability $AL$ depend on the financial market. By the same argument, the normal cost satisfies the same SDE (4) but with initial condition $NC(0) = NC_0 = \psi_{NC} P_0$.

### 2.3 The fund wealth

The manager builds a portfolio based on the financial market and designs an amortization scheme varying with time. The amount of fund invested in time $t$ in the risky asset $S^i$ is denoted by $\pi_i(t)$, $i = 1, \ldots, n$. The remainder, $F(t) - \sum_{i=1}^n \pi_i(t)$, is invested in the bond. Borrowing and shortselling are allowed. A negative value of $\pi_i$ means that the manager sells a part of his risky asset $S^i$ short while, if $\pi_i$ is larger than $F$, he or she then gets into debt to purchase the corresponding stock, borrowing at the riskless interest rate $r$. $\pi$ denotes $(\pi_1, \ldots, \pi_n)^\top$. We suppose the contribution rate $\{C(t) : t \geq 0\}$ and the investment strategy $\{\pi(t) : t \geq 0\}$ are control processes adapted to filtration $\{\mathscr{F}_t\}_{t \geq 0}$, $\mathscr{F}_t$-measurables, markovian and stationary, satisfying

$$\mathbb{E}_{F_0, AL_0} \int_0^T SC^2(t) dt < \infty, \tag{5}$$

and

$$\mathbb{E}_{F_0, AL_0} \int_0^T \pi^\top(t) \pi(t) dt < \infty, \tag{6}$$

for every $T < \infty$. In the above, $\mathbb{E}_{F_0, AL_0}$ denotes conditional expectation with respect to the initial conditions $(F_0, AL_0)$.

The dynamic fund evolution under the investment policy $\pi$ is:

$$dF(t) = \sum_{i=1}^{n} \pi_i(t) \frac{dS^i(t)}{S^i(t)} + \left( F(t) - \sum_{i=1}^{n} \pi_i(t) \right) \frac{dS^0(t)}{S^0(t)} + (C(t) - P(t))\, dt. \tag{7}$$

By substituting (1) and (2) in (7), we obtain the SDE that determines the fund evolution,

$$dF(t) = \Big( rF(t) + \pi^\top(t)(b - r\bar{1}) + C(t) - P(t) \Big) dt + \pi^\top(t)\sigma\, dW(t), \tag{8}$$

with initial condition $F(0) = F_0 > 0$.

### 2.4 The optimization problem

The manager, at every time $\tau$, $\tau \in [0, \infty)$, wishes to minimize a convex combination of the contribution rate risk and the solvency risk in a infinite horizon, but discounted at a non-constant rate of discount. Following Josa-Fombellida and Rincón-Zapatero (2004), at every instant of time $t$, $t > \tau$, we define the solvency risk as the quadratic deviation of the fund assets $F(t)$ with respect to its ideal value $AL\,(t)$, instead of its expected value $\mathbb{E}_{F_\tau} F(t)$, that is to say $\mathbb{E}_{F_\tau, AL_\tau}\, UAL(t)^2$. Analogously, we define the contribution rate risk as $\mathbb{E}_{F_\tau, AL_\tau} SC(t)^2$, and then the functional objective as

$$J((F_\tau, AL_\tau); (SC, \pi)) = \mathbb{E}_{F_\tau, AL_\tau} \int_\tau^\infty e^{-\int_\tau^s \tilde{\rho}(v-\tau)dv} \left( \beta SC^2(s) + (1-\beta)(AL\,(s) - F(s))^2 \right) ds. \tag{9}$$

Thus, from the perspective of the fund manager at $\tau = 0$, the optimization problem that we consider is to minimize $J((F_0, AL_0); (SC, \pi))$ subject to (3), (4) and (8), over the class of admissible controls $\mathcal{A}_{F_0, AL_0}$.

Note that we choose $SC = C - NC$ as the control variable instead of $C$, leading to an equivalent control problem. Here, $\mathcal{A}_{F_0, AL_0}$ is the set of Markovian processes $(SC, \pi)$, adapted to the filter $\{\mathscr{F}_t\}_{t \geq 0}$ where $C$ satisfies (5), $\pi$ satisfies (6), and where $F$ and $AL$ satisfy (8) and (4), respectively. The parameter $\beta$, $0 < \beta \leq 1$, is a weighting factor reflecting the relative importance for the manager of the two different types of risks.

In (9), we consider an instantaneous positive non-constant impatience rate $\tilde{\rho}(\cdot)$ for the manager. We also assume it is a decreasing and bounded function in $[0, \infty)$. A decreasing rate of impatience implies that the decision maker is more concerned with the present than with the distant future, in the sense of being more impatient in the short-run decisions compared with similar decisions in the long-run.

In the non-constant discounting literature the main feature is that temporal preferences depend on the temporal position of the decision maker. A fund manager taking decisions at $\tau$, the $\tau$-manager, will discount future payments at an instant $s$, with $\tau < s$, by using $D(s,\tau) = D(s-\tau) = e^{-\int_\tau^s \tilde{\rho}(v-\tau)dv} = e^{-\int_0^{s-\tau} \tilde{\rho}(v)dv}$. When the decision maker moves to a different instant of time $\tau' \neq \tau$ over the time horizon, the discount function becomes $D(s,\tau') = D(s-\tau') = e^{-\int_{\tau'}^s \tilde{\rho}(v-\tau')dv}$. As a consequence of this dependence on the temporal position of the fund manager, $\tau$ or $\tau'$, it arises the time inconsistency of the temporal preferences. A clarifying discussion on the problem of time consistency can be found in Caputo (2005), Chapter 12. To better see this, note that if $d\tilde{\rho}(t)/dt \neq 0$, we have that

$$e^{-\int_\tau^s \tilde{\rho}(v-\tau)dv} = e^{-\int_\tau^{\tau'} \tilde{\rho}(v-\tau)dv} e^{-\int_{\tau'}^s \tilde{\rho}(v-\tau)dv} \neq e^{-\int_\tau^{\tau'} \tilde{\rho}(v-\tau)dv} e^{-\int_{\tau'}^s \tilde{\rho}(v-\tau')dv},$$

for $\tau < \tau' < s$, which is a key point in the derivation of the standard HJB equation, and the reason that standard optimization techniques fail in characterizing time consistent policies in our setting. With non-constant discounting, and also with other general temporal preferences as, for instance, heterogeneous discounting (e.g., Marín-Solano and Patxot (2012)) and stochastic hyperbolic discounting (e.g., Zou *et al.* (2014)) is required a modified HJB equation in order to characterize time consistent policies. The non-constant discounting case has been studied in different works in the literature, as Karp (2007) in the deterministic case, and Marín-Solano and Navas (2010) in the stochastic case, between others.

Note also that $D(t = s-\tau)$ satisfies $\tilde{\rho}(t) = -\dot{D}(t)/D(t)$, and denote $\rho = \lim_{t\to\infty} \tilde{\rho}(t)$ with the assumption that $\tilde{\rho}(t) \geq \rho$, for all $t \geq 0$.

We can find several functional specifications for general discounts functions with non-constant instantaneous rates of time preference, being one of them a convex linear combination of exponential functions. An economic motivation for this particular discount function is the following: consider the case that there exist $N$ different participants in the plan that exhibit constant but different rates of time preference, and consider that in this collective there exist $m$ ($m \leq N$) different profiles or subgroups in terms of their time preference, which can be motivated by several factors (age, personal wealth, genre, life expectancy, etc.). If each profile has a rate of time preference of $\rho_i$, $i = 1, ..., m$, and with $\rho_1 < \ldots < \rho_m$, we can define:

$$D(t) = \sum_{i=1}^{m} \lambda_i e^{-\rho_i t}$$

where $\lambda_i$, $0 < \lambda_i < 1$, $i = 1, ..., m$, $\sum_{i=1}^{m} \lambda_i = 1$, represents the weight of the corresponding subgroup in the whole collective. Then, the discount function $D(t)$ has a decreasing instantaneous rate of time preference

$$\tilde{\rho}(t) = -\dot{D}(t)/D(t) = \frac{\lambda_1 \rho_1 e^{-\rho_1 t} + \ldots + \lambda_m \rho_m e^{-\rho_m t}}{\lambda_1 e^{-\rho_1 t} + \ldots + \lambda_m e^{-\rho_m t}},$$

that satisfies $d\tilde{\rho}(t)/dt < 0$ and $\lim_{t \to \infty} \tilde{\rho}(t) = \rho_1$.

The dynamic programming approach will be used to solve the problem. The value function is defined as

$$\widehat{V}(F, AL) = \min_{(SC,\pi) \in \mathcal{A}_{F,AL}} \left\{ J((F, AL); (SC, \pi)) \ : \ \text{s.t. } (4), (8) \right\}.$$

Since the problem is autonomous and the horizon unbounded, we may suppose that $\widehat{V}$ is time independent. The value function so defined is non-negative and strictly convex. The connection between value functions and time consistent feedback controls in stochastic control theory with non-constant discount is accomplished by a modified HJB. In order to obtain a sophisticated solution, Marín-Solano and Navas (2010) analyse the finite horizon case, that is easily translated to the unbounded case as follows. The modified HJB equation is

$$\begin{aligned} -\rho V - K + \min_{SC,\pi} \Big\{ & \beta SC^2 + (1-\beta)(F - AL)^2 + \big(rF + \pi^\top (b - r\bar{1}) + SC + NC - P\big) V_F \\ & + \mu AL\, V_{AL} + \frac{1}{2}\pi^\top \Sigma^{-1} \pi V_{FF} + \frac{1}{2}\eta^2 AL^2 V_{AL,AL} + \eta \pi^\top \sigma q AL\, V_{F,AL} \Big\} = 0, \quad (10) \end{aligned}$$

where

$$K(F, AL) = \int_0^\infty e^{-\int_0^s \tilde{\rho}(v) dv} (\tilde{\rho}(s) - \rho) \mathbb{E}_{F,AL} \left\{ \beta \widehat{SC}^2(s) + (1-\beta)(AL(s) - \widehat{F}(s))^2 \right\} ds, \quad (11)$$

where $\mathbb{E}_{F,AL}$ denotes the conditional expectation to $F(0) = F$ and $AL(0) = AL$, and where $\widehat{SC}$ and $\widehat{F}$ denote $SC$ and $F$ with $\widehat{C}$ and $\widehat{\pi}$, which are the arguments minimizing in (10).

**Remark 2.1** *In the definition of the $K(F, AL)$ function, control variables $SC$ and $\pi$ are non-local terms when solving (10) leading to functional equations in the solution procedure. In some cases it is possible to simplify the process working with the total differential of $K$ with respect to time (see for instance Remark 2 in Karp (2007)). For instance, if the discount function is a linear combination of two exponential discount functions, $D(t) = \lambda e^{-\rho_1 t} + (1-\lambda)e^{-\rho_2 t}$,with $\lambda \in [0,1]$ and $0 < \rho_1 < \rho_2$, by differentiating (11) we obtain*

$$\rho_2 K = (\rho_2-\rho_1)(1-\lambda)\left(\beta \widehat{SC}^2(s) + (1-\beta)(AL(s) - \widehat{F}(s))^2\right) + \left(r\widehat{F} + \widehat{\pi}^\top(b - r\bar{1}) + \widehat{SC} + NC - P\right)K_F + \mu AL\, K_{AL} + \frac{1}{2}\widehat{\pi}^\top \Sigma^{-1}\widehat{\pi} K_{FF} + \frac{1}{2}\eta^2 AL^2 K_{AL,AL} + \eta\widehat{\pi}^\top \sigma q AL\, K_{F,AL}, \quad (12)$$

*so, the time consistent solution can be characterized as the solution of the system* (10), (12). *This property will be used in the numerical illustration in Section 4.*

# 3 The time consistent strategies

In this section we show how the fund manager may select the rate of contribution and the proportion of fund assets put into the risky assets. We analyze some properties of these time consistent strategies and study the associated fund evolution. We have the following result.

**Theorem 3.1** *Suppose that Assumption 1 holds and the inequalities*

$$2\mu + \eta^2 < \rho, \quad (13)$$

$$2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta < \rho, \quad (14)$$

*are satisfied. The time consistent rate of contribution and investments in the risky assets are given by*

$$C^* = NC - \frac{\alpha_{FF}}{\beta}F - \frac{\alpha_{F,AL}}{2\beta}AL\,, \quad (15)$$

$$\pi^* = -\Sigma^{-1}(b - r\bar{1})F - \frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\Sigma^{-1}(b - r\bar{1}) + \eta\sigma^{-\top}q\right)AL\,, \quad (16)$$

*respectively, where $\alpha_{FF}$ is a positive solution of the equation*

$$-\frac{\alpha_{FF}^2}{\beta} + \left(-\rho + 2r - \theta^\top\theta\right)\alpha_{FF} + (1-\beta) - \kappa_{FF} = 0, \quad (17)$$

*where*

$$\kappa_{FF}=\left(\frac{\alpha_{FF}^2}{\beta}+1-\beta\right)\int_0^\infty e^{-\int_0^s\tilde{\rho}(v)dv}(\tilde{\rho}(s)-\rho)e^{\left(2r-2\frac{\alpha_{FF}}{\beta}-\theta^\top\theta\right)s}ds, \tag{18}$$

*and* $\alpha_{F,AL}$ *is the unique solution of the equation*

$$-\frac{\alpha_{FF}}{\beta}\alpha_{F,AL}+\left(-\rho+r-\theta^\top\theta-\eta q^\top\theta+\mu\right)\alpha_{F,AL}+2(\mu-\delta)\alpha_{FF}-2(1-\beta)-\kappa_{FAL}=0, \tag{19}$$

*where*

$$\begin{aligned}\kappa_{FAL}=&\frac{\left(\frac{\alpha_{FF}^2}{\beta}+1-\beta\right)\left(\frac{\alpha_{F,AL}}{\beta}+2(\delta-\mu)\right)}{-r+\frac{\alpha_{FF}}{\beta}+\mu-\eta q^\top\theta}\int_0^\infty e^{-\int_0^s\tilde{\rho}(v)dv}(\tilde{\rho}(s)-\rho)e^{\left(2r-2\frac{\alpha_{FF}}{\beta}-\theta^\top\theta\right)s}ds\\&+\left(\frac{\alpha_{FF}}{\beta}\alpha_{F,AL}-2(1-\beta)-\frac{\left(\frac{\alpha_{FF}^2}{\beta}+1-\beta\right)\left(\frac{\alpha_{F,AL}}{\beta}+2(\delta-\mu)\right)}{-r+\frac{\alpha_{FF}}{\beta}+\mu-\eta q^\top\theta}\right)\\&\cdot\int_0^\infty e^{-\int_0^s\tilde{\rho}(v)dv}(\tilde{\rho}(s)-\rho)e^{\left(r-\theta^\top\theta-\frac{\alpha_{FF}}{\beta}+\mu-\eta q^\top\theta\right)s}ds.\end{aligned} \tag{20}$$

*The time consistent fund is the solution of the system given by* (4) *and*

$$\begin{aligned}dF^*(t)=&\left(\left(r-\theta^\top\theta-\frac{\alpha_{FF}}{\beta}\right)F^*(t)-\left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\theta^\top\theta+\eta q^\top\theta+\frac{\alpha_{FF}}{\beta}\right)+\delta-\mu\right)AL(t)\right)dt\\&-\left(\theta^\top F^*(t)+\frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\theta^\top+\eta q^\top\right)AL(t)\right)dW(t),\end{aligned} \tag{21}$$

*with* $F(0)=F_0,\ AL(0)=AL_0$.

The time consistent strategies $C^*$ and $\pi^*$ are linear functions of the fund assets $F$ and the actuarial liability $AL$, and depend on the parameters of the financial market and the benefit process, and also, through $\alpha_{F,AL}$, depend on the rate of discount $\tilde{\rho}$, the technical rate of interest $\delta$ and the benefit drift parameter $\mu$.

The investment decisions $\pi^*$, (16), are composed by two terms. The first is proportional to $F$, with coefficient proportional to the market price of risk $\theta$, but the second is proportional to $AL$ and depends on the rate of discount, and the parameters containing the correlation between benefit and risky assets. The constant of proportionality in the first term, $\Sigma^{-1}(b-r\bar{1})$, is the so called optimal-growth portfolio strategy, that appears in the Merton model where a CRRA utility of consumption is maximized.

An interesting consequence is that there exists a linear relationship between the supplementary cost and the investment strategy,

$$\pi^* = \Sigma^{-1}(b - r\bar{1})\frac{\beta}{\alpha_{FF}}SC^* - \eta\sigma^{-\top}q\frac{\alpha_{F,AL}}{2\alpha_{FF}}AL\,,$$

thus for each unit of additional amortization with respect to the normal cost the manager must invest $\Sigma^{-1}(b - r\bar{1})\frac{\beta}{\alpha_{FF}}$ units in the risky assets, plus an additional quantity of $\eta\sigma^{-\top}q\frac{-\alpha_{F,AL}}{2\alpha_{FF}}AL$ units. Both quantities depend on the rate of discount through $\alpha_{FF}$ and $\alpha_{FAL}$. Note that if there is no correlation between the benefit and the risky asset, $q = 0$, or the benefit is deterministic, $\eta = 0$, then the investment strategy $\pi^*$ is proportional to the supplementary cost $SC^*$.

**Remark 3.1** The manager must borrow money at rate $r$ to invest in the risky asset $S^i$, that is to say $\pi_i^* > F^*$, when the level of the fund is below $\lambda_i AL$, where the constant $\lambda_i$ is defined as

$$\lambda_i = \frac{\bar{e}_i\Sigma^{-1}(b - r\bar{1}) + \eta\bar{e}_i\sigma^{-\top}\bar{q}}{1 + \bar{e}_i\Sigma^{-1}(b - r\bar{1})}\frac{\alpha_{F,AL}}{2\alpha_{FF}}, \qquad \bar{e}_i = (0, \ldots, \overset{i)}{1}, 0, \ldots, 0),$$

for all $i = 1, 2, \ldots, n$, and he/she need to short sell asset, that is to say $\pi_i^* < 0$, when the fund is above the value $\lambda_i' AL$, where

$$\lambda_i' = \frac{\bar{e}_i\Sigma^{-1}(b - r\bar{1}) + \eta\bar{e}_i\sigma^{-\top}\bar{q}}{\bar{e}_i\Sigma^{-1}(b - r\bar{1})}\frac{\alpha_{F,AL}}{2\alpha_{FF}}.$$

Thus the manager does not need short-selling neither borrowing, $0 \leq \pi_i^* \leq F^*$, when the fund $F^*$ is between $\lambda_i AL$ and $\lambda_i' AL$.

**Remark 3.2** The model can be observed in some particular cases. For instance, the model analyzed in Josa-Fombellida and Rincón-Zapatero (2004) can be recovered by setting $\tilde{\rho}(t) = \rho$, $\forall t \in [0, \infty)$, where in this case $\kappa_{FF} = \kappa_{FAL} = 0$. Other non-constant discount functions, as the hyperbolic, where $D(t) = (1 + k_1 t)^{-k_2/k_1}$, with $k_2 > k_1 > 0$, are included in the general framework also, but they will not be studied in this paper. It is straightforward to particularize the model to the case of a deterministic benefit, where $\eta = 0$, or to a constant benefit, where $\mu = \eta = 0$. The extreme cases where benefit and financial market are uncorrelated or are perfectly correlated are obtained by taking $q = 0$ or $q = \pm 1$, respectively. The case where the fund is only invested in the riskless asset can also be analyzed with a similar methodology, although note that this is not a particular case, since now the unique control variable is $SC$.

In order to maintain in the long term the fund assets near the actuarial liability and the rate of contribution near the normal cost, we give a valuation of the technical rate of actualization $\delta$, consisting in a spread method of fund amortization, as in Josa-Fombellida and Rincón-Zapatero (2004). These spread methods, widely used in pension funding (see Owadally and Haberman (1999)), assume that the supplementary cost $SC$ is proportional to the unfunded actuarial liability $UAL$.

From (15), in order to achieve a spread method the identity $\alpha_{F,AL} = -2\alpha_{FF}$ must be satisfied. Substituting in (19) we obtain

$$\begin{aligned}
&-\frac{\alpha_{FF}^2}{\beta} + \left(-\rho + r - \theta^\top\theta - \eta q^\top\theta + \delta\right)\alpha_{FF} + 1 - \beta - \frac{\frac{\alpha_{FF}}{\beta} + \ \mu - \delta}{\frac{\alpha_{FF}}{\beta} + \mu - r - \eta q^\top\theta}\kappa_{FF} \\
&+ \frac{\left(\frac{\alpha_{FF}^2}{\beta} + 1 - \beta\right)\left(r + \eta q^\top\theta - \delta\right)}{\frac{\alpha_{FF}}{\beta} + \mu - r - \eta q^\top\theta}\int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv}(\tilde{\rho}(s) - \rho)e^{\left(r - \frac{\alpha_{FF}}{\beta} - \theta^\top\theta + \mu - \eta q^\top\theta\right)s}ds = 0, \quad (22)
\end{aligned}$$

and comparing (22) with (17), we obtain that the technical interest rate must coincide with the rate of return of the bond modified to get rid of the sources of uncertainty. Specifically we assume that the valuation of liabilities $\delta$ is the risk free interest rate $r$ plus the product of the Sharpe ratio of the portfolio $\theta$ and a term depending on the parameters containing the correlations $q$ and the diffusion parameter of the benefit process $\eta$, as in Josa-Fombellida and Rincón-Zapatero (2004).

**Assumption 2** *The technical rate of actualization is $\delta = r + \eta q^\top\theta$.*

Note that $\delta$ does not depend on the parameter $\mu$ associated to the benefit $P$. If there is no correlation between the benefit and the financial market, or if the benefit is deterministic, then $\delta$ is the risk-free rate of interest $r$. We can also observe that the existence of a non-constant discount rate does not influence in the selection of the technical rate of interest.

Besides this valuation it provides, this selection of $\delta$ will allow us to simplify the explicit solution of the problem in the following result.

**Corollary 3.1** *Suppose that Assumptions 1 and 2 hold, and the inequalities* (13), (14), *are satisfied. The time consistent rate of contribution and time consistent investments in the risky assets are given by*

$$C^* = NC + \frac{\alpha_{FF}}{\beta}\, UAL, \tag{23}$$

$$\pi^* = \Sigma^{-1}(b - r\bar{1})\, UAL + \eta\sigma^{-\top} q AL\,, \tag{24}$$

*respectively, where $\alpha_{FF}$ is a positive solution of the equation* (17)*. The associated fund evolution is the solution of the system given by* (4) *and*

$$\begin{aligned} dF^*(t) = {} & \Big(\Big(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta}\Big)F^*(t) - \Big(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} - \mu\Big)AL\,(t)\Big)\, dt \\ & + \Big( -\theta^\top F^*(t) + \big(\theta^\top + \eta q^\top\big)AL\,(t)\Big)\, dW(t), \end{aligned} \tag{25}$$

*with* $F(0) = F_0$, $AL\,(0) = AL_0$.

The supplementary cost $SC^*$ is proportional to the unfunded actuarial liability $UAL$, with constant of proportionality depending on the rate of discount. The investment decisions $\pi^*$, (24), are similar to those in Josa-Fombellida and Rincón-Zapatero (2004), since they do not depend on the rate of discount. They are composed by two terms. The first is again proportional to $UAL$, but the second is a correction term, depending on the risk parameters of the model and $AL$. This second term is zero when there is no uncertainty in the benefits, as in Josa-Fombellida and Rincón-Zapatero (2001), and when there is no correlation between benefit and risky asset.

We also obtain that the rate of contribution $C^*$ and the investment $\pi^*$ do not depend on $\mu$. However, from (25), all parameters of the benefit process influence linearly in the optimal fund evolution. Also we observe that the manager takes a greater risk when the wealth of the fund is far below the actuarial liability than when it is closer. The optimal rate of contribution $C^*$ and the fund evolution $F^*$ depend on the rate of discount function $\tilde{\rho}$ through $\alpha_{FF}$.

Next proposition shows that this selection of $\delta$ allows, in expected values, to fulfill in the long term one of the objectives of the pension plan manager, which is the maintenance of the fund $F^*$ and the contribution $C^*$ close to their ideal values $AL$ and $NC$, respectively.

**Proposition 3.1** *Suppose that Assumptions 1, 2 and the inequalities* (13), (14) *and*

$$\alpha_{FF} > \beta\big(r - \theta^\top \theta\big), \tag{26}$$

*are satisfied. Then the expected unfunded actuarial liability and the expected supplementary cost converge in the long term to zero, that is to say,*

$$\lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} UAL^*(t) = \lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} SC^*(t) = 0.$$

*When the pension plan is underfunded, the total expected supplementary cost is*

$$\overline{SC} = \int_0^\infty \mathbb{E}_{F_0,AL_0} SC^*(t)dt = \frac{\alpha_{FF}/\beta}{\alpha_{FF}/\beta + \theta^\top\theta - r} UAL_0.$$

The fulfilment of (26) is necessary to guarantee a finite total expected supplementary cost. However, by (23), the total expected contribution rate is infinite because the normal cost $NC$ is a geometric Brownian motion with positive drift $\mu$. Notice that though the unfunded actuarial liability $UAL^*$ is not a geometric Brownian motion, because the term $\eta\sqrt{1 - q^\top q} AL\, dW_0$ appears in its SDE (see proof of Proposition 3.1), it can be possible to obtain the accumulated expected contribution rate along the interval $[0, T]$, $\int_0^T \mathbb{E}_{F_0,AL_0} C^*(t)$, using (23). When the benefit is deterministic, $\eta = 0$, by (24) and (26), the expected investment strategies converge in the long term to zero, i.e.,

$$\lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} \pi^*(t) = 0.$$

Additionally, since $UAL^*$ is a geometric Brownian motion, consequently $(UAL^*)^2$ too. In this case the pension plan intensifies its stability and security qualities in the long term, that is to say, the solvency and contribution rate risks converge in the long term to zero, i.e.,

$$\lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} (UAL^*(t))^2 = \lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} (SC^*(t))^2 = 0,$$

whenever the additional condition $\alpha_{FF} > \beta\big(r - \theta^\top\theta/2\big)$ holds. These convergence properties of the actuarial risks of the pension plan are also achieved when the benefit and the financial market are perfectly correlated, $q^\top q = 1$, since then $UAL^*$ is a geometric Brownian motion. The convergence to zero of the expected investment strategies holds also when the benefit and the financial market are uncorrelated, $q = 0$, because the investment strategy $\pi^*$ is proportional to the supplementary cost $SC^*$ or also by (24).

## 4 A numerical illustration

Next, we numerically illustrate the dynamic behaviour of the pension fund and the time consistent strategies (contribution rate and investment polices) by conducting some simulations for a specific example. After the analysis of the baseline case we will include a sensibility analysis with respect to several parameters: the risk preference parameter, the financial market regime, the correlation between the benefit and the risky asset, and the diffusion parameter of the benefit. When fixing the parameter values, we have taken data from several sources. All the data refers to the U.S. Economy and we take as reference point the end of 2018. The financial and macroeconomic data are obtained from Yahoo! Finance, the U.S. Department of Treasury and the Bureau of Labor Statistics, and the actuarial data are obtained from the report *Corporate Pension Funding Study*, by Wadia *et al.* (2019).

We consider that benefits have mean return $\mu = 0.018$ and standard deviation $\eta = 0.05$. For the benefits mean return we have taken as a proxy (a fraction of the) the U.S. Employment Cost Index that rose 2.8% in 2018, which includes retirement and saving benefits between other benefits as for instance they are other supplemental cash payments, insurance benefits as health or temporal disability, etc. By taking a 65% of this increase, we cover the inflation rate for the previous year that rose 1.8%, considering that it could exist some indexation over past inflation rates. Although other functions may be considered, we assume that the distribution function $M$ is uniform in $[a, d]$, that is to say, $M(u) = \frac{u-a}{d-a}$, for $a \leq u \leq d$.

Without loss of generality, we consider that there is one risky asset with mean rate of return $b = 0.115$ and volatility deviation $\sigma = 0.167$ (this implies a Sharpe ratio of $\theta = 0.52994$), which has a correlation coefficient with benefits of $q = 0.5$, while the risk-free rate of interest is equal to $r = 0.0265$. Parameters $b$ and $\sigma$ has been estimated from daily variations of the S&P 500 index from 2009 to 2018, while $r$ is obtained from the U.S. ten-year Treasury note at the end of 2018. From Wadia *et al.* (2019), it is reported that at year 2018 the funded ratio equals 87.1%, while the discount rate was a 4.01%. Inspired from these data, initial values for the actuarial liability and the fund are set, respectively, to $AL_0 = 100$ and $F_0 = 87.1$, so we consider that at $t = 0$ the actuarial liability is not totally covered by the fund. For the objective function we initially take as risk preference parameter $\beta = 0.5$, i.e., the manager gives equal value to the contribution rate risk and the solvency risk to be minimized, and we will perform a sensibility analysis on $\beta$.

Finally, we select the technical rate of interest leading a spread method of funding ($\delta = 0.0397485$), and as a discount function for the manager in the objective function a linear combination between two exponential functions, i.e., $D(t) = \lambda e^{-0.04t} + (1-\lambda)e^{-0.3t}$, which will allow us to obtain parameters of the value function as described in Remark 2.1. Note that for this discount function, our instantaneous rate of time preference decreases with the discount period from $\tilde{\rho}(0) = 0.17$, for $\lambda = 0.5$, to $\rho = 0.04$, so short-term values are discounted at a higher rate than long-term values. In contrast, with a standard discount function, all values would be discounted at a given rate, independently of the discount distance.

These parameters satisfy the transversality conditions (13), (14), and the convergence condition (26), assuring the stability of the pension plan, that we are going to check below.

We consider several cases for the weight parameter $\lambda$. In the baseline case, $\lambda = 0.5$, we consider two kinds of participants: patient participants with $\rho_1 = 0.04$ and impatient participants with $\rho_2 = 0.3$, with equal weight. The case with $\lambda = 0.9$ corresponds to situations where patient participants are majority while in the case with $\lambda = 0.1$ the impatient agents are majority. Tables 1 and 2 below also consider the extreme cases with $\lambda = 0$ and $\lambda = 1$, that correspond to the constant discounts $D(t) = e^{-0.3t}$ and $D(t) = e^{-0.04t}$, represent the less and more patience cases, respectively. They are not included in the figures in order to simplify them. A look at the tables allows to verify that there is continuity between the results corresponding to the cases of constant discount and those of the exponential combination. Then the comparison with the constant cases is direct.

Figure 1 shows, for $\lambda = 0.5$, two paths of the fund process $F$ and the actuarial liability process $AL$, over a time horizon of 20 years (=240 months) and with a step of 1/12, together with the expected fund assets $\mathbb{E}F(= \mathbb{E}_{F_0,AL_0}F)$ and the expected actuarial liability $\mathbb{E}AL\,(= \mathbb{E}_{F_0,AL_0}AL\,)$ for a total of 1000 realizations. We observe that the fund get moving near the the actuarial liability, and its expected value approaches to the expected actuarial liability in the long term. It is possible to verify that this behaviour holds with other values of the weight $\lambda$. Additionally, Figure 2 includes two overlapped histograms collecting all the realizations at time $t = 60$ (the end of 5th year) for the fund assets and the actuarial liability. We can observe $F(60)$ and $AL(60)$ showing a similar distribution pattern, and the same results are obtained at other time moments.

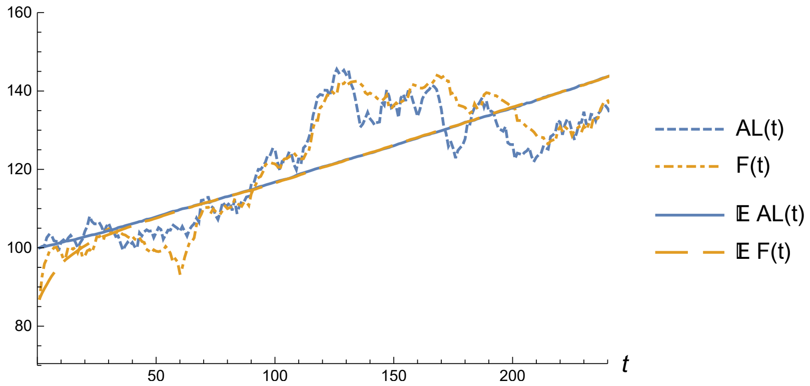


Figure 1: Actuarial liability, fund assets and its expected values in the baseline case

It is interesting to observe that the expected unfunded actuarial liability $\mathbb{E}\,UAL$ is positive, because the pension plan is underfunded, by (33) in the Appendix. However this property is not satisfied by a particular realization of $UAL$. In term of expected values, $\mathbb{E}AL(t)$ is above $\mathbb{E}F(t)$ for all $t \in [0, 240]$. Other interesting fact is that the fund $F$ is positive along the planning horizon. Figure 3 shows $\mathbb{E}F$, the minimum and the maximum paths of $F$ (the paths attaining the minimum and the maximum value $F(t)$, for all $t$) and the upper and lower bounds of $F$, for the 1000 realizations considered.

Figure 4 shows the expected actuarial liability $\mathbb{E}AL$ and the expected fund assets $\mathbb{E}F$ for $\lambda = 0.5$ (baseline case: $F$), $\lambda = 0.9$ ($F1$) and $\lambda = 0.1$ ($F2$) for the same 1000 realizations but over the third year (subperiods ranging from 24 to 36) with a step of 1/12. Despite the patient or impatient majority in the whole group, we can observe that the fund dynamics are very similar for all three cases and are approaching to the actuarial liability. More specifically, observing with more detail the same graph, it can be seen that $\mathbb{E}F1(t) > \mathbb{E}F(t) > \mathbb{E}F2(t)$, for all $t$, i.e., in the case of more patient participants the value of the fund is the higher one, and it

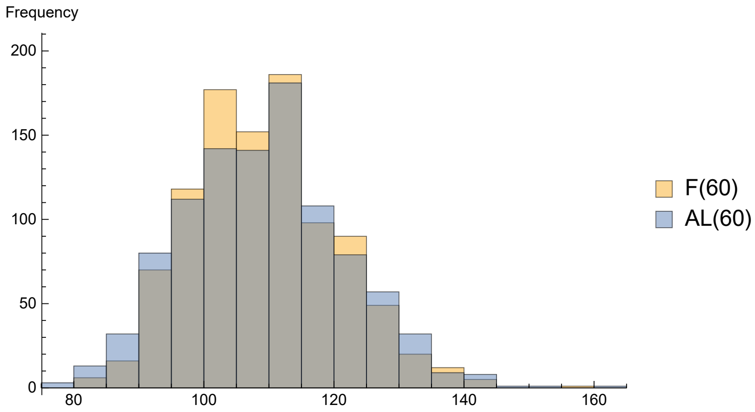


Figure 2: Histogram of values $F(60)$ and $AL(60)$ for the 1000 paths of $F$ and $AL$

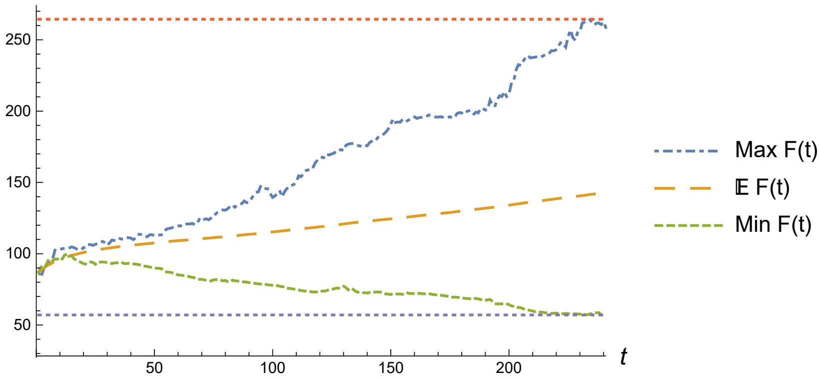


Figure 3: Expected fund assets, minimum fund path and maximum fund path in the baseline case

decreases as impatience increases ($\lambda = 0.9 \to \lambda = 0.5 \to \lambda = 0.1$). This has, as a consequence, that the unfunded actuarial liability $\mathbb{E}\,UAL$ is inversely related with $\lambda$.

From Corollary 3.1 we have that contributions $C^*$ are proportional to the unfunded actuarial liability and the value of $\alpha_{FF}$, while investments $\pi^*$ are proportional to $UAL$ and $AL$. We next compute, at Table 1, some values of $\alpha_{FF}$ and $\alpha_{F,AL}$ for the case of applying a spread method ($\delta = 0.0397485$) where it holds that $\alpha_{F,AL} = -2\alpha_{FF}$ and for the general case where $\alpha_{FF}$ and $\alpha_{F,AL}$ are obtained as the solution of (17) and (19), where we have chosen as the technical rate of interest $\delta = 0.06$.

From Table 1, we can observe that $\alpha_{FF}$ increases with $\lambda$. As a consequence, for a given value

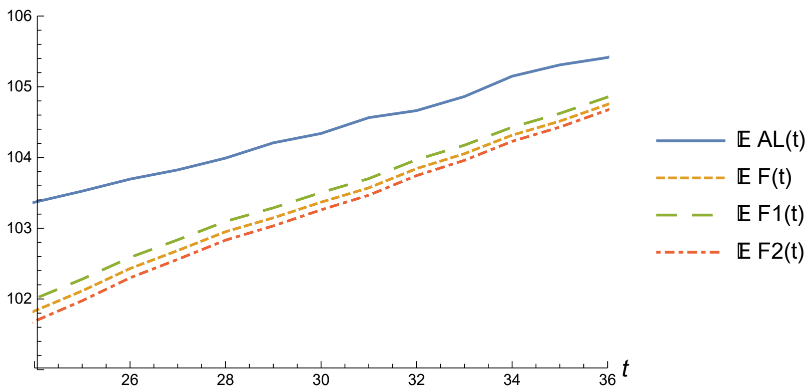


Figure 4: Expected actuarial liability and expected fund assets dynamics for different values of $\lambda$

| $\delta = 0.0397485$ | $\lambda = 1$ | $\lambda = 0.9$ | $\lambda = 0.5$ | $\lambda = 0.1$ | $\lambda = 0$ |
|---|---|---|---|---|---|
| $\alpha_{FF}$ | 0.437504 | 0.432491 | 0.412003 | 0.390661 | 0.385161 |
| $\alpha_{F,AL}$ | -0.875009 | -0.864982 | -0.824007 | -0.781322 | -0.770322 |
| $\delta = 0.06$ | $\lambda = 1$ | $\lambda = 0.9$ | $\lambda = 0.5$ | $\lambda = 0.1$ | $\lambda = 0$ |
| $\alpha_{FF}$ | 0.437504 | 0.432491 | 0.412003 | 0.390661 | 0.385161 |
| $\alpha_{F,AL}$ | -0.890225 | -0.879883 | -0.837603 | -0.793514 | -0.782141 |

Table 1: $\alpha_{FF}$ and $\alpha_{F,AL}$ for different values of $\lambda$ under the spread method $\delta = 0.0397485$ and general case $\delta = 0.06$

of $UAL$, contributions will be higher in the case of having a majority of patient participants. Moreover, as in our model we have that $UAL(0)$ and $AL(0)$, initial investments will be equal for our three studied cases ($\lambda = 0.9 \to \lambda = 0.5 \to \lambda = 0.1$). For these reasons, at the beginning of the time horizon $\mathbb{E}F1(t)$ will overcome $\mathbb{E}F(t)$ and $\mathbb{E}F2(t)$, and the corresponding unfunded actuarial liability will become the lowest one. While this last fact holds, investments will be the lowest ones for $\lambda = 0.9$, while contributions will depend on the cross effect between $\alpha_{FF}$ and the $UAL$. We next focus on the investment strategies.

Figure 5 shows the time evolution of the expected time consistent investment relative to fund size, $\mathbb{E}(\pi^*/F)(= \mathbb{E}_{F_0,AL_0}(\pi^*/F))$. We first mention that borrowing and shortselling are not necessary. In our three cases the global behaviour is similar, starting at the highest level and stabilizing in the long run. We can also see that relative investments are higher for more impatient participants, i.e., the higher the level of impatience, the larger the fraction of the fund invested in risky assets, not only for initial periods as explained above, but also for the whole simulated time horizon, what is related to the fact that in this case the fund $F2$ has the lowest value and, consequently, the associated $UAL$ is the highest.

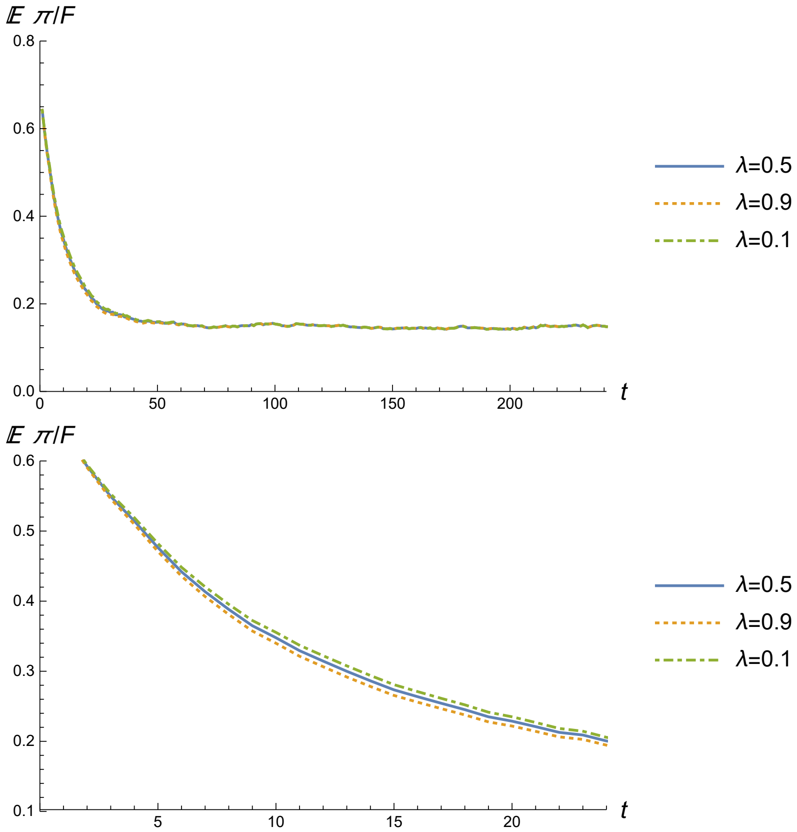


Figure 5: Expected relative investments dynamics for different values of $\lambda$

Looking now at the contributions $C^*$, as mentioned before, its value will depend not only on $\alpha_{FF}$ but also on $UAL$. In Figure 6, we first note that initially they start at a high level from which decrease up to converge the normal cost $NC$. Focusing on our three studied cases, we can observe that in the case of having more patient participants (large $\lambda$) initial contributions are higher, and the expected value of the fund increases faster. This has however, as a consequence,

| $\beta = 0.5$ | $\lambda = 1$ | $\lambda = 0.9$ | $\lambda = 0.5$ | $\lambda = 0.1$ | $\lambda = 0$ |
|---|---|---|---|---|---|
| $\overline{SC}$ | 9.99483 | 9.96881 | 9.85742 | 9.73202 | 9.69801 |
| $\mathbb{E}F(60)$ | 109.26 | 109.286 | 109.276 | 109.269 | 109.257 |
| $\mathbb{E}SC^2(60)$ | 8.88787 | 8.70457 | 8.28373 | 7.785 | 7.65756 |
| $\mathbb{E}\,UAL^2(60)$ | 11.6084 | 11.6341 | 12.2001 | 12.7526 | 12.9046 |

Table 2: Expected accumulated contribution, and expected value of the fund, contribution rate risk and solvency risk at the end of the fifth year, for several values of $\lambda$

that the unfunded actuarial liability becomes smaller. As the contributions are proportional to $UAL$ and $\alpha_{FF}$, it is not clear how contributions will be depending on the value of $\lambda$. This is what we observe in Figure 6 when looking at subperiods 0 to 12 (first year) and 24 to 48 (third year). In the first case, we have that initial contributions effectively are higher for $\lambda = 0.9$, but after some moment they reverse as a consequence that despite of having the largest $\alpha_{FF}$, this cannot be compensated by the lower $UAL$.

Table 2 collects the expected accumulated contributions $\overline{SC}$ and the expected value of the fund at the end of the fifth year $\mathbb{E}F(60)(= \mathbb{E}_{F_0,AL_0}F(60))$, which are negatively related to participants' impatience. The same property satisfies the contribution rate risk $\mathbb{E}SC^2(60)(= \mathbb{E}_{F_0,AL_0}SC^2(60))$. However, the solvency risk, $\mathbb{E}\,UAL^2(60)(= \mathbb{E}_{F_0,AL_0}\,UAL^2(60))$, grows with the impatience.

A sensibility analysis of the time-consistent strategies and the fund with respect to the risk preference parameter $\beta$ shows a similar behaviour in the approaching to the ideal values than in the baseline case, but it is faster with small $\beta$. As expected, the risk preference parameter has a greater effect on the actuarial risks. If the importance given to the contribution rate risk in the minimization process increases, then the contribution rate risk and the expected accumulated contribution decrease, but the solvency risk increases. Thus, for instance, for $\lambda = 0.5$, at the end of the fifty year, the contribution rate risk decreases from 17.0043, for $\beta = 0.25$, to 8.28373, for $\beta = 0.5$, and to 3.73199, for $\beta = 0.75$. When the weight $\beta$ increases, we also observe a slight decline of the expected fund at the end of the fifth year. Table 3 includes the values at the end

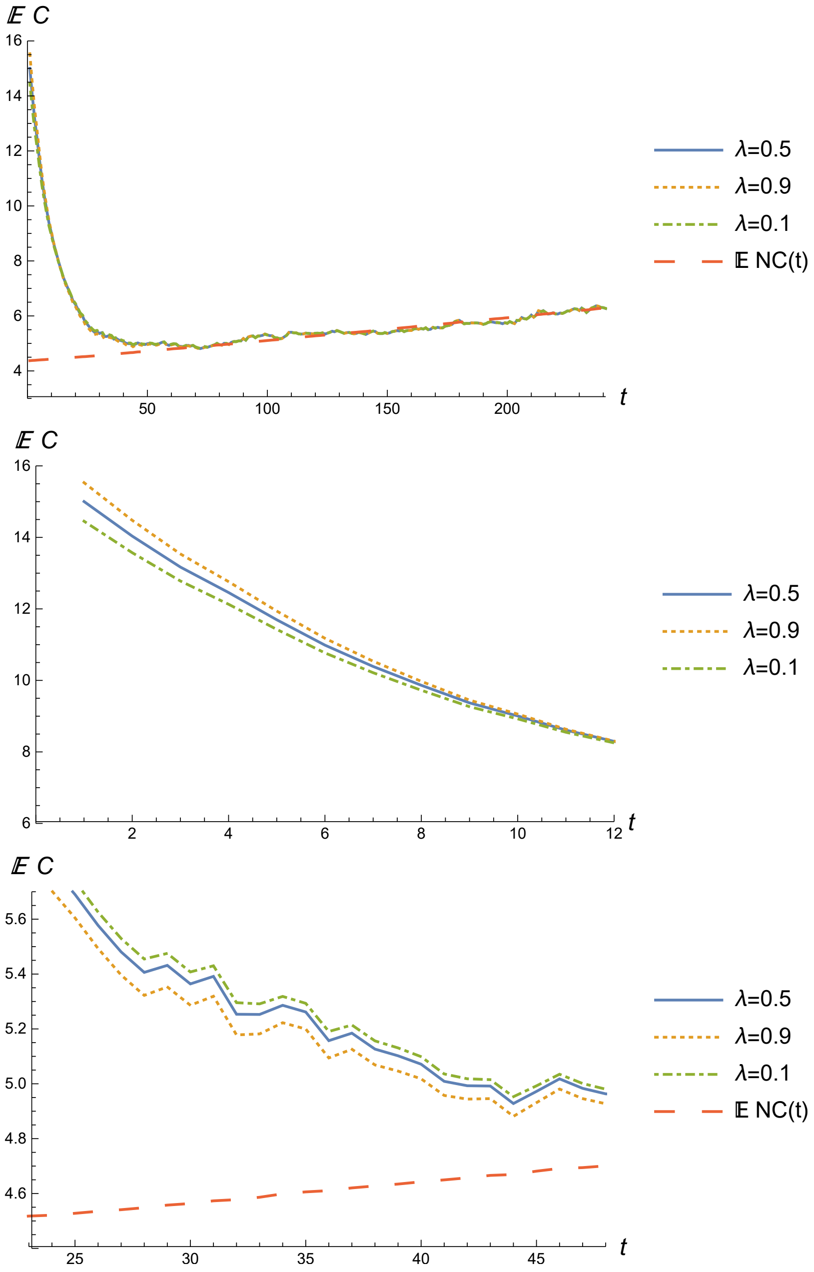


Figure 6: Expected contributions for different values of $\lambda$ and expected normal cost dynamics

of the fifth year for $\beta = 0.25$ and $0.75$.

Now we compare the current bull financial regime where $\theta = 0.52994$ with a hypothetical bear regime with drift parameter of the risky asset $b = 0.03$, diffusion parameter $\sigma = 0.2$ (this implies a Sharpe ratio of $\theta = 0.1$), and risk-free rate of interest $r = 0.01$. The remaining

| $\beta = 0.25$ | $\lambda = 1$ | $\lambda = 0.9$ | $\lambda = 0.5$ | $\lambda = 0.1$ | $\lambda = 0$ |
|---|---|---|---|---|---|
| $\overline{SC}$ | 11.1338 | 11.1231 | 11.0783 | 11.0298 | 11.0171 |
| $\mathbb{E}F(60)$ | 109.41 | 109.474 | 109.331 | 109.25 | 109.226 |
| $\mathbb{E}SC^2(60)$ | 17.7377 | 17.5603 | 17.0043 | 16.4042 | 16.252 |
| $\mathbb{E}\,UAL^2(60)$ | 6.90026 | 6.92764 | 7.10831 | 7.29071 | 7.33885 |
| $\beta = 0.75$ | $\lambda = 1$ | $\lambda = 0.9$ | $\lambda = 0.5$ | $\lambda = 0.1$ | $\lambda = 0$ |
| $\overline{SC}$ | 8.29902 | 8.2449 | 8.00847 | 7.73053 | 7.65243 |
| $\mathbb{E}F(60)$ | 108.99 | 109.229 | 109.162 | 109.078 | 108.964 |
| $\mathbb{E}SC^2(60)$ | 4.18845 | 3.99624 | 3.73199 | 3.35334 | 3.25566 |
| $\mathbb{E}\,UAL^2(60)$ | 19.6934 | 19.9014 | 21.5235 | 23.181 | 23.6669 |

Table 3: Expected accumulated contribution, and expected value of the fund, contribution rate risk and solvency risk at the end of the fifth year, for several values of $\lambda$ and $\beta$

of parameters rare unchanged. The behaviour of the time consistent strategies with respect to time is similar to the bull case but there exist some differences. The expected investment strategy is time decreasing as in the bull case, but it is under 0.2 from the second month unlike the bull case that needs four years to get off it (see Figure 7). The opposite effect holds with respect to the close up of the time consistent contribution to the normal cost. The expected accumulated contribution is greater than in the bull regime: $\overline{SC} = 12.9$ vs 9.8574, for $\lambda = 0.5$. However, the contribution rate risk is a little higher in the bear case than in the bull case. The opposite holds with respect to the solvency risk.

It is interesting to observe the effect of the correlation $q$ between the risky asset and the benefit. With a negative correlation, $q = -0.5$, the investment strategy is time decreasing, and also, from the second year, it is necessary to short selling to invest the fund in the risky asset (see Figure 8). Moreover, the convergence of $F^*$ to $AL$ and of $C^*$ to $NC$ is a slightly faster. Finally, the expected fund is lower than with positive correlation ($\mathbb{E}F^*(60) = 108.541$ vs 109.276, for $\lambda = 0.5$, at the end of the fifth year), and the actuarial risks, the solvency risk and

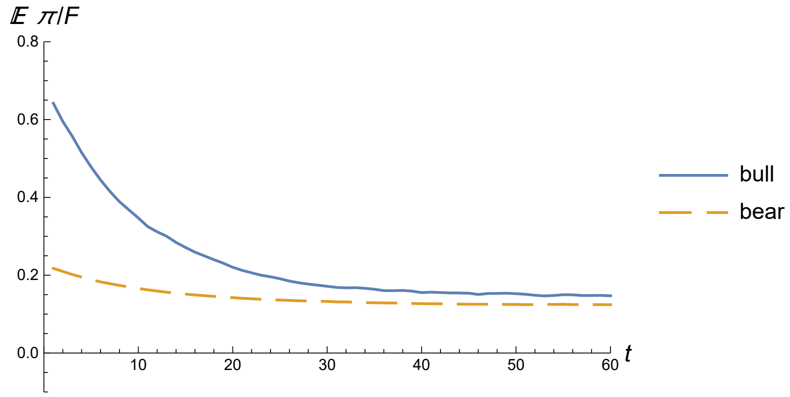


Figure 7: Expected relative investments dynamics for bull and bear regimes

the contribution rate risk, are also slightly lower. Next we analyze the cases of uncorrelation and perfect correlation. When $q = 0$, the expected investment strategies converge to zero in the long term (see Figure 8). The actuarial risks are higher than the obtained with other values of correlation. For instance, for $\lambda = 0.5$ the solvency risk at the end of the fifth year decreases from 17.677 for $q = 0$, to 12.8949 for $q = -0.5$, to 13.9364 for $q = 0.5$, to 0.0188358 for $q = 1$, and to 0.0116807 for $q = -1$. The convergence to zero in the long term, for both actuarial risks holds when $q = \pm 1$. Note that, at the end of the fifth year, the contribution rate risks are 0.0127893 for $q = 1$ and 0.00793102 for $q = -1$, near zero similarly to the solvency risks previously mentioned. This trend is maintained in the long term.

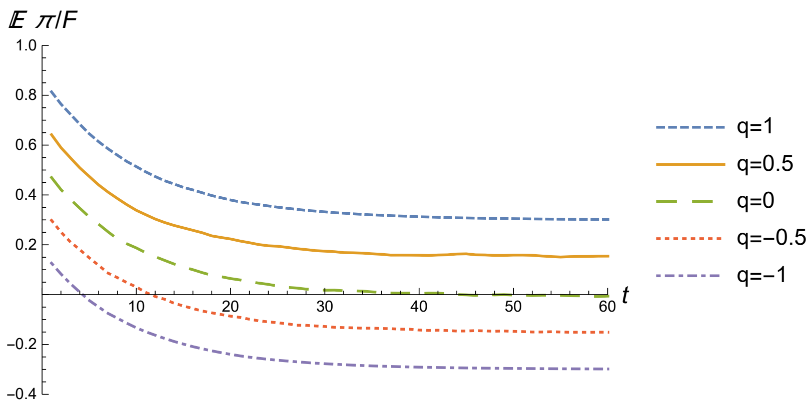


Figure 8: Expected relative investments dynamics for different values of $q$

Finally, a sensitivity analysis on $\eta$ confirms that the time consistent contribution does not depend on $\eta$, and that this parameter has little influence on the fund evolution (note that only appears in the diffusion term in (25)). The investment strategy is smaller when the volatility of the benefit is smaller; see Figure 9. When the benefit is not a stochastic process, $\eta = 0$, the investment strategy is time decreasing and converges to zero in the long term, and the actuarial risks converge to zero in the long term.

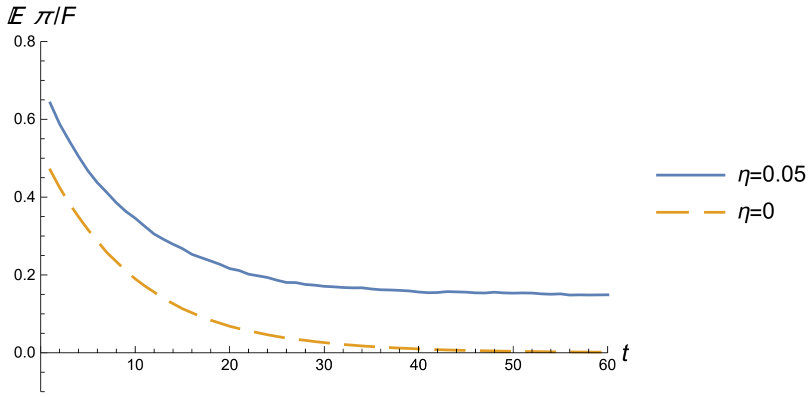


Figure 9: Expected relative investments dynamics for different values of $\eta$

## 5 Conclusions

We have analyzed by means of dynamic programming techniques the management of an aggregated defined benefit pension plan where the rate of discount is non-constant and the benefit is stochastic. The objective is to determine the contribution rate and the investment strategy minimizing both the contribution and the solvency risk in a infinite time horizon. We have found that the weight of the patient participants in the whole of participants intervenes in the time consistent strategies and in the associated fund evolution.

It is possible to select the technical rate of interest such that the investment strategy does not depend on the rate of discount (only depends on the diffusion parameter of the benefit and the financial market). In this case, however, the time consistent contribution does depend on the discount rate but not on the parameters of the benefit process, getting a spread amortization and the plan stability in the long term.

A numerical illustration shows how, when the discount function is a convex combination between two exponential functions, in the long term, the expected values of the fund and the time consistent contribution are closed, respectively, to the expected values of the actuarial liability and to the normal cost. More patient participants will lead to a higher value of the fund, and consequently, a lower unfunded actuarial liability (a greater close up between fund and actuarial). That is to say, the convergence of the fund to the actuarial liability is faster with more patience. In the case of the investment strategies, we have observed that the more impatient is the majority of participants, the higher are the investments in risky assets. However, there is not a constant relationship between the contributions and the degree of impatience, changing their ordering at some intermediate point in the planning horizon. We have also checked that the participants' patience diminishes the expected accumulated contribution and, in the medium-term, the expected fund and the contribution rate risk, but increases the solvency risk. Finally we have analyzed the sensibility of the results with respect to the risk preference parameter, the correlation between the benefit and the financial market, the volatility of the benefit and the regime of the financial market.

Further research can be guided to include Poisson jumps in the financial market, to consider a regime switching model, and to analyze other non-exponential discount functions.

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# A Appendix

**Proof of Theorem 3.1.**

If there is a smooth solution $V$ of the equation (10), strictly convex, then the minimizers values of the contribution rate and the investment rates are given by

$$\widehat{SC}(V_F) = -\frac{V_F}{2\beta}, \tag{27}$$

$$\widehat{\pi}(V_F, V_{FF}, V_{F,AL}) = -\Sigma^{-1}(b - r\bar{1})\frac{V_F}{V_{FF}} - \eta AL\,\sigma^{-\top} q \frac{V_{F,AL}}{V_{FF}}, \tag{28}$$

respectively.

From (27) and (28), the structure of the HJB equation obtained once we have substituted these values for $SC$ and $\pi$ in (10), suggests a quadratic homogeneous solution

$$V(F, AL) = \alpha_{FF}F^2 + \alpha_{F,AL}F\,AL + \alpha_{AL,AL}AL^2.$$

Imposing this solution in (27) and (28), we obtain

$$\begin{aligned}\widehat{SC} &= -\frac{\alpha_{FF}}{\beta}F - \frac{\alpha_{F,AL}}{2\beta}AL\,,\\ \widehat{\pi} &= -\Sigma^{-1}(b - r\bar{1})F - \frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\Sigma^{-1}(b - r\bar{1}) + \eta\sigma^{-\top}q\right)AL\,.\end{aligned}$$

Now we are going to obtain the explicit form of $K(F, AL)$ in (11). Taking into account that process $AL$ satisfies the SDE (4), after substitution in (8) of expressions for $\widehat{SC}$ and $\widehat{\pi}$, process $F$ satisfies the SDE (21). Following Arnold (1974), we can apply the Itô's formula to the processes $F^2, FAL$ and $AL^2$. Taking expected values, the functions defined by $\phi(t) = \mathbb{E}_{F_0,AL_0}F^2(t)$, $\psi(t) = \mathbb{E}_{F_0,AL_0}(FAL)(t)$ and $\xi(t) = \mathbb{E}_{F_0,AL_0}AL^2(t)$ satisfy the linear differential equations

$$\phi'(t) = \left(2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta\right)\phi(t) - \left(\frac{\alpha_{F,AL}}{\beta} + 2(\delta - \mu)\right)\psi(t) + \frac{\alpha_{F,AL}^2}{4\alpha_{FF}^2}\left(\theta^\top\theta + 2\eta q^\top\theta + \eta^2 q^\top q\right)\xi(t),$$

$$\psi'(t) = \left(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta\right)\psi(t) - \left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}(\theta^\top\theta + 2\eta q^\top\theta + \frac{\alpha_{FF}}{\beta} + \eta^2 q^\top q) + \delta - \mu\right)\xi(t),$$

$$\xi'(t) = (2\mu + \eta^2)\xi(t),$$

with initial conditions $\phi(0) = F^2$, $\psi(0) = FAL$ and $\xi(0) = AL^2$, respectively. Thus the explicit expressions for these functions are:

$$\xi(t) = AL^2 e^{(2\mu+\eta^2)t},$$

$$\psi(t) = \left(FAL - \frac{b}{2\mu+\eta^2-a}AL^2\right)e^{at} + \frac{b}{2\mu+\eta^2-a}AL^2 e^{(2\mu+\eta^2)t},$$

$$\phi(t) = \left(F^2 - \frac{d}{a-c}FAL + \frac{bd}{(2\mu+\eta^2-a)(a-c)}AL^2 - \frac{1}{2\mu+\eta^2-c}\left(\frac{bd}{2\mu+\eta^2-a}+m\right)AL^2\right)e^{ct}$$
$$+\left(\frac{d}{a-c}FAL - \frac{bd}{(2\mu+\eta^2-a)(a-c)}AL^2\right)e^{at} + \left(\frac{bd}{2\mu+\eta^2-a}+m\right)\frac{AL^2}{2\mu+\eta^2-c}e^{(2\mu+\eta^2)t}$$

where $a = r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta$, $b = -\frac{\alpha_{FAL}}{2\alpha_{FF}}\left(\theta^\top\theta + 2\eta q^\top\theta + \frac{\alpha_{FF}}{\beta} + \eta^2 q^\top q\right) - \delta + \mu$, $c = 2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta$, $d = -\frac{\alpha_{FAL}}{\beta} - 2(\delta-\mu)$ and $m = \left(\theta^\top\theta + 2\eta q^\top\theta + \eta^2 q^\top q\right)\frac{\alpha_{FAL}^2}{4\alpha_{FF}^2}$. Substituting the minimizers,

$$\begin{aligned}
&\mathbb{E}_{F,AL}\left\{\beta SC^2(s) + (1-\beta)(AL(s) - F(s))^2\right\}\\
&= \mathbb{E}_{F,AL}\left\{\frac{\alpha_{FF}^2}{\beta}\left(F(s) + \frac{\alpha_{FAL}}{2\alpha_{FF}}AL(s)\right)^2 + (1-\beta)(AL(s)-F(s))^2\right\}\\
&= \frac{\alpha_{FF}^2}{\beta}\left(\phi(s) + \frac{\alpha_{FAL}}{\alpha_{FF}}\psi(s) + \frac{\alpha_{FAL}^2}{4\alpha_{FF}^2}\xi(s)\right) + (1-\beta)\left(\phi(s) - 2\psi(s) + \xi(s)\right),
\end{aligned}$$

and then $K(F, AL) = \kappa_{FF}F^2 + \kappa_{FAL}FAL + \kappa_{AL,AL}AL^2$, where $\kappa_{FF}$ is given by (18), $\kappa_{F,AL}$ is given by (20) and $\kappa_{AL,AL}$ is another constant what does not need to be determined. Note that the improper integral that appears in (18) is well defined by condition (14) y because the rate of discount function $\tilde{\rho}$ is time-decreasing. Analogously with the improper integral in (20), by (32); see below.

Substituting the minimizers in (10) and using (3), the following set of three equations for the coefficients is obtained: (17), (19) and

$$\begin{aligned}
(-\rho + 2\mu - \eta^2)\alpha_{AL,AL} - \left(\theta^\top\theta + \eta^2 q^\top q + 2\eta q^\top\theta\right)\frac{\alpha_{F,AL}^2}{4\alpha_{FF}}&\\
-(\delta-\mu)\alpha_{F,AL} - \frac{\alpha_{F,AL}^2}{4\beta} + 1 - \beta - \kappa_{AL,AL} = 0.& \qquad (29)
\end{aligned}$$

In order to prove that the solution of (10) is the value function and $C^*$ and $\pi^*$, given by (15) and (16), respectively, are the time consistent strategies of the stochastic control problem, it is sufficient to check that the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0} V(F^*(t), AL\,(t)) = \lim_{t\to\infty} e^{-\rho t}\left(\alpha_{FF}\phi(t) + \alpha_{F,AL}\psi(t) + \alpha_{AL,AL}\xi(t)\right) = 0$$

holds, where $AL$ and $F^*$ satisfy, respectively, (4) and (21). By our previous calculations,

$$\xi(t) = AL\,_0^2 e^{(2\mu+\eta^2)t},$$

hence $\lim_{t\to\infty} e^{-\rho t}\xi(t) = 0$ if and only if (13) holds. On the other hand,

$$\psi(t) = (F_0 - a_1 AL\,_0)AL\,_0 e^{at} + a_1\xi(t),$$

where $a_1 = \frac{b}{2\mu+\eta^2-a}$ is a constant depending on parameters of model. Then $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0}\psi(t) = 0$, if and only if, both (13) and the inequality

$$r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta < \rho \tag{30}$$

simultaneously hold. On the other hand,

$$\phi(t) = \left(F_0^2 + a_2 AL\,_0^2 - a_3 F_0 AL\,_0\right)e^{ct} + a_2\psi(t) - a_3\xi(t),$$

where $a_2 = \frac{d}{a-c}$ and $a_3 = a_1 a_2 - \frac{ba_1+m}{2\mu+\eta^2-c}$ are constants. Hence $\lim_{t\to\infty} e^{-\rho t}\phi(t) = 0$ if and only if (13), (30) and

$$2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta < \rho \tag{31}$$

hold. Now we prove that the conditions (13) and (31) imply (30). Observe that

$$0 \le \left(\theta^\top + \eta q^\top\right)(\theta + \eta q) = \theta^\top\theta + 2\eta q^\top\theta + \eta^2 q^\top q \le \theta^\top\theta + 2\eta q^\top\theta + \eta^2,$$

implies

$$-\eta q^\top\theta \le \frac{1}{2}\theta^\top\theta + \frac{1}{2}\eta^2.$$

We have used $q^\top q \le 1$. Thus

$$r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta \le r - \frac{1}{2}\theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu + \frac{1}{2}\eta^2 < \frac{\rho}{2} + \frac{\rho}{2} = \rho. \tag{32}$$

Since $V$ is a homogeneous quadratic polynomial in $F$ and $AL$, $e^{-\rho t}\mathbb{E}_{F_0,AL_0}V(F^*(t), AL(t))$ converges to 0 when $t$ goes to $\infty$. Finally, applying the analogous theorem to the verification Theorem 8.1, chapter 3, in Fleming and Soner (1993), we conclude that $V$ is the value function and $C^*$, given by (15), and $\pi^*$, given by (16), are the time consistent controls. $\square$

**Proof of Proposition 3.1.** From (21), using $\alpha_{F,AL} = -2\alpha_{FF}$ and Assumption 2 we obtain (25) and then

$$dUAL^*(t) = \left(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta}\right) UAL^*(t)dt + \eta\sqrt{1 - q^\top q}AL\,(t)\,dW_0(t) - \theta^\top UAL^*(t)dW(t).$$

Thus

$$\mathbb{E}_{F_0,AL_0} UAL^*(t) = \mathbb{E}_{F_0,AL_0} AL\,(t) - \mathbb{E}_{F_0,AL_0} F^*(t) = (AL_0 - F_0)e^{\left(r-\theta^\top\theta-\frac{\alpha_{FF}}{\beta}\right)t}, \tag{33}$$

converges to zero when $t$ goes to $\infty$, by (26). Analogously

$$\mathbb{E}_{F_0,AL_0} SC^*(t) = \frac{\alpha_{FF}}{\beta}\mathbb{E}_{F_0,AL_0} UAL^*(t),$$

converges to zero when $t$ goes to $\infty$.

On the other hand, by Corollary 3.1,

$$\begin{aligned}\overline{SC} &= \int_0^\infty \mathbb{E}_{F_0,AL_0} SC(t)dt = \frac{\alpha_{FF}}{\beta}\int_0^\infty \mathbb{E}_{F_0,AL_0} UAL(t)dt \\ &= \frac{\alpha_{FF}}{\beta} UAL_0 \int_0^\infty e^{(r-\theta^\top\theta-\alpha_{FF}/\beta)t}dt = \frac{\alpha_{FF}/\beta}{\alpha_{FF}/\beta + \theta^\top\theta - r} UAL_0 > 0,\end{aligned}$$

by (26), and because the plan is underfunded, $UAL_0 = AL_0 - F_0 > 0$. □

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