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Non-finite axiomatizability of first-order Peano Arithmetic

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Abstract

The system of *Peano Arithmetic* is a system more than enough for proving almost all statements of the natural numbers. We will work with a version of this system adapted to first-order logic, denoted as *PA*. The aim of this work will be showing that there is no equivalent finitely axiomatizable system. In order to do this, we will introduce some concepts about the complexity of formulas and codification of sequences to prove *Ryll-Nardzewski's theorem*, which states that there is no consistent extension of *PA* finitely axiomatized.

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Introduction

Whenever I explain that I am studying a degree in Mathematics people tend to ask me the same question: What do mathematicians study? Their first thought is that we spend four years in University learning and developing new techniques for proving theorems, which is not much far from truth. However, what they never expect is that we, mathematicians, do not have the answers to everything, that there are things that can not been proved.

Ironically, this is a statement that has already been proved. Formally, is what we know as *Gödel's incompleteness theorem* [4], which states that there is no consistent recursively axiomatized theory T capable of proving all truths about the arithmetic of natural numbers. In other words, that there are some statements about the natural numbers that are true, but that can not been proved.

Even though, there is a system more than enough for proving statements about the natural numbers: the system of *Peano Arithmetic*. It is a hard work the one of finding a statement that can not been proved in this system, though it exists.

Peano produced his postulates in 1889 and they were first presented in a short work under the title *"Arithmetices principia nova methodo exposita" - "The principles of arithmetic, presented by a new method"* [6], originally writen in latin. He formulated his axioms with the objective of giving a clear and rigorous presentation of arithmetic and of mathematics in general. In fact, he believed that an accurate presentation of arithmetic would avoid errors and ease the mathematics development.

The system of *Peano Arithmetic* will be one of the topics addressed in this project, but I will not use it just as it is, I will introduce a version of the system adapted to first-order logic, which I will denote as *PA*. This system consists of 16 axioms that are clearly true in $\mathbb{N}$ and an axiom of induction.

As you will see later, *Peano Arithmetic* is a system constructed with infinitely many axioms, since the axiom of induction is not given by a single sentence, but by an axiom scheme. So, the natural question that comes to mind is the following:

Is there an equivalent finite system?

Obviously, we are not the first to wonder this. In the twentieth century, the polish mathematician Ryll-Nardzewski asked himself the same question and was even able to answer it. In 1952, Ryll-Nardzewski published an article in *Fundamenta Mathematicae* called

"The role of the axiom of induction in elementary arithmetic" [7] proving the following stronger statement:

Theorem *(Ryll-Nardzewski's theorem) No consistent extension of PA is finitely axiomatized.*

The main objective of this project will be giving an accurate proof of *Ryll-Nardzewski's theorem* including all the previous concepts and theorems required. To do so, I consulted mainly three books; two basic manuals, to acquire the essential background in logic [3] and model theory [2], and Richard Kaye's book [5], which has been the basis of my project. Additionally, I also looked up the notes of the subject of *Mathematical Logic* from my tutor Enrique Casanovas [1].

Memoir structure

This work is more than just a summary of Richard Kaye's book, however it is true that most of the information given can be found there. I spent the last 9 months reading, understanding, reordering and sometimes even correcting his book in order to write this project as clear and rigorous as possible. To achieve this, I decided to structure the work as follows:

The first chapter is a brief introduction to logic and model theory. It was written to give a background, I hope more than enough, for those which are not familiar with mathematical logic.

Chapter 2 is probably the most important one, since is where the *standard model* $\mathbb{N}$ and *Peano Arithmetic* are presented. Here is where I introduce the theory we will be working with during the whole project.

In the next chapter, I give one of the most relevant definitions of the project, the definition of the Σ_n *class*, used as a measure of the complexity of formulas and sentences. The Σ_n *class* will appear constantly in the following chapters. The last section of chapter 3 is dedicated to study the possible extensions of the language $\mathcal{L}_{\mathcal{A}}$ and its properties. This section will play an important role in chapter 4.

The main objective of chapter 4 is showing that *PA* can handle syntax and semantics adequately to end up giving a definition of truth provable in *PA*. To achieve this objective, I introduced first some concepts about codification of sequences.

Finally, the last chapter is the one dedicated to prove *Ryll-Nardzewski's theorem*. But before doing this, I present the set of definable elements of a model and its properties, which will be essential in the proof of the theorem.

Chapter 1

Preliminaries

In first-order logic, a **language** $\mathcal{L}$ is a collection of three kinds of symbols: function symbols $(F_0, F_1, \ldots)$, relation symbols $(R_0, R_1, \ldots)$ and constant symbols $(c_0, c_1, \ldots)$. Each relation and function symbol is related to a natural number $n \geq 1$, we call this number the **arity** of the symbol. We define then the language $\mathcal{L}$ as $\mathcal{F}_{\mathcal{L}} \cup \mathcal{R}_{\mathcal{L}} \cup \mathcal{C}_{\mathcal{L}}$ for $\mathcal{F}_{\mathcal{L}}$ the set of function symbols, $\mathcal{R}_{\mathcal{L}}$ of relation symbols and $\mathcal{C}_{\mathcal{L}}$ of constant symbols.

To complement the language we need the following logical symbols: connectives $(\neg, \vee, \wedge, \rightarrow, \leftrightarrow)$, quantifiers $(\forall, \exists)$, variables (x,y,z, ...), brackets and a relation symbol $\doteq$ for equality. We denote the set of variables as $\mathcal{V}$.

A **term** of $\mathcal{L}$ is a finite sequence of variables, function symbols and constants of $\mathcal{L}$ constructed with the following rules:

- Any constant $c \in \mathcal{C}_{\mathcal{L}}$ is a term.
- Any variable $x \in \mathcal{V}$ is a term.
- If $t_1, \ldots, t_n$ are terms and $F \in \mathcal{F}_{\mathcal{L}}$ is an n-ary function symbol, then $F(t_1, \ldots, t_n)$ is also a term.

We write $t(\bar{x})$ with $\bar{x} = (x_0, \ldots, x_n)$ to say that all the variables that appear in the term t are in $\bar{x}$.

An **atomic formula** of $\mathcal{L}$ is a finite sequence of terms and relation symbols constructed with the following rules:

- If t_1 and t_2 are terms of $\mathcal{L}$ then $t_1 \doteq t_2$ is an atomic formula.
- If $t_1, \ldots, t_n$ are terms and $R \in \mathcal{R}_{\mathcal{L}}$ is an n-ary relation symbol, then $R(t_1, \ldots, t_n)$ is an atomic formula.

Finally, a **formula** of $\mathcal{L}$ is a finite sequence of atomic formulas, connectives and variables given by the following rules:

- Any atomic formula is a formula.
- If φ and ψ are formulas and $* \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$, then $(\varphi * \psi)$ is a formula.
- If φ is a formula then $\neg\varphi$ is a formula.
- If φ is a formula and $x \in \mathcal{V}$ then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

When a quantifier $Q = \{\forall, \exists\}$ appears in an $\mathcal{L}$-formula φ it is always followed by a variable $x \in \mathcal{V}$ and a subformula ψ. We denote the subformula $Qx\psi$ as the **scope** of the quantifier Q and we say then that all appearances of the variable x in the subformula $Qx\psi$ are **bounded** by this quantifier. If one appearance of a variable in the formula φ is not bounded we say that this variable is **free**. We write then $\varphi(\bar{x})$ with $\bar{x} = (x_0, \ldots, x_n)$ to say that all free variables of the formula φ are in the list $\bar{x}$.

Sometimes we will reduce the set of connectives to $(\neg, \wedge)$ and the set of quantifiers to $(\exists)$ defining the others by the sentences $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \to \psi) := \neg(\varphi \wedge \neg\psi)$, $(\varphi \leftrightarrow \psi) := (\neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi))$ and $\forall x\varphi := \neg\exists x\neg\varphi$ for φ and ψ $\mathcal{L}$-formulas and $x \in \mathcal{V}$. We will use this notation to reduce the cases in induction proofs.

A **universe** A for $\mathcal{L}$ is a nonempty set such that each n-ary $\mathcal{L}$-function symbol F corresponds to a function $F^M : A^n \to A$ on A, each m-ary $\mathcal{L}$-relation symbol R, to a relation $R^M \subseteq A^m$ on A and each constant symbol c, to a constant $c^M \in A$. This correspondences are given by a function $\mathfrak{I}$ mapping the symbols of $\mathcal{L}$ to relations, functions and constants in A. Now we can define a **model** for $\mathcal{L}$ as a pair $M = \langle A, \mathfrak{I} \rangle$. We also denote A as the **domain** of M. In the practice we will use the same notation for the model as for the domain.

Given an $\mathcal{L}$-term $t(\bar{x})$ with $\bar{x} = (x_0, \ldots, x_n)$ and some $\bar{a} = (a_0, \ldots, a_n) \in M$ for a model M for $\mathcal{L}$, we define the **value of** $t(\bar{x})$ **at** $\bar{a}$ by:

- $t^M[\bar{a}] = c^M$ for c^M the interpretation of c in M, if $t = c$ for $c \in \mathcal{C}_{\mathcal{L}}$.
- $t^M[\bar{a}] = a_i$, if $t = x_i$ for $i \in \{0, \ldots, n\}$.
- $t^M[\bar{a}] = F^M(t_1^M[\bar{a}], \ldots, t_m^M[\bar{a}])$ where F^M is the interpretation of the symbol F in M, if $t = F(t_1, \ldots, t_m)$ for $F \in \mathcal{F}_{\mathcal{L}}$ an m-ary function symbol and $t_1(\bar{x}), \ldots, t_m(\bar{x})$ terms.

We say then that an $\mathcal{L}$-formula $\varphi(\bar{x})$ with $\bar{x} = (x_0, \ldots, x_n)$ **is true in** M with the assignation $\bar{a} = (a_0, \ldots, a_n) \in M$ for M a model for $\mathcal{L}$ and write it as $M \vDash \varphi(\bar{a})$ if it satisfies the following rules:

- If φ is an atomic formula $t_1 \doteq t_2$ for terms $t_1(\bar{x})$ and $t_2(\bar{x})$, then $M \vDash \varphi(\bar{a})$ iff $t_1^M[\bar{a}] = t_2^M[\bar{a}]$.
- If φ is an atomic formula $R(t_1, \ldots, t_m)$ for $R \in \mathcal{R}_{\mathcal{L}}$ an m-ary relation symbol and $t_1(\bar{x}), \ldots, t_m(\bar{x})$ terms, then $M \vDash \varphi(\bar{a})$ iff $R^M(t_1^M[\bar{a}], \ldots, t_m^M[\bar{a}])$ where R^M is the interpretation of R in M.
- If φ is $\psi_1 \wedge \psi_2$ for ψ_1 and ψ_2 $\mathcal{L}$-formulas then $M \vDash \varphi(\bar{a})$ iff $M \vDash \psi_1(\bar{a})$ and $M \vDash \psi_2(\bar{a})$.
- If φ is $\neg\psi$ for an $\mathcal{L}$-formula ψ then $M \vDash \varphi(\bar{a})$ iff $M \nvDash \psi(\bar{a})$.
- If φ is $\exists x_i\psi$ for an $\mathcal{L}$-formula ψ and $i \in \{0, \ldots, n\}$ then $M \vDash \varphi(\bar{a})$ iff exists some $b \in M$ such that $M \vDash \psi(a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$.
- If φ is $\exists y\psi$ for an $\mathcal{L}$-formula ψ and $y \notin \{x_0, \ldots, x_n\}$ then $M \vDash \varphi(\bar{a})$ iff exists some $b \in M$ such that $M \vDash \psi(a_0, \ldots, a_n, b)$.

We also say then that M **satisfies** φ with the assignation $\bar{a}$. We can extend this definition to a set of formulas Σ and say that M **satisfies** Σ with the assignation $\bar{a} \in M$, $M \vDash \Sigma(\bar{a})$, if $M \vDash \psi(\bar{a})$ for each ψ in Σ. A set of formulas Σ is **satisfiable** if $M \vDash \Sigma(\bar{a})$ for some model M and some assignation $\bar{a} \in M$. Respectively, a formula φ is **satisfiable** if $M \vDash \varphi(\bar{a})$ for some model M and some assignation $\bar{a} \in M$.

Two $\mathcal{L}$-formulas $\varphi(\bar{x}), \psi(\bar{x})$ with $\bar{x} = (x_0, \ldots, x_n)$ are **equivalent**, $\varphi \equiv \psi$, iff for each model M of $\mathcal{L}$ and each $\bar{a} = (a_0, \ldots, a_n) \in M$ we have $M \models \varphi(\bar{a})$ iff $M \models \psi(\bar{a})$.

We say that two models M and N for the same language $\mathcal{L}$ are **elementarily equivalent**, and we write it as $M \equiv N$, iff every sentence that holds in M also holds in N, and the other way round.

Given two models M and N for the same language $\mathcal{L}$, we say that M is a **submodel or substructure** of N, $M \subseteq N$, iff:
(a) The domain of M is a subset of the domain of N.
(b) The domain of M contains the constants of N and is closed under the functions of N.
(c) Each non-logical symbol of $\mathcal{L}$ is interpreted in M according to the restriction of its interpretation in N.
(c.1) $F^N \restriction_{M^n} = F^M$ for F a n-ary function symbol.
(c.2) $R^N \cap M^n = R^M$ fo R a n-ary relation symbol.
(c.3) $c^N = c^M$.
We say then that N is an **extension** of M.

M is an **elementary submodel** of N, $M \preceq N$, iff $M \subseteq N$ and for each formula $\varphi(\bar{x})$ and each $\bar{a} \in M$,

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a}).$$

If $M \preceq N$ then M and N satisfy the same sentences; the converse may not be true, even if $M \subseteq N$.

Theorem 1.1. *(Tarski-Vaught test) Let $M \subseteq N$ be models for the same language $\mathcal{L}$. Then the following are equivalent:*
(a) $M \preceq N$.
(b) For each $\mathcal{L}$-formula $\varphi(\bar{x}, y)$ and for each $\bar{a} \in M$

$$N \models \exists y \varphi(\bar{a}, y) \Rightarrow \textit{there exists } b \in M \textit{ s.t. } N \models \varphi(\bar{a}, b).$$

Proof. The proof of **Tarski-Vaught test** is similar to the one of proposition 3.1.2. of [2]. □

Given a set of $\mathcal{L}$-formulas Σ and an $\mathcal{L}$-formula φ, we say that φ **is a consequence of** Σ, $\Sigma \models \varphi$, if for each model M of $\mathcal{L}$ and each $\bar{a} \in M$ such that $M \models \Sigma(\bar{a})$ we have $M \models \varphi(\bar{a})$.

We write $\Sigma \vdash \varphi$ to denote that there is a proof of φ from Σ, Λ and some rules of inference. The rules of inference and the set Λ, formed by some formulas called logical axioms, will depend on the deductive calculus we are working with. As an example of deductive calculus you can see section 2.4 of Enderton's book [3].

Theorem 1.2. *(Completeness theorem). Let Σ be a set of $\mathcal{L}$-formulas and let φ be an $\mathcal{L}$-formula. Then $\Sigma \vdash \varphi$ iff $\Sigma \models \varphi$.*

Proof. You can find a proof of the **Completeness theorem** in page 135 of [3]. □

An $\mathcal{L}$-sentence is a formula with no free variables. We say then that an $\mathcal{L}$-sentence σ is **satisfiable** if $M \vDash \sigma$ for some model M of $\mathcal{L}$, respectively a set of sentences Σ is **satisfiable** if $M \vDash \sigma$ for each $\sigma \in \Sigma$ and some model M. Moreover, $\Sigma \vDash \sigma$ if $M \vDash \sigma$ for each model M such that $M \vDash \Sigma$.

A **theory** T of the language $\mathcal{L}$ is a collection of $\mathcal{L}$-sentences closed under logical consequence, i.e. if $T \vDash \sigma$ for σ an $\mathcal{L}$-sentence then $\sigma \in T$. There are many ways of defining a theory T, but we will mostly use two.

One is by listing its set of axioms. A **set of axioms** of a theory T is a set of sentences with the same consequences as T, this consequences are called theorems. In other words, a set Γ of sentences of $\mathcal{L}$ is a set of axioms of T if $T = \{\sigma | \Gamma \vDash \sigma\}$.

Some theories can be defined by more than one set of axioms, we will see an example of this in section 2.3. Given a theory T, the intriguing issue will be to find its most simple set of axioms and, if possible, finite. If the set of axioms of a theory T is finite we say that T is **finitely axiomatizable**.

Theorem 1.3. *(Completeness theorem for theories). Let T be a theory in the language $\mathcal{L}$ with set of axioms Σ and let σ be an $\mathcal{L}$-sentence. Then $\sigma \in T$ iff $\Sigma \vdash \sigma$ iff $\Sigma \vDash \sigma$.*

The other way is defining T as the set of all sentences which hold in M, for M a model of $\mathcal{L}$. In this case, we denote $T = Th(M)$ as **the theory of** M and we say that M models T.

Some theories can be defined by more than one model and there are also theories that can not be defined by any model. If there is some model M for $\mathcal{L}$ satisfying all sentences of T we say that T is **satisfiable** and write $T \subseteq Th(M)$.

We say that a theory T of $\mathcal{L}$ is **complete** if for each sentence σ of $\mathcal{L}$, either $\sigma \in T$ or $\neg\sigma \in T$. The theory of a model is always complete and satisfiable. Moreover, we can easily see that a theory is complete if and only if all its models are elementarily equivalent.

A theory T is **inconsistent** if there is some $\mathcal{L}$-sentence σ such that $\sigma \in T$ and $\neg\sigma \in T$. If a theory T is not inconsistent we say that T is **consistent**. Every consistent theory is satisfiable.

Given a complete and consistent theory T and an $\mathcal{L}$-sentence σ then $\sigma \notin T \Leftrightarrow \neg\sigma \in T$.

Theorem 1.4. *(Compactness) A set Σ of $\mathcal{L}$-formulas is satisfiable iff every finite subset $S \subseteq \Sigma$ is satisfiable.*

Proof. You can find a proof of the **Compactness theorem** in page 142 of [3]. □

Chapter 2

Peano Arithmetic

We will work in the language $\mathcal{L}_{\mathcal{A}} = \{0, 1, +, \cdot, <\}$ where 0, 1 are constants, $+, \cdot$ binary function symbols and $<$ a binary relation symbol. Each symbol of $\mathcal{L}_{\mathcal{A}}$ is meant to represent its common interpretation, 0 for the natural number zero, 1 for the one, $+$ and $\cdot$ for the addition and the product and $<$ for the linear order.

Notation 2.1. All the $\mathcal{L}_{\mathcal{A}}$ -formulas will be written in the "natural" way, instead of writing $+(x, y)$ or $\cdot(x, y)$ we will write $x + y$ and $x \cdot y$.

Notation 2.2. Given an $\mathcal{L}_{\mathcal{A}}$-term t and an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(\bar{x}, y)$, we will use $\forall y < t\varphi(\bar{x}, y)$ as an abbreviation for $\forall y(y < t \rightarrow \varphi(\bar{x}, y))$ and $\exists y < t\varphi(\bar{x}, y)$ for $\exists y(y < t \wedge \varphi(\bar{x}, y))$, to say that the quantifier is bounded by t in φ. Similarly, we will write $\forall y \leq t\varphi(\bar{x}, y)$ for $\forall y(y \leq t \rightarrow \varphi(\bar{x}, y))$ and $\exists y \leq t\varphi(\bar{x}, y)$ for $\exists y(y \leq t \wedge \varphi(\bar{x}, y))$.

Notation 2.3. Given an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(x, \bar{y})$, we will write $\exists! x\varphi(x, \bar{y})$ as an abbreviation for $\exists x\varphi(x, \bar{y}) \wedge \forall x, z(\varphi(x, \bar{y}) \wedge \varphi(z, \bar{y}) \rightarrow x \doteq z)$, to say that there is a unique x satisfying the formula φ .

2.1 The standard model

To introduce **Peano Arithmetic** we need to start by presenting the structure $\mathbb{N}$ and some of its characteristics. The structure $\mathbb{N}$, also denoted as **the standard model** is an $\mathcal{L}_{\mathcal{A}}$-structure with domain the set of non-negative integers and with the common interpretation for the symbols in $\mathcal{L}_{\mathcal{A}}$.

Notation 2.4. We will denote the complete $\mathcal{L}_{\mathcal{A}}$-theory of the standard model as $Th(\mathbb{N})$.

To give a more precise definition of $\mathbb{N}$ we will focus on those $\mathcal{L}_{\mathcal{A}}$-structures that are not isomorphic to $\mathbb{N}$, called **nonstandard structures**.

Notation 2.5. For each $n \in \mathbb{N}$ we denote the numeral of n, given by the closed term $\underbrace{(\ldots(((1+1)+1)+1)+\ldots+1)}_{n}$ of $\mathcal{L}_{\mathcal{A}}$, as $\underline{n}$.

Let us expand the language $\mathcal{L}_{\mathcal{A}}$ to a language $\mathcal{L}_{\mathcal{C}}$ by adding a new constant symbol c and consider then the $\mathcal{L}_{\mathcal{C}}$-theory T_c given by the axioms of $Th(\mathbb{N})$ and the axioms

$$c > \underline{n} \;\; for \;\; each \;\; n \in \mathbb{N}.$$

Proposition 2.6. *The theory T_c is consistent.*

Proof. For each finite subset S of T_c exists some k such that $k > n$ for all $n \in S$.

Let us define the $\mathcal{L}_{\mathcal{C}}$-structure $(\mathbb{N}, k)$ with domain $\mathbb{N}$ and $0, 1, +, \cdot, <$ interpreted naturally and c interpreted by k. This structure satisfies S.

We have found then a model for every subset of T_c. Hence, by the compactness theorem, the theory T_c is consistent. $\square$

As a corollary, T_c has a model M_c. Since $M_c \vDash c > \underline{n}$ for all $n \in \mathbb{N}$ we can say that M_c contains an "infinite" integer. Let us reduce M_c to the original language $\mathcal{L}_{\mathcal{A}}$ and denote this model by M.

Proposition 2.7. *M is not isomorphic to the standard model $\mathbb{N}$, so M is nonstandard.*

Proof. Let us suppose that there is an isomorphism $h : \mathbb{N} \to M$. This isomorphism should send each $n \in \mathbb{N}$ to an element of M, $h(n) = \underline{n}^M$.

Since $\mathbb{N} \vDash \forall x, y(x > y \to \neg x \doteq y)$, then $M \vDash \forall x, y(x > y \to \neg x \doteq y)$. Therefore, the element realizing c in M can not be in the image of h. $\square$

From now on we will identify $\mathbb{N}$ with the image of h in M, so $\mathbb{N}$ is a substructure of every model $M \vDash Th(\mathbb{N})$. We will denote the elements of M that are not in $\mathbb{N}$ as **nonstandard elements**.

We say then that $\mathbb{N}$ is an **initial segment** of M and M an **end-extension** of $\mathbb{N}$, $\mathbb{N} \subseteq_e M$, since $\mathbb{N} \subseteq M$ and for all $n \in \mathbb{N}$ and all $b \in M$ such that $M \vDash b < \underline{n}$ we have $b \in \mathbb{N}$.

2.2 The axioms of PA

To present the **system of Peano Arithmetic** we first need to define the theory PA^-, a theory defined by 16 axioms given by sentences that are obviously true in $\mathbb{N}$.

Notation 2.8. We will omit some parentheses in the axioms to ease the reading of the sentences.

The first four axioms of PA^- state the basic properties of the binary functions $\cdot$ and $+$: the commutative and the associative properties. Moreover, the fifth axiom says that $+$ and $\cdot$ satisfy the distributive law.

Axiom 2.9. $\forall x, y(x + y \doteq y + x)$

Axiom 2.10. $\forall x, y(x \cdot y \doteq y \cdot x)$

Axiom 2.11. $\forall x, y, z((x + y) + z \doteq x + (y + z))$

Axiom 2.12. $\forall x, y, z((x \cdot y) \cdot z \doteq x \cdot (y \cdot z))$

Axiom 2.13. $\forall x, y, z(x \cdot (y + z) \doteq x \cdot y + x \cdot z)$

The next two axioms state that 0 is the identity for + and a zero for $\cdot$, and that 1 is the identity for $\cdot$.

Axiom 2.14. $\forall x((x + 0 \doteq x) \wedge (x \cdot 0 \doteq 0))$

Axiom 2.15. $\forall x(x \cdot 1 \doteq x)$

The following axioms make reference to the linear order in $\mathbb{N}$ given by the relation symbol $<$. The first three state that $<$ is transitive and irreflexive and that satisfies the trichotomy law.

Axiom 2.16. $\forall x, y, z((x < y \wedge y < z) \rightarrow x < z)$

Axiom 2.17. $\forall x \neg x < x$

Axiom 2.18. $\forall x, y(x < y \vee x \doteq y \vee y < x)$

From this three axioms we can also deduce the asymmetric property, which says that $\forall x, y(x < y \rightarrow \neg y < x)$. We can use $x \leq y$ to express $x < y \vee x \doteq y$ and rewrite then axiom 2.18 as $\forall x, y(x \leq y \vee y \leq x)$ and the asymmetric property as $\forall x, y(x \leq y \leftrightarrow \neg y < x)$.

The next two axioms state that the operations $+$ and $\cdot$ respect the order.

Axiom 2.19. $\forall x, y, z(x < y \rightarrow x + z < y + z)$

Axiom 2.20. $\forall x, y, z(0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z)$

The thirteenth axiom is similar to the idea of subtraction in $\mathbb{N}$ and says that for $x < y$, x can be subtracted from y.

Axiom 2.21. $\forall x, y(x < y \to \exists z \quad x + z \doteq y)$

The order in $\mathbb{N}$ is also a discrete order and we state this with the next axiom.

Axiom 2.22. $0 < 1 \wedge \forall x(0 < x \to 1 \leq x)$

To finish, the last axiom says that 0 is the least natural number.

Axiom 2.23. $\forall x(0 \leq x)$

Now that the theory PA^- has been described we can define Peano Arithmetic. The axioms of Peano Arithmetic are those of PA^- together with the second-order induction axiom,

$$\forall X(0 \in X \wedge \forall x(x \in X \to x + 1 \in X) \to \forall y(y \in X)).$$

With the incorporation of this last axiom, Peano Arithmetic characterizes the standard model $\mathbb{N}$ up to isomorphism. But we are not interested in working with second-order logic, since there is no Completeness Theorem for second-order logic. Therefore, we will restrict the induction axiom to subsets X defined by a first-order $\mathcal{L}_{\mathcal{A}}$-formula, obtaining so a weaker theory, PA, which no longer characterizes $\mathbb{N}$. The restricted induction axiom, $I_x\varphi$, is given by the sentence

$$\forall \bar{y}(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \varphi(x + 1, \bar{y})) \to \forall x \varphi(x, \bar{y}))$$

with $\varphi(x, \bar{y})$ an $\mathcal{L}_{\mathcal{A}}$-formula, x the induction variable and $\bar{y}$ the parameters.

So, finally, we define PA as the first-order theory axiomatized by PA^- together with the induction axioms $I_x\varphi$ over all $\mathcal{L}_{\mathcal{A}}$-formulas φ.

Remark 2.24. *PA* is a recursively axiomatized theory (even though non-finite), which means that there is a recursive procedure (an algorithm) to decide if a given sentence is an axiom of *PA*.

2.3 Alternative induction schemes

In this section we will show alternative sets of axioms which can also define *PA*. In particular, we will be interested in changing the induction scheme by others that can be justified in *PA*. At the same time, we will also develop new techniques for proving theorems in *PA*.

2.3.1 Principle of induction up to z

When working in $\mathbb{N}$, if we wish to show $\mathbb{N} \vDash \forall x \leq \underline{n}\varphi(x)$ for $n \in \mathbb{N}$ and $\varphi(x)$ a formula, is clearly enough to show that $\mathbb{N} \vDash \varphi(0) \wedge \forall x < \underline{n}(\varphi(x) \rightarrow \varphi(x+1))$, even if the stronger statement $\mathbb{N} \vDash \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)$ might not be true. The idea is to find an equivalent principle proved by *PA*.

We can express this principle by the scheme

$$\forall \bar{y}, z(\varphi(0,\bar{y}) \wedge \forall x < z(\varphi(x,\bar{y}) \rightarrow \varphi(x+1,\bar{y})) \rightarrow \forall x \leq z\varphi(x,\bar{y}))$$

over all $\mathcal{L}_\mathcal{A}$-formulas $\varphi(x,\bar{y})$, denoted by $U_x\varphi$.

Proposition 2.25. *PA proves all instances of* $U_x\varphi$.

Proof. Let M be an arbitrary model of *PA* and let $\varphi(x,\bar{y})$ be any $\mathcal{L}_\mathcal{A}$-formula. By the completeness theorem, it will be enough to prove that $M \vDash U_x\varphi$.

Let us assume $\bar{a}, b \in M$ and $M \vDash \varphi(0,b) \wedge \forall x < b(\varphi(x,\bar{a}) \rightarrow \varphi(x+1,\bar{a}))$ to show $M \vDash \forall x \leq b\varphi(x,\bar{a})$.

If we define the $\mathcal{L}_\mathcal{A}$-formula

$$\psi(x,\bar{y},z) := (x \leq z \wedge \varphi(x,\bar{y})) \vee (x > z),$$

clearly $M \vDash \forall x > b\psi(x,\bar{a},b)$.

From the assumption of $M \vDash \varphi(0,b) \wedge \forall x < b(\varphi(x,\bar{a}) \rightarrow \varphi(x+1,\bar{a}))$ follows that $M \vDash \psi(0,\bar{a},b) \wedge \forall x < b(\psi(x,\bar{a},b) \rightarrow \psi(x+1,\bar{a},b))$.

So $M \vDash \psi(0,\bar{a},b) \wedge \forall x(\psi(x,\bar{a},b) \rightarrow \psi(x+1,\bar{a},b))$ and, by induction, $M \vDash \forall x\psi(x,\bar{a},b)$. Hence, by the definition of $\psi(x,\bar{y},b)$, $M \vDash \forall x \leq b\varphi(x,\bar{a})$ as required. □

2.3.2 Least number principle

As $\mathbb{N}$ is a well-ordered set, it is true that every non-empty set of $\mathbb{N}$ has a least element. Since we are working with a first-order language $\mathcal{L}_\mathcal{A}$, we need to find an aproximate principle proved by *PA*. The scheme

$$\forall \bar{y}(\exists x\varphi(x,\bar{y}) \rightarrow \exists z(\varphi(z,\bar{y}) \wedge \forall w < z\neg\varphi(w,\bar{y})))$$

over all $\mathcal{L}_\mathcal{A}$-formulas $\varphi(x,\bar{y})$, denoted by L_φ, states this principle.

Proposition 2.26. *PA proves all instances of* L_φ.

Proof. Let M be an arbitrary model of *PA* and let $\varphi(x,\bar{y})$ be any $\mathcal{L}_\mathcal{A}$-formula.

For $\bar{a}, b \in M$ we will assume that $M \vDash \varphi(b,\bar{a})$ and $M \nvDash \exists z(\varphi(z,\bar{a}) \wedge \forall w < z\neg\varphi(w,\bar{a}))$, i.e. $M \vDash \forall z(\varphi(z,\bar{a}) \rightarrow \exists w < z\varphi(w,\bar{a}))$, to arrive to a contradiction.

Let us define the $\mathcal{L}_\mathcal{A}$-formula

$$\theta(x,\bar{y}) := \forall z(z < x \rightarrow \neg\varphi(z,\bar{y})).$$

Notice that $M \vDash \theta(0, \bar{a})$, since 0 is the smallest element of M. Now suppose $c \in M$ and $M \vDash \theta(c, \bar{a})$, to show that $M \vDash \theta(c+1, \bar{a})$.

If $d \in M$ and $M \vDash d < c+1$ we can consider two cases:

1. $M \vDash d < c$: So $M \vDash \neg\varphi(d, \bar{a})$ since $M \vDash \theta(c, \bar{a})$.

2. $M \vDash d \doteq c$: Then $M \vDash \forall w < c \neg\varphi(w, \bar{a})$, i.e. $M \vDash \neg(\exists w < c)\varphi(w, \bar{a})$, and by the assumption of $M \vDash \forall d(\varphi(d, \bar{a}) \rightarrow \exists w < d\varphi(w, \bar{a}))$, $M \vDash \neg\varphi(d, \bar{a})$.

In both cases $M \vDash \neg\varphi(d, \bar{a})$, which implies that $M \vDash \theta(c+1, \bar{a})$. As a result,

$$M \vDash \theta(0, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \rightarrow \theta(x+1, \bar{a}))$$

and, by $I_x\theta$, $M \vDash \forall x\theta(x, \bar{a})$, i.e. $M \vDash \forall z\neg\varphi(z, \bar{a})$, contradicting so the existence of some $b \in M$ such that $M \vDash \varphi(b, \bar{a})$ as required. □

2.3.3 Principle of complete induction

The last induction principle is a formulation of the principle of complete induction for $\mathbb{N}$, which states that for proving $\mathbb{N} \vDash \forall x\varphi(x)$, for $\varphi(x)$ a formula, it is enough to prove $\mathbb{N} \vDash \forall x(\forall z < x\varphi(z) \rightarrow \varphi(x))$. We usually define this principle for all sets of $\mathbb{N}$, but since we are working with first-order logic we will enunciate it for those sets that can be defined with $\mathcal{L}_\mathcal{A}$-formulas.

The principle of complete induction for first-order logic is the one given by the scheme

$$\forall \bar{y}(\forall x(\forall z < x\varphi(z, \bar{y}) \rightarrow \varphi(x, \bar{y})) \rightarrow \forall x\varphi(x, \bar{y}))$$

over all $\mathcal{L}_\mathcal{A}$-formulas $\varphi(x, \bar{y})$, denoted by $T_x\varphi$.

Proposition 2.27. *PA proves all instances of $T_x\varphi$.*

Proof. Let M be an arbitrary model of *PA* and $\varphi(x, \bar{y})$ any $\mathcal{L}_\mathcal{A}$-formula.

Let $\bar{a} \in M$ and suppose $M \vDash \forall x(\forall z < x\varphi(z, \bar{a}) \rightarrow \varphi(x, \bar{a}))$ and $M \nvDash \forall x\varphi(x, \bar{a})$, i.e. $M \vDash \neg\varphi(b, \bar{a})$ for some $b \in M$, to arrive to a contradiction.

Since $M \vDash L_{\neg\varphi}$, there is a least $b \in M$ such that $M \vDash \neg\varphi(b, \bar{a})$, contradicting the hypothesis of $M \vDash \forall x(\forall z < x\varphi(z, \bar{a}) \rightarrow \varphi(x, \bar{a}))$. Hence $M \vDash \forall x\varphi(x, \bar{a})$, as required.

□

Chapter 3

Complexity of formulas

3.1 The arithmetic hierarchy

Definition 3.1. *An $\mathcal{L}_{\mathcal{A}}$-formula φ is Δ_0 iff all its quantifiers are bounded.*

We also denote Δ_0 by Σ_0 and Π_0. With the initial case defined, we can now define the classes Σ_n and Π_n for all $n \in \mathbb{N}$.

Definition 3.2. *An $\mathcal{L}_{\mathcal{A}}$-formula φ is Σ_{n+1} iff it is of the form $\exists \bar{x} \psi(\bar{x}, \bar{y})$ for ψ a Π_n $\mathcal{L}_{\mathcal{A}}$-formula, which means that φ looks like*

$$\exists \bar{x}_1 \forall \bar{x}_2 \exists \bar{x}_3 ... Q \bar{x}_n \psi(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{y})$$

where all quantifiers in ψ are bounded and Q is $\exists$ if n is even or $\forall$ if n is odd.

Definition 3.3. *An $\mathcal{L}_{\mathcal{A}}$-formula φ is Π_{n+1} iff it is of the form $\forall \bar{x} \psi(\bar{x}, \bar{y})$ for ψ a Σ_n $\mathcal{L}_{\mathcal{A}}$-formula which means that φ looks like*

$$\forall \bar{x}_1 \exists \bar{x}_2 \forall \bar{x}_3 ... Q \bar{x}_n \psi(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{y})$$

where all quantifiers in ψ are bounded and Q is $\exists$ if n is odd or $\forall$ if n is even.

Notation 3.4. We will write $\varphi \in \Sigma_n$ if there is a Σ_n $\mathcal{L}_{\mathcal{A}}$-formula ψ equivalent to φ in the model M or the theory T, and $\varphi \in \Pi_n$ if there is an equivalent $\mathcal{L}_{\mathcal{A}}$-formula Π_n. When it is important to remark the model or theory in which this equivalence takes place, we will write $\Sigma_n(T)$, $\Pi_n(T)$, $\Sigma_n(M)$ or $\Pi_n(M)$.

Definition 3.5. *An $\mathcal{L}_{\mathcal{A}}$-formula φ is Δ_n iff it is equivalent to both, a Σ_n and a Π_n formula.*

Notation 3.6. As before, if it is important to remark the model or theory where the equivalence takes place, we will write $\Delta_n(M)$ or $\Delta_n(T)$.

Definition 3.7. *An $\mathcal{L}_{\mathcal{A}}$-formula φ is* ***provably*** *$\Delta_n(T)$ if there are formulas $\psi \in \Sigma_n$ and $\chi \in \Pi_n$ such that*

$$T \vdash \varphi \leftrightarrow \psi \quad \text{and} \quad T \vdash \varphi \leftrightarrow \chi.$$

Remark 3.8. Every $\mathcal{L}_{\mathcal{A}}$-formula is equivalent to a Σ_n or Π_n $\mathcal{L}_{\mathcal{A}}$-formula for some $n \in \mathbb{N}$.

Since blocks of quantifiers are allowed to be empty, any Π_n formula is both, a Σ_{n+1} and a Π_{n+1} formula. The same happens for Σ_n formulas. We have then the obvious inclusions $\Sigma_n \subseteq \Delta_{n+1} \subseteq \Sigma_{n+1}$ and $\Pi_n \subseteq \Delta_{n+1} \subseteq \Pi_{n+1}$.

Proposition 3.9. *The classes Σ_n and Π_n are closed under conjunctions and disjunctions.*

Proof. Let us assume $\theta_1(\bar{x}), \theta_2(\bar{x}) \in \Sigma_n$ for $n \in \mathbb{N}$ and prove $\theta_1(\bar{x}) \wedge \theta_2(\bar{x}) \in \Sigma_n$.

We can write $\theta_1(\bar{x})$ as $\exists \bar{y}_1 \forall \bar{y}_2 .... Q\bar{y}_n \varphi_1(\bar{x}, \bar{y})$ and $\theta_2(\bar{x})$ as $\exists \bar{z}_1 \forall \bar{z}_2 .... Q\bar{z}_n \varphi_2(\bar{x}, \bar{z})$, with $\bar{y}_i \cap \bar{z}_j = \emptyset$ for all $i, j \in \{1, \dots, n\}$, $\varphi_1, \varphi_2 \in \Delta_0$ and $Q = \exists$ if n is even or $Q = \forall$ if n is odd, and hence $\theta_1(\bar{x}) \wedge \theta_2(\bar{x})$ as

$$\exists \bar{y}_1 \forall \bar{y}_2 .... Q\bar{y}_n \varphi_1(\bar{x}, \bar{y}) \wedge \exists \bar{z}_1 \forall \bar{z}_2 .... Q\bar{z}_n \varphi_2(\bar{x}, \bar{z})$$

Notice that this formula is equivalent to $\exists \bar{y}_1 \bar{z}_1 \forall \bar{y}_2 \bar{z}_2 .... Q\bar{y}_n \bar{z}_n (\varphi_1(\bar{x}, \bar{y}) \wedge \varphi_2(\bar{x}, \bar{z}))$ if we re-order the quantifiers, so $\theta_1(\bar{x}) \wedge \theta_2(\bar{x}) \in \Sigma_n$.

Following the same arguments we can prove the same for $\theta_1(\bar{x}), \theta_2(\bar{x}) \in \Pi_n$ and for the disjunction. $\square$

Proposition 3.10. *If $\theta(\bar{x}) \in \Sigma_n$ then $\neg\theta(\bar{x}) \in \Pi_n$. Similarly if $\theta(\bar{x}) \in \Pi_n$, $\neg\theta(\bar{x}) \in \Sigma_n$.*

Remark 3.11. This proposition proves that the class Δ_n is closed under negations, even if Σ_n and Π_n are not.

3.2 The collection axiom

The aim of this section will be showing that the classes Σ_n, Π_n and Δ_n are closed under bounded quantification in *PA*. To do so, we need to define the **collection axiom**.

Given an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(x, \bar{y}, \bar{z})$, the **collection axiom** for φ is the sentence

$$\forall \bar{z}, t (\forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{z}) \rightarrow \exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{z}))$$

denoted by B_φ. Since the converse of B_φ is true for all $\mathcal{L}_{\mathcal{A}}$-structures, we have

$$\forall \bar{z}, t (\forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{z}) \leftrightarrow \exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{z})),$$

which means that we can sometimes transform a formula Π_2 into a Σ_1.

If we consider as well the collection axiom for $\neg\varphi$, we obtain

$$\forall \bar{z}, t (\exists x < t \forall \bar{y} \varphi(x, \bar{y}, \bar{z}) \leftrightarrow \forall s \exists x < t \forall \bar{y} < s \varphi(x, \bar{y}, \bar{z})),$$

transforming so a Σ_2 formula into a Π_1.

Notation 3.12. We will denote $\{B_\varphi \mid \varphi \text{ a } \Sigma_n\ \mathcal{L}_{\mathcal{A}}\text{-formula}\}$, with φ a Σ_n formula in the strict sense, which means φ in Σ_n form, not equivalent to a Σ_n formula, by $B\Sigma_n$ and similarly $\{B_\varphi \mid \varphi \text{ a } \Pi_n\ \mathcal{L}_{\mathcal{A}}\text{-formula}\}$, with φ a Π_n formula in the strict sense, by $B\Pi_n$.

We can define now a new theory $Coll = PA^- + \{B_\varphi | \varphi \text{ is an } \mathcal{L}_\mathcal{A}\text{-formula}\}$, free of induction, and a subtheory $Coll_n$ given by the axioms of $PA^- + B\Sigma_n$.

Proposition 3.13. *For all $n \in \mathbb{N}$, let $\varphi(x,\bar{y}) \in \Sigma_n$ and $\psi(x,\bar{y}) \in \Pi_n$ be $\mathcal{L}_\mathcal{A}$-formulas and $t(\bar{z})$ an $\mathcal{L}_\mathcal{A}$-term with $x \notin \bar{z}$. Then $\forall x < t(\bar{z})\varphi(x,\bar{y}) \in \Sigma_n(Coll_n)$ and $\exists x < t(\bar{z})\psi(x,\bar{y}) \in \Pi_n(Coll_n)$. Therefore $\Sigma_n(Coll_n)$, $\Pi_n(Coll_n)$ and $\Delta_n(Coll_n)$ are closed under bounded quantification for all $n \in \mathbb{N}$.*

Proof. We will prove it by induction on n.

Initial case: $\Sigma_0 = \Pi_0 = \Delta_0$ are clearly closed under bounded quantification.

Let us prove now the induction case. We will assume $\Sigma_{n-1}(Coll_{n-1})$, $\Pi_{n-1}(Coll_{n-1})$ and $\Delta_{n-1}(Coll_{n-1})$ closed under bounded quantification to show that $\Sigma_n(Coll_n)$, $\Pi_n(Coll_n)$ and $\Delta_n(Coll_n)$ are also closed under bounded quantification.

Let M be such that $M \vDash Coll_n$ and let $\varphi(x,\bar{y})$ be an $\mathcal{L}_\mathcal{A}$-formula of the form $\exists \bar{z}\theta(x,\bar{y},\bar{z})$ for $\theta(x,\bar{y},\bar{z}) \in \Pi_{n-1}$ and $n \geq 1$. Applying the collection axiom to θ we have

$$M \vDash \forall \bar{y}(\forall x < t\varphi(x,\bar{y}) \leftrightarrow \exists s \forall x < t \exists \bar{z} < s\theta(x,\bar{y},\bar{z})). \qquad (1)$$

Notice that, by the induction hypothesis and since $\theta(x,\bar{y},\bar{z})$ is Π_{n-1}, we can conclude that $\exists \bar{z} < s\theta(x,\bar{y},\bar{z})) \in \Pi_{n-1}(Coll_{n-1})$, i.e that exists a Π_{n-1} formula $\chi(x,\bar{y},s)$ such that

$$Coll_{n-1} \vdash \forall x,\bar{y},s(\chi(x,\bar{y},s) \leftrightarrow \exists \bar{z} < s\theta(x,\bar{y},\bar{z})). \qquad (2)$$

Since $Coll_n \vdash Coll_{n-1}$ we have $M \vDash Coll_{n-1}$, and hence by (1) and (2),

$$M \vDash \forall y(\forall x < t\varphi(x,\bar{y}) \leftrightarrow \exists s \forall x < t\chi(x,\bar{y},s)).$$

The formula $\exists s \forall x < t\chi(x,\bar{y},s)$ is clearly Σ_n, since $\chi(x,\bar{y},s)$ is Π_{n-1}, so we have then that $\forall x < t\varphi(x,\bar{y}) \in \Sigma_n(Coll_n)$ as we wanted to show.

That $\Pi_n(Coll_n)$ is closed under bounded quantification can be proved in a similar way. □

We have used the collection axioms to prove that the classes Σ_n, Π_n and Δ_n are closed under bounded quantification. But now we need to show that the collection axioms are, in fact, provable in PA.

To do so, we will define a new theory called $I\Sigma_n$, resulting from the axioms of PA^- and induction for all Σ_n formulas. We can define also the theory $I\Pi_n$ in a similar way. Then PA is equivalent to $I\Sigma_1 + I\Sigma_2 + I\Sigma_3 + \ldots$, which is also equivalent to the theory $I\Pi_1 + I\Pi_2 + I\Pi_3 + \ldots$.

Proposition 3.14. *$I\Sigma_n \vdash Coll_n$ for all $n \geq 1$. Hence $PA \vdash Coll$.*

Proof. We will prove it by induction on n. We will see first the induction case and then the initial case.

For $n \geq 2$ let us suppose $I\Sigma_{n-1} \vdash Coll_{n-1}$ and show that $I\Sigma_n \vdash Coll_n$. To prove this we will assume $M \vDash I\Sigma_n$, $\varphi(x,\bar{y},\bar{z}) \in \Sigma_n$ and $\bar{a}, b \in M$ such that

$$M \vDash \forall x < b \exists \bar{y}\varphi(x,\bar{y},\bar{a})$$

and show then that $M \vDash \exists c \forall x < b \exists \bar{y} < c\varphi(x, \bar{y}, \bar{a})$.

Since $\varphi(x, \bar{y}, \bar{a}) \in \Sigma_n$, we can write it as $\exists \bar{u}\theta(x, \bar{y}, \bar{a}, \bar{u})$ for $\theta(x, \bar{y}, \bar{a}, \bar{u}) \in \Pi_{n-1}$ some $\mathcal{L}_{\mathcal{A}}$-formula and hence $\forall x < b \exists \bar{y}\varphi(x, \bar{y}, \bar{a})$ as $\forall x < b \exists \bar{y}, \bar{u}\theta(x, \bar{y}, \bar{a}, \bar{u})$ which is equivalent to $\forall x < b \exists \bar{z}\theta(x, \bar{z}, \bar{a})$ for $\theta(x, \bar{z}, \bar{a}) \in \Pi_{n-1}$ and $\bar{z} = \bar{y}\bar{u}$.

Let us consider the formula

$$\psi(u, \bar{a}) := (\exists c \forall x < u \exists \bar{z} < c\theta(x, \bar{z}, \bar{a})) \vee u > b.$$

Since $\theta(x, \bar{z}, \bar{a}) \in \Pi_{n-1}$, we can use the previous proposition and obtain then that $\forall x < u \forall \bar{z} < c\theta(x, \bar{z}, \bar{a}) \in \Pi_{n-1}(Coll_{n-1})$. By the induction hypothesis, we also have $\forall x < u \forall \bar{z} < c\theta(x, \bar{z}, \bar{a}) \in \Pi_{n-1}(I\Sigma_{n-1})$.

Notice that $I\Sigma_n \vdash I\Sigma_{n-1}$, since $\Sigma_{n-1} \subseteq \Sigma_n$, therefore $\psi(u, \bar{a}) \in \Sigma_n(I\Sigma_n)$. So we can apply induction on ψ.

Clearly $M \vDash \psi(0, \bar{a})$, so there is only left to show that $M \vDash \forall x(\psi(x, \bar{a}) \to \psi(x+1, \bar{a}))$. Let us suppose $M \vDash \psi(w, \bar{a})$ for $w \in M$ and prove then that $M \vDash \psi(w+1, \bar{a})$. We will consider two cases:

1. Case $M \vDash w \geq b$: Then $M \vDash w + 1 > b$ and $M \vDash \psi(w+1, \bar{a})$.

2. Case $M \vDash w < b$: We shall show that $M \vDash \exists c \forall x < w + 1 \exists \bar{z} < c\theta(x, \bar{z}, \bar{a})$. Since $M \vDash \psi(w, \bar{a})$, there is some $v_1 \in M$ such that $M \vDash \forall x < w \exists \bar{z} < v_1\theta(x, \bar{z}, \bar{a})$. Let us define $v_2 = max(\bar{z}) + 1$ and $v = max(v_1, v_2)$. Since $M \vDash \forall x < b \exists \bar{z}\theta(x, \bar{z}, \bar{a})$ and $M \vDash w < b$, we have found some $v \in M$ such that $M \vDash \forall x < w + 1 \exists \bar{z} < v\theta(x, \bar{z}, \bar{a})$ as required.

We have seen that $M \vDash \forall u\psi(u, \bar{a})$, so in particular, $M \vDash \psi(b, \bar{a})$, which means that $M \vDash \exists c \forall x < b \exists \bar{z} < c\theta(x, \bar{z}, \bar{a})$ and hence $M \vDash \exists c \forall x < b \exists \bar{y} < c\varphi(x, \bar{y}, \bar{a})$.

Let us prove now the initial case, $I\Sigma_1 \vdash Coll_1$. We will assume $M \vDash I\Sigma_1$, $\varphi(x, \bar{y}, \bar{z}) \in \Sigma_1$ and $\bar{a}, b \in M$ such that $M \vDash \forall x < b \exists \bar{y}\varphi(x, \bar{y}, \bar{a})$ to show that there is some $c \in M$ such that $M \vDash \forall x < b \exists \bar{y} < c\varphi(x, \bar{y}, \bar{a})$. Since $\psi(w, \bar{a}) \in \Sigma_1$ we can apply induction on ψ as before and obtain $M \vDash \exists c \forall x < b \exists \bar{y} < c\varphi(x, \bar{y}, \bar{a})$. □

Now we will see that, in fact, collection is actually equivalent to PA over the theory $I\Delta_0$, i.e. that PA is equivalent to the theory $I\Delta_0 + Coll$. For showing this we will need two previous lemmas.

Lemma 3.15. *For each $n \geq 0$ we have $I\Pi_n + Coll_{n+2} \vdash I\Sigma_{n+1}$.*

Proof. For M such that $M \vDash I\Pi_n + Coll_{n+2}$, we want to show that $M \vDash I\Sigma_{n+1}$.

To do so we will assume $M \vDash \theta(0, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \to \theta(x+1, \bar{a}))$ for some $\theta(x, \bar{y}) \in \Sigma_{n+1}$ and some $\bar{a} \in M$ and prove that $M \vDash \theta(b, \bar{a})$ for all $b \in M$.

Since $\theta(x, \bar{a}) \in \Sigma_{n+1}$ we can write it as $\exists \bar{y}\psi(x, \bar{y}, \bar{a})$ for $\psi(x, \bar{y}, \bar{a}) \in \Pi_n$ an $\mathcal{L}_{\mathcal{A}}$-formula. Let us define the formula

$$\chi(x, \bar{y}, \bar{a}) := \psi(x, \bar{y}, \bar{a}) \vee \forall \bar{z} \neg \psi(x, \bar{z}, \bar{a}),$$

which can also be written as $\exists \bar{z}\psi(x, \bar{z}, \bar{a}) \to \psi(x, \bar{y}, \bar{a})$.

The negation of a formula Π_n is Σ_n, so $\neg\psi \in \Sigma_n$. If we add the quantifier we have, $\forall \bar{z} \neg \psi(x, \bar{z}, \bar{a}) \in \Pi_{n+1}$ and hence $\chi \in \Pi_{n+1} \subseteq \Pi_{n+2}$. So we can apply collection to χ, since $M \vDash Coll_{n+2}$.

Notice that $M \vDash \exists \bar{y} \chi(x, \bar{y}, \bar{a})$, in particular $M \vDash \forall x < b + 1 \exists \bar{y} \chi(x, \bar{y}, \bar{a})$ for any arbitrary $b \in M$, and by the collection axiom, $M \vDash \exists c \forall x < b + 1 \exists \bar{y} < c \chi(x, \bar{y}, \bar{a})$. So there is a $c \in M$ such that

$$M \vDash \forall x \leq b \exists \bar{y} < c(\exists \bar{z} \psi(x, \bar{z}, \bar{a}) \to \psi(x, \bar{y}, \bar{a})).$$

Reordering the formula we obtain that there is some $c \in M$ such that

$$M \vDash \forall x \leq b(\exists \bar{y} \psi(x, \bar{y}, \bar{a}) \to \exists \bar{y} < c \psi(x, \bar{y}, \bar{a})).$$

Since the other implication is clearly true, there is some $c \in M$ such that

$$M \vDash \forall x \leq b(\exists \bar{y} \psi(x, \bar{y}, \bar{a}) \leftrightarrow \exists \bar{y} < c \psi(x, \bar{y}, \bar{a})).$$

Let us define a formula $\varphi(x, c, \bar{a}) := \exists \bar{y} < c \psi(x, \bar{y}, \bar{a})$. From the hypothesis follows that $M \vDash \theta(0, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \to \theta(x + 1, \bar{a}))$ and since $M \vDash \theta(x, \bar{a}) \leftrightarrow \exists \bar{y} \psi(x, \bar{y}, \bar{a})$ and $M \vDash \varphi(x, c, \bar{a}) \leftrightarrow \exists \bar{y} < c \psi(x, \bar{y}, \bar{a})$ we have

$$M \vDash \varphi(0, c, \bar{a}) \wedge \forall x < b(\varphi(x, c, \bar{a}) \to \varphi(x + 1, c, \bar{a})).$$

The formula φ is clearly Π_n and since $M \vDash I\Pi_n$, we can apply induction up to b to φ, obtaining so $M \vDash \forall x \leq b \varphi(x, c, \bar{a})$. In particular, $M \vDash \varphi(b, c, \bar{a})$ and hence $M \vDash \theta(b, \bar{a})$ as required. □

Lemma 3.16. *For all $n \geq 0$ we have $I\Sigma_n \vdash I\Pi_n$ and $I\Pi_n \vdash I\Sigma_n$.*

Proof. To prove $I\Pi_n \vdash I\Sigma_n$ we will assume $M \vDash I\Pi_n$ and show $M \vDash I\Sigma_n$. To do so, let $\varphi(x, \bar{y})$ be a Σ_n formula and $\bar{a} \in M$ such that

$$M \vDash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \to \varphi(x + 1, \bar{a}))$$

and prove then that $M \vDash \varphi(b, \bar{a})$ for each $b \in M$.

We will assume the opposite, i.e that there is some $b \in M$ such that $M \vDash \neg\varphi(b, \bar{a})$, to arrive to a contradiction. Let us define the formula

$$\psi(x, b, \bar{a}) := x > b \vee (x \leq b \wedge \forall y(y + x \doteq b \to \neg\varphi(y, \bar{a})))$$

which is Π_n, as $\neg\varphi \in \Pi_n$.

Notice that $M \vDash \psi(0, b, \bar{a})$, since $M \vDash \neg\varphi(b, \bar{a})$, $0 \leq b$ and $M \vDash y + 0 \doteq b$ only if $M \vDash y \doteq b$. By the initial assumption of $M \vDash \forall x(\varphi(x, \bar{a}) \to \varphi(x + 1, \bar{a}))$, we have $M \vDash \forall x(\psi(x, b, \bar{a}) \to \psi(x + 1, b, \bar{a}))$ and therefore

$$M \vDash \psi(0, b, \bar{a}) \wedge \forall x(\psi(x, b, \bar{a}) \to \psi(x + 1, b, \bar{a}))$$

Since $M \vDash I\Pi_n$ and $\psi \in \Pi_n$ we can deduce that $M \vDash \forall x \psi(x, b, \bar{a})$ by applying induction to the formula ψ. In particular, $M \vDash \psi(b, b, \bar{a})$ and hence $M \vDash \neg\varphi(0, a)$, contradicting so $M \vDash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \to \varphi(x + 1, \bar{a}))$.

To prove $I\Sigma_n \vdash I\Pi_n$ we shall follow the same arguments but considering $\psi(x, b, \bar{a})$ as $x > b \vee (x \leq b \wedge \exists y(y + x \doteq b \wedge \neg\varphi(y, \bar{a})))$. □

Theorem 3.17. *PA is equivalent to* $I\Delta_0 + Coll$.

Proof. To show that PA is equivalent to $I\Delta_0 + Coll$, we need to prove both directions, $PA \vdash I\Delta_0 + Coll$ and $I\Delta_0 + Coll \vdash PA$.

$PA \vdash I\Delta_0 + Coll$ follows from proposition 3.14. So there is only left showing that $I\Delta_0 + Coll \vdash PA$. Since PA is equivalent to $I\Sigma_1 + I\Sigma_2 + I\Sigma_3 + \dots$ it will be enough to show that $I\Delta_0 + Coll_{n+1} \vdash I\Sigma_n$ for all $n \in \mathbb{N}$. We will do it by induction on n.

The initial case is clearly true since $I\Sigma_0$ is equivalent to $I\Delta_0$. Let us suppose now that $I\Delta_0 + Coll_{n+1} \vdash I\Sigma_n$ and show $I\Delta_0 + Coll_{n+2} \vdash I\Sigma_{n+1}$. By lemma 3.16 we have $I\Delta_0 + Coll_{n+1} \vdash I\Pi_n$ and by lemma 3.15, $I\Delta_0 + Coll_{n+1} + Coll_{n+2} \vdash I\Sigma_{n+1}$.

Since $Coll_{n+2} \vdash Coll_{n+1}$ we can omit $Coll_{n+1}$ and obtain $I\Delta_0 + Coll_{n+2} \vdash I\Sigma_{n+1}$ as required. □

Notation 3.18. We denote $\{L_\varphi \mid \varphi \text{ a } \Sigma_n\ \mathcal{L}_\mathcal{A}\text{-formula}\}$, with φ a Σ_n formula in the strict sense, which means φ in Σ_n form, not equivalent to a Σ_n formula, by $L\Sigma_n$ and similarly $\{L_\varphi \mid \varphi \text{ a } \Pi_n\ \mathcal{L}_\mathcal{A}\text{-formula}\}$, with φ a Π_n formula in the strict sense, by $L\Pi_n$.

Proposition 3.19. *For all* $n \geq 0$ *we have* $L\Sigma_n \Leftrightarrow I\Sigma_n \Leftrightarrow I\Pi_n \Leftrightarrow L\Pi_n$.

Proof. $I\Sigma_n \Leftrightarrow I\Pi_n$ is proved by lemma 3.16. We will only prove $L\Sigma_n \Leftrightarrow I\Pi_n$, since the proof of $L\Pi_n \Leftrightarrow I\Sigma_n$ is similar. We can prove $I\Pi_n \Rightarrow L\Sigma_n$ following the proof of proposition 2.26, since if $\varphi \in \Sigma_n$ then $\theta \in \Pi_n$. So there is only left to show that $L\Sigma_n \Rightarrow I\Pi_n$.

Let M be an arbitrary model of PA such that $M \vDash L\Sigma_n$ and show then that $M \vDash I\Pi_n$. For $\bar{a} \in M$ and for an $\mathcal{L}_\mathcal{A}$-formula $\varphi(x, \bar{y}) \in \Pi_n$ we will suppose that $M \vDash \varphi(0, \bar{a})$ and $M \vDash \varphi(x, \bar{a}) \to \varphi(x+1, \bar{a})$ and prove then that $M \vDash \forall x \varphi(x, \bar{a})$. Let us assume that there is some $b \in M$ such that $M \vDash \neg\varphi(b, \bar{a})$ to arrive to a contradiction.

Since $\varphi(x, \bar{y}) \in \Pi_n$ we have that $\neg\varphi(x, \bar{y}) \in \Sigma_n$. By the least number principle there is some $d \in M$ such that $M \vDash \neg\varphi(d, \bar{a}) \wedge \forall z < d \varphi(z, \bar{a})$, and $M \vDash \neg d \doteq 0$ since $M \vDash \varphi(0, \bar{a})$. We can write then $d = c + 1$ for some $c \in M$. As $M \vDash c < d$ we also have $M \vDash \varphi(c, \bar{a})$ and by the hypothesis of $M \vDash \varphi(x, \bar{a}) \to \varphi(x+1, \bar{a})$ we can conclude that $M \vDash \varphi(c+1, d)$ arriving so to a contradiction. □

3.3 Extensions of $\mathcal{L}_\mathcal{A}$

In this section we will show how to extend a language and a theory by introducing new symbols and its applications in the study of the complexity of formulas.

Definition 3.20. *We say that an* $\mathcal{L}_\mathcal{A}$*-formula* $\varphi(x_1, \dots, x_n, y)$ *is* ***functional*** *in PA if*

$$PA \vdash \forall x_1, \dots, x_n \exists! y \varphi(x_1, \dots, x_n, y).$$

Proposition 3.21. *For all* $n \in \mathbb{N}$, *if* $\varphi(x_1, \dots, x_m, y) \in \Sigma_n$ *is functional in PA, then* φ *is provably* $\Delta_n(PA)$.

Proof. Let $\varphi(x_1,\ldots,x_m,y) \in \Sigma_n$ be an $\mathcal{L}_\mathcal{A}$-formula and functional in PA, we have then $PA \vdash \forall x_1,\ldots,x_m \exists! y \varphi(x_1,\ldots,x_m,y)$ and hence,

$$PA \vdash \forall z(\neg z \doteq y \to \neg\varphi(x_1,\ldots,x_m,z)) \leftrightarrow \varphi(x_1,\ldots,x_m,y).$$

Since the negation of a Σ_n formula is a Π_n, $\neg\varphi(x_1,\ldots,x_m,y) \in \Pi_n$ and therefore $\forall z(\neg z \doteq y \to \neg\varphi(x_1,\ldots,x_m,z)) \in \Pi_n$. We have shown then that $\varphi(x_1,\ldots,x_m,y)$ is equivalent to a formula Π_n, so $\varphi(x_1,\ldots,x_m,y)$ is equivalent to both, a Σ_n and a Π_n formula, as required. $\square$

Definition 3.22. *A function $f : \mathbb{N}^m \to \mathbb{N}$ is* ***provably recursive*** *if there is a Σ_1 functional formula $\varphi(x_1,\ldots,x_m,y)$ defining f in $\mathbb{N}$. In other words, if*

$$f(a_1,\ldots,a_m) = \text{ the unique } b \text{ satisfying } \varphi(a_1,\ldots,a_m,b)$$

such that $\varphi(x_1,\ldots,x_m,y)$ is functional in PA.

Let us extend the language $\mathcal{L}_\mathcal{A}$ to ${\mathcal{L}_\mathcal{A}}' = \mathcal{L}_\mathcal{A} \cup \{F\}$ by introducing a new function symbol F, given by some provably recursive function $f : \mathbb{N}^m \to \mathbb{N}$ defined by a Σ_1 formula $\varphi_f(x_1,\ldots,x_m,y)$. Now we can extend the theory PA to a theory $PA^+ = PA + \theta_F$ where θ_F is defined as $\forall x_1,\ldots,x_m \varphi_f(x_1,\ldots,x_m,F(x_1,\ldots,x_m))$.

Our goal will be proving that for each ${\mathcal{L}_\mathcal{A}}'$-formula $\chi(y_1,\ldots,y_n) \in \Sigma_m$ exists some $\mathcal{L}_\mathcal{A}$-formula $\psi(y_1,\ldots,y_n) \in \Sigma_m$ equivalent in PA^+. The same will happen for Π_m formulas. To show this we need four previous lemmas.

Lemma 3.23. *Let $t(y_1,\ldots,y_n)$ be an ${\mathcal{L}_\mathcal{A}}'$-term. Then there is some functional Σ_1 formula of $\mathcal{L}_\mathcal{A}$, $\psi(y_1,\ldots,y_n,y)$, such that $PA^+ \vdash t \doteq y \leftrightarrow \psi(y_1,\ldots,y_n,y)$.*

Proof. We will prove it by induction on t.

Case $t = y_1$: We can define then $\psi(y_1,y)$ as $y_1 \doteq y$.

Case $t = 0$: Defining $\psi(y)$ as $0 \doteq y$.

Case $t = 1$: Defining $\psi(y)$ as $1 \doteq y$.

Case $t = t_1 + t_2$: By induction hypothesis there are some $\psi_1, \psi_2 \in \Sigma_1(PA)$ functional formulas such that

$$PA^+ \vdash t_1 \doteq y \leftrightarrow \psi_1(y_1,\ldots,y_n,y)$$

and

$$PA^+ \vdash t_2 \doteq y \leftrightarrow \psi_2(y_1,\ldots,y_n,y).$$

Let us define $\psi(y_1,\ldots,y_n,y)$ as

$$\exists z_1,z_2(\psi_1(y_1,\ldots,y_n,z_1) \wedge \psi_2(y_1,\ldots,y_n,z_2) \wedge y \doteq z_1 + z_2)$$

and show then that $\psi(y_1,\ldots,y_n,y)$ is $\Sigma_1(PA)$ and functional. Since ψ_1 and ψ_2 are functional, z_1 and z_2 are unique, therefore $y = z_1 + z_2$ is also unique and hence ψ functional. The formula ψ is clearly Σ_1 because ψ_1 and ψ_2 are Σ_1.

Case $t \doteq t_1 \cdot t_2$: We can prove it as before considering $\psi(y_1,\ldots,y_n,y)$ as

$$\exists z_1,z_2(\psi_1(y_1,\ldots,y_n,z_1) \wedge \psi_2(y_1,\ldots,y_n,z_2) \wedge y \doteq z_1 \cdot z_2).$$

Case $t \doteq F(t_1, \ldots, t_m)$: By induction hypothesis there are $\psi_1, \ldots \psi_m \in \Sigma_1(PA)$ functional formulas such that

$$PA^+ \vdash t_i \doteq y \leftrightarrow \psi_i(y_1, \ldots, y_n, y) \qquad for \qquad i = 1, \ldots, m.$$

Let us define $\psi(y_1, \ldots, y_n, y)$ as

$$\exists z_1, \ldots, z_m(\psi_1(y_1, \ldots, y_n, z_1) \wedge \ldots \wedge \psi_m(y_1, \ldots, y_n, z_m) \wedge \varphi_f(z_1, \ldots, z_m, y))$$

and show then that $\psi(y_1, \ldots, y_n, y)$ is $\Sigma_1(PA)$ and functional. Since $\psi_1, \ldots, \psi_m, \varphi_f$ are functional, $z_1, \ldots, z_m$ are unique, therefore y is also unique and hence ψ functional. The formula ψ is clearly Σ_1 because $\psi_1, \ldots \psi_m, \varphi_f$ are Σ_1. $\square$

Lemma 3.24. *Given $t_1 = t_1(y_1, \ldots, y_n)$ and $t_2 = t_2(y_1, \ldots, y_n)$ terms of $\mathcal{L_A}'$. There is some provably Δ_1 $\mathcal{L_A}$-formula $\psi(y_1, \ldots, y_n)$ such that $PA^+ \vdash t_1 \doteq t_2 \leftrightarrow \psi(y_1, \ldots, y_n)$.*

Proof. By the previous lemma there are $\psi_1(y_1, \ldots, y_n, y), \psi_2(y_1, \ldots, y_n, y) \in \Sigma_1$ functional formulas of $\mathcal{L_A}$ such that

$$PA^+ \vdash y \doteq t_1 \leftrightarrow \psi_1(y_1, \ldots, y_n, y)$$

and

$$PA^+ \vdash y \doteq t_2 \leftrightarrow \psi_2(y_2, \ldots, y_n, y).$$

Let us define the formula $\psi(y_1, \ldots, y_n)$ as

$$\exists y(\psi_1(y_1, \ldots, y_n, y) \wedge \psi_2(y_1, \ldots, y_n, y))$$

and show then that $\psi(y_1, \ldots, y_n)$ is Δ_1. We can clearly see that $\psi(y_1, \ldots, y_n) \in \Sigma_1$. So thee is only left to show that $\psi(y_1, \ldots, y_n) \in \Pi_1$, which is true since

$$PA \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \forall z_1, z_2(\psi_1(y_1, \ldots, y_n, z_1) \wedge \psi_2(y_1, \ldots, y_n, z_2) \rightarrow z_1 \doteq z_2).$$

$\square$

Lemma 3.25. *Given $t_1 = t_1(y_1, \ldots, y_n)$ and $t_2 = t_2(y_1, \ldots, y_n)$ terms of $\mathcal{L_A}'$. There is some provably Δ_1 $\mathcal{L_A}$-formula $\psi(y_1, \ldots, y_n)$ such that $PA^+ \vdash t_1 < t_2 \leftrightarrow \psi(y_1, \ldots, y_n)$.*

Proof. Similar to the previous one, defining $\psi(y_1, \ldots, y_n)$ as

$$\exists z_1, z_2(z_1 < z_2 \wedge \psi_1(y_1, \ldots, y_n, z_1) \wedge \psi_2(y_1, \ldots, y_n, z_2)).$$

To see that $\psi(y_1, \ldots, y_n) \in \Pi_1$ we will use that

$$PA \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \forall z_1, z_2(\psi_1(y_1, \ldots, y_n, z_1) \wedge \psi_2(y_1, \ldots, y_n, z_2) \rightarrow z_1 < z_2).$$

$\square$

Lemma 3.26. *Given $\chi(y_1, \ldots, y_n)$ an $\mathcal{L}_\mathcal{A}{}'$-formula Δ_0. There is some provably $\Delta_1(PA)$ formula of $\mathcal{L}_\mathcal{A}$ $\psi(y_1, \ldots, y_n)$ such that $PA^+ \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \chi(y_1, \ldots, y_n)$.*

Proof. We will prove it by induction on χ.

Case $\chi := t_1 \doteq t_2$: lemma 3.24.

Case $\chi := t_1 < t_2$: lemma 3.25.

Case $\chi(y_1, \ldots, y_n) := \neg\chi'(y_1, \ldots, y_n)$: By the induction hypothesis there is some provably $\Delta_1(PA)$ formula of $\mathcal{L}_\mathcal{A}$, $\psi'(y_1, \ldots, y_n)$ such that

$$PA^+ \vdash \psi'(y_1, \ldots, y_n) \leftrightarrow \chi'(y_1, \ldots, y_n).$$

If we define $\psi(y_1, \ldots, y_n)$ as $\neg\psi'(y_1, \ldots, y_n)$ we have a $\Delta_1(PA)$ formula of $\mathcal{L}_\mathcal{A}$ such that $PA^+ \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \chi(y_1, \ldots, y_n)$.

Case $\chi(y_1, \ldots, y_n) := \chi_1(y_1, \ldots, y_n) \wedge \chi_2(y_1, \ldots, y_n)$: By the induction hypothesis there are some provably $\Delta_1(PA)$ formulas of $\mathcal{L}_\mathcal{A}$ $\psi_1(y_1, \ldots, y_n)$ and $\psi_2(y_1, \ldots, y_n)$ such that

$$PA^+ \vdash \psi_1(y_1, \ldots, y_n) \leftrightarrow \chi_1(y_1, \ldots, y_n)$$

and

$$PA^+ \vdash \psi_2(y_1, \ldots, y_n) \leftrightarrow \chi_2(y_1, \ldots, y_n).$$

Defining $\psi(y_1, \ldots, y_n)$ as $\chi_1(y_1, \ldots, y_n) \wedge \chi_2(y_1, \ldots, y_n)$ we have a $\Delta_1(PA)$ $\mathcal{L}_\mathcal{A}$-formula such that $PA^+ \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \chi(y_1, \ldots, y_n)$.

Case $\chi(y_1, \ldots, y_n) := \exists x < t(y_1, \ldots, y_n)\chi'(y_1, \ldots, y_n, x)$: By the induction hypothesis there is some provably $\Delta_1(PA)$ formula of $\mathcal{L}_\mathcal{A}$, $\psi'(y_1, \ldots, y_n, x)$, such that

$$PA^+ \vdash \psi'(y_1, \ldots, y_n, x) \leftrightarrow \chi'(y_1, \ldots, y_n, x).$$

By lemma 2.18 exists a Σ_1 functional formula of $\mathcal{L}_\mathcal{A}$, $\psi_t(y_1, \ldots, y_n, x)$, such that

$$PA^+ \vdash y \doteq t \leftrightarrow \psi_t(y_1, \ldots, y_n, y).$$

Let us define $\psi(y_1, \ldots, y_n)$ as $\exists x, y(x < y \wedge \psi_t(y_1, \ldots, y_n, x) \wedge \psi'(y_1, \ldots, y_n, x))$. Clearly $PA^+ \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \chi(y_1, \ldots, y_n)$.

As both, $\psi_t(y_1, \ldots, y_n, x)$ and $\psi'(y_1, \ldots, y_n, x)$, are Σ_1 we have $\psi(y_1, \ldots, y_n) \in \Sigma_1$.

For proving that $\psi(y_1, \ldots, y_n)$ is also Π_1 it will be enough to show that it is equivalent to the formula

$$\forall y(\psi_t(y_1, \ldots, y_n, y) \to \exists x < y\psi'(y_1, \ldots, y_n, x)).$$

□

Proposition 3.27. *For all $k \geq 1$, given $\chi(y_1, \ldots, y_n)$ a Σ_k (or Π_k) formula of $\mathcal{L}_\mathcal{A}{}'$. There is some Σ_k (or Π_k) formula $\psi(y_1, \ldots, y_n)$ of $\mathcal{L}_\mathcal{A}$ such that*

$$PA^+ \vdash \psi(y_1, \ldots, y_n) \leftrightarrow \chi(y_1, \ldots, y_n).$$

Proof. We will prove the case of $\chi(y_1,\dots,y_n)$ being Σ_k. The other case is similar. Since $\chi(y_1,\dots,y_n)$ is a Σ_k formula we can write it as

$$\exists\bar{x_1},\forall\bar{x_2}\dots\exists\bar{x_k}\chi'(y_1,\dots,y_n,x_1,\dots,x_k) \quad if \quad k \quad even$$

or

$$\exists\bar{x_1},\forall\bar{x_2}\dots\forall\bar{x_k}\chi'(y_1,\dots,y_n,x_1,\dots,x_k) \quad if \quad k \quad odd$$

with $\chi'(y_1,\dots,y_n,x_1,\dots,x_k)\in\Delta_0$ an $\mathcal{L_A}'$-formula.

By the previous lemma, there is some $\mathcal{L_A}$-formula $\psi'(y_1,\dots,y_n,x_1,\dots,x_k)$ and provably Δ_1 equivalent to $\chi'(y_1,\dots,y_n,x_1,\dots,x_k)$. Since $\psi'(y_1,\dots,y_n,x_1,\dots,x_k)$ is Δ_1 there are some equivalent $\mathcal{L_A}$-formulas Σ_1 and Π_1.

For k even we will replace $\chi'(y_1,\dots,y_n,x_1,\dots,x_k)$ with the equivalent $\mathcal{L_A}$-formula Σ_1 and for k odd, with the Δ_1. Obtaining so, in both cases, an $\mathcal{L_A}$-formula Σ_k. □

Remark 3.28. All statements will still be true if we extend the language adding more than one function or a relation.

Chapter 4

Satisfaction

In this chapter we will show that PA can handle syntax and semantics adequately to give a definition of truth in PA.

We will constantly define new functions, most of them used to code sequences, to extend the language $\mathcal{L}_\mathcal{A}$ to a language $\mathcal{L}_\mathcal{A}'$ and the theory PA to a theory PA^+. Because of section 3.3, it will be important that all the new functions defined are provably recursive.

Notation 4.1. To introduce less notation we will use the same notation for the function symbol as for the corresponding operation.

4.1 Coding sequences

We will start the chapter introducing a method for coding sequences of natural numbers. To do so, we need to define some previous concepts of arithmetic in PA.

Proposition 4.2. $PA \vdash \forall x, y(\neg x \doteq 0 \rightarrow \exists! d, r(y \doteq x \cdot d + r \wedge r < x))$.

Proof. Let M be an arbitrary model of PA.

To prove the existence we will define the formula

$$\varphi(y) = \forall x(\neg x \doteq 0 \rightarrow \exists d, r(y \doteq x \cdot d + r \wedge r < x))$$

and use induction to show that $M \vDash \forall y \varphi(y)$.

Clearly $M \vDash \varphi(0)$, since for each $a \in M$ with $a \neq 0$, $M \vDash 0 \doteq a \cdot 0 + 0$ and $M \vDash 0 < a$.

Let us suppose now $M \vDash \varphi(w)$ and show that $M \vDash \varphi(w + 1)$. Notice that $M \vDash \varphi(w)$ implies that for each $b \in M$, with $b \neq 0$, exist some $c, m \in M$ such that $M \vDash w \doteq b \cdot c + m$ and $M \vDash m < b$. We want to prove that there are also some elements $c', m' \in M$ such that $M \vDash a + 1 \doteq b \cdot c' + m'$ and $M \vDash m' < b$.

We will consider two cases:

1. $M \vDash m + 1 < b$: We can take c' as c and m' as $m + 1$.

2. $M \vDash m + 1 \doteq b$: We have $M \vDash w + 1 \doteq b \cdot c + m + 1$, since $M \vDash w \doteq b \cdot c + m$, and then $M \vDash w + 1 \doteq b \cdot c + b$. By axiom 2.18, $M \vDash w + 1 \doteq b \cdot (c + 1)$ and we can choose hence $c' = c + 1$ and $m' = 0$.

We have found in both cases some values for m' and c' such that $M \vDash m' < b$ and $M \vDash w + 1 \doteq b \cdot c' + m'$. So $M \vDash \varphi(w) \to \varphi(w+1)$ and by induction $M \vDash \forall y \varphi(y)$ as required.

To prove the uniqueness we will suppose that there are $d_1, d_2, r_1, r_2 \in M$ such that $M \vDash d_1 \cdot b + r_1 \doteq a$, $M \vDash r_1 < b$, $M \vDash d_2 \cdot b + r_2 \doteq a$ and $M \vDash r_2 < b$ for some $a, b \in M$. Since $M \vDash d_1 \cdot b + r_1 \doteq a$ and $M \vDash d_2 \cdot b + r_2 \doteq a$ we have

$$M \vDash d_1 \cdot b + r_1 \doteq d_2 \cdot b + r_2. \qquad (*)$$

Let us suppose $M \vDash d_1 < d_2$ to arrive to a contradiction. So there is some $n \in M$ such that $M \vDash n > 0$ and $M \vDash d_2 \doteq d_1 + n$ and hence we can follow from (*) that $M \vDash d_1 \cdot b + r_1 \doteq (d_1 + n) \cdot b + r_2$. By axiom 2.18, $M \vDash d_1 \cdot b + r_1 \doteq d_1 \cdot b + n \cdot b + r_2$, so $M \vDash r_1 \doteq n \cdot b + r_2$. Since $M \vDash n > 0$, we have $M \vDash r_1 \doteq n \cdot b + r_2$ iff $r_1 \geq b$, contradicting the hypothesis of $M \vDash r_1 < b$.

If we assume $M \vDash d_1 > d_2$ we will arrive to the same contradiction. So necessarily $M \vDash d_1 \doteq d_2$ and $M \vDash r_1 \doteq r_2$. □

Definition 4.3. *Let $M \vDash PA$ and $x, y \in M$. We define then the binary function that gives the **remainder** on dividing y by x as*

$$rem(y,x) := \begin{cases} z \ s.t. \ \exists w \leq y(x \cdot w + z \doteq y \wedge z < x), \ if \ \ x \neq 0 \\ 0 \ otherwise \end{cases}$$

or, which is the same, by the Δ_0 $\mathcal{L}_{\mathcal{A}}$-formula

$$\varphi(x,y,z) := [(\neg x \doteq 0 \wedge \exists w \leq y(x \cdot w + z \doteq y \wedge z < x)) \vee (x \doteq 0 \wedge z \doteq 0)].$$

We have then that

$$PA \vdash rem(y,x) \doteq z \leftrightarrow [(\neg x \doteq 0 \wedge \exists w \leq y(x \cdot w + z \doteq y \wedge z < x)) \vee (x \doteq 0 \wedge z \doteq 0)]$$

so the formula $rem(y,x) \doteq z$ is $\Delta_0(PA)$ and in particular $\Sigma_1(PA)$. By proposition 4.2, $PA \vdash \forall x, y \exists! z(rem(y,x) \doteq z)$. So by section 3.3, the function that gives the remainder is a provably recursive function and we can add the symbol *rem* to the extended language $\mathcal{L}_{\mathcal{A}}{}'$.

Definition 4.4. *Let $M \vDash PA$ and $x, y, z \in M$. We say that x is **congruent to** y **modulo** z, and denote it by $x \equiv y \ mod \ z$, if they satisfy the three-place relation given by*

$$x \equiv y \ mod \ z \leftrightarrow (\neg z \doteq 0 \wedge rem(x,z) \doteq rem(y,z)).$$

Notation 4.5. We write $x|y$ to say that x divides y.

Definition 4.6. *Let $M \vDash PA$ and $x, y \in M$. We say that x and y are **coprime**, and denote it by $coprim(x,y)$, if they satisfy the binary relation given by*

$$coprim(x,y) \leftrightarrow (x \geq 1 \wedge y \geq 1 \wedge \forall u(u|x \wedge u|y \to u \doteq 1)).$$

Definition 4.7. *We define the **function** β **of Gödel**, $\beta : \mathbb{N}^3 \to \mathbb{N}$ as*

$$\beta(a, b, i) := \text{ the least } z \text{ s.t. } z \equiv a \; mod(b \cdot (i+1)+1).$$

Theorem 4.8. *(Chinese remainder theorem) Given $m_0, \ldots, m_{n-1} \in \mathbb{N}$ pairwise coprimes and $a_0, \ldots, a_{n-1} \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that*

$$k \equiv a_i \; mod \; m_i \qquad for \; each \; 0 \leq i < n.$$

The Chinese reminder theorem is also provable in PA and we will use it to code a finite sequence $x_0, \ldots, x_{n-1}$ of elements of M, for $M \vDash PA$. To do so, let us define $m = c!$ for $c = max(n, x_0, \ldots, x_{n-1})$.

Proposition 4.9. *The sequence of numbers $m+1, 2\cdot m+1, 3\cdot m+1, \ldots, n\cdot m+1$ is pairwise coprime.*

Proof. We want to show that $M \vDash coprim(i\cdot m+1, j\cdot m+1)$ for $0 < i < j \leq n$. Let us assume $M \vDash u|(i\cdot m+1)$ and $M \vDash u|(j\cdot m+1)$ for some $u \in M$ and prove then that $M \vDash u|1$. If $M \vDash u|(i\cdot m+1)$ and $M \vDash u|(j\cdot m+1)$ we have $M \vDash u|((i\cdot m+1)-(j\cdot m+1))$ and hence $M \vDash u|(i-j)\cdot m$.

Since $i - j < n \leq c$ and $m = c!$ we have $M \vDash (i-j)|m$ and then $M \vDash u|m$. So as $M \vDash u|i\cdot m$ and $M \vDash u|(i\cdot m+1)$, $M \vDash u|(i\cdot m-(i\cdot m+1))$ and $M \vDash u|1$ as required. □

By the Chinese remainder theorem, we can find some $k \in M$ such that

$$\beta(x_i, m, i) = k$$

for each $i < n$, and say then that the pair (k, m) codes the sequence $x_0, \ldots, x_{n-1}$. Finally, to reduce this pair to a single number we need to define a pairing function.

Definition 4.10. *For $M \vDash PA$ we define the **pairing function** $\langle , \rangle : M \times M \to M$ as*

$$\langle x, y \rangle := \frac{(x+y+1)(x+y)}{2} + y$$

Remark 4.11. Notice that either $2|(x+y+1)$ or $2|(x+y)$, so $2|(x+y+1)(x+y)$.

For each $z \in M$ exists a unique pair (x, y) with $x, y \in M$ such that $z = \langle x, y \rangle$ which implies that $PA \vDash \forall z \exists! x, y \langle x, y \rangle \doteq z$. Moreover the formula $\langle x, y \rangle \doteq z$ is clearly Δ_0, since

$$PA \vdash \langle x, y \rangle \doteq z \leftrightarrow 2z \doteq ((x+y+1)(x+y)+2y).$$

So, the pairing function is provably recursive and we can add then the symbol $\langle , \rangle$ to the language $\mathcal{L_A}'$.

Lemma 4.12. *PA proves the following:*
(a) $\forall z \exists x, y \langle x, y\rangle \doteq z$.
(b) $\forall x, y, u, v(\langle x, y\rangle \doteq \langle u, v\rangle \to x \doteq u \wedge y \doteq v)$.

Proof. (a) Let M be an arbitrary model of PA and $\varphi(z)$ the formula $\exists x, y\langle x, y\rangle \doteq z$. We will show that $M \vDash \forall z \varphi(z)$ by induction.

The initial case is clear, since $M \vDash \langle 0, 0\rangle \doteq 0$.

Let us suppose now that $M \vDash \varphi(w)$ for $w \in M$ and prove then that $M \vDash \varphi(w+1)$. To do so we will assume that exist some $a, b \in M$ such that $M \vDash \langle a, b\rangle \doteq w$ and show that there are also some $c, d \in M$ such that $M \vDash \langle c, d\rangle \doteq w+1$. We will consider two cases.

1. Case $M \vDash a \doteq 0$: We have

$$w+1 = \frac{(a+b+1)(a+b)}{2} + b + 1 = \frac{(a+1)b}{2} + b + 1 = \langle b+1, 0\rangle.$$

So we can take $c = b+1$ and $d = 0$.

2. Case $M \vDash a > 0$: We have

$$w+1 = \frac{(a+b+1)(a+b)}{2} + b + 1 =$$

$$= \frac{((a-1)+(b+1)+1)((a-1)+(b+1))}{2} + b + 1 = \langle a-1, b+1\rangle.$$

So we can choose $c = a-1$ and $d = b+1$.

We have shown that $M \vDash \varphi(0) \wedge \forall z(\varphi(z) \to \varphi(z+1))$ and by induction we can conclude $M \vDash \forall \varphi(z)$.

(b) Let M be an arbitrary model of PA. We will start by showing that for $a, b, c, d \in M$, if $M \vDash \langle a, b\rangle \doteq \langle c, d\rangle$ then $M \vDash a + b \doteq c + d$. To do so we will suppose $M \vDash a + b < c + d$ to arrive to a contradiction. If $M \vDash a + b < c + d$ we have $M \vDash a + b + 1 \leq c + d$. Then

$$\langle a, b\rangle = \frac{(a+b+1)(a+b)}{2} + b < \frac{(a+b+1)(a+b)}{2} + a + b + 1 =$$

$$= \frac{(a+b+1)(a+b+2)}{2} \leq \frac{(c+d)(c+d+1)}{2} = \langle c, d\rangle$$

and hence $M \vDash \langle a, b\rangle < \langle c, d\rangle$ contradicting so the hypothesis.

Assuming $M \vDash c + d < a + b$ we would arrive to a similar contradiction. Therefore we can suppose $M \vDash a + b \doteq c + d$.

Let us see now that $M \vDash b \doteq d$. We have

$$b = \langle a, b\rangle - \frac{(a+b+1)(a+b)}{2} = \langle c, d\rangle - \frac{(c+d+1)(c+d)}{2} = d.$$

Since $M \vDash a + b \doteq c + d$, also $M \vDash a \doteq c$ as required. □

Notation 4.13. We will denote as z_L the unique $x \in M$ such that $z = \langle x, y\rangle$ and as z_R the unique $y \in M$ such that $z = \langle x, y\rangle$.

Proposition 4.14. *PA proves the following:*
(a) $\forall z(z_L \leq z)$
(b)$\forall z(z_R \leq z)$

Notation 4.15. We will write $z = \langle x_0, \ldots, x_{n-1} \rangle$ to say that z codes the sequence $x_0, \ldots, x_{n-1}$ of elements of M using the function beta and the pairing function.

For any arbitrary model M of PA, given some $z \in M$ such that $z = \langle x_0, \ldots, x_{n-1} \rangle$ we can recover each x_i with

$$x_i = rem(z_L, z_R(i+1)+1).$$

Definition 4.16. *Given $z \in M$ such that $z = \langle x_0, \ldots, x_{n-1} \rangle$ we define the binary function $(z)_i$ for $i < n$ as*

$$(z)_i := rem(z_L, z_R(i+1)+1)$$

or which is the same, as

$$(z)_i := \beta(z_L, z_R, i).$$

Proposition 4.17. *PA proves the following:*
(a) $\forall z, i \exists! x (z)_i \doteq x$
(b) $\forall z, i (z)_i \leq z$
(c) $\forall x \exists z (z)_0 \doteq x$
(d) $\forall x, i, z \exists w (\forall j < i ((w)_j \doteq (z)_j) \wedge (w)_i \doteq x)$

Proof. (a) By lemma 4.12 we have $PA \vdash \forall z \exists! z_L, z_R \langle z_L, z_R \rangle \doteq z$ and by proposition 4.2, $PA \vdash \forall x, y \exists! z (rem(x, y) \doteq z)$, so it is clear that $PA \vdash \forall z, i \exists! x (z)_i \doteq x$

(b) Notice that $PA \vdash \forall x, y (rem(y, x) \leq y)$ and hence $PA \vdash \forall z, i (z)_i \leq z_L$. So since $PA \vdash \forall z (z_L \leq z)$ we can conclude that $PA \vdash \forall z, i (z)_i \leq z$.

(c) Let M be an arbitrary model of PA. We want to prove that $M \vDash \exists z (z)_0 \doteq a$ for each $a \in M$. Let us define z as $\langle a, a \rangle$, then $(z)_0 = rem(a, a+1)$ and hence $M \vDash (z)_0 \doteq a$ as required. □

The formula, $(z)_i \doteq x$ is clearly $\Delta_0(PA^+)$ and hence, by section 3.3, $\Delta_1(PA)$. Moreover, since $PA \vdash \forall z, i \exists! x (z)_i \doteq x$, the function $(z)_i$ is provably recursive and we can add its symbol to the extended language $\mathcal{L}_{\mathcal{A}}'$.

For the rest of the section let M be an arbitrary model of PA.

Definition 4.18. *For $z \in M$, we define the* ***length*** *of the sequence coded by z as*

$$len(z) := (z)_0.$$

The formula $len(z) \doteq n$ is clearly $\Delta_0(PA^+)$ and hence $\Delta_1(PA)$. We also have that $PA \vDash \forall x \exists! n (len(x) \doteq n)$. Therefore $len(z)$ is a provably recursive function and we can add the symbol len to the extended language $\mathcal{L}_{\mathcal{A}}'$.

Definition 4.19. *For $z, i \in M$, we define the function*

$$[z]_i := \begin{cases} (z)_{i+1} & if \quad i < len(z) \\ 0 & otherwise \end{cases}$$

then $PA \vdash [z]_i \doteq x \leftrightarrow (i \geq len(z) \wedge x \doteq 0) \vee (i < len(z) \wedge (z)_{i+1} \doteq x)$.

As before, the function $[z]_i$ is also provably recursive and we can add its symbol to the language $\mathcal{L_A}'$.

Definition 4.20. *For $n \in \mathbb{N}$ and $x_0, \ldots, x_{n-1} \in M$ we define*

$$[x_0, \ldots, x_{n-1}] := \text{ the least } z \text{ s.t. } len(z) \doteq n \wedge \bigwedge_{i<n} ([z]_i \doteq x_i).$$

For all $n \in \mathbb{N}$ the function $[x_0, \ldots, x_{n-1}]$ is provably recursive, since we can write $[x_0, \ldots, x_{n-1}] \doteq z$ as

$$len(z) \doteq n \wedge \bigwedge_{i<n} ([z]_i \doteq x_i \wedge \forall w < z(\neg len(w) \doteq n \vee \bigvee_{i<n} \neg [w]_i \doteq x_i)$$

which is clearly $\Delta_0(PA^+)$ and $\Delta_1(PA)$.

For the following definitions we will consider $x, y \in M$ such that x codes the sequence $[x]_0, \ldots, [x]_{len(x)-1}$ and y the sequence $[y]_0, \ldots, [y]_{len(y)-1}$

Definition 4.21. *For $x, y \in M$ we define the function*

$$x * y := \text{ the least } z \text{ s.t. } len(z) \doteq len(x) + len(y) \wedge$$

$$\forall i < len(x)([z]_i \doteq [x]_i) \wedge \forall j < len(y)([z]_{len(x)+j} \doteq [y]_j).$$

The idea is that $x * y$ codes the sequence $[x]_0, \ldots, [x]_{len(x)-1}, [y]_0, \ldots, [y]_{len(y)-1}$ of length $len(x * y) = len(x) + len(y)$.

Clearly the formula $x * y \doteq z$ is $\Delta_0(PA^+)$, since all quantifiers are bounded, and hence $\Delta_1(PA)$.

Notation 4.22. We can omit the parenthesis when using the operation $*$, since it satisfies the associative property, i.e. $PA \vdash \forall x, y, z((x * y) * z \doteq x * (y * z))$.

Definition 4.23. *For $x, w \in M$, we define*

$$x \restriction w := \text{ the least } z \text{ s.t. } len(z) \doteq w \wedge \forall i < len(z)([z]_i \doteq [x]_i).$$

The idea is that, if $w \leq len(x)$, then $x \restriction w$ codes the sequence $[x]_0, \ldots, [x]_{w-1}$ and, if $w > len(x)$, the sequence $[x]_0, \ldots, [x]_{len(x)-1}, 0, \ldots, 0$ of length n.

Once again, the formula $x \restriction w \doteq z$ is $\Delta_1(PA)$.

Definition 4.24. *For $x, y, w \in M$, we define*

$$x[y|w] := \text{ the least } z \text{ s.t. } len(z) \doteq max(len(x), w + 1) \wedge$$

$$\forall i < len(z)[(i \doteq w \rightarrow [z]_i \doteq y) \wedge (\neg i \doteq w \rightarrow [z]_i \doteq [x]_i)].$$

Intuitively, $x[y|w]$ codes $[x]_0, \dots, [x]_{w-1}, y, [x]_{w+1}, \dots, [x]_{len(x)-1}$ if $w < len(x)$ and the sequence $[x]_0, \dots, [x]_{len(x)-1}, 0, \dots, 0, y$ of length w if $w \geq len(x)$.

Notice that the formula $x[y|w] \doteq z$ is also $\Delta_1(PA)$.

In fact, the last three functions are provably recursive, so we can add its symbols to $\mathcal{L}_{\mathcal{A}}{}'$.

4.2 Gödel-numbering

Having shown how to code sequences of elements of any model of PA we are ready to introduce the Gödel-numbering and a method for coding strings $\sigma = s_0 \dots s_{n-1}$ of $\mathcal{L}_{\mathcal{A}}$-symbols.

The first step will be to assign a unique natural number $v(s)$ to each symbol s of the first order language $\mathcal{L}_{\mathcal{A}}$. We will use the following table.

s	0	1	$+$	$\cdot$	$<$	$\doteq$	$\wedge$	$\vee$	$\neg$	$\exists$	$\forall$	(	)	v_i
$v(s)$	0	1	2	3	4	5	6	7	8	9	10	11	12	$\langle 13, i \rangle$

Remark 4.25. Notice that $\langle 13, i \rangle \geq 13$ for each i, so the value of $v(s)$ is unique for each $\mathcal{L}_{\mathcal{A}}$-symbol s.

Definition 4.26. *We define $\#\sigma$ as the least $x \in \mathbb{N}$ coding the sequence $v(s_0), \dots, v(s_{n-1})$.*

Definition 4.27. *We define the formula $GN(x)$*

$$\forall i < len(x)([x]_i \leq 12 \vee \exists j \leq x [x]_i \doteq \langle 13, j \rangle) \wedge$$

$$\forall w < x(\neg len(w) \doteq len(x) \vee \exists i < len(x) \neg [x]_i \doteq [w]_j)$$

*to say that x is a **Gödel-number**.*

It is easy to check that $GN(x)$ is $\Delta_1(PA)$. Moreover, any string σ has a unique Gödel-number $\#\sigma$.

Notation 4.28. We will use $\ulcorner \sigma \urcorner$ to denote the numeral of the Gödel-number of a string σ, i.e. the numeral of $\#\sigma$.

Notation 4.29. We will write $var(i)$ to denote $[\langle 13, i \rangle]$.

4.2.1 Syntax

Now we are ready to give some syntactic notions in PA, such as the definition of the Gödel-number of a formula or a term.

To define the **Gödel-number of a term** we need to introduce the previous formula $termseq(x)$. Given some x coding a sequence $[x]_0, \dots, [x]_{len(x)-1}$ the formula $termseq(x)$

says that for each $i < len(x)$ the element $[x]_i$ is either the Gödel-number of a constant $(0,1)$, of a variable or of the addition or product of two previous elements of the sequence. In other words, it says that x codes a sequence where each element codes a step of the construction of a term, following the rules given in chapter 1.

Definition 4.30. *termseq(x) denotes the $\mathcal{L_A}'$-formula*

$$\forall i < len(x)\{[x]_i \doteq \ulcorner 0 \urcorner \vee [x]_i \doteq \ulcorner 1 \urcorner \vee \exists j \leq x([x]_i \doteq var(j))$$
$$\vee \exists j,k \leq i([x]_i \doteq \ulcorner (\urcorner * [x]_j * \ulcorner + \urcorner * [x]_k * \ulcorner) \urcorner)$$
$$\vee \exists j,k \leq i([x]_i \doteq \ulcorner (\urcorner * [x]_j * \ulcorner \cdot \urcorner * [x]_k * \ulcorner) \urcorner)\}$$

The formula $termseq(x)$ is clearly $\Delta_0(PA^+)$ and hence $\Delta_1(PA)$.

Definition 4.31. *We define the formula $term(x) := \exists y(termseq(y * [x]))$ to denote that x is the **Gödel-number of a term**.*

The formula $term(x)$ has an existential quantifier. To transform this quantifier into a bounded one, we will define a provably recursive function $g : \mathbb{N} \to \mathbb{N}$, given by a Σ_1-formula $\varphi_g(x,y)$ and represented by the symbol *bound*, such that

$$PA^+ \vdash \forall x \varphi_g(x, bound(x))$$

and

$$PA^+ \vdash term(x) \leftrightarrow \exists y \leq bound(x) termseq(y * [x])$$

and add it to $\mathcal{L_A}'$.

Definition 4.32. *Let us define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that*

$$f(m,n) := the\ least\ k\ such\ that\ len(k) = n\ and\ for\ all\ i < n\ [k]_i = m.$$

Formally we define f as the function given by the formula

$$\varphi_f(m,n,x) := len(x) \doteq n \wedge \forall i < n[x]_i \doteq m \wedge \forall z < x(\neg len(z) \doteq n \vee \exists i < n \neg [z]_i \doteq m).$$

Lemma 4.33. *$PA \vdash \forall m, n \exists! x \varphi_f(m,n,x)$.*

Proof. The uniqueness is proved by the definition of x as the least number.

To prove the existence we will define the formula $\psi(n) = \forall m \exists x \varphi_f(m,n,x)$ and show $M \vDash \forall n \psi(n)$, for an arbitrary model M of *PA*, by induction.

Initial case: If we take x as 0 we have $M \vDash \psi(0)$.

Let us suppose now that $M \vDash \psi(w)$ and prove that $M \vDash \psi(w+1)$. So assume that there is some $a \in M$ such that $M \vDash \forall m \varphi_f(m,w,a)$ to show that there is some $b \in M$ such that $M \vDash \forall m \varphi_f(m,w+1,b)$. If we take b as $a * [m]$ we have $M \vDash len(b) = len(a) + 1 = w + 1$ and $M \vDash \forall i < w + 1[b]_i = m$ for all $m \in M$, since $M \vDash \forall i < w[b]_i = m$ by hypothesis and $M \vDash [b]_w = m$ by definition of b. So $M \vDash \psi(w+1)$.

We have shown then that $M \vDash \psi(0) \wedge \forall n(\psi(n) \to \psi(n+1))$ and by induction we have $M \vDash \forall n \psi(n)$ as required. $\square$

Since all quantifiers in φ_f are bounded, we have that φ_f is a $\Delta_0(PA^+)$ formula and hence $\Delta_1(PA)$. By lemma 4.33, f is a provably recursive function.

Definition 4.34. *Let us define now the function $g : \mathbb{N} \to \mathbb{N}$ as $g(n) := f(len(n), n)$, given by the formula $\varphi_g(x, bound(x)) := \varphi_f(len(x), x, bound(x))$.*

This function is clearly provably recursive and therefore we can add the symbol *bound* to the language $\mathcal{L}_{\mathcal{A}}{}'$.

Since $PA^+ \vdash term(x) \leftrightarrow \exists y \leq bound(x) termseq(y * [x])$, the formula $term(x)$ is $\Delta_0(PA^+)$ and $\Delta_1(PA)$.

We have already defined the Gödel-number of a term, now we will do similar for formulas. As before, we will need to define a previous formula $formseq(x)$ to define the **Gödel-number of an $\mathcal{L}_{\mathcal{A}}$-formula**.

Definition 4.35. *$formseq(x)$ denotes the $\mathcal{L}_{\mathcal{A}}{}'$-formula*

$$\forall i < len(x)\{\exists u, v \leq x[term(u) \wedge term(v) \wedge$$

$$([x]_i \doteq \ulcorner(\urcorner * u * \ulcorner\doteq\urcorner * v * \ulcorner)\urcorner \vee [x]_i \doteq \ulcorner(\urcorner * u * \ulcorner<\urcorner * v * \ulcorner)\urcorner)]$$

$$\vee \exists j, k < i([x]_i \doteq \ulcorner(\urcorner * [x]_j * \ulcorner\wedge\urcorner * [s]_k * \ulcorner)\urcorner)$$

$$\vee \exists j < i([x]_i \doteq \ulcorner\neg\urcorner * [x]_j)$$

$$\vee \exists j < i \exists k \leq x([x]_i \doteq \ulcorner\exists\urcorner * var(k) * [x]_j)\}$$

The formula $formseq(x)$ has been constructed considering the different ways of building up a formula and is $\Delta_0(PA)$ since all quantifiers are bounded.

Definition 4.36. *We define the formula $form(x) := \exists y(formseq(y * [x]))$ to denote that x is the **Gödel-number of a formula**.*

We can show with the symbol $bound(x)$ that $form(x) \in \Delta_1(PA)$, since

$$PA+ \vdash form(x) \leftrightarrow \exists y \leq bound(x) formseq(y * [x]).$$

Now we will define the **Gödel-number of Σ_n and Π_n-formulas**. Once again, we will need to define the previous functions $formseq_{\Delta_0}(x)$, $formseq_{\Sigma_n}(x)$ and $formseq_{\Pi_n}(x)$. Let us start with the initial case Δ_0.

Definition 4.37. *$formseq_{\Delta_0}(x)$ is the formula*

$$\forall i < len(x)\{\exists u, v \leq x(term(u) \wedge term(v) \wedge$$

$$([x]_i \doteq \ulcorner(\urcorner * u \ulcorner\doteq\urcorner * v * \ulcorner)\urcorner \vee [x]_i \doteq \ulcorner(\urcorner * u * \ulcorner<\urcorner * v * \ulcorner)\urcorner))$$

$$\vee \exists j, k < i([x]_i \doteq \ulcorner(\urcorner * [x]_j * \ulcorner\wedge\urcorner * [x]_k * \ulcorner)\urcorner) \vee \exists j < i([x]_j \doteq \ulcorner\neg\urcorner * [x]_j)$$

$$\vee \exists j < i \exists k, u \leq x(term(u) \wedge \forall l < len(u) \neg [u]_l \doteq \langle 13, k \rangle \wedge$$

$$[x]_i \doteq \ulcorner\exists\urcorner * var(k) * \ulcorner(\urcorner * \ulcorner(\urcorner * var(k) * \ulcorner<\urcorner * u * \ulcorner)\urcorner * \ulcorner\wedge\urcorner * [x]_j * \ulcorner)\urcorner)\}$$

The formula $formseq_{\Delta_0}(x)$ is clearly $\Delta_0(PA^+)$ and hence $\Delta_1(PA)$.

Definition 4.38. *We define then the formula* $form_{\Delta_0}(x) := \exists s(formseq_{\Delta_0}(s * [x])$ *to say that* x *is the* ***Gödel-number of a*** Δ_0***-formula***.

The formula $form_{\Delta_0}(x)$ has an existential quantifier, but we can transform it into a bounded one with the symbol *bound*, so the formula $form_{\Delta_0}(x)$ is also $\Delta_1(PA)$.

Notation 4.39. We also write $form_{\Delta_0}$ as $form_{\Sigma_0}$ or $form_{\Pi_0}$.

With the initial case defined we can define now the cases Σ_n and Π_n by recursion.

Definition 4.40. *For each* $n \in \mathbb{N}$ $formseq_{\Sigma_{n+1}}(x)$ *is the formula*

$$\forall i < len(x)\{(form_{\Pi_n}([x]_i) \wedge i \doteq 0)$$

$$\vee(i > 0 \wedge \exists k \leq x([x]_i \doteq \ulcorner\exists\urcorner * var(k) * [x]_{j-1}))\}$$

and $formseq_{\Pi_{n+1}}(x)$ *the formula*

$$\forall i < len(x)\{(form_{\Sigma_n}([x]_i) \wedge i \doteq 0)$$

$$\vee(i > 0 \wedge \exists k \leq x([x]_i \doteq \ulcorner\forall\urcorner * var(k) * [x]_{j-1}))\}.$$

Definition 4.41. *For each* $n \in \mathbb{N}$, *we define the formula*

$$form_{\Sigma_n}(x) := \exists s(formseq_{\Sigma_n}(s * [x])$$

to denote that x *is the* ***Gödel-number of a*** Σ_n***-formula*** *and the formula*

$$form_{\Pi_n}(x) := \exists s(formseq_{\Pi_n}(s * [x])$$

to denote that x *is the* ***Gödel-number of a*** Π_n***-formula***.

Using the symbol bound we can also see that the formulas $formseq_{\Sigma_n}(x)$, $formseq_{\Pi_n}(x)$, $form_{\Sigma_n}(x)$ and $form_{\Pi_n}(x)$ are $\Delta_1(PA)$ for each $n \in \mathbb{N}$.

4.2.2 Semantics

We have almost all the tools we need to give a definition of truth in PA.

The next step will be defining a function that gives the value of a term. To define this function we will need the previous formula $valseq(y, x, r)$, where y codes the assignation of the variables, x an $\mathcal{L}_{\mathcal{A}}$-term and r the values of each $[x]_i$ with the assignation y.

Definition 4.42. $valseq(y, x, r)$ *denotes the formula*

$$termseq(x) \wedge len(r) \doteq len(x) \wedge \forall i < len(x)\{$$

$$([x]_i \doteq \ulcorner 0\urcorner \wedge [r]_i \doteq 0) \vee ([x]_i \doteq \ulcorner 1\urcorner \wedge [r]_i \doteq 1) \vee \exists j \leq x([x]_i \doteq var(j) \wedge [r]_i \doteq [y]_j)\vee$$

$$\exists j,k < i([x]_i \doteq \ulcorner(\urcorner * [x]_j * \ulcorner+\urcorner * [x]_k * \ulcorner)\urcorner \wedge [r]_i \doteq [r]_j + [r]_k)\vee$$

$$\exists j,k < i([x]_i \doteq \ulcorner(\urcorner * [x]_j * \ulcorner\cdot\urcorner * [x]_k * \ulcorner)\urcorner \wedge [r]_i \doteq [r]_j \cdot [r]_k)\}.$$

The formula $valseq(y, x, r)$ is $\Delta_0(PA^+)$ and therefore $\Delta_1(PA)$.

Definition 4.43. *Given x, the Gödel-number of a term, and y coding a sequence, we define* $val(x,y) = z \leftrightarrow \exists s, r(valseq(y, s * [x], r * [z])) \vee (\neg term(x) \wedge z \doteq 0)$.

Intuitively, $val(x,y)$ is the value of a term $t(v_0, \dots, v_k)$ coded by x when each variable v_i is given the value $[y]_i$. If $i \geq len(y)$, we define $[y]_i = 0$, so $val(x,y)$ is a well-defined function.

Proposition 4.44. *The function $val(x,y)$ is a provably recursive function, i.e. the formula $val(x,y) = z$ is $\Sigma_1(PA)$ and $PA \vdash \forall x, y \exists ! z val(x,y) \doteq z$.*

Proof. It can be proved by complete induction in the variable x. □

Proposition 4.45. *For any $\mathcal{L}_{\mathcal{A}}$-term $t(v_0, \dots, v_k)$,*

$$PA \vdash \forall v_0, \dots, v_k (t(v_0, \dots, v_k) \doteq val(\ulcorner t \urcorner, [v_0, \dots, v_k])).$$

Proof. We can prove it by induction in the construction of t.

Case $t(\bar{v}) = 0$: Notice that $PA \vdash \forall v_0, \dots, v_k (0 \doteq val(\ulcorner 0 \urcorner, [v_0, \dots, v_k]))$. That is because, in PA we can prove that exist r, s such that $valseq([v_0, \dots, v_k], s * [\ulcorner 0 \urcorner], r * [0])$, since we can take $r = s = 0$.

Case $t(\bar{v}) = 1$: Similar to the previous case.

Case $t(\bar{v}) = v_i$: We want to check that $PA \vdash \forall v_0, \dots, v_k (v_i \doteq val(\ulcorner v_i \urcorner, [v_0, \dots, v_k]))$. Again, that is because if we choose $r = s = 0$, we have $valseq([v_0, \dots, v_k], s * [\ulcorner v_i \urcorner], r * [v_i])$.

Case $t(\bar{v}) = t_1(\bar{v}) + t_2(\bar{v})$: From the induction hypothesis follows that

$$PA \vdash \forall v_0, \dots, v_k (t_i(v_0, \dots, v_n) \doteq val(\ulcorner t_i \urcorner, [v_0, \dots, v_k])) \quad for \quad i = 1, 2,$$

so, working in PA, we can find some values r_1, s_1, r_2, s_2 such that

$$valseq([v_0, \dots, v_k], s_i * [\ulcorner t_i \urcorner], r_i * [t_i(v_0, \dots, v_k)]) \quad for \quad i = 1, 2.$$

If we take $s = s_1 * [\ulcorner t_1 \urcorner] * s_2 * [\ulcorner t_2 \urcorner]$ and $r = r_1 * [t_1(v_0, \dots, v_k)] * r_2 * [t_2(v_0, \dots, v_k)]$, then

$$valseq([v_0, \dots, v_k], s * [\ulcorner t \urcorner], r * [t(v_0, \dots, v_k)]).$$

Case $t(\bar{v}) = t_1(\bar{v}) \cdot t_2(\bar{v})$: Analog to the previous one. □

Now we have all we need to formalize a truth definition for formulas. We will start with the **truth definition for Δ_0-formulas**, denoted by the formula $Sat_{\Delta_0}(x,y)$.

We will need to define a previous formula $satseq_{\Delta_0}(x,t)$. In $satseq_{\Delta_0}(x,t)$ x codes the construction of a Δ_0-formula and t a sequence of triples $\langle i, z, w \rangle$ where i is the index for the sequence x, z the assignation for the variables of the formula and w a truth value.

Notation 4.46. We will use 0 for false and 1 for true.

Definition 4.47. *$satseq_{\Delta_0}(x,t)$ is the formula*

$$formseq_{\Delta_0}(x) \wedge \forall l < len(t) \exists i,z,w \leq t([t]_l \doteq \langle i,z,w\rangle \wedge i < len(x) \wedge w \leq 1) \wedge$$

$$\forall l < len(t) \forall i,z,w \leq t\{[t]_l \doteq \langle i,z,w\rangle \rightarrow$$

$$[\forall u,u' \leq x((term(u) \wedge term(u') \wedge [x]_i \doteq u * \ulcorner \doteq \urcorner * u') \rightarrow$$

$$(w \doteq 1 \leftrightarrow val(u,z) \doteq val(u',z))) \wedge$$

$$\forall u,u' \leq x((term(u) \wedge term(u') \wedge [x]_i \doteq u * \ulcorner < \urcorner * u') \rightarrow$$

$$(w \doteq 1 \leftrightarrow val(u,z) < val(u',z))) \wedge$$

$$\forall j,k < i([x]_i \doteq \ulcorner (\urcorner * [x]_j * \ulcorner \wedge \urcorner * [x]_k * \ulcorner) \urcorner \rightarrow$$

$$\exists l_1,l_2 < l \exists w_1,w_2 \leq 1([t]_{l_1} \doteq \langle j,z,w_1\rangle \wedge [t]_{l_2} \doteq \langle k,z,w_2\rangle \wedge$$

$$(w \doteq 1 \leftrightarrow w_1 \doteq 1 \wedge w_2 \doteq 1))) \wedge$$

$$\forall j < i([x]_i \doteq \ulcorner \neg \urcorner * [x]_j \rightarrow$$

$$\exists l_1 < l \exists w_1 \leq 1([t]_{l_1} \doteq \langle j,z,w_1\rangle \wedge (w \doteq 1 \leftrightarrow w_1 \doteq 0))) \wedge$$

$$\forall j < i \forall k,u \leq s((term(u) \wedge \forall m < len(u) \neg [u]_m \doteq \langle 13,k\rangle \wedge$$

$$[x]_i \doteq \ulcorner \exists \urcorner * var(k) * \ulcorner (\urcorner * var(k) * \ulcorner < \urcorner * u * \ulcorner \wedge \urcorner * [x]_j * \ulcorner) \urcorner) \rightarrow$$

$$\forall r < val(u,z) \exists l_1 < l \exists w_1 \leq 1([t]_{l_1} \doteq \langle j,z[r|k],w_1\rangle) \wedge$$

$$(w \doteq 1 \leftrightarrow \exists r < val(u,z) \exists l_1 < l([t]_{l_1} \doteq \langle j,z[r|k],1\rangle))))]\}.$$

Since the operation $val(x,y)$ is provably recursive and the formulas $formseq_{\Delta_0}(x)$ and $term(x)$ are $\Delta_1(PA)$, we can see that the formula $satseq_{\Delta_0}(x,t)$ is $\Delta_1(PA)$.

Definition 4.48. *$Sat_{\Delta_0}(x,y)$ is the formula*

$$\exists s,t[satseq_{\Delta_0}(s * [x],t) \wedge \exists l < len(t)([t]_l \doteq \langle len(s),y,1\rangle)].$$

The idea is that given some x coding the construction of a Δ_0-formula and some t coding a sequence of triples $\langle i,y,w\rangle$ as in $satseq_{\Delta_0}$, the formula $Sat_{\Delta_0}(x,y)$ denotes that the formula is true with the assignation y.

The formula $Sat_{\Delta_0}(x,y)$ is $\Delta_1(PA)$, but to prove it we will need two previous lemmas. The first lemma states that the value of a bounded formula, in PA, only depends on the information coded by y, that is, the assignation of the free variables. And the second one, that all bounded formulas can be evaluated.

Lemma 4.49. *$PA \vdash \forall x,t,x',t',w,w'[satseq_{\Delta_0}(x,t) \wedge satseq_{\Delta_0}(x',t') \wedge \exists l < len(t) \exists l' < len(t')$ $\exists i < len(x) \exists i' < len(x')([x]_i \doteq [x']_{i'} \wedge [t]_l \doteq \langle i,y,w\rangle \wedge [t']_{l'} \doteq \langle i',y,w'\rangle) \rightarrow w \doteq w']$.*

Proof. It can be proved by complete induction on the variable x. □

Lemma 4.50. *$PA \vdash \forall x, y[formseq_{\Delta_0}(x) \rightarrow \exists s, t, w(satseq_{\Delta_0}(s * x, t) \wedge \exists l < len(t)[t]_l \doteq \langle len(s), y, w\rangle)]$*

Proof. It can be proved by complete induction on the variable x. □

Theorem 4.51. *The formula $Sat_{\Delta_0}(x, y)$ is $\Delta_1(PA)$ and PA proves the following:*
(a) $Sat_{\Delta_0}(r * \ulcorner \doteq \urcorner * s, y) \leftrightarrow val(r, y) \doteq val(s, y)$
(b) $Sat_{\Delta_0}(r * \ulcorner < \urcorner * s, y) \leftrightarrow val(r, y) < val(s, y)$
(c) $Sat_{\Delta_0}(\ulcorner (\urcorner * u * \ulcorner \wedge \urcorner * v * \ulcorner) \urcorner, y) \leftrightarrow Sat_{\Delta_0}(u, y) \wedge Sat_{\Delta_0}(v, y)$
(d) $Sat_{\Delta_0}(\ulcorner \neg \urcorner * u, y) \leftrightarrow \neg Sat_{\Delta_0}(u, y)$
(e) $Sat_{\Delta_0}(\ulcorner \exists \urcorner * var(i) * \ulcorner (\urcorner * var(i) * \ulcorner < \urcorner * r * \ulcorner \wedge \urcorner * u \ulcorner) \urcorner, y)$
$\leftrightarrow \exists x < val(r, y) Sat_{\Delta_0}(u, y[x|i])$
for all y, i, r, s, u, v.

Proof. We will only prove that $Sat_{\Delta_0}(x, y)$ is $\Delta_1(PA)$, since the other properties are straightforward.

From the definition of Sat_{Δ_0} is clear that $Sat_{\Delta_0}(x, y)$ is $\Sigma_1(PA)$, so we only need to show that $PA \vdash \forall x, y(sat_{\Delta_0}(x, y) \leftrightarrow \psi(x, y))$ for some Π_1-formula ψ. Let us define the formula $\psi(x, y)$ as

$$form_{\Delta_0}(x) \wedge \forall t, s \forall w \leq 1[(satseq_{\Delta_0}(s * [x], t) \wedge \forall l < len(t)[t]_l \doteq \langle len(s), y, x\rangle) \rightarrow w \doteq 1].$$

We will show both directions separately. The direction $PA \vdash \forall x, y(sat_{\Delta_0}(x, y) \rightarrow \psi(x, y))$ follows from lemma 4.49 and from $PA \vdash \forall x(\exists s, t(satseq_{\Delta_0}(s * [x], t)) \rightarrow form_{\Delta_0}(x))$. While the other direction, $PA \vdash \forall x, y(\psi(x, y) \rightarrow sat_{\Delta_0}(x, y))$, can be proved with lemma 4.50.

We have seen that the formula $Sat_{\Delta_0}(x, y)$ is equivalent to a Π_1-formula in PA, so $Sat_{\Delta_0}(x, y)$ is $\Delta_1(PA)$. □

Proposition 4.52. *For any Δ_0-formula $\theta(v_0, \ldots, v_k)$,*

$$PA \vdash \forall v_0, \ldots, v_k[\theta(v_0, \ldots, v_k) \leftrightarrow Sat_{\Delta_0}(\ulcorner \theta \urcorner, [v_0, \ldots, v_k])].$$

Proof. It can be proved by induction in the construction of θ. □

Notation 4.53. We also write Sat_{Δ_0} as Sat_{Σ_0} or Sat_{Π_0}.

So we are finally prepared to give a **truth definition for Σ_n and Π_n formulas** for all $n \in \mathbb{N}$, denoted by the formulas $Sat_{\Sigma_n}(x, y)$ and $Sat_{\Pi_n}(x, y)$ respectively.

Definition 4.54. *For $n \in \mathbb{N}$, $Sat_{\Sigma_{n+1}}(x, y)$ is the formula*

$$form_{\Sigma_{n+1}}(x) \wedge$$
$$\exists s, t[len(t) \doteq len(s) \wedge formseq_{\Sigma_{n+1}}(s) \wedge [s]_{len(s)-1} \doteq x \wedge [t]_{len(t)-1} \doteq y \wedge$$
$$\forall i < len(s)(i > 0 \rightarrow \exists k \leq s \exists z \leq t([s]_i \doteq \ulcorner \exists \urcorner * var(k) * [s]_{i-1} \wedge [t]_{i-1} \doteq [t]_i[z|k]))$$
$$\wedge Sat_{\Pi_n}([s]_0, [t]_0)]$$

and $Sat_{\Pi_{n+1}}(x,y)$ the formula

$$form_{\Pi_{n+1}}(x)\wedge$$
$$\forall s,t[len(t)\doteq len(s)\wedge formseq_{\Pi_{n+1}}(s)\wedge [s]_{len(s)-1}\doteq x\wedge [t]_{len(t)-1}\doteq y\wedge$$
$$\forall i<len(s)(i>0\rightarrow \exists k\leq s\exists z\leq t([s]_i\doteq \ulcorner\forall\urcorner * var(k) * [s]_{i-1}\wedge [t]_{i-1}\doteq [t]_i[z|k]))$$
$$\wedge Sat_{\Sigma_n}([s]_0,[t]_0)].$$

Theorem 4.55. *For each $n\geq 1$, $Sat_{\Sigma_n}(x,y)$ is $\Sigma_n(PA)$, $Sat_{\Pi_n}(x,y)$ is $\Pi_n(PA)$ and PA proves the following:*

(a) $\forall s[form_{\Sigma_{n-1}}(x)\rightarrow \forall y(Sat_{\Sigma_n}(x,y)\leftrightarrow Sat_{\Sigma_{n-1}}(x,y))\wedge$
$\forall y(Sat_{\Pi_n}(x,y)\leftrightarrow Sat_{\Sigma_{n-1}}(x,y))]$

(b) $\forall s[form_{\Pi_{n-1}}(x)\rightarrow \forall y(Sat_{\Sigma_n}(x,y)\leftrightarrow Sat_{\Pi_{n-1}}(x,y))\wedge$
$\forall y(Sat_{\Pi_n}(x,y)\leftrightarrow Sat_{\Pi_{n-1}}(x,y))]$

(c) $\forall x,y,k(Sat_{\Sigma_n}(\ulcorner\exists\urcorner * var(k) * x,y)\leftrightarrow \exists z Sat_{\Sigma_n}(x,y[z|k]))$

(d) $\forall x,y,k(Sat_{\Pi_n}(\ulcorner\forall\urcorner * var(k) * x,y)\leftrightarrow \forall z Sat_{\Pi_n}(x,y[z|k]))$

Proof. $Sat_{\Sigma_n}(x,y)$ being Σ_n and $Sat_{\Pi_n}(x,y)$ being Π_n can be proved by induction on $n\in\mathbb{N}$. The initial case follows from theorem 4.51.

The other properties are straightforward. □

Proposition 4.56. *For any Σ_n-formula $\theta(v_0,\ldots,v_k)$ and any Π_n-formula $\psi(v_0,\ldots,v_k)$,*

$$PA\vdash \forall v_0,\ldots,v_k(\theta(v_0,\ldots,v_k)\leftrightarrow Sat_{\Sigma_n}(\ulcorner\theta\urcorner,[v_0,\ldots,v_k]))$$

and

$$PA\vdash \forall v_0,\ldots,v_k(\psi(v_0,\ldots,v_k)\leftrightarrow Sat_{\Pi_n}(\ulcorner\psi\urcorner,[v_0,\ldots,v_k])).$$

Proof. It can be proved by induction on $n\in\mathbb{N}$. The initial case is given by proposition 4.52. □

Chapter 5

Ryll-Nardzewski's theorem

We are coming closer to our main objective. In this final chapter we will give the last definitions needed to prove **Ryll-Nardzewski's theorem**.

5.1 Definable elements

Let M be an arbitrary model of PA and A a subset of M.

Definition 5.1. *An element $b \in M$ is **definable in** M **over** A iff there is an $\mathcal{L}_A$-formula $\theta(x, \bar{y})$ and a tuple $\bar{a} \in A$ such that $M \models \exists! x \theta(x, \bar{a})$ and b is this unique element.*

Notation 5.2. We denote the set of all elements of M definable over A as $K(M; A)$. If $A = \{a_1, \ldots, a_n\}$ is finite we can denote $K(M; A)$ by $K(M; a_1, \ldots, a_n)$ and if it is empty, we will denote it just $K(M)$.

We want to show now that $K(M; A)$ is the universe of a model that satisfies PA. We will denote this model also as $K(M; A)$. To show this, we will prove that $K(M; A) \preceq M$.

Theorem 5.3. *If $M \models PA$ and $A \subseteq M$ then $A \subseteq K(M; A) \preceq M$.*

Proof. To show that $A \subseteq K(M; A)$ it is enough to see that each $a \in A$ is definable over A by the formula $x \doteq a$.

Let us prove now that $K(M; A) \preceq M$. To do so we will first show that $K(M; A)$ is a substructure of M. Clearly all elements of $K(M; A)$ are elements of M and also $0, 1 \in K(M; A)$ since they can be defined by the formulas $x \doteq 0$ and $x \doteq 1$. Suppose now some $c, d \in K(M; A)$ defined by the formulas $\theta_1(x, \bar{a})$ and $\theta_2(y, \bar{b})$ with $\bar{a}, \bar{b} \in A$. Then $c + d$ and $c \cdot d$ are defined by

$$\exists u, v(\theta_1(u, \bar{a}) \land \theta_2(v, \bar{b}) \land z \doteq u + v)$$

and

$$\exists u, v(\theta_1(u, \bar{a}) \land \theta_2(v, \bar{b}) \land z \doteq u \cdot v).$$

So $c + d, c \cdot d \in K(M; A)$ and hence $K(M; A)$ is a substructure of M.

Now we will use Tarski-Vaught test to show that $K(M;A) \preceq M$. To do so we will prove that for each $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(x,\bar{y})$ with $\bar{y} = (y_0, \dots, y_n)$ and each $\bar{c} = (c_0, \dots, c_n) \in K(M;A)$ such that $M \vDash \exists x\varphi(x,\bar{c})$, exists some $d \in K(M;A)$ such that $M \vDash \varphi(b,\bar{a})$.

Since $\bar{c} \in K(M;A)$ there is a formula $\eta_i(x,\bar{a})$ defining c_i for each $i \in \{0,\dots,n\}$ and for some $\bar{a} \in A$. Then

$$M \vDash \exists x, \bar{y}(\bigwedge_{i=1}^{n} \eta_i(y_i,\bar{a}) \wedge \varphi(x,\bar{y})).$$

By the least number principle,

$$M \vDash \forall \bar{y}[\exists x\varphi(x,\bar{y}) \rightarrow \exists z(\varphi(z,\bar{y}) \wedge \forall w < z \neg\varphi(w,\bar{y}))]$$

and in particular,

$$M \vDash \exists x\varphi(x,\bar{c}) \rightarrow \exists z(\varphi(z,\bar{c}) \wedge \forall w < z \neg\varphi(w,\bar{c})).$$

Since $M \vDash \exists x\varphi(x,\bar{c})$, we have

$$M \vDash \exists z[\exists \bar{y}(\bigwedge_{i=1}^{n} \eta_i(y_i,\bar{a}) \wedge \varphi(z,\bar{y})) \wedge \forall w < z \forall \bar{y}(\bigwedge_{i=1}^{n} \eta_i(y_i,\bar{a}) \rightarrow \neg\varphi(w,\bar{y}))].$$

Notice that the formula in square brackets defines an element $d \in K(M;A)$ such that $M \vDash \varphi(d,\bar{c})$, as required. $\square$

Definition 5.4. *For each complete consistent theory T extending PA with the language $\mathcal{L}_{\mathcal{A}}$, we define the **prime model for** T as $K_T = K(M)$ for $M \vDash T$ an arbitrary model.*

With the next theorem we will show that the definition of K_T only depends on T and not on the choice of M.

Theorem 5.5. *Let T be a complete consistent extension of PA with the language $\mathcal{L}_{\mathcal{A}}$ and N a model of T. Then there is a unique elementary embedding $h : K_T \hookrightarrow N$ and the image of this embedding is $K(N)$.*

Proof. Let us consider N and M such that $N \vDash T$, $M \vDash T$ and $K_T = K(M)$. Since $K_T \vDash T$ and T complete, by Theorem 5.3 we have,

$$N \vDash \exists! x\theta(x) \Leftrightarrow T \vdash \exists! x\theta(x) \Leftrightarrow K_T \vDash \exists! x\theta(x)$$

for any $\mathcal{L}_{\mathcal{A}}$-formula $\theta(x)$.

For each $a \in K_T$ let $\theta_a(x)$ be a formula defining a in M and $h : K_T \rightarrow N$ the function defined by $h(a) =$ the unique element of N satisfying $\theta_a(x)$.

Let us see now that h is an embedding.

• Injective: Let $a, b \in K_T$ such that $a \neq b$, then

$$T \vdash \forall x, y(\theta_a(x) \wedge \theta_b(y) \rightarrow \neg x \doteq y)$$

and hence $h(a) \neq h(b)$.

- Respects addition: Let $a, b \in K_T$ and $a + b = c$ in K_T, then

$$T \vdash \forall x, y, z(\theta_a(x) \wedge \theta_b(y) \wedge \theta_c(z) \rightarrow x + y \doteq z)$$

and hence $h(a) + h(b) = h(c)$ in N.

- Respects product: Let $a, b \in K_T$ and $a \cdot b = c$ in K_T, then

$$T \vdash \forall x, y, z(\theta_a(x) \wedge \theta_b(y) \wedge \theta_c(z) \rightarrow x \cdot y \doteq z)$$

and hence $h(a) \cdot h(b) = h(c)$ in N.

- Respects order: Let $a, b \in K_T$ such that $a < b$, then

$$T \vdash \forall x, y(\theta_a(x) \wedge \theta_b(y) \rightarrow x < y)$$

and hence $h(a) < h(b)$.

Thus h is an embedding $h : K_T \hookrightarrow N$. Let us see now that it is unique.

Let $k : K_T \to N$ be an arbitrary elementary embedding. Then for each $a \in K_T$, we have $K_T \vDash \theta_a(x)$ and since k is an elementary embedding also $N \vDash \theta_a(k(a))$. Moreover, since $T \vdash \exists! x\theta_a(x)$ then $N \vDash \exists! x\theta_a(x)$. So $k(a) = h(a)$ for all a.

There is only left to show that $K(N)$ is the image of h. We will prove both inclusions.

$\supseteq$: All elements of the image of h are clearly defined in N by the formula $\theta_a(x)$.

$\subseteq$: Suppose $b \in K(N)$, then b is defined in N by some $\mathcal{L}_\mathcal{A}$-formula $\varphi(x)$. Hence, $N \vDash \exists! x\varphi(x)$, so $K_T \vDash \exists! x\varphi(x)$ which means that there is some $a \in K_T$ such that $K_T \vDash \varphi(a)$. Since $M \vDash \theta_a(a)$ we have $M \vDash (\theta_a(x) \leftrightarrow \varphi(x))$ and then $N \vDash (\theta_a(x) \leftrightarrow \varphi(x))$, so $N \vDash \theta_a(b)$ and hence $h(a) = b$. □

Remark 5.6. Let T be the complete theory of $\mathbb{N}$, then the model K_T is precisely $\mathbb{N}$.

5.2 Σ_n - definable elements

Let M be a model such that $M \vDash PA^-$ and A a subset of M.

Definition 5.7. *For $n \geq 1$, we denote the set $K^n(M; A)$ as the* ***elements in** M **defined by** Σ_n **formulas and*** *$\bar{a} \in A$. In other words, the subset of M consisting of all $b \in M$ such that*

$$M \vDash \theta(b, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \rightarrow x \doteq b)$$

for some $\theta(x, \bar{y}) \in \Sigma_n$ and $\bar{a} \in A$.

Notation 5.8. As in the previous section, we denote $K^n(M; A)$ by $K^n(M; a_1, \ldots, a_n)$ if $A = \{a_1, \ldots, a_n\}$ is finite and by $K^n(M)$ if it is empty.

Proposition 5.9. *$K^n(M;A)$ is a substructure of M.*

Proof. Clearly all elements of $K^n(M;A)$ are elements of M and also $0,1 \in K^n(M;A)$ since they can be defined by the formulas $x \doteq 0$ and $x \doteq 1$. So there is only left to show that $b+c \in K^n(M;A)$ and $b \cdot c \in K^n(M;A)$ for each $b,c \in K^n(M;A)$.

Let $\varphi(y,\bar{a})$ and $\psi(z,\bar{a})$ be the Σ_n formulas defining $b,c \in K^n(M;A)$ respectively. Then $b+c$ is defined by the formula $\exists y,z(\varphi(y,\bar{a}) \wedge \psi(z,\bar{a}) \wedge x \doteq y+z)$ and $b \cdot c$, by the formula $\exists y,z(\varphi(y,\bar{a}) \wedge \psi(z,\bar{a}) \wedge x \doteq y \cdot z)$, both Σ_n formulas. □

Definition 5.10. *For Γ a class of $\mathcal{L}_{\mathcal{A}}$-formulas we say that N is a Γ-**elementary extension** of M, $M \preceq_\Gamma N$, iff $M \subseteq N$ and for each $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(\bar{x}) \in \Gamma$ and each $\bar{a} \in M$,*

$$M \vDash \varphi(\bar{a}) \Leftrightarrow N \vDash \varphi(\bar{a}).$$

Lemma 5.11. *Given some $M \vDash PA^-$ and an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(x,\bar{y})$ such that:*
(1) $M \vDash L_\varphi$.
(2) $\varphi(x,\bar{y}) \wedge \forall u < x \neg\varphi(u,\bar{x}) \in \Sigma_n$.
(3) For all $a,\bar{b} \in K^n(M;A)$, $K^n(M;A) \vDash \varphi(a,\bar{b}) \Leftrightarrow M \vDash \varphi(a,\bar{b})$.
Then for all $\bar{b} \in K^n(M;A)$,

$$K^n(M;A) \vDash \exists x \varphi(x,\bar{b}) \Leftrightarrow M \vDash \exists x \varphi(x,\bar{b}).$$

Proof. We will prove the two directions of the implication separately.

$\Rightarrow$: This direction is clear from assumption (3) and proposition 5.9.

$\Leftarrow$: Let $\bar{b} = \{b_0,\dots,b_m\} \in K^n(M;A)$ be arbitrary and suppose $M \vDash \exists x \varphi(x,\bar{b})$. Since $M \vDash L_\varphi$, there is some $c \in M$ such that

$$M \vDash \varphi(c,\bar{b}) \wedge \forall w < c \neg\varphi(w,\bar{b}).$$

The formula $\psi(x,\bar{b}) = \varphi(x,\bar{b}) \wedge \forall w < x \neg\varphi(w,\bar{b}) \in \Sigma_n$ defines c in M over $\bar{b}$, therefore $c \in K^n(M; A \cup \bar{b})$. Let us define then the formula

$$\theta(x,\bar{a}) = \bigwedge_{i=0}^{m} \eta_i(y_i,\bar{a}) \wedge \psi(x,\bar{y})$$

with $\eta_i \in \Sigma_n$ defining b_i in M over A. The formula θ is also Σ_n and defines c in M over A. So $c \in K^n(M;A)$ and by assumption (3), $K^n(M;A) \vDash \varphi(c,\bar{b})$ which implies that $K^n(M;A) \vDash \exists x \varphi(x,\bar{b})$. □

Lemma 5.12. *Let $n \geq k \geq 1$ and $M \vDash I\Sigma_{k-1}$ such that $K^n(M;A) \preceq_{\Pi_{k-1}} M$, then $K^n(M;A) \preceq_{\Sigma_k} M$.*

Proof. Let us assume $M \vDash I\Sigma_{k-1}$, $K^n(M;A) \preceq_{\Pi_{k-1}} M$ and an $\mathcal{L}_{\mathcal{A}}$-formula $\psi(\bar{y}) \in \Sigma_k$ and show then that for any $\bar{b} \in K^n(M;A)$ we have

$$K^n(M;A) \vDash \psi(\bar{b}) \Leftrightarrow M \vDash \psi(\bar{b}).$$

Since $\psi(\bar{y}) \in \Sigma_k$, we can write ψ as $\exists \bar{x}\varphi(\bar{x},\bar{y})$ for some $\varphi(\bar{x},\bar{y}) \in \Pi_{k-1}$. Let us define then the formula

$$\theta(z,\bar{y}) = \exists \bar{x} < z\varphi(\bar{x},\bar{y}).$$

This formula is Π_{k-1}, since $I\Sigma_{k-1} \vdash Coll_{k-1}$ and hence by section 3.2 the class Π_{k-1} is closed under bounded quantification.

Now we will verify that all assumptions of lemma 5.11 hold to show that for any $\bar{b} \in K^n(M;A)$, $K^n(M;A) \vDash \exists z\theta(z,\bar{b}) \Leftrightarrow M \vDash \exists z\theta(z,\bar{b})$.

Assumption (1) holds by proposition 3.19 and assumption (3) follows from the hypothesis of $K^n(M;A) \preceq_{\Pi_{k-1}} M$. So there is only left to show that

$$\theta(z,\bar{y}) \wedge \forall u < z\neg\theta(u,\bar{y}) \in \Sigma_n,$$

which is true since the negation of a Π_{k-1} formula is Σ_{k-1} and $\Sigma_{k-1} \subseteq \Sigma_n$.

We can assume now that $K^n(M;A) \vDash \exists z\theta(z,\bar{b}) \Leftrightarrow M \vDash \exists z\theta(z,\bar{b})$ for each $\bar{b} \in K^n(M;A)$. Since $\exists z\theta(z,\bar{y})$ is the same as $\exists \bar{x}\varphi(\bar{x},\bar{y})$, we get

$$K^n(M;A) \vDash \psi(\bar{b}) \Leftrightarrow M \vDash \psi(\bar{b})$$

for all $\bar{b} \in K^n(M;A)$, as required. □

Lemma 5.13. *Given some $M \vDash L\Delta_0$ then $K^n(M;A) \preceq_{\Delta_0} M$.*

Proof. We will prove it by induction in the construction of an $\mathcal{L}_A$-formula $\varphi(\bar{x})$ Δ_0. To do so we will show that for all $\bar{a} \in K^n(M;A)$,

$$K^n(M;A) \vDash \varphi(\bar{a}) \Leftrightarrow M \vDash \varphi(\bar{a}).$$

Let us consider the following cases:

For φ an atomic formula it follows from $K^n(M;A)$ being a substructure of M.

Case $\varphi := \neg\psi$ for some $\psi \in \Delta_0$: We have $K^n(M;A) \vDash \varphi(\bar{a})$ iff $K^n(M;A) \nvDash \psi(\bar{a})$ and $M \vDash \varphi(\bar{a})$ iff $M \nvDash \psi(\bar{a})$ for each $\bar{a} \in K^n(M;A)$. By the induction hypothesis, we can conclude $K^n(M;A) \nvDash \psi(\bar{a}) \Leftrightarrow M \nvDash \psi(\bar{a})$ and hence $K^n(M;A) \vDash \varphi(\bar{a}) \Leftrightarrow M \vDash \varphi(\bar{a})$.

Case $\varphi := \psi_1 \wedge \psi_2$ for some $\psi_1, \psi_2 \in \Delta_0$: We have that for each $\bar{a} \in K^n(M;A)$, $K^n(M;A) \vDash \varphi(\bar{a})$ iff $K^n(M;A) \vDash \psi_1(\bar{a})$ and $K^n(M;A) \vDash \psi_2(\bar{a})$ and $M \vDash \varphi(\bar{a})$ iff $M \vDash \psi_1(\bar{a})$ and $M \vDash \psi_1(\bar{a})$.

From the induction hypothesis follows then that $K^n(M;A) \vDash \varphi(\bar{a}) \Leftrightarrow M \vDash \varphi(\bar{a})$.

Case $\varphi(z,\bar{y}) := \exists x < z\psi(x,z,\bar{y})$: We will use lemma 5.11. Assumption (1) follows from $M \vDash L\Delta_0$ and assumption (3) from the induction hypothesis. We just need to show that $\varphi(z,\bar{y}) \wedge \forall u < z\neg\varphi(u,\bar{y}) \in \Sigma_n$, which is true since it is Δ_0.

So, by induction we can conclude that $K^n(M;A) \preceq_{\Delta_0} M$. □

Theorem 5.14. *Let $n \geq k \geq 1$ and $A \subseteq M \vDash I\Sigma_{k-1}$, then $A \subseteq K^n(M;A) \preceq_{\Sigma_k} M$.*

Proof. To show that $A \subseteq K^n(M;A)$ it is enough to see that each $a \in A$ can be defined by the formula $x \doteq a$.

Let us prove now that $K^n(M;A) \preceq_{\Sigma_k} M$. We will do it by induction on k.

Initial case: $M \vDash I\Sigma_0$ and hence $M \vDash L\Delta_0$. So by lemma 5.13, $K^n(M;A) \preceq_{\Delta_0} M$, which is the same as $K^n(M;A) \preceq_{\Pi_0} M$. Therefore, from lemma 5.12 follows $K^n(M;A) \preceq_{\Sigma_1} M$.

Let us do now the induction case. Assume it true for $k = r$ and prove it for $k = r+1$. For $n \geq r+1$ and $M \vDash I\Sigma_r$ we want to show that $K^n(M;A) \preceq_{\Sigma_{r+1}} M$.

$I\Sigma_{r-1} \subseteq I\Sigma_r$, so $M \vDash I\Sigma_{r-1}$ and by the induction hypothesis $K^n(M;A) \preceq_{\Sigma_r} M$. Notice that $K^n(M;A) \preceq_{\Sigma_r} M$ is the same as $K^n(M;A) \preceq_{\Pi_r} M$, so from lemma 5.12 we can conclude $K^n(M;A) \preceq_{\Sigma_{r+1}} M$ as required. □

Proposition 5.15. *Let M be an $\mathcal{L}_{\mathcal{A}}$-structure, if $\mathbb{N} \subseteq_e M$ then $\mathbb{N} \preceq_{\Delta_0} M$.*

Proof. We will prove it by induction in the construction of an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(\bar{x})$ Δ_0. We need to show that for any $\bar{a} \in \mathbb{N}$,

$$\mathbb{N} \vDash \varphi(\bar{a}) \Leftrightarrow M \vDash \varphi(\bar{a}).$$

The cases $\varphi := t_1 \doteq t_2$ and $\varphi := t_1 < t_2$ for $t_1(\bar{x})$ and $t_2(\bar{x})$ $\mathcal{L}_{\mathcal{A}}$-terms are also clear, since $t_1[\bar{a}], t_2[\bar{a}] \in \mathbb{N}$.

Case $\varphi := \neg\psi$ for some $\psi \in \Delta_0$: By the induction hypothesis $\mathbb{N} \vDash \psi(\bar{a}) \Leftrightarrow M \vDash \psi(\bar{a})$, so $\mathbb{N} \nvDash \psi(\bar{a}) \Leftrightarrow M \nvDash \psi(\bar{a})$ and hence $\mathbb{N} \vDash \varphi(\bar{a}) \Leftrightarrow M \vDash \varphi(\bar{a})$.

Case $\varphi := \psi_1 \wedge \psi_2$ for some $\psi_1, \psi_2 \in \Delta_0$: From the induction hypothesis follows that $\mathbb{N} \vDash \psi_1(\bar{a}) \wedge \psi_2(\bar{a}) \Leftrightarrow M \vDash \psi_1(\bar{a}) \wedge \psi_2(\bar{a})$.

Case $\varphi(z,\bar{y}) := \exists x < z\psi(\bar{y})$ for some $\psi \in \Delta_0$: Let us assume that $M \vDash \exists x < b\psi(\bar{a})$ for some $b, \bar{a} \in \mathbb{N}$, i.e. there is some $c \in M$ such that $M \vDash c < b$ and $M \vDash \psi(\bar{a})$. By the hypothesis of $\mathbb{N} \subseteq_e M$ we have $c \in \mathbb{N}$ and by induction hypothesis $\mathbb{N} \vDash \psi(\bar{a})$, so $\mathbb{N} \vDash \varphi(b,\bar{a})$. The other direction follows from $\mathbb{N} \subseteq M$ and the induction hypothesis. □

Remark 5.16. $K^n(M;A)$ may be nonstandard.

For example, let us assume $n = 1$, $A = \emptyset$ and $M \vDash PA + \exists x\chi(x)$ for some formula $\chi \in \Delta_0$ such that $\mathbb{N} \vDash \forall x\neg\chi(x)$. Such a formula exists as a consequence of Gödel's incompleteness theorem. It is clear that $K^1(M)$ contains the least element $c \in M$ such that $M \vDash \chi(c)$. Moreover since $M \vDash PA$, we have $\mathbb{N} \subseteq_e M$ and by proposition 5.15, $\mathbb{N} \preceq_{\Delta_0} M$. Therefore, we can not have $c \in \mathbb{N}$, since otherwise $\mathbb{N} \vDash \chi(c)$, contradicting $\mathbb{N} \vDash \forall x\neg\chi(x)$. So $c \in K^1(M)$ is a nonstandard element.

Lemma 5.17. *For M a model of PA, if $K^n(M;A)$ is nonstandard, $n \geq 1$ and A is finite (i.e $A = \{\bar{a}\}$), then $K^n(M;A) \nvDash PA$.*

Proof. Let us define $\varphi(x,y) := Sat_{\Sigma_n}(y,[\bar{a},x]) \wedge \forall z(Sat_{\Sigma_n}(y,[\bar{a},z]) \rightarrow z \doteq x)$ and assume $K^n(M;A) \vDash PA$.

For each $c \in K^n(M;A)$ there exists a formula $\theta(\bar{x}) \in \Sigma_n$ defining c in M over A. By proposition 4.56 $M \vDash \varphi(c,b)$ for some $b \in \mathbb{N}$ coding θ. We can say then that for each $c \in K^n(M;A)$ exists some $b \in \mathbb{N}$ such that $M \vDash \varphi(c,b)$. Since $\varphi(c,b)$ is the conjunction of a formula Σ_n and a formula Π_n, in particular $\varphi \in \Sigma_{n+1}$. Notice that $M \vDash PA$ implies $M \vDash I\Sigma_n$, so theorem 5.14 gives us $K^n(M;A) \vDash \varphi(c,b)$.

Thus for any nonstandard $d \in K^n(M;A)$, which means $d > \mathbb{N}$, we have

$$K^n(M;A) \vDash \forall c \exists b < d\varphi(c,b),$$

and by the least number principle, there is a least $d_0 \in K^n(M;A)$ such that

$$K^n(M;A) \vDash \forall c \exists b < d_0\varphi(c,b).$$

This d_0 must be standard, since otherwise we could define w as $d_0 - 1$ and have then $w \in K^n(M;A)$ nonstandard and $K^n(M;A) \vDash \forall c \exists b < w\varphi(c,b)$, so d_0 would not be the least one. But if $d_0 \in \mathbb{N}$ we will have d_0 finite, so the possible values for b will also be finite and we would have a finite number of formulas defining an infinite numbers of elements, which is impossible. So necessarily $K^n(M;A) \nvDash PA$. $\square$

We have acquired now all knowledge required to prove **Ryll-Nardzewski's theorem** and conclude, as a corollary, that there is no finitely axiomatizable system equivalent to PA.

Theorem 5.18. *(Ryll-Nardzewski) No consistent extension of PA is finitely axiomatized.*

Proof. Let us assume that there is a finitely axiomatized theory T of $\mathcal{L}_\mathcal{A}$ such that $PA \subseteq T$. Then all axioms of T are Π_n for some $n \in \mathbb{N}$. Consider now a nonstandard model $M \vDash T$ and a nonstandard $a \in M$ (i.e $a > \mathbb{N}$). Notice that $M \vDash I\Sigma_{n-1}$, so by theorem 5.14, $K^n(M;a) \preceq_{\Sigma_n} M$. Then, $K^n(M;a) \vDash T$ and hence $K^n(M;a) \vDash PA$, contradicting lemma 5.17. $\square$

Conclusion

In this project we have presented a system enough for proving almost all statements in $\mathbb{N}$, the system of *Peano Arithmetic*. We have learned how to measure the complexity of a formula and given a definition of truth in *PA*. All this for proving *Ryll-Nardzewski's theorem* and coming to the conclusion of *Peano Arithmetic* not being finitely axiomatizable.

But in addition to this conclusion, from this project I also learned the importance of being rigorous. Personally, I really enjoyed writing chapter 4, since it was a challenge for me; writing it as plain as possible and choosing a clear notation was a really hard work.

Moreover, while doing this project I discovered the close relation between Model Theory and Arithmetic and how beautiful these fields can be. For this reason I have decided to keep studying them next year.

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