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Rubinstein's bargaining model

Francisca Gayà Torres

Advisors: **Dr. Xavier Jarque**
Dept. de Matemàtiques i Informàtica
Dr. Javier Martínez de Albéniz
Dept. de Matemàtica Econòmica, Financera i Actuarial

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Abstract

Rubinstein's bargaining model defines a multi-stage non-cooperative game in extensive form with complete information. It is applied to two-person games that feature alternating offers through an infinite time horizon. We study the process of bargaining due to Rubinstein (1982) — from his seminal paper *Perfect equilibrium in a bargaining model.*

Firstly, we present in detail this model. The fundamental assumption is that the players are impatient and the main result provides conditions under which the game has a unique subgame perfect equilibrium. The result gives a characterization of this equilibrium and features the fact that bargaining implies costs for the agents (time and money). In addition, we introduce a variation of the model which was revisited some years later (1988). To do it, it uses new utility functions which are used to arrive to the same conclusion of the original model. Finally, we present an extension of the model of bargaining to the war of attrition (Ponsati and Sákovics, 1995), using games with incomplete information. They introduce the deadline effect.

Resum

El model de negociació de Rubinstein defineix un joc no cooperatiu format per diverses etapes en forma extensiva amb informació completa. S'aplica a jocs de dos agents en els quals es presenten ofertes alternades al llarg d'un període de temps eventualment infinit. Estudiem el procés de negociació ideat per Rubisntein (1982) — del seu article decisiu *Perfect equilibrium in a bargaining model.*

En primer lloc, presentam detalladament aquest model. La suposició fonamental és que els jugadors són impacients i el resultat final proporciona condicions sota les quals el joc té un únic equilibri perfecte en subjocs. El resultat dona una caracterització d'aquest equilibri i mostra el fet que la negociació suposa uns costos per als agents (temps i diners). A més a més, introduïm una variació del model que va ser revisada alguns anys després (1988). Per fer-ho, s'utilitzen noves funcions d'utilitat per arribar a la mateixa conclusió del model original. Finalment, presentam una extensió del model de negociació a la guerra de desgast (Ponsati i Sákovics, 1995), mitjançant jocs amb informació incompleta. Introdueixen l'efecte de la data límit.

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Contents

Introduction

Bargaining is clearly present in our interactions with partners, family, lovers or everyday business. It is thought as a psychological play with rewards, menaces, and final agreements. The analysis of bargaining from a theoretical point of view is recent and uses Game Theory. The goal of Game Theory is to study the behavior of decision-makers, called players, whose decisions affect each other. To date, economics has been the largest area of application but it also has other applications with political science, evolutionary biology, computer science, the foundations of mathematics, statistics,...

In Game Theory we need to distinguish between two approaches: the non-cooperative and the cooperative. These two theories have quite different characters. The non-cooperative theory concentrates on the strategic choices of the individual (how each player plays the game and what strategies he chooses to achieve his goals) while the cooperative theory deals with the options available to the group (what coalitions form and how the available payoff is divided). The non-cooperative theory focuses on the details of the processes and rules defining a game; and the cooperative theory usually abstracts away from such rules, and looks only at more general descriptions that specify only what each coalition can get, without saying how (is left unmodeled). However, there is a close relation between the two approaches: they complement and strengthen one another.

Non-cooperative Game Theory, as in one-person decision theory, makes the analysis from a rational, rather than a psychological or sociological viewpoint. This assumption was not so clear at the beginning of the study of this science. For years, economists tend to agree that further specification of a bargaining solution would need to depend on the vague notion of bargaining ability and so, they regarded the bargaining problem as indeterminate. Even von Neumann and Morgenstern (1944) [30] suggested that the bargaining outcome would necessarily be determined by unmodeled psychological properties of the players.

Nash (1950, 1953) [11, 13] broke away from this tradition. His agents are fully rational and once their preferences are given, other psychological issues are irrelevant. The bargaining outcome in Nash's model is determined by the players' attitudes towards risk.

> *"A two-person bargaining situation involves two individuals who have the opportunity to collaborate for mutual benefit in more than one way. The two individuals are highly rational, each can accurately compare his desires for various things and they are equal in bargaining skill."*
>
> (John Nash, 1950)

John Forbes Nash was born on June 13, 1928, in Bluefield, West Virginia. He got the Nobel Memorial Prize in Economic Sciences in 1994, joint with John Harsanyi and Reinhard Selten, "for their pioneering analysis of equilibria in the theory of non-cooperative games". He was a mathematician and he got the Abel Prize in 2015, joint with Louis Nirenberg, "for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis". On May 23, 2015, Nash and his wife died in a traffic accident coming back from Oslo on their way home from the airport, after receiving the Abel Prize.

Nearly all human interaction can be seen as bargaining of one form or another. This type of problem is analyzed in this work as a non-cooperative game. The target of such a non-cooperative theory of bargaining is to find theoretical predictions of what agreement, if any, will be reached by the bargainers. One hopes thereby to explain the manner in which the bargaining outcome depends on the parameters of the bargaining problem and to shed light on the meaning of some of the verbal concepts that are used when bargaining is discussed in ordinary language. It was Nash himself (1950)[11] who felt the need to add the axiomatic approach for this type of game. An axiomatic approach involves abstracting away the details of the process of bargaining and consider only the set of outcomes or agreements that satisfy reasonable properties. In particular, the Nash program consists of studying cooperative solutions such that they are equilibria of some non-cooperative game. Nash was the first in adopting a systematic theoretical approach to the bargaining problem, using an axiomatic approach. Along the pass of time, the theoretical interest has progressively shifted towards a different approach, the strategic one, in which, unlike the axiomatic approach, does explicitly take into account the procedure and the context of the negotiation. This theory of strategic negotiation attempts to resolve the indeterminacy through the explicit modeling of the negotiation procedure.

Nash (1951) [12] began with another brilliant result regarding a game of two-player bargaining. Here too, he drew from von Neumann and Morgenstern's derivation of utility given in 1947. He modeled the situation where two-players are bargaining over an issue of mutual interest. Nash produced an unprecedented result here that has become a workhorse model for bargaining in various disciplines of economics. Following bargaining, he went on to produce the result in which he is mostly known for: the formulation of *Nash Equilibrium* along with its existence. A Nash Equilibrium is the situation in which no player can reach a better output without using the equilibrium strategy.

In his second paper (1953)[13], he demonstrated that the solution of a non-cooperative game is the limit of a sequence of equilibrium of bargaining games. This analysis in non-cooperative games is important because it explains, within the theory, why bargaining is a problem, and thus provides a framework in which the influence of the environment on bargaining outcomes can be evaluated. After his work, there are some other conclusions wherein the bargaining is represented by a multi-stage game.

A sequential bargaining theory attempts to resolve the indeterminacy by explicitly modeling the bargaining procedure as a sequence of offers and counteroffers. In the context of such models, Cross (1965) [4] suggests that the players' time preferences may be highly relevant to the outcome. In what follows, who gets what depends exclusively on how patient each player is.

Schelling (1960) [22] was skeptic about the extent to which such commitments can genuinely be made. A player may make threats about his last offer being final, but the opponent can dismiss such threats unless it would actually be in the interests of the threatening player to carry out his threat if his implicit ultimatum were disregarded. In such situations, where threats need to be credible to be effective, we must replace Nash equilibrium by Selten's notion of subgame perfect equilibrium.

The first to investigate the alternating offer procedure was Ståhl (1972) [26] in several papers, and he studied the subgame perfect equilibria of such time-structured models by using backwards induction in finite horizon models. Where the horizons in his models are infinite, he postulates nonstationary time preferences that lead to the existence of a "critical period" at which one player prefers to yield rather than to continue, independently of what might happen next. This creates a *last interesting period* from which one can start the backwards induction. In the infinite horizon models studied below, which were first investigated by Rubinstein (1982) [21], different techniques are required to establish the existence of a unique subgame perfect equilibrium.

Ståhl (1972) [26] and Rubinstein (1982) provided the first negotiation model that reflected the fact that negotiation is a typically dynamic process that involves offers and counteroffers. In this game with complete information it can be seen that almost any division of the pie can be obtained as a Nash equilibrium.

Rubinstein's model consists in a two-person game where player 1 starts the negotiation making an offer about the partition of a pie and, after that, player 2 has to accept it or reject it. Then, if the offer is not accepted, it is players 2's time to make an offer. Whenever an offer is accepted, the bargaining ends. And so on until an agreement is reached.

Non-cooperative bargaining theory has been deeply influenced by Rubinstein's paper (1982) which has provided the basic framework for an enormous and still growing literature. Rubinstein's theory embraces both slight impatience (frequent offers) and discounting (significant delay) between offers. Then, the importance of the Rubinstein's model lies in the fact that he proved that a Subgame Perfect Equilibrium (S.P.E.) does exist if the Nash equilibria that are sustained by non-credible threats are eliminated. This single distribution of the cake is determined by the temporal preferences of the agents, by the interval between offers and by the specification of who makes the first offer. The solution is efficient, since in the S.P.E. the agreement occurs in the first period, that is, player 2 accepts the first offer made by player 1.

Unfortunately, neither Nash's axiomatic solution nor the Rubinstein model provide an explanation for the delays that occur in reaching an agreement. Incomplete information games explain this type of problem. Ponsati and Sákovics [20], 1995, presented an analysis of the war of attrition about reservation values where we can see reflected this type of characterization of a game.

About this work

The main part of this project is devoted to study the existence of subgame perfect equilibria in repeated games based in alternating offers through an infinite time horizon.

In Chapter 1 we present some basic concepts of Game Theory focused mainly on non-cooperative games needed for the development of the following chapters. Chapter 2 introduces Rubinstein's bargaining model, following the original Rubinstein's paper (1982), which it has a key role in our study. We adapt some concepts to this specific model and we present the main theorem. We also introduce two applications based on the main theorem about the solution in two different types of models: fixed bargaining cost and fixed discounting factor. In Chapter 3 we review a later version of the original model of Rubinstein. Introducing some new concepts we conclude that the bargaining game has one unique subgame perfect equilibrium. Finally, Chapter 4 provide us a brief extension with incomplete information of an specific type of game, the war of attrition.

Chapter 1

Preliminaries

In this chapter we will find some definitions and basic explanations about Game Theory that we use in the following chapters. Most of these preliminaries are well-known, but they allow to fix a notation and to put in context the following chapters. These concepts are used in the next chapters because they appear during the negotiation between agents and according to their behaviours. For the basic notions we follow Watson (2002) [31] and Gibbons (1992) [8].

1.1 What is a game?

The object of study in game theory is the *game*, which is a formal model of an interactive situation. It usually involves a group of individuals called *players*. The formal definition of game lays out the list of players, their preferences, their information (a description of what players know when they act), the strategic actions available to them (a complete description of what the players can do) and how these influence the outcome.

Game theory can be classified into two different big groups:

1. *Cooperative game theory.* Cooperative game theory assumes that groups of players, called *coalitions*, are the primary units of decision-making, and may enforce cooperative behaviour. The basic assumption in cooperative game theory is that the grand coalition, that is the group consisting of all players, will form. As an example, the players may be several parties in parliament. Each party has a different strength, based upon the number of seats occupied by party members. The game describes which coalition of parties can form a majority, but does not delineate, for example, the negotiation process through which an agreement to vote en bloc is achieved.

2. *Non-cooperative game theory.* Non-cooperative game theory treats all of the players' actions as individual actions done in strategic settings. So, each player makes an individual action from a set of options, called *strategies*. By individually, we mean that no agreement is established between players and so each agent acts in his or her own interest. The procedure is a constant negotiation where every agent bargain the different options that they have, including offers and counteroffers. A player cannot think about carrying out an optimal decision because it depends on what the other agents will do in the game.

1.2 Non-cooperative game: extensive form and normal form

We focus at non-cooperative games and therefore we take a detailed look in them. We assume that there is a finite number, n, of players with $n \geq 2$. Let $N = \{1, ..., n\}$ be the set of all players and let $i \in N$ denote an arbitrary player. The strategies chosen by each player determine the outcome of the game. Associated to each possible outcome there is a collection of numerical payoffs, one for each player. These payoffs represent the utility that the outcome or result gives to each agent. We assume these utilities are personal and generally speaking non-comparable across agents. There are no binding agreements between agents, each one looks after his or her utility.

There are two common forms in which non-cooperative games are represented mathematically: the *extensive form* and the *normal form*.

We will start explaining the extensive form.[1] The extensive form representation of a game it is a most detailed description of a game and specifies: (1) the players in the game; (2) what moves are possible for each player, the order in which they have to play and the information available to them from others' previous moves; and (3) the payoffs received by each player for each possible combination of moves.

The graph that represents the strategic interaction between the players is a *tree*. A *tree* is an undirected graph and consists of a finite set of nodes V together with a set A of unordered pairs of distinct members of V. An element $v \in V$ is called *vertex or node* and represents a decision point by one of the players. Every extensive-form game has one distinguished node which is unique. This node is the initial one and it is called the *root*. Then, a *rooted tree* is a tree in which one vertex has been designated the root. In a tree, if two nodes are connected, they are by exactly one path. In other words, a tree is a connected acyclic undirected path. Each $\{v_1, v_2\} \in A$ is a *branch* that connects the vertices v_1 and v_2 and each branch indicates the various actions that players can choose. A path connecting nodes v_1 and v_m is a sequence $v_1, v_2, \ldots, v_m$ of distinct vertices such that $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{m-1}, v_m\}$ are branches of the tree. Finally, the *leaves* represent the final outcomes over which player has a utility function.

To represent the knowledge available at each stage we will use the concept of *information set*. We will indicate that a collection of decision nodes constitutes an information set by connecting the nodes by a dotted line. The nodes in an information set are indistinguishable to the agent, so all have the same set of actions.

Nodes are represented by solid circle and branches by arrows connecting nodes. It is usually most convenient to represent the players' preferences ranking with numbers, which are called *payoffs* or *utilities*.

In Figure 1.1 we depict a tree with $V = \{a, b, c, d, e\}$, $A = \{\{a, b\}\{a, c\}\{c, d\}\{c, e\}\}$ and a as the root.

Let us formalize the definition of the game in extensive form.

Definition 1.1. A *n-person game in extensive form*, Γ, is defined as consisting of the following:

i) A set $N = \{1, 2, \ldots, n\}$ of players.

ii) A rooted tree, T, called the game tree where $L(T)$ is the set of terminal nodes.

[1] The extensive form was defined in von Neumann and Morgenstern (1944) [30]. It was originally in von Neumann (1928) [29], translated in [28].

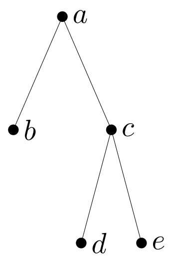


Figure 1.1: A tree

iii) A partition of the set of non-terminal nodes of T, which is called *move*, into $n+1$ subsets denoted $P^0, P^1, P^2, \ldots, P^n$. The members of P^0 are called *chance nodes*; for each $i \in N$, the members of P^i are called the nodes of player i.

iv) For each node in P^0, a probability distribution over its outgoing branches.

v) For each $i \in N$, a partition of P^i into $k(i)$ *information sets,* $U_1^i, U_2^i, \ldots, U_{k(i)}^i$, such that, for each $j = 1, 2, \ldots, k(i)$:

   a) all nodes in U_j^i have the same number of outgoing branches, and there is given one-to-one correspondence between the sets of outgoing branches of different nodes in U_j^i;

   b) every path in the tree from the root to a terminal node can cross each information set U_j^i at most once.

vi) For each terminal node $t \in L(T)$, an n-dimensional vector $g(t) = (g^1(t), g^2(t), \ldots, g^n(t))$ of payoffs.

vii) The complete description (i)-(vi) is *common knowledge* among the players. This means that all players know it and each one knows that everyone knows it.

An *n-person game* Γ (in extensive form) is a game of perfect information if all information sets are singletons, i.e., $|U^i| = 1$ for each player $i \in N$ and each information set $U^i \in I^i$. Thus, in a game of perfect information, every player, whenever called upon to make a choice, always knows exactly where he is in the game tree.

We introduce strategies to define later a game in normal form. Let $I^i := \{U_1^i, U_2^i, \ldots, U_{k(i)}^i\}$ be the set of information sets of player i. For each information set U^i of player i, $i \in N$, let $\nu \equiv \nu(U^i)$ be the number of branches going out of each node in U^i. Thus, we denote $C(U^i) := \{1, 2, \ldots, \nu(U^i)\}$ the set of choices available to player i at any node in U^i.

Definition 1.2. A *strategy s_i of player i, $i \in N$,* is a function defined on the set of information sets

$$s_i : I^i \to \{1, 2, \ldots\},$$

such that $s_i(U^i) \in C(U^i)$ for all $U^i \in I^i$. That is, it assigns a branch going out of each node in U^i.

Let S^i denote the set of strategies available of player i, which is called its *strategy space.* So, we can write it as $S^i := \Pi_{U^i \in I^i} C(U^i)$.

Then $S := S^1 \times S^2 \times \ldots \times S^n$ is the set of $n - tuples$ of strategies of the players and let $s = (s_1, \ldots, s_n) \in S$ denote a *strategy profile*, where s_i is the strategy chosen by player $i \in N$.

The extensive form is one straightforward way of representing a game. Another way of formally describing games is based on the idea of strategies. It is called the normal form (or strategic form) representation of a game. This alternative representation is more compact than the extensive form in some settings. Now, we can proceed to define the game in normal form.

Definition 1.3. An *n-person game G in normal (or strategic) form* Γ is described as a triple $G = (N, S, u)$ and consists of the following:

i) A set $N = \{1, 2, \ldots, n\}$ of players.

ii) For each player $i \in N$, a set S^i of strategies. Let $S := S^1 \times S^2 \times \ldots \times S^n$ denote the set of n-tuples of strategies.

iii) And $u = (u_i)_{i \in \mathbb{N}}$. For each player $i \in N$, a function $u_i : S \to \mathbb{R}$, called the *payoff function of player i,* based on the strategies chosen by all players.

Whenever the strategy set for any player is finite, a natural way to represent games is via a table, especially for two players. The cells of the table are the payoffs for all players in any combination of strategies.

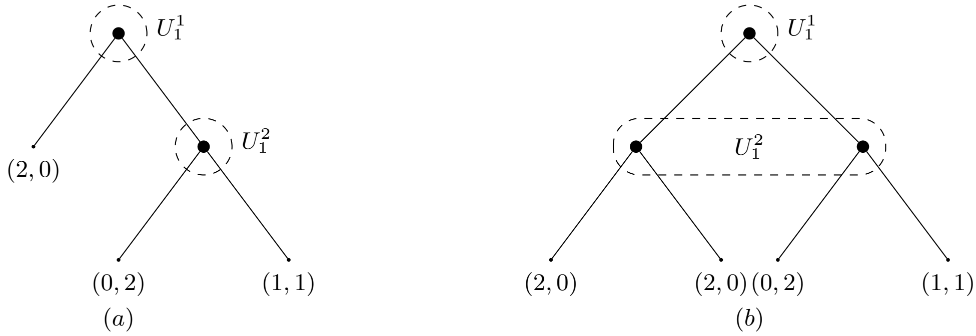
Example 1.1. *Prisoner's Dilemma.* To illustrate our definition we use a well-known example. Two members of a gang of bank robbers have been arrested and are being interrogated in separate rooms. The authorities have no other witnesses, and can only prove the case against them if they can convince at least one of the robbers to betray his accomplice and testify to the crime. Each bank robber is faced with the choice to cooperate with his accomplice and remain silent or to defect from the gang and testify for the prosecution. If they both cooperate and remain silent, then the authorities will only be able to convict them on a lesser charge of loitering, which will mean one year in jail each (1 year for each one). If one testifies and the other does not, then the one who testifies will go free and the other will get nine years. However if both testify against the other, each will get six years in jail for being partly responsible for the robbery. In terms of the game, we have two players and each player has two strategies: to confess or not to confess. The payoffs explained before depending on the strategy chosen are represented in Table 1.1.

1/2	Confess	Don't confess
Confess	(-6, -6)	(0, -9)
Don't confess	(-9, 0)	(-1, -1)

Table 1.1

In the table, each row represents the possible actions for player 1 and each column represents the possible actions for player 2. Furthermore, each cell represents the different payoffs of the players depending on the actions chosen by each player where the left number represents the payoff of player 1 and the right number represents the payoff of player 2. We can notice that if each prisoner thinks about his or her own benefit, the best payoff is when he or she confesses and the other prisoner remains silent. On the other hand, if prisoners seek the best outcome as a group, the best they can do is to not confess.

Therefore, it is easy to check that the normal form can be derived from the extensive form and also, given a game in normal form, we can construct an extensive form to represent it. Thus, any game can be represented in either normal or extensive form, although for some games one of the two forms is more convenient to analyze. The normal form is most convenient for games in which all of the players' decisions are made simultaneously and independently.



(2, 0)	(2, 0)
(0, 2)	(1, 1)

(c)

Figure 1.2: Two games in extensive form, (a) and (b), with the same strategic form (c).

Example 1.2. Considering the example just explained before, the *Prisoner's Dilemma*, we can get the extensive form from the normal form. So, it can be represented with a tree diagram:

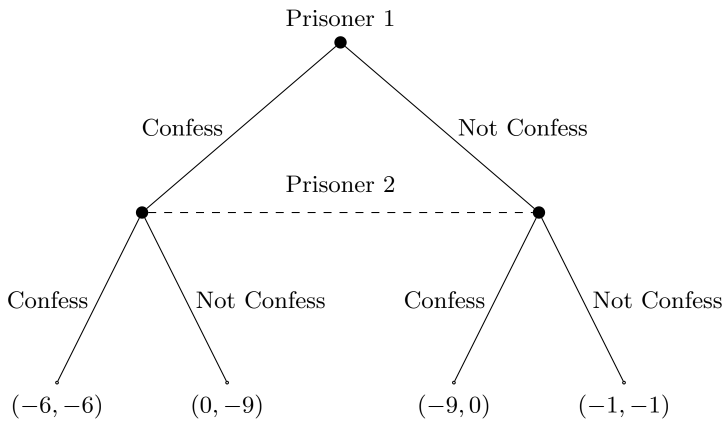


Figure 1.3: Prisoner's Dilemma in extensive form

1.3 Nash Equilibrium

A player with a rational behaviour wants to maximize the payoff that he/she expects to obtain. So, the player should select the strategy that yields the greatest expected payoff if he/she could know what were the strategies of the other players. This strategy is called a *best response,* to the strategy profile of the other players.

In a setting of strategic certainty, the players are best responding and the players' beliefs and behaviour are consistent. So, the player's strategies are mutual best responses. The idea of mutual best response is one of the many contributions of Nobel laureate John Nash to the field of game theory. Nash [12] used the term *equilibrium* to refer to this term and it is what now we call *Nash equilibrium.*

Informally, a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ and its corresponding payoffs represent a Nash equilibrium if no player can increase his or her payoff by changing his or her strategy, as long as the other players keep their strategies unchanged. That is, if no player has the incentive to deviate from their chosen strategy. Formally, the Nash equilibrium is defined as follows.

Definition 1.4. Given a normal form game $G = (N, S, u)$, a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S_1 \times \ldots \times S_n$ is a *Nash equilibrium* if and only if

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)$$

for every $s_i \in S_i$ and each player $i \in N$.

The existence and number of Nash equilibria in a game depends on the characteristics of this game. In fact, there are games without any Nash equilibrium and others that have an infinite amount of them.

Continuance of Example 1.1. In the prisoner's dilemma we find one Nash equilibrium: it is the unique one. The prisoner 1 spends less time in jail if he confesses, regardless of whether prisoner 2 confesses or remains silent. The analogous situation happens to prisoner 2. Therefore, if both prisoners confess, there is no incentive for players to change their strategy. Thus, the only best response is to play the strategy Confess. So, (Confess, Confess) is the unique Nash equilibrium of the game. We have that the best response for both players is Confess, and so the strategy profile (Confess, Confess) is the Nash equilibrium of the prisoner's dilemma.

It is interesting to remark that the Nash equilibrium of the Prisoner's Dilemma is not the best outcome prisoners can get as a group, since if neither of them confesses they get a better payoff as a pair. However, when both prisoners look after his or her own interest and confess, they get a worse payoff as a group.

1.4 Subgame Perfect Equilibrium

In this section, we consider games in which players make choices in sequence. We assume that the moves in all previous stages are known before the next stage begins, and we allow simultaneous moves within each stage. This type of games are better represented by the extensive form explained before. We will assume that the players are sequentially rational, which means that an optimal strategy for a player should maximize the expected payoff, conditional on every information set at which this player has to move. We make this assumption because if we only assume that player select best responses *ex ante*, from the point of view of the beginning of the game, there would appear Nash equilibria which are not consistent with the rationality of the players' information sets. This type of Nash equilibria appear when we use the normal form and forget the underlying decision tree.

Notice that we suppose that the players do not select their strategies in real time (as the game progresses), but rather choose their strategies in advance of playing the game as suggested by the normal form (perhaps they write their strategies on slips of paper, as instructions to their managers regarding how to behave). So, in this case, the player could act sequentially rationally if player $i's$ strategy specifies an optimal action from each of player $i's$ information sets, even those that player i does not believe (*ex ante*) will be reached in the game.

Whenever the information is perfect, we follow another procedure to find the Nash equilibria. It is called *Backward induction procedure* and it is the process of analyzing a game from the end to the beginning. At each decision node, one strikes from consideration any actions that are dominated, given the terminal nodes that can be reached through the play of the actions identified at successor nodes. The backward induction procedure identifies a unique strategy profile for every game of perfect information that has no payoff ties (where a player would get the same payoff at multiple terminal nodes). So, for each game, we obtain a subset of strategy profiles, at least one of which is a Nash equilibrium.

To find the Nash equilibria of a game and then remove the ones that violate sequential rationality we need to introduce a new concept: the concept of subgame. A subgame is a stage within a game which begins at any point where a player has to make a decision.

Definition 1.5. Given an extensive form game Γ, a node x in the tree is said to *initiate a subgame* if neither x nor any of its successors are in an information set that contains nodes that are not successors of x. A *subgame* of Γ rooted at node x, Γ_x, is the tree structure defined by such a node x and its successors. Whenever y is a decision node following x, and z is in the information set containing y, then z also follows x.

Subgames are self-contained extensive forms, which means that they are meaningful trees on their own. Subgames that start from nodes other than the initial node are called *proper subgames.* Notice that in a game of perfect information, every node initiates a subgame.

Applying the concept of Nash equilibrium on each subgame of an extensive form game, we get the notion of a *subgame perfect equilibrium.* This concept was introduced by Selten [23, 24].

Definition 1.6. Given an extensive form game Γ, a strategy profile s is called a *subgame perfect Nash equilibrium (SPE)* of Γ if s induces a Nash equilibrium in every subgame of Γ, the original game.

Since Γ is in particular its own subgame, every subgame perfect equilibrium is also a Nash equilibrium. That is, subgame perfect Nash equilibrium is a refinement of Nash equilibrium. We assume that a rational player, confronted to any stage of the game, will select only a Nash equilibrium. Therefore, any equilibrium which involves unbelievable threats should be discarded. Then, the basic idea behind subgame perfection is that a solution concept should be consistent with its own application from anywhere in the game where it can be applied. Because Nash equilibrium can be applied to all well-defined extensive-form games, subgame perfect Nash equilibrium requires the Nash condition to hold on all subgames.

We can also define the SPE from another point of view, but we need to introduce some concepts to define it.

Definition 1.7. For a given node x in a game, a player's *continuation value*, or also called *continuation payoff* is the payoff that this player will eventually get contingent on the path of play passing through node x.

In some games, the players discount the amounts that they receive over time. So, in this case it's convenient to apply the continuation value discounted to the start of the continuation point. Note that for any strategy profile s, we can calculate the continuation values associated with s. To do it, we simply construct, from any particular node, the path to a unique terminal node that s implies. With these concepts we can introduce the new and equivalent SPE definition.

Definition 1.8. Consider any finite extensive form game Γ. A strategy profile s^* is a subgame perfect Nash equilibrium if and only if, for each player i, $i \in N$, and for each subgame, no single deviation would raise player $i's$ payoff in the subgame.

Example 1.3. *A centipede game.* Consider the game depicted in the following tree (Figure 1.4):

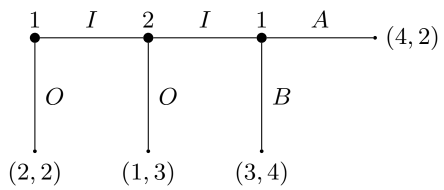


Figure 1.4

The strategies of player 1 are $S_1 = \{OA, OB, IA, IB\}$ and the strategies of player 2 are $S_2 = \{O, I\}$. The backward induction leads to the strategy profile (OA,O), and this is the unique subgame perfect equilibrium. To put sequential rationality into words, we have the following story: if player 1's second decision node is reached, then this player will definitely select action A. Player 2 can anticipate this (knowing that player 1 is sequentially rational), so player 2 would definitely select O at her decision node. Finally, because player 1 knows that player 2 is rational, player 1 optimally selects O at the initial node.

Let us illustrate all the information of this section with an example.

Example 1.4. *The centipede game.* It is an extensive form game in which two players, 1 and 2, intervene. There is an increasing pot with money. In each turn, player 1 or 2 (depending on who has to act), has to choose either to take a slightly larger share of the pot or to pass the pot to the other player. Since it is an increasing pot, if one player passes the pot to one's opponent and the opponent takes the pot on the next round, one receives slightly less than if one had taken the pot on this round. In our case, we will do it with 4 different rounds. In a first stage, player 1 must decide whether to take the money or to wait one period more. The best outcome for player 2 is that player 1 waits one more period. If it happens, then there is a second stage where player 2 must decide whether to take the money or also wait one more period. If player 1 prefers to take the money, the game will finish. After that, if player 2 decides to wait one more period, is player 1's turn to decide if take the money or wait for the last period. If he decides to wait, then it's player 2 turn to decide whether to take the money or split the money between the two players. So, players 1 and 2 alternate, starting with player 1, and may on each turn play a move from $\{take, wait\}$. In the last turn, player 2 has to decide between $\{take, split\}$. The game finishes if *take* is played for the first time before the fourth round or if this not happens, the game finishes in the forth round.

Suppose the game ends on round $t \in \{0, 1, 2, 3\}$ with player $i \in \{1, 2\}$. Suppose also that $m_0, m_1 \in \mathbb{N}$ and $m_0 > m_1$. The outcomes of the game is defined as follows:

- If player i *takes*, then player i gains $2^t m_0$ coins and the other player gains $2^t m_1$.
- If player i *waits*, then player i gains $2^{t+1} m_1$ coins and the other player gains $2^{t+1} m_0$.

The extensive form representation of this game is depicted in Figure 1.5.

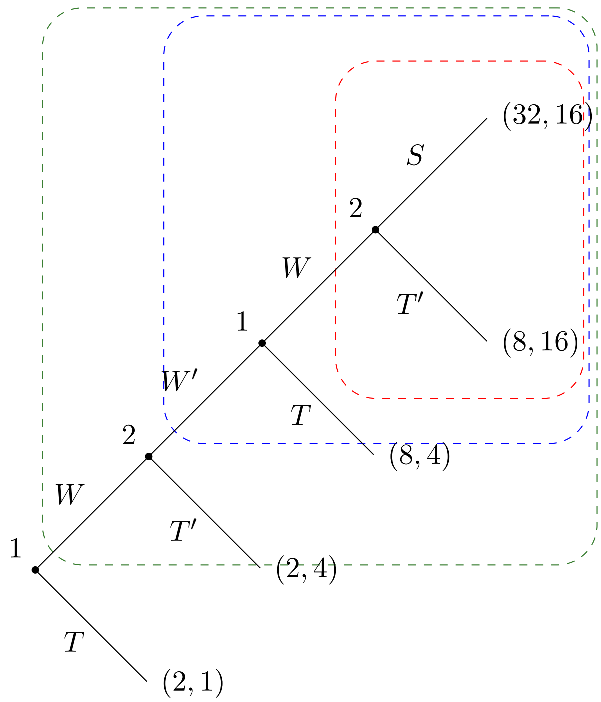


Figure 1.5: Centipede game in extensive form. Subgame 1 is the one in the red box, subgame 2 is the one in the blue box, subgame 3 is the one in the green box and subgame 4 is the entire game.

A strategy consists of a set of actions, one for each choice point in the game. There must be actions even at the choice points that will never be reached. So, strategies for player 1 are given by (Wait, Wait), (Take, Wait), (Wait, Take), (Take, Take), whereas strategies for player 2 are (Wait, Split), (Take, Split), (Wait, Take), (Take, Take).

We can observe that there are 4 different subgames that can be seen in Figure 1.5.

There is only one subgame perfect equilibrium. In this unique subgame perfect equilibrium, each player chooses to take the money at every opportunity. Then, SPE=((Take, Take), (Take, Take)). So, take the money by the first player is the unique subgame perfect equilibrium and is required by any Nash equilibrium. It can be established by backward induction. Suppose two players reach the final round of the game; player 2 will do better by taking the money because if not, player 1 receives a larger share of the pot. Since we suppose that player 2 will defect, player 1 does better by taking the money in the second to last round, because if not player 2 receives a higher payoff than he would have received by allowing player 2 to take the money in the last round. But knowing this, player 2 ought to take the money in the third to last round. This reasoning proceeds backwards through the game tree until we conclude that the best action is for player 1 to take the money in the first round.

Finally, we represent this example using the normal form as follows:

Player 1/Player 2	Wait, Split	Take, Split	Wait, Take	Take, Take
Wait, Wait	<u>(32, 16)</u>	(2, 4)	<u>(8, 16)</u>	(2, 4)
Take, Wait	(2, 1)	<u>(2, 1)</u>	(2, 1)	<u>(2, 1)</u>
Wait, Take	(3, 1)	<u>(2, 4)</u>	<u>(8, 4)</u>	<u>(2, 4)</u>
Take, Take	(2, 1)	<u>(2, 1)</u>	(2, 1)	<u>(2, 1)</u>

Table 1.2: Centipede game in normal form, where the underlined pairs are the Nash equilibria of the game.

In order to determine the Nash equilibria of the representation above, we highlight in blue the response which leads to the best outcome for player 1, taking player 2's strategies as given. The best actions for player 2, given what player 1 is doing, are in green. Therefore, there are 9 Nash equilibria of the game (the payoffs underlined above). In each, player 1 takes the money on the first round and player 2 also takes the money in the next round. However, all of them except ((Take, Take), (Take, Take)) are not subgame perfect equilibria because there are actions that are not considered credible threats. Being in a Nash equilibrium does not require that strategies be rational at every point in the game as in the subgame perfect equilibrium. For example, one Nash equilibrium is for both players to take the money on each round (even in the later rounds that are never reached).

Chapter 2

Rubinstein's bargaining model

Ariel Rubinstein [21] specifies a model of bargaining to explain how negotiation works. He applies subgame perfect equilibrium in this model. The Rubinstein bargaining model is applied to a class of bargaining games that feature alternating offers through an infinite time horizon. Before his contribution, the bargaining procedure was thought as a black box; so, Rubinstein's conclusions are one of the most influential findings in game theory.

Rubinstein's model is an influential model that has the same basis and the same direction as the Nash program. This model is dynamic, instead of static, because it considers the different strategies adopted by the players over the periods. Dynamic models force players to take into account the responses of the other players of the previous periods when they have to choose an action. Rubinstein's model shows that any bargaining result with two players can be obtained by Nash equilibria of non-cooperative games in sequential form, i.e, in extensive form.

2.1 Rubinstein's model

Rubinstein defines the *Bargaining Problem* as the situation where two individuals have different possibilities of contractual agreements. He assumes that both parties behave rationally and individually, and both have interests in reaching an agreement even though their interests are not identical. In an ideal situation, the agreed contract, if any, should be Pareto optimal; i.e. there is no other agreement that both would prefer. However, most of the contracts satisfying this condition are not agreed because to agree on a contract depends on the bargaining ability of both parts.

To make it clear, we will consider the following situation: two players (player 1 and player 2) have to achieve an agreement on the partition of a pie of size 1. Each of them has to make a proposal during his turn as to how it should be divided. When one player has made an offer, the other has to decide between two options:

i) Accept the offer.

ii) Reject the offer and continue the game.

Then, the dynamic of the game is as follows. Firstly, player 1 makes an offer to player 2. Player 2 has to accept it or reject it. If he accepts the offer, the game has finished with the proposal of player 1. If player 2 rejects the offer, then he has to make a proposal.

After that, is player 1's turn to accept the new offer or reject it. And so on. Therefore, the game finishes when one player accepts the offer of the other player if this eventually happens.

The set of ordered pairs is defined as (s,t), $0 \leq s \leq 1$ and $t \in \mathbb{N} \cup \{0\}$, where s represents the portion made of a pie of size 1 and t represents the time in the discrete format. Therefore, $t \geq 0$. So, the pair describes that "player 1 receives s and player 2 receives $1-s$ at time t." The preferences on (s,t) of any player should satisfy the following conditions:

a) *More pie is better.* In mathematical terms, if $x > y$, then $(x,t) \succ (y,t)$.

b) *Time* is valuable. This means that if $x > 0$ and $t_2 > t_1$, then $(x,t_1) \succ (x,t_2)$.

c) *Continuity.* This means that there are no jumps in people's preferences. In mathematical terms, a preference relation is continuous if given a sequence $\{x^n\}_{n=1}^{\infty}$ with $x^n \to x$ and $x^n \geq y \ \forall n \in \mathbb{N}$, then $x \geq y$. In other words, if we prefer a point A along a preference curve to point B, points very close to A will also be preferred to B.

d) *Stationarity.* This means that the preference of (x,t) over $(y,t+1)$ is independent of t. In other words, if player i, $(i = 1,2)$, prefers (x,t) to $(y,t+1)$ then he should prefer (x,t') to $(y,t'+1)$, $\forall t' \in \mathbb{N} \cup \{0\}$.

e) *More compensation for more waiting time.* For a bigger portion, the player needs more compensation for waiting one period and being indifferent to him. So, if (x,t) is equivalent to $(y,t+1)$ then y needs to be bigger than x to continue one more period with the bargaining and being immaterial to him.

The differences between the players are the negotiating order (depending on who has the "first turn") and the preferences. According to this, there are two sub-families of models:

i) *Fixed bargaining cost*: every player assumes a different and fixed cost for each period. Let c_i be the fixed cost of player i, $i = 1,2$. Fixed costs are defined in the unit interval, $c_i \in [0,1]$. So, the preference of player 1 comes from the function $s - c_1 \cdot t$ and the preference of player 2 comes from $1 - s - c_2 \cdot t$, where s is the portion that player 1 receives. Player 1 prefers the agreement (s,t) than (r,t') if and only if $s - c_1 \cdot t \geq r - c_1 \cdot t'$. Observe that player 1 is indifferent between these two agreements if and only if $s - c_1 \cdot t = r - c_1 \cdot t'$.

   For example, suppose that $c_1 = 0.1$ and $c_2 = 0.15$. The preference of player 1 comes from $s - 0.1 \cdot t$ and the preference of player 2 comes from $1 - s - 0.15 \cdot t$. For a higher t, the utility of the final partition will decrease because the functions $c_1 \cdot t$ and $c_2 \cdot t$ are increasing functions of t. Then, as time goes by they have to assume a bigger cost. If $s = 0.5$ in the first period, at time $t = 0$, player 1 and 2 would get a utility of 0.5 each one; in the second period, at time $t = 1$, player 1 would get a utility of 0.4 and player 2 will get a utility of 0.35; at $t = 2$, player 1 would get a utility of 0.3 and player 2 would get a utility of 0.2 only; at $t = 3$, player 1 would get a utility of 0.2 and player 2 would get a utility of 0.05; and so on.

ii) *Fixed discounting factor*: every player i $(i = 1,2)$ has a fixed discounting factor, δ_i, where $\delta_i \in (0,1]$. As δ_i approaches to 1, the player discounts less the future. If δ_i is equal to 1, for the player it would indifferent to receive the same partition at different moments of time. So, the preference of player 1 comes from the function $s \cdot \delta_1^t$ and the preference of player 2 comes from the function $(1-s) \cdot \delta_2^t$. Let player 1 with discounting factor δ_1. In other words, player 1 prefers the agreement (s,t) than (r,t') if and only if $s \cdot \delta_1^t \geq r \cdot \delta_1^{t'}$. Observe that player 1 is indifferent between these two agreements if and only if $s \cdot \delta_1^t = r \cdot \delta_1^{t'}$.

Continuing with the same example where $s = 0.5$, but now $\delta_1 = 0.80$ and $\delta_2 = 0.75$, a similar situation happens here as time goes by. At time $t = 0$ player 1 and 2 also would get a utility of 0.5 each one; at $t = 1$ player 1 would get a utility of 0.4 and player 2 would get a utility of 0.38; at time $t = 2$, player 1 would get a utility of 0.32 and player 2 would get a utility of 0.28 only; at time $t = 3$, player 1 would get a utility of 0.26 and player 2 would get a utility of 0.21; and so on. In this case, time is less of a penalty than in the previous case.

Now we can interpret the fixed discounting factors. Assume player 1 and player 2 are bargaining over one dollar. Each offer takes one period, and the players are impatient: they discount payoffs received in later periods by the factor δ_i. In this case, δ_i reflects the time-value of money. That's because, for instance, a dollar received at the beginning of one period can be put in a bank to earn interest.

Observations. *Notice that in the case of a fixed bargaining cost it is better for the player to have a cost close to 0, independently of the cost of the other player. In the case of a fixed discounting factor, as close to 1 is better for the player, also independently of the cost of the other player.*

A priori one can think that both models are equivalent, but they are not. As we will see at the end of this chapter, there exist some conclusions about the *perfect equilibrium partition* (P.E.P.)[1] depending on the model used. In a fixed bargaining cost model, it turns out that if $c_1 > c_2$, player 1 receives c_2 only; if $c_1 < c_2$, player 1 receives all the pie; and, if $c_1 = c_2$ any partition of the pie from which player 1 receives at least c_1 is a P.E.P. In conclusion, the weaker player (the one with a higher fixed cost) gets almost nothing. In a fixed discounting factor model there is only one P.E.P., where player 1 obtains $(1-\delta_2)/(1-\delta_1\delta_2)$. In this case, the player who starts the bargaining is the one who has more advantages.

Ståhl (1972, 1977) [26, 27] investigated a similar bargaining situation where there was a finite and known time horizon. He studied different cases where P.E.P. exists, independently of who has the first move.

2.2 The model in mathematical terms

Let us introduce here all mathematical notation in order to be able to rigorously describe the game and finally give the solution. With this aim, we assume the particular cases of bargaining fixed cost and fixed discount factor.

Let's define the bargaining model with the example described before. The two players, 1 and 2, are negotiating on the partition of a pie of size 1. The pie will be only distributed

[1]The P.E.P. will provide us the partition of the pie that player 1 will receive (and indirectly the portion that player 2 will receive). Notice also that we discuss about the proposal and not about the discounted payoffs.

after the players reach an agreement. In turn, every player has to offer a partition and the other player may agree to the offer, "Y", or reject it, "N". When one of the players agree with an offer, the bargaining ends. Otherwise, the rejecting player has to make a counteroffer and continue with these dynamics without any given limit.

Let $S = [0, 1]$ be the complete pie and $s_i \in [0, 1]$ the portion of the pie that player i receives in the partition. So, with the notation before, we have that $s = s_1$ and $s_2 = 1 - s_1$. For example, the ordered pair $(0.6, 2)$ means that player 1 receives 0.6 of the pie at time $t = 2$, and so player 2 receives 0.4.

The negotiation starts at $t = 0$, and so if they agree in this period, the first one, they have no discount.

Now, we will define $\mathcal{F}$ and $\mathcal{G}$ to start fixing the notation of this bargaining model. Let $\mathcal{F}$ denote the collection of all strategies available to player 1, which is player who starts the bargaining. And so, let $\mathcal{G}$ be the set of all strategies of player 2, the player who has to reply to the first offer of the negotiation made by player 1. In other words, every action made by the player 1 who has to start making an offer to player 2 is modeled by $\mathcal{F}$ and every action made by player 2 whose first move is a response to player's 1 offer is modeled by $\mathcal{G}$. A strategy specifies the offer that a player makes whenever it is his turn to make an offer and his reaction to any offer made by his opponent. A strategy includes the player's plan even after a series of moves that are inconsistent with the strategy itself.

From now on we will use s_i^t, where $i = 1, 2$ refers to the player who makes the offer and t refers to the time in which is the offer made. We assume that both players offer the portion that will receive player 1. So, all the offers made, make reference to the portion of the pie that will receive player 1.

Before describing mathematically these two sets it is important to understand the procedure of the bargaining:

1. In the first period, at time $t = 0$, player 1 starts the bargaining by making an offer to player 2. To make an offer, he has to make a proposal about the partition of the pie. So, he has to offer a partition s_1 ($s_1^0 \in [0, 1]$), defined before. After his offer, player 2 has to decide if accepts it or rejects it. If player 2 accepts the offer, the bargaining ends and player 1 receives s_1^0 and player 2 receives $1 - s_1^0$. Otherwise, the negotiation continues.

2. If there were no agreement at time $t = 0$ then in the second period, at time $t = 1$, it's player 2's turn to make a new offer. In the same way, he has to offer a partition s_2^1. Now is player 1's time to reply to the offer of player 2. Proceeding as before, if player 1 accepts it, the bargaining ends and so, player 1 receives s_2^1 and player 2 receives $1 - s_2^1$. If player 1 rejects the offer, the negotiation continues.

3. If there were no agreement in the second period then at time $t = 2$ happens the same as time $t = 0$. Player 1 makes a new offer, s_1^2, and player 2 to replies the offer. The procedure about continuing or not the negotiation is the same as before.

4. And so on.

Observations. *Notice that we cannot assume that the game should finish before any time period. In other words, the time variable of the game might tend to infinity and so, a strategy, in general, should contain movements (offers) for every time $t \in \mathbb{N}$.*

Now, we can describe formally $\mathcal{F}$ and $\mathcal{G}$. Firstly, we need to know where the functions come from and where they go. S^t is the set of all sequences of length from 0 to t of elements of S. Formally, $S^t = [0,1] \times [0,1] \times \overset{(t+1}{\cdots} \times [0,1]$. Moreover $\{Y, N\}$ is the set of this two elements: *accept the offer* (Y, which comes from YES) and *reject the offer* (N, which comes from NO).

Let $\mathcal{F}$ be the set of all strategies of player 1. Each strategy $f \in \mathcal{F}$ is given by a sequence of functions $f = \{f^t\}_{t=0}^{\infty}$ defined as follows: for every t even, $t \geq 2$, $f^t : S^t \to S$, and $f^0 \in S$; for t odd, $f^t : S^{t+1} \to \{Y, N\}$. Similarly, let $\mathcal{G}$ be the set of all strategies of player 2. Each strategy $g \in \mathcal{G}$ is given by a sequence of functions $g = \{g^t\}_{t=0}^{\infty}$ defined as follows: when t is odd $g^t : S^t \to S$; and when t is even, $g^t : S^{t+1} \to \{Y, N\}$.

Notice that the strategies of player 1, $\mathcal{F}$, start making a proposal about the partition of the pie while the strategies of player 2, $\mathcal{G}$, start answering "Y" or "N".

Let's now explain the bargaining step by step with this notation:

1. At time $t = 0$, the involved functions are $f^0 \in S$ (notice that it should be $f^t : S^t \to S$ if it was t even at a different time) and $g^0 : [0,1] \to \{Y, N\}$. As player 1 starts making an offer, firstly intervenes the function f^0. It gives the guidelines to start making a proposal when nothing happened before, but recall that $f^0 \in S$ and then it is not a function in this case. Then, it's player 2's turn to reply any offer and so, the function g^0 intervenes. This one goes from $s_1^0 \in [0,1]$ to $\{Y, N\}$. It's a function that, given an offer, will return Y or N depending if he has to accept the offer or reject it.

2. At time $t = 1$, the involved functions are $g^1 : [0,1] \to [0,1]$ and $f^1 : [0,1] \times [0,1] \to \{Y, N\}$. In this case, first intervenes the function g^1 because now it's player 2's turn to make a proposal and then intervenes f^1 because player 1 has to reply the offer. The function g^1 gives the new offer that will use player 2 for any proposal made before by player 1, s_1^0. Notice that g^1 makes sense only if $g^0(s_1^0) = N$. After that, the function f^1 will return Y or N for any first proposal at time 0 of player 1, s_1^0, and for any proposal of player 2 at time 1, s_2^1. For example, $f^1(s_1^0, s_2^1)$ is the answer of player 1 at time 1 assuming that he offered s_1^0 at time 0, his opponent reject it and made the offer s_2^1.

3. At time $t = 2$ happens a similar situation as in time $t = 0$. In this case, the involved functions are $f^2 : [0,1] \times [0,1] \to [0,1]$ and $g^2 : [0,1] \times [0,1] \times [0,1] \to \{Y, N\}$. First intervenes the function f^2 because it's player 1's turn to make a proposal and then intervenes g^2 because player 2 has to reply the offer. The function f^2 gives the new offer that will use player 1 for any proposal made before by player 1 in the first period, s_1^0, and any proposal made by player 2 in the second period, s_2^1. Notice that f^2 makes sense only if $f^1(s_1^0, s_2^1) = N$. After that, the function g^2 will return Y or N for any first proposal at time $t = 0$ of player 1, s_1^0; for any proposal of player 2 at time $t = 1$, s_2^1; and for any proposal of player 1 at time $t = 2$, s_1^2.

4. And so on.

Once the set of functions have been fixed, we need to detail which is an strategy of a player. Each strategy of one of the players must contain all the information that this player eventually need for any possible move that the other player makes. Formally, a strategy of player 1 is a function f, which is a particular and infinite strip of $f's$: $f = \{f^j\}_{j=0}^{\infty} = \{f^0, f^1, f^2, f^3, \ldots\}$, where f^j indicates what player 1 has to do at time j according to what has happened until time j. For example, f^2, that goes from $[0,1] \times [0,1]$

to $[0,1]$ is conditioned by all the offers made before, s_1^0 and s_2^1. Then, the strategy is an infinite strip because we don't know when the game will finish. In other words, an strategy is a set of functions defined in different domains that are incorporated as the negotiation goes on. For example, an infinite strategy could be the one in which the answer of player 1 will be N if the offer of player 2 is smaller than 0.5. Otherwise, if it is higher than 0.5, player 1 will answer Y. So, if the offer of player 2 is compatible with player 1's strategies, the game will be finite. Otherwise, the game will be infinite.

Observations. *Notice that an strategy of any player may include histories which are not consistent with his own plans. For example, for player 1 $f^2(s_1, s_2)$ is required to be defined even when $f^0 \neq s_1^0$ and when $f^1(s_1, s_2) = Y$.*

Now, we define how to deal with a game without end. Let (f, g) be the sequence of offers given by players 1 and 2 by choosing $f \in \mathcal{F}$ by player 1 and $g \in \mathcal{G}$ by player 2.[2] So, (f, g) is a strategy profile and let $T(f, g)$ be the length of this strategy profile, that can be ∞. Finally, let $D(f, g)$ be the last element of (f, g) which is known as the *partition.* This partition specifies which is the part of S assigned to player 1 and the rest that will be assigned to player 2.

The outcome function of the game is defined by

$$P(f,g) = \begin{cases} (D(f,g), T(f,g)), & T(f,g) < \infty, \\ \\ (0, \infty), & T(f,g) = \infty. \end{cases}$$

The result of this function is the pair (s, t) and it means achieve the agreement s at time t. The outcome $(0, \infty)$ shows a perpetual disagreement. The outcome function of the game $P(f, g)$, then, takes the value (s, t) if two players who adopt strategies f and g reach an agreement s at time t.

We assume that players have preference relations[3] $\succcurlyeq_1$ and $\succcurlyeq_2$ on the set of pairs (part, time) from $S \times \mathbb{N} \cup \{0\} \cup \{(0, \infty)\}$, and we suppose that both are complete, reflexive and transitive. The preferences of the players satisfy the next five conditions.

For all $r, s \in S$, $t, t_1, t_2 \in \mathbb{N} \cup \{0\}$, and $i \in \{1, 2\}$:

a) if $r_i > s_i$, then $(r_i, t) \succ_i (s_i, t)$;

b) if $s_i > 0$ and $t_2 > t_1$, then $(s_i, t_1) \succ_i (s_i, t_2) \succ_i (0, \infty)$;

c) $(r, t_1) \succsim_i (s, t_1 + 1)$ if and only if $(r, t_2) \succsim_i (s, t_2 + 1)$;

d) if $r_n \to r$ and for all $n \in \mathbb{N}$, $(r_n, t_1) \succsim_i (s, t_2)$, then $(r, t_1) \succsim_i (s, t_2)$; if $r_n \to r$ and $(r_n, t_1) \succsim_i (0, \infty)$, then $(r, t_1) \succsim_i (0, \infty)$;

e) if $(s + \varepsilon, 1) \backsim_1 (s, 0)$, $(\bar{s} + \bar{\varepsilon}, 1) \backsim_1 (\bar{s}, 0)$, and $s < \bar{s}$, then $\varepsilon \leq \bar{\varepsilon}$ for $\varepsilon, \bar{\varepsilon} > 0$.

if $(s + \varepsilon, 1) \backsim_2 (s, 0)$, $(\bar{s} + \bar{\varepsilon}, 1) \backsim_2 (\bar{s}, 0)$, and $s < \bar{s}$, then $\varepsilon \leq \bar{\varepsilon}$ for $\varepsilon, \bar{\varepsilon} < 0$.

[2] In the case where the negotiation order is reversed, we will use the same notation but changing the order of f and g.

[3] We should use $\succcurlyeq_i^t$ instead of $\succcurlyeq_i$, where the first one indicates player i's preference on the result when the players haven't reached an agreement before (first $t-1$ periods). In this chapter we will assume that $\succsim_i^t \equiv \succsim_i$.

The first condition, a), is a simple result of the complete preferences and the stationarity assumption. It is clear that if $r_i > s_i$ and it is offered at the player at the same time, he will prefer the bigger portion because we know that pie is desirable. About the second one, b), it comes from the assumptions that pie is desirable and time is valuable. Following with the next one, c), is linked to the stationarity assumption. In d), we assume that exists a sequence of proposals which converges to r over time. If all of this r_n is preferred to s, r is also preferred to s. About the last one, e), tries to explain that if one player is indifferent between two different partitions at $t = 0$ and $t = 1$, a partition increase in 0 requires an increase in 1 of at least the same as in 0 for the player to be indifferent.

2.3 Subgame Perfect Equilibrium

In this section we ask about the outcomes of the bargaining. The main concept is that of Nash Equilibrium, see Chapter 1.

Definition 2.1. The ordered pair $(\hat{f}, \hat{g}) \in \mathcal{F} \times \mathcal{G}$ is called a *Nash Equilibrium* if there is no $f \in \mathcal{F}$ such that $P(f, \hat{g}) \succ_1 P(\hat{f}, \hat{g})$ and there is no $g \in \mathcal{G}$ such that $P(\hat{f}, g) \succ_2 P(\hat{f}, \hat{g})$.

To simplify the notation, we will start to use $s^0, s^1, \dots, s^t$ to refer to the offers of the players, without the subscript referring to the player that is doing it, where the superscript refers to the temporal part of the offer. So, $s^0, s^1, \dots, s^t \in [0, 1]$ only represent the offer that each player proposes at every time, ignoring the part of the strategy of the player which refers to answer "Y" or "N" to the other player. We have to take into account that we will only propose s^t, for all $t \geq 0$, if the game hasn't finished yet, i.e., if there has been no agreement before. So, it's clear that $s^0, s^2, s^4, s^6, \dots$ will be the offers made by player 1 and $s^1, s^3, s^5, \dots$ will be the offers made by player 2. Then, player 1 makes the offer in even-numbered periods and player 2 in odd-numbered periods. We have to consider that s^t is the partition in which player 1 receives s^t and player 2 receives $1 - s^t$.

The following proposition justifies that it is not appropriate to use the concept of Nash equilibrium because everything could happen, i.e, it not discriminate much.

Proposition 2.1. *For all $s \in S$, s is a partition induced by some Nash equilibrium.*

Proof. Define $\hat{f} \in \mathcal{F}$ and $\hat{g} \in \mathcal{G}$ as follows:

- For t even, player 1 (using the strategy $\hat{f}$) proposes a partition $s^t \equiv s$. Player 2, using the strategy $\hat{g}$, has to reply to the offer. He will reject it if the partition s^t of player 1 is strictly bigger than the equilibrium partition (because player 2 would receive a smaller portion than $1 - s$). Player 2 will accept the offer if he obtains at least the equilibrium partition, where player 2's partition would be $(1 - s^t) \geq s$. The strategies are defined as

$$\hat{f}^t \equiv s,\ \hat{g}^t(s^0, s^1, \dots, s^t) = \begin{cases} Y, & s^t \leq s, \\ \\ N, & s^t > s; \end{cases} \text{, for } t \text{ even.}$$

- For t odd, player 2 makes an offer to player 1. Depending on the values of $s^0, s^1 \dots, s^t$, player 1 will accept it (N) or reject it(Y):

$$\hat{g}^t \equiv s, \ \hat{f}^t(s^0, s^1, \dots, s^t) = \begin{cases} Y, & s^t \geq s, \\ \\ N, & s^t < s. \end{cases}$$

On one hand, suppose that the strategy of player 1 is s at any time except in $t = t'$ that is $s + \varepsilon, \varepsilon > 0$, and the strategy of player 2 is s at any time. At time $t = t'$ player 2 will answer N to the offer because $s + \varepsilon > s$. At time $t = t' + 1$ player 2 will offer s and player 1 will answer Y. Then, in this case $P(f, g) = (s, t' + 1)$ which is not a Nash Equilibrium because both players prefer (s, t') than $(s, t'+1)$. On the other hand, suppose that the strategy of player 1 is s at any time except in $t = t'$ that is $s - \varepsilon, \varepsilon > 0$, and the strategy of player 2 is s at any time. At time $t = t'$ player 2 will answer Y to the offer because $s - \varepsilon < s$. But this will not be a Nash Equilibrium because if player 1 offers s instead of $s - \varepsilon$ player 2 will also answer Y and player 1 prefers (s, t') instead of $(s - \varepsilon, t')$.

Then, we have proved that $(\hat{f}, \hat{g})$ is a Nash equilibrium strategy and $P(\hat{f}, \hat{g}) = (s, 0)$. So, player 1 proposes Nash equilibrium portion in the first period and player 2 accepts it. □

Rubinstein's strategic approach concludes in a unique equilibrium in which an offer is proposed and accepted in the first period. This result does not contribute much to the model because it simply tell us that anything can happen.

With the proposition before we can observe that the concept of Nash equilibrium is very soft, and sometimes there appear some difficulties. For example, if we assume that player 1 demands $s + \varepsilon$, where $\varepsilon > 0$, and player 2 tries to reach an agreement on the originally planned contract there appear some of these difficulties. That's because depending on the value of ε (if it is sufficiently small) player 2 will prefer to agree to the new proposal, $s + \varepsilon$, instead of reject it. This situation happens when $(s, 1) \prec_2 (s + \varepsilon, 0)$.

In this case, $(s+\varepsilon, 0)$ is not a Nash equilibrium because it does not satisfy the Definition 2.1. Player 2 would have $1 - s - \varepsilon$ that is less than having $1 - s$. Then, in this situation there would be a deviation of player 2: accept the offer $s + \varepsilon$.

From this example, we can conclude that the concept of Nash equilibrium used in the Proposition 2.1 is not adequate since it is not what the players would do in the case they were in other histories, that is, it is not a subgame perfect equilibrium.

Rubinstein uses Selten's [23, 24] definition of the subgame perfect equilibrium to solve the above difficulties and he writes his own definition adapting the concept to his setting. Before defining it, we introduce some notation. Let $s^0, s^1, \dots, s^T \in S$. Let define $f|s^0, s^1, \dots, s^T$ and $g|s^0, s^1, \dots, s^T$ as the strategies that come from f and g after the offers have been announced and later rejected.

Notice that if T is odd it is player 1's turn to propose a partition of the pie and player 2's first move is a response to player 1's offer. Thus, $f|s^0, \dots, s^T \in \mathcal{F}$ and $g|s^0, \dots, s^T \in \mathcal{G}$. For example, if $T = 3$ then the offers made and already rejected are s^0, s^1, s^2 and s^3. So, now player 1 has to make a new proposal of the partition of the pie and $f|s^0, s^1, s^2, s^3 \in \mathcal{F}$. When T is even, it is player 2's turn to make an offer and therefore $g|s^0, \dots, s^T \in \mathcal{F}$ and $f|s^0, \dots, s^T \in \mathcal{G}$. For example, if $T = 2$ then the offers made and already rejected are s^0, s^1 and s^2. So, now player 2 has to make a new proposal of the partition of the pie and

$f|s^0, s^1, s^2 \in \mathcal{G}$. In conclusion, time T gives us information about which is the player who has to continue offering a partition.

That is, for T even and t odd,

$$(f|s^0, s^1, \ldots, s^T)^t(r^0, r^1, \ldots, r^{t-1}) = f^{T+t}(s^0, s^1, \ldots, s^T, r^0, r^1, \ldots, r^{t-1})$$

$$(g|s^0, s^1, \ldots, s^T)^t(r^0, r^1, \ldots, r^t) = g^{T+t}(s^0, s^1, \ldots s^T, r^0, r^1, \ldots, r^t)$$

So, the left members are a prediction made by each player at the moment t of the future strategies $r^0, \ldots, r^{t-1}$ (or $r^0, \ldots, r^t$), when they have already observed the rejected offers $s^0, s^1, \ldots, s^T$. The right members are the strategy of each player at the moment $T+t$ after seen the rejected offers $s^0, s^1, \ldots, s^T, r^0, r^1, \ldots, r^{t-1}$ (or $s^0, s^1, \ldots, s^T, r^0, r^1 \ldots, r^t$). The equality between the right member and the left member comes from he assumption of rationality. So, every player knows the past results and has perfect foresight on the progression of the game in future subgames. Then, for the following definition we have to suppose that the strategies predicted for future periods are equal to those that ara going to be played once the game reaches t.

Definition 2.2. $(\hat{f}, \hat{g})$ is a *Subgame Perfect Equilibrium* (S.P.E.) if for all $s^0, s^1, \ldots, s^T$, when T is even:

i) there is no $f \in \mathcal{F}$ such that $P(\hat{f}|s^0, s^1, \ldots, s^T, f) \succ_2 P(\hat{f}|s^0, s^1, \ldots, s^T, \hat{g}|s^0, s^1, \ldots, s^T)$;

ii) if $\hat{g}^T(s^0, s^1, \ldots, s^T) = Y$, there is no $f \in \mathcal{F}$ such that $P(\hat{f}|s^0, s^1, \ldots, s^T, f) \succ_2 (s^T, 0)$;

iii) if $\hat{g}^T(s^0, s^1, \ldots, s^T) = N$, $P(\hat{f}|s^0, s^1, \ldots, s^T, \hat{g}|s^0, s^1, \ldots, s^T) \succsim_2 (s^T, 0)$;

and when T is odd:

iv) there is no $f \in \mathcal{F}$ such that $P(f, \hat{g}|s^0, s^1, \ldots, s^T) \succ_1 P(\hat{f}|s^0, s^1, \ldots, s^T, \hat{g}|s^0, s^1, \ldots, s^T)$;

v) if $\hat{f}^T(s^0, s^1, \ldots, s^T) = Y$, there is no $f \in \mathcal{F}$ such that $P(f, \hat{g}|s^0, s^1, \ldots, s^T) \succ_1 (s^T, 0)$;

vi) if $\hat{f}^T(s^0, s^1, \ldots, s^T) = N$, $P(\hat{f}|s^0, s^1, \ldots, s^T, \hat{g}|s^0, s^1, \ldots, s^T) \succsim_1 (s^T, 0)$;

Notice that in Definition 2.2, conditions i) and iv) make sure that after a succession of proposals and rejections $(s^0, s^1, \ldots, s^T)$ the best strategy of the player who has to continue the negotiation is to continue with the original and planned strategy. In particular, i) tries to explain that there is no better deviation for player 2, which gives him a better partition, than the one that gets with $\hat{g}|s^0, s^1, \ldots, s^T$. This is precisely the concept of subgame perfection since it is compatible with sequential rationality. With the simple example in the previous page, if player 1 offers $s + \varepsilon$, player 2 would accept it and it would be a subgame perfect equilibrium. If player 2 rejects it, it would not be a subgame perfect equilibrium because it would not satisfy i). Then, ii) and v) ensure that the best strategy of a player who has the intention to accept the offer s^T is to accept it. In particular, ii) explains that if player 2 accepts $\hat{g} \in \mathcal{G}$, then player 1 can't made a better counteroffer $f \in \mathcal{F}$ for player 2. Finally, iii) and vi) ensure that if a player has the intention to reject an offer, the best for him is not to accept the offer. In particular, iii) means that if player 2 rejects an offer $\hat{g} \in \mathcal{G}$, there exists another offer that would be preferred at that moment.

Example 2.1. Suppose that $(\hat{f}, \hat{g})$ is a Nash equilibrium where $D(\hat{f}, \hat{g}) = s = 0.5$ (half of the pie for each). Let's also suppose that both players have fixed bargaining costs and they are respectively $c_1 = 0.1$ and $c_2 = 0.2$. We want to check if it also is a subgame perfect equilibrium. If it is a Nash equilibrium, as we told before, and we suppose that player 1 at $t = 0$ offers $s^0 = 0.6$ then player 2 will reject it (with the notation we can write $\hat{g}^0(0.6) = N$) because $s^0 > s$. Then, the bargaining follows at $t = 1$. If we suppose that player 2 offers the equilibrium portion $s = 0.5$ then player 1 will accept it. So, $P(\hat{f}|0.6, \hat{g}|0.6) = (0.5, 1)$, but player 2 prefers $(0.6, 0)$ than $(0.5, 1)$ because in this case he will obtain $1 - 0.6 - c_2 \cdot 0 = 0.4$. With the pair $(0.5, 1)$ he will only obtain $1 - 0.5 - c_2 \cdot 1 = 0.3$. In conclusion, the equilibrium $(\hat{f}, \hat{g})$ will not satisfy condition iii) of Definition 2.2 because player 2 would prefer 0.4 at $t = 0$ than 0.3 at $t = 1$ and so, rejecting the offer $s^0 = 0.6$ it is not acceptable for a subgame perfect equilibrium.

2.4 The main theorem

In this section we introduce some concepts that we will use before to announce some lemmas and propositions. The lemmas help us to demonstrate easier the propositions and the sum of all the propositions is what we will call the main theorem.

Until now, we have considered that player 1 was the player who starts the bargaining. We have to take into account that player 2 could also start the bargaining, at we will have to consider this in this section. Let's define two sets that will have an important role in this section. Let A be the collection of all P.E.P.'s in a game in which player 1 starts the negotiation and let B be the collection of all P.E.P.'s in which player 2 starts the negotiation.

Formally, this two sets can be written as $A := \{s \in S|$ there is a S.P.E. $(f, g) \in \mathcal{F} \times \mathcal{G}$ such that $s = D(f, g)\}$ and $B := \{s \in S|$ there is a S.P.E. $(g, f) \in \mathcal{G} \times \mathcal{F}$ such that $s = D(g, f)\}$. Now, we will introduce some lemmas that show the connections between A and B.

Lemma 2.1. *Let $a \in A$. For all $b \in S$ such that $b > a$, there is $c \in B$ such that $(c, 1) \succsim_2 (b, 0)$.*[4]

Proof. Suppose that $(\hat{f}, \hat{g})$ is a S.P.E. that satisfies $D(\hat{f}, \hat{g}) = a$. Let $b \in S$ with $b > a$. Since $(\hat{f}, \hat{g})$ is a S.P.E., Definition 2.2 is to be ensured. To complete i), we have that $\hat{g}^0(b) = N$. If this is not true, there would be a contradiction with i) because if $f^0 = b$ then $P(f, \hat{g}) = (b, 0) \succ_1 (a, 0) \succsim_1 (a, T(\hat{f}, \hat{g})) = P(\hat{f}, \hat{g})$. In addition, to complete iii) we have that $P(\hat{f}|b, \hat{g}|b) \succsim_2 (b, 0)$. So, with the definition of $P(f, g)$, we can write it as $(D(\hat{f}|b, \hat{g}|b), T(\hat{f}|b, \hat{g}|b)) \succsim_2 (b, 0)$. Now, using the affirmation ii) from the preferences of the players we conclude that $(D(\hat{f}|b, \hat{g}|b), 1) \succsim_2 (b, 0)$ and, consequently, $D(\hat{f}|b, \hat{g}|b) = c$. □

Lemma 2.2. *For all $a \in B$ and for all $b \in S$ such that $b < a$, there is $c \in A$ such that $(c, 1) \succsim_1 (b, 0)$.*

Proof. It can be shown by a similar argument used in the proof of Lemma 2.1. □

[4]What Lemma 2.1 implies is that $a \in A$ has to be safeguarded from the possibility that player 1 could achieve a better contract. Player 1 would do it if there is $b \in S$ satisfying $b > a$ in which player 2 would accept it if player 1 offers it. Player 2 must reject such an offer, because if not, $a \notin A$. To be optimal for player 2 to reject the offer, player 2 has to expect to achieve a better partition in the future; that is, there must be a P.E.P. $c \in B$ that takes places after player 2 rejection and in which $(c, 1) \succsim_2 (b, 0)$).

Lemma 2.3. *Let $a \in A$. Then for all $b \in S$ such that $(b,1) \succ_2 (a,0)$ there is $c \in A$ such that $(c,1) \succsim_1 (b,0)$.*[5]

Proof. Let $(\hat{f},\hat{g})$ be a S.P.E. such that $D(\hat{f},\hat{g}) = a$.

A) Let $\hat{f}^0 = s$ and suppose that $\hat{g}^0(\hat{f}^0) = N$. Then $D(\hat{f}|s,\hat{g}|s) = a$ and $a \in B$. Now, applying the affirmations i) and ii) of the preferences of the players we have that if $(b,2) \succ_2 (a,1)$ then $b < a$. Using Lemma 2.2 we know that there is $c \in A$ that $(c,1) \succsim_1 (b,0)$.

B) Suppose that player 1 offers a and player 2 accepts it. So, with the notation it can be written as $\hat{f}^0 = a$ and $\hat{g}^0(a) = Y$. Let $b \in S$ satisfy $(b,1) \succ_2 (a,0)$. So, if player 2 rejects the first offer of player 1 and offers b, then player 1 would reject it ($\hat{f}^1(a,b) = N$) because otherwise there would be a contradiction with the affirmation ii) of the Definition 2.2. Let's see the contradiction. For any $f \in \mathcal{F}$ satisfying $f^0 = b$, $P(\hat{f}|a,f) = (b,1) \succ_2 (a,0)$ in contradiction with ii). Then, applying the affirmation vi) of the Definition 2.2, we have that $P(\hat{f}|a,b,\hat{g}|a,b) \succsim_1 (b,0)$. This is the same as $(D(\hat{f}|a,b,\hat{g}|a,b),1) \succsim_1 (b,0)$ and so $D(\hat{f}|a,b,\hat{g}|a,b) \in A$.

□

Lemma 2.4. *For all $a \in B$ and for all $b \in S$ such that $(b,1) \succ_1 (a,0)$ there is $c \in B$ such that $(c,1) \succsim_2 (b,0)$.*

Proof. It can be shown by a similar argument used in the proof of Lemma 2.3. □

In the previous lemmas we have seen two different results: (1) there is no better partition than a S.P.E. to offer at time $t = 0$ because there is always another partition that the other player prefers at time $t = 1$, and (2) it is not possible to have a partition at $t = 1$ without having a better partition at $t = 0$ for the player who starts the bargaining. Now we will enunciate and demonstrate the main theorem with some propositions. To this end we need several definitions:

$$\Delta = \left\{ (x,y) \in S \times S \mid \begin{array}{l} y \text{ is the smallest number such that } (y,0) \succsim_1 (x,1); \\ x \text{ is the largest number such that } (x,0) \succsim_2 (y,1); \end{array} \right\},$$

$$\Delta_1 = \{x \in S \mid \text{there is } y \in S, \text{ such that } (x,y) \in \Delta\},$$

$$\Delta_2 = \{y \in S \mid \text{there is } x \in S, \text{ such that } (x,y) \in \Delta\}.$$

Let's start with some propositions that prove the relations between A, B, Δ_1 and Δ_2.

Proposition 2.2. *If $(x,y) \in \Delta$, then $x \in A$ and $y \in B$.*

Proof. Consider the following $(\hat{f},\hat{g})$:

[5] What Lemma 2.3 implies is that if a is a P.E.P., player 1 should usually not agree to any offer of player 2 which is preferred by player 2 to accepting player 1's original offer. Assume that in a certain P.E., player 2 plans to agree to a in the first period (case B of the demonstration). If we consider that $(b,1) \succ_2 (a,0)$, player 2 will reject a if he thinks that player 1 would agree to b. In conclusion, player 1 must threaten to reject any offer b. So that this threat is credible, there must be a P.E. in the subgame beginning with the offer of player 1 that will produce a new agreement c such that $(c,1) \succsim_1 (b,0)$, where c must be a member of A.

$$\text{for } t \text{ even, } \hat{f}^t \equiv x,\ \hat{g}^t(s^0 \dots s^t) = \begin{cases} Y, & s^t \leq x, \\ N, & x < s^t; \end{cases}$$

$$\text{for } t \text{ odd, } \hat{g}^t \equiv y,\ \hat{f}^t(s^0 \dots s^t) = \begin{cases} Y, & y \leq s^t, \\ N, & s^t < y. \end{cases}$$

Clearly, $(\hat{f}, \hat{g})$ is a subgame perfect equilibrium. We use a similar argument than in Proposition 2.1 and we also use the previous lemmas. □

Proposition 2.3. *$\Delta \neq \varnothing$ and therefore A and B are not empty.*

Proof. We consider the following functions, $d_1(\cdot)$ and $d_2(\cdot)$, defined in the following way: if $(x, y) \in \Delta$, $d_1(x) = y$ and $d_2(y) = x$.

Then,

$$\Delta = \{(x, y) \mid y = d_1(x) \text{ and } x = d_2(y)\}.$$

Because of the way Δ is defined, $d_1(\cdot)$ and $d_2(\cdot)$ are well defined. Notice that $d_1(x)$ is the smallest y such that $(x, 1) \precsim_1 (y, 0)$ and $d_2(y)$ is the largest x such that $(y, 1) \precsim_2 (x, 0)$.

Once we have the functions $d_1(\cdot)$ and $d_2(\cdot)$ defined for each $(x, y) \in \Delta$, we need to define them in other cases $(x, y) \notin \Delta$. These two functions can be described as:

$$d_1(x) = \begin{cases} 0 \text{ if for all } y \in S \text{ we have } (y, 0) \succ_1 (x, 1), \\ y \text{ if there exists } y \text{ satisfying } (y, 0) \sim_1 (x, 1), \end{cases}$$

$$d_2(y) = \begin{cases} 1 \text{ if for all } x \in S \text{ we have } (x, 0) \succ_2 (y, 1), \\ x \text{ if there exists } x \text{ satisfying } (x, 0) \sim_2 (y, 1). \end{cases}$$

Therefore, $d_1(x)$ and $d_2(y)$ are well defined, continuous and increasing. They also satisfy that $0 \leq d_1(x) < 1$ and $0 < d_2(y) \leq 1$.

Let's define $D(x)$ as the function $D(x) = d_2(d_1(x))$. Therefore if $y = d_1(x)$ we obtain that $D(x) = d_2(d_1(x)) = d_2(y) = x$. So, now we can write $\Delta = \{(x, y) | y = d_1(x)$ and $x = D(x)\}$. Since D is continuous, there exists at least one fixed point $x_* \in S$ such that $D(x_*) = x_*$. Thus, $(x_*, d_1(x_*)) \in \Delta$ and consequently $\Delta \neq \varnothing$. We can visualize it in Figure 2.1.

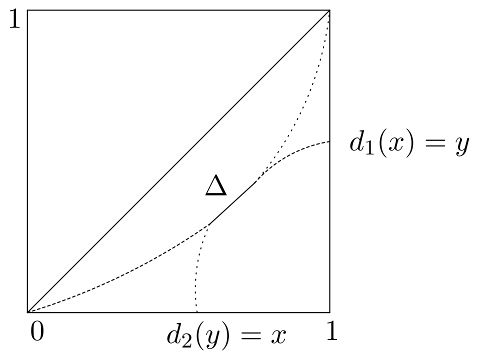


Figure 2.1

□

Proposition 2.4. *The graph of Δ is a closed line segment which lies parallel to the diagonal $y = x$.*

Proof. Recall from the proof of Proposition 2.3, that the functions d_1 and d_2 defined are continuous, therefore Δ is closed. We also know that d_1 is an increasing function. Denote by x_0 the value of S which satisfies $(0, 0) \sim_1 (x_0, 1)$.

- For $x \leq x_0$ we know that $d_1(x) = 0$. So, $x - d_1(x) = x$.
- For $x_0 \leq x_1 < x_2$ we know that $(d_1(x_2), 0) \sim_1 (x_2, 1)$ and $(d_1(x_1), 0) \sim_1 (x_1, 1)$. Using that d_1 is an increasing function and using the fifth affirmation of the player's preference (if $(s + \varepsilon, 1) \backsim_1 (s, 0)$, $(\bar{s} + \bar{\varepsilon}, 1) \backsim_1 (\bar{s}, 0)$, and $s < \bar{s}$, then $\varepsilon \leq \bar{\varepsilon}$), we find out that $x_2 - d_1(x_2) \geq x_1 - d_1(x_1)$.

Then, we have just shown that the function $x - d_1(x)$ is an increasing function.

We need to show that $x - y$ is constant for all $(x, y) \in \Delta$. Suppose now that $(x_1, y_1) \in \Delta$ and $(x_2, y_2) \in \Delta$. Then, $x_2 - d_1(x_2) \geq x_1 - d_1(x_1)$, that is $x_2 - y_2 \geq x_1 - y_1$. Similarly $d_2(y_2) - y_2 \leq d_2(y_1) - y_1$, that is $x_2 - y_2 \leq x_1 - y_1$. Thus, $x_1 - y_1 = x_2 - y_2$. We conclude that $x - y$ is constant for all $(x, y) \in \Delta$. □

Proposition 2.5. *If $a \in A$, then $a \in \Delta_1$, and if $b \in B$, then $b \in \Delta_2$.*

Proof. Since Δ is a segment, suppose $\Delta_1 = [x_1, x_2]$ and $\Delta_2 = [y_1, y_2]$. Let $s = supA$ and so, it satisfies, $x_2 < s$. Then $d_2(d_1(s)) < s$. Let $a \in A$, such that $r = d_2(d_1(s)) < a < s$. Let $b \in S$, such that $d_2^{-1}(a) > b > d_1(s)$. Then $a > d_2(b)$ and $(b, 1) \succ_2 (a, 0)$. From Lemma 2.3, there exists $c \in A$ such that $(c, 1) \succsim_1 (b, 0)$. Therefore there exists $c \in A$ satisfying $d_1(c) \geq b$. Using that d_1 is an increasing function and using that $d_1(c) \geq b > d_1(s)$, we have that $c > s$. This contradicts the definition of s. After that, using Lemma 2.4 we can see that $y_1 = infB$. Now, using Lemma 2.1 and Lemma 2.2 we obtain that $x_1 = infA$ and $y_2 = supB$. □

Joining all the propositions before, we obtain the final result which is the Rubinstein's theorem.

Theorem $A = \Delta_1 \neq \varnothing$, $B = \Delta_2 \neq \varnothing$. A and B are closed intervals and there exists $\varepsilon \geq 0$ such that $B = A - \varepsilon$.[6]

Proof. From the previous propositions we know that $A = \Delta_1$ and $B = \Delta_2$. Using Proposition 2.3 we know that A and B are not empty and so, $A = \Delta_1 \neq \varnothing$, $B = \Delta_2 \neq \varnothing$. Now, using Proposition 2.4, we know that A and B are closed intervals and there exists $\varepsilon \geq 0$ where $B = A - \varepsilon$.

[6] This expression implies that for all $a \in A$ minus ε gives us a value $b \in B$ and vice versa.

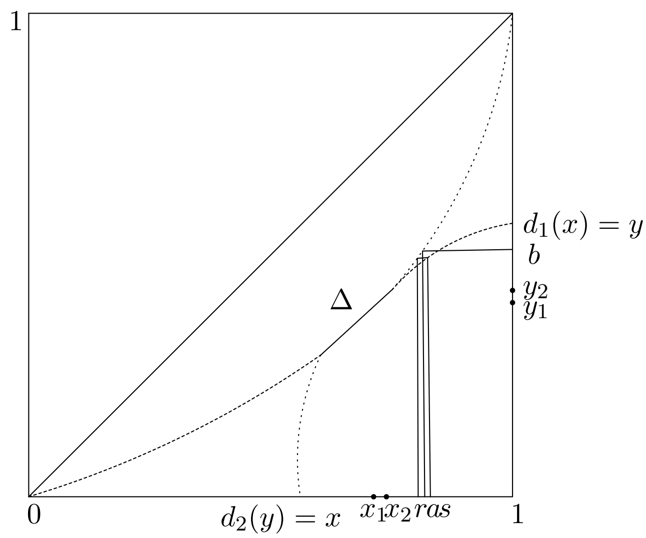


Figure 2.2

□

2.5 Applications

Rubinstein in 1982 gave two main applications of the main theorem about what we have discussed before: the fixed bargaining cost and the fixed discounting factors.

APPLICATION 1

In the first application we will consider that the players have fixed bargaining costs, as we have seen in page 16. The fixed bargaining costs of player 1 and 2 are c_1 and c_2 respectively, where $c_1, c_2 \in [0, 1]$.

1. If $c_1 > c_2$, c_2 is the only P.E.P.
2. If $c_1 = c_2$, every $x \in [0, 1]$, $c_1 \leq x \leq 1$ is a P.E.P.
3. If $c_1 < c_2$, 1 is the only P.E.P.

Recall that in the case of fixed bargaining costs, the preferences of each player come from the function $x - c_1 \cdot t$ for player 1 and $1 - x - c_2 \cdot t$ for player 2. Recall also that $d_1(x)$ is the smallest y such that $(y, 0) \succsim_1 (x, 1)$ and $d_2(y)$ is the largest x such that $(x, 0) \succsim_2 (y, 1)$. So, given $d_1(x) > 0$ and $d_2(y) < 1$ we have that $d_1(x) = max\{x - c_1, 0\}$ and $d_2(y) = min\{y + c_2, 1\}$. Then, $x - c_1$ is the smallest portion that player 1 can obtain in the second period and $y + c_2$ is the maximum portion that player 2 can obtain in the second period.

As a result, Δ is the set of all solutions to the set of equations $y = max\{x - c_1, 0\}$ and $x = min\{y + c_2, 1\}$. We can observe quickly this final conclusion in the three diagrams of Figure 2.3.

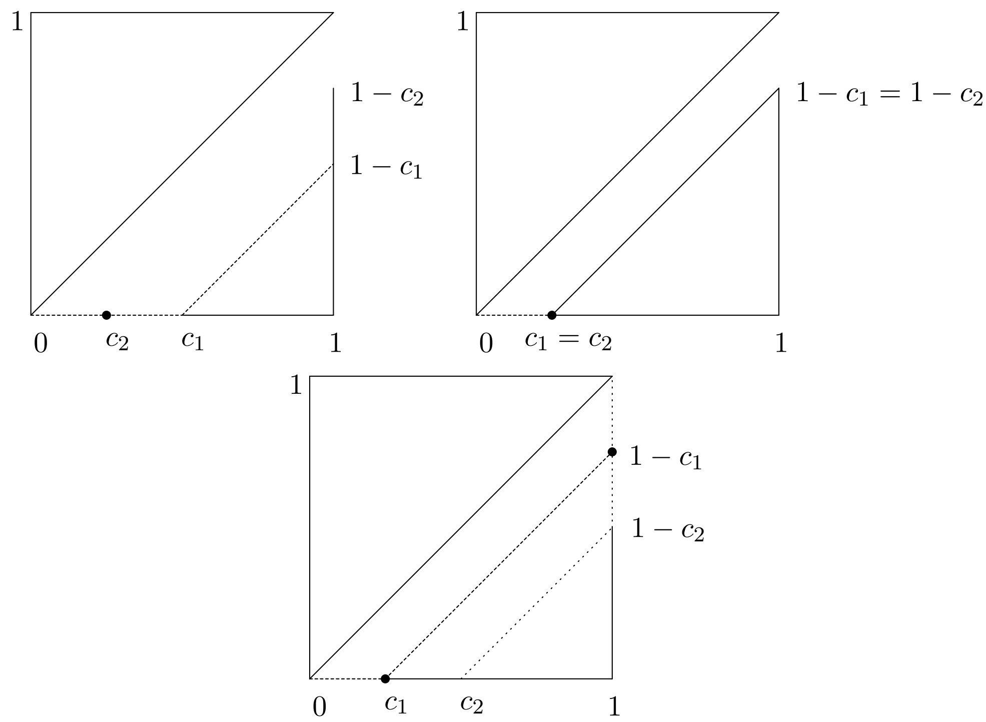


Figure 2.3

Notice that when $A \cap B = \phi$, S.P.E. is an agreement reached immediately after the first offer. If $(\hat{f}, \hat{g})$ is a S.P.E. and $T(\hat{f}, \hat{g}) > 0$, then $D(\hat{f}, \hat{g})$ is an element of A and also of B. Then, except if $c_1 = c_2$ the bargaining ends in the first period.

APPLICATION 2

Now, in application 2, we will consider that players 1 and 2 have fixed discounting factors (explained in page 16) δ_1 and δ_2 respectively, which satisfy $\delta_1, \delta_2 \in (0, 1]$. If at least one of the fixed discounting factors match $\delta_i < 1$ and at least one of them match $\delta_i > 0$, then exists only one P.E.P. and is $M = (1 - \delta_2)/(1 - \delta_1\delta_2)$.

Observations. *Notice that when $\delta_1 = 0$, player 1 can gain $1 - \delta_2$ because it's the portion of pie that 2 will lose if he rejects 1's offer and gets 1 in the second period. If $\delta_2 = 0$, player 2 has no threat because it has no value for him after the first period. When $0 < \delta_1 = \delta_2 = \delta < 1$, player 1 gets $1/1 + \delta > 1/2$. The fact that player 1 starts the negotiation, his gains decreases while δ tends to 1.*

Recall that in the case of fixed discounting factors, the preferences of each player came from the function $x \cdot \delta_1^t$ for player 1 and $(1 - x) \cdot \delta_2^t$ for player 2. Recall also that $d_1(x)$ is the smallest y such that $(y, 0) \succsim_1 (x, 1)$ and $d_2(y)$ is the largest x such that $(x, 0) \succsim_2 (y, 1)$. So, now we can define the functions as $d_1(x) = x \cdot \delta_1$ and $d_2(y) = 1 - \delta_2 + \delta_2 \cdot y$. Mathematically, $M = (1 - \delta_2)/(1 - \delta_1\delta_2)$ is the solution of $d_2(d_1(x)) = x$. So, the intersection of d_1 and d_2 is where $1 - \delta_2 + \delta_2 x \delta_1 = x$, that is the place where $x = M$. We can observe this in Figure 2.4.

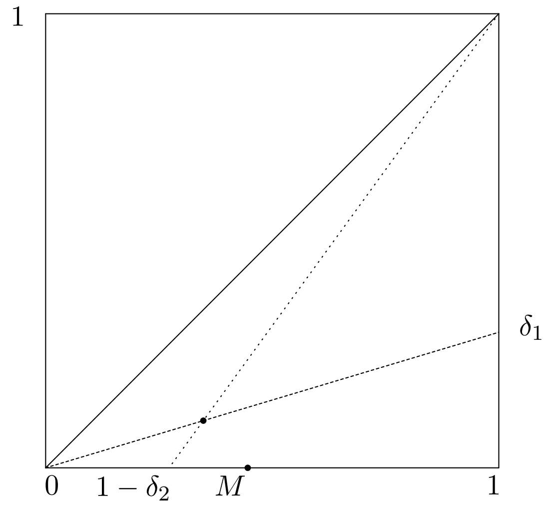


Figure 2.4

2.6 Final comments

In the case of Application 2, we can make some comments about the application that can help us to understand better how it can be applied in the real negotiation:

1. Being patient helps. The P.E.P. where player 1 obtains $(1-\delta_2)/(1-\delta_1\delta_2)$ is growing up in δ_1 and decreasing in δ_2. If players are patient they can wait until they have the power of the bargaining. In other words, they can wait until they have the opportunity to make their offer.

2. The player who starts the bargaining has an advantage. With identical discounting factors, $\delta=\delta_1=\delta_2$, player 1 obtains $1/(1+\delta)$ and player 2 obtains $\delta/(1+\delta)$, which is better the portion of player 1. However, while δ tends to 1, this advantage for the player who starts the bargaining it disappears. At $\delta=1$ the portion for player 1 and player 2 are the same, 1/2.

Chapter 3

Rubinstein's model revisited

This chapter will help us to clarify the previous chapter and highlight its meaning. Rubinstein, together with Ken Binmore and Martin J. Osborne, (1988)[1] rewrote his first version some years later and he expanded it. In the expanded model they eliminate the part of the original model when the players have to answer Y or N. So, now we have a more clear and transparent model. Then, during this chapter we will follow a similar bargaining procedure.

We will consider the case of fixed discounting factors. As we have seen in page 16, we will use δ_1 and δ_2, where $\delta_1, \delta_2 \in (0, 1]$. The preference of player 1 comes from the function $s \cdot \delta_1^t$ and the preference of player 2 comes from the function $(1 - s) \cdot \delta_2^t$.

3.1 The bargaining procedure

The archetypal bargaining problem is that of dividing a pie between two players. However, the discussion can be easily interpreted broadly to fit a large class of bargaining situations, as dividing the dollar (this example is used by Rubinstein in some explanations). The set of feasible agreements is identified with $S = [0, 1]$. The two bargainers, players 1 and 2, have opposing preferences over S. If $s, r \in S$ we know that player 1 will receive s and player 2 will receive $1 - s$, and the same with r. Then, when $s > r$, player 1 prefers s to r and player 2 prefers r to s.

The following procedure is familiar from street markets and bazaars all over the world. The bargaining consists simply of a repeated exchange of offers. Formally, we study a model in which all events take place at one of the times t in a prespecified set $T = (0, t_1, t_2, \ldots)$, where (t_n) is strictly increasing. The players alternate in making offers, starting with player 1. An offer s, made at time t_n, may be accepted or rejected by the other player. If it is accepted, the game ends with the agreed deal being implemented at time t_n. This outcome is denoted by (s, t_n). If the offer is rejected, the rejecting player makes a counteroffer at time t_{n+1}. And so on. Nothing binds the players to offers they have made in the past, and no predetermined limit is placed on the time that may be expended in bargaining. In principle, a possible outcome of the game is therefore perpetual disagreement or *impasse*, which is denoted by D.

Suppose that, in this model, player 1 could make a commitment to hold out for s or more. Player 2 could then do not better than to make a commitment to hold out for $1 - s$ or better. In this case, it is clear that the result would be a Nash equilibrium sustaining an agreement on s because no player could improve their outcome unilaterally.

Players are assumed to be impatient with the unproductive passage of time. The times in the set T at which offers are made are restricted to $t_n = n \cdot \tau$, where $n = 0, 1, \dots$ and $\tau > 0$ is the length of one period of negotiation.

Once we know from the previous chapter the preferences of the players, we can make some conclusions. The conditions on the preferences of the players are sufficient to imply the existence of continuous functions Φ_1 and Φ_2 which represents those functions. More precisely we can write $\Phi_1(s,t) = \phi_1(s)\delta_1^t$ and $\Phi_2(s,t) = \phi_2(1-s)\delta_2^t$, where the functions $\phi_i : [0,1] \to [0,1]$ are strictly increasing functions of their variable, and $\delta_1, \delta_2 \in (0,1)$. They satisfy that $\Phi_1(D) = \Phi_2(D) = 0$.

Notice that the function Φ_2 is a decreasing function with respect to the first variable, s, since ϕ_2 is increasing but applied to $1-s$. The second player values the agreement (s,t) which is the offer to player 1.

3.2 The result

We will now introduce some claims that we will later use to announce the final result.

Without loss of generality we can assume that for $i \in \{1,2\}$:

$$\text{for each } s \in S \text{ there exists } r \in S \text{ such that } (r,0) \backsim_i (s,\tau). \tag{3.1}$$

This is a consequence of that the functions which define the preferences of the players are strictly increasing and defined in the unit interval.

With the claims (3.1) we have that $\phi_i(0) = 0$. Then, we can assume that $\phi_i(1) = 1$.[1] Let's introduce a function that will be useful to continue with this explanation. This function $y(\cdot)$, where $y : [0,1] \to [0,1]$, will return the payoff or utility of player 2. So, $y^{-1}(\cdot)$ will return the payoff or utility of player 1. We can define it by $y(u_1) = \phi_2(1 - \phi_1^{-1}(u_1))$. Then, a deal reached at time 0 that assigns utility u_1 to player 1 assigns $u_2 = y(u_1)$ to player 2. In other words, the function $y(\cdot)$ for each eventual utility of player 1, u_1, returns the utility that player 2 will receive if they agree on s. Using this function we simplify the way of explain this process.

Observations. *Notice that u_1 is the utility or happiness that player 1 receives when players accord s. Then, $\phi_1(s) = u_1$ is a function which returns the utility (or happiness) for player 1 according to a partition s. In other words, u_1 is like replacing the happiness that causes partition s to player 1 and $1-s$ for player 2.*

More generally, the set U^t of utility pairs available at time t is

$$U^t = \{(u_1\delta_1^t, y(u_1)\delta_2^t) : 0 \le u_1 \le 1\}.$$

Let's see what is $y(u_1) = \phi_2(1 - \phi_1^{-1}(u_1))$. From the fact that player 1 has utility u_1 with the agreed agreement, we need to know which is the utility of player 2. Then, we need to calculate $\phi_2(1-s)$, where s is the agreement that gives utility u_1 to player 1. To calculate this value we will do it using ϕ_1, without using s.

[1]Observe that if we consider player 2, $\phi_2(0)$ implies $s = 1$ and $\phi_2(1)$ implies $s = 0$.

Notice that a positive aspect of this game is that all subgames in which a player makes the first offer have the same strategic structure. Then, we only need to characterize the subgame perfect equilibria of this game. Firstly, we will examine a pair of strategies in which both players always plan to do the same in strategically equivalent subgames. We will also assume that they do it regardless of the history events that must have taken place for the subgame to have been reached. Let's consider two possible agreements s^* and r^*, and let u^* and v^* be the utility pairs that result from the implementation of these agreements at time 0. Then $u^*, v^* \in U^0$. So, $u^* = (u_1^*, u_2^*)$ will be the utilities that players 1 and 2 will receive respectively if they agree on s^*, and $v^* = (v_1^*, v_2^*)$ will be the utilities that players 1 and 2 will receive respectively if they agree on r^*. Let $\hat{f}$ be the strategy of player 1 in which consists of offer always s^* and accept an offer of player 2 if and only if the offer is greater or equal than r^*. Similarly, let $\hat{g}$ be the strategy of player 2 which consists in offer always to offer r^* and accept an offer of player 1 if and only if the offer is smaller or equal than s^*.

Then, the pair $(\hat{f}, \hat{g})$ is a subgame perfect equilibrium if and only if

$$v_1^* = \delta_1 u_1^* \text{ and } u_2^* = \delta_2 v_2^*. \tag{3.2}$$

The above affirmation it is clear because if we suppose that ϕ_1 and ϕ_2 are the identity, $\phi_1(s) = s$ and $\phi_2(s) = s$ for all $s \in [0,1]$, we have that:

- From $v_1^* = \delta_1 u_1^*$ we have that $r^* = \delta_1 s^*$.
- From $u_2^* = \delta_2 v_2^*$ we have that $(a - s^* = \delta_2(1 - r^*)$.

Then, substituting we obtain the expression that we obtained in APPLICATION 2:

$$1 - s^* = \delta_2(1 - \delta_1 s^*) \to 1 - s^* = \delta_2 - \delta_2\delta_1 s^* \to s^* = (1 - \delta_2)/(1 - \delta_1\delta_2).$$

Observations. *Observe that each player is always offered the utility that he will get if he refuses the offer and $\hat{f}$ and $\hat{g}$ continue to be used in the subgame that ensues.*

Notice that the above affirmation admits a solution if and only if the equation

$$y(s) = \delta_2 y(s\delta_1) \tag{3.3}$$

has a solution. We can afirm that it has a solution because y is continuous, $y(0) = 1$ and $y(1) = 0$. Each solution of (3.2) generates a different subgame perfect equilibrium. Thus, the uniqueness of a solution to (3.2) is a necessary condition for the uniqueness of a subgame perfect equilibrium in the game.

Let's assume that (3.2) has a unique solution. A condition that ensures this is

$$(s + \alpha, \tau) \sim_i (s, 0), (r + \beta, \tau) \sim_i (r, 0), \text{ and } s < r \text{ imply that } \alpha < \beta. \tag{3.4}$$

The interpretation of (3.4) is that the more you get, the more you have to be compensated for delay in getting it.

Theorem 3.1 (Rubinstein (1982))**.** *Under assumptions (3.1)-(3.4) the bargaining game has a unique subgame perfect equilibrium. In this equilibrium, the agreement is reached immediately, and players' utilities satisfy (3.2).*[2]

[2] Alternative versions of Rubinstein's proof appear in Binmore (1987b) and Shaked and Sutton [25] (1984).

Proof. [3] For the argument of the proof, we take $\tau = 1$. Let M_1 be the supremum of all subgame-perfect equilibrium payoffs to player 1 and let m_1 be the infimum. We will fix the same notation for player 2, which make reference to the companion game where the roles of player 1 and 2 are reversed, M_2 and m_2. The aim is to show that $m_1 = u_1^*$ and $M_2 = v_2^*$, where u_1^* and v_2^* are uniquely defined by (3.2). Then, using an analogous argument it will be easy to show that $M_1 = u_1^*$ and $m_2 = v_2^*$. To finish the demonstration we will see that the equilibrium payoffs are uniquely determined.

We know that u^* is a subgame perfect equilibrium pair of payoffs. Thus, $m_1 \leq u_1^*$ and $M_2 \geq v_2^*$. We will show that (i) $\delta_2 M_2 \geq y(m_1)$ and (ii) $M_2 \leq y(\delta_1 m_1)$:

(i) Observe that if player 2 rejects the opening offer, then the companion game is played from time 1. If equilibrium strategies are played in this game, player 2 gets no more than $\delta_2 M_2$. Therefore, in any equilibrium, player 2 must accept at time $t = 0$ any offer that assigns him a payoff strictly greater than $\delta_2 M_2$. Thus player 1 can guarantee himself any payoff less than $y^{-1}(\delta_2 M_2)$. Hence, $m_1 \geq y^{-1}(\delta_2 M_2)$.

(ii) In the companion game, player 1 can guarantee himself any payoff less than $\delta_1 m_1$ by rejecting player 2's opening offer. Thus, $M_2 \leq y(\delta_1 m_1)$.

The uniqueness of (u_1^*, v_2^*) satisfying (3.2) is shown in Figure 3.1 by the fact that the curves $y(\delta_1 u_1) = u_2$ and $y^{-1}(\delta_2 u_2) = u_1$ intersect only at (u_1^*, v_2^*). From (i) and $m_1 \leq u_1^*$, we have that (m_1, M_2) belongs to region (i). From (ii) and $M_2 \geq v_2^*$, we have that (m_1, M_2) belongs to region (ii). Hence $(m_1, M_2) = (u_1^*, v_2^*)$. Similarly, $(M_1, m_2) = (u_1^*, v_2^*)$.

This implies that the equilibrium strategies are unique because after every history, the proposer's offer must be accepted in equilibrium. If for example player 1's demand of u_1^* were rejected, he would get at most $\delta_1 v_1^* < u_1^*$.

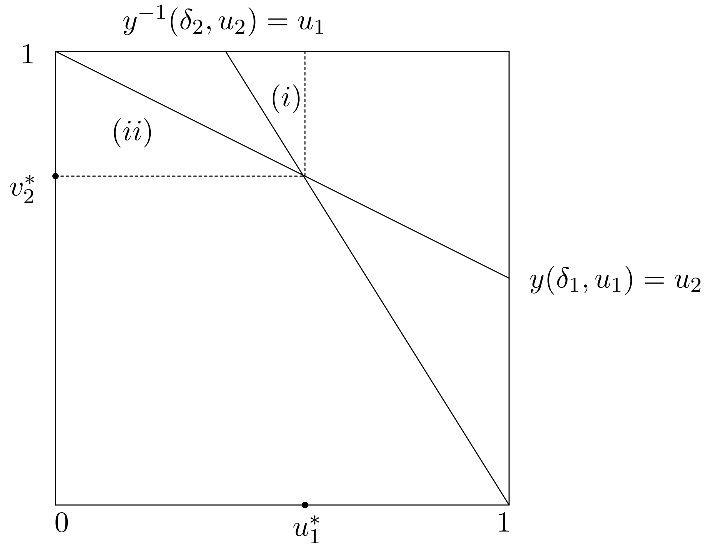


Figure 3.1

□

[3] The following proof of Shaked and Sutton is especially useful for extensions and modifications of the theorem.

3.3 Fixed costs

In this case, the results are the same obtained in the previous chapter in Application 1. We can notice that if c is small, $c < 1/3$, some of these equilibria involve delay in agreement being rejected. That is, equilibria exist in which one or more offers get rejected. Even when the interval τ between successive proposals become negligible, the equilibrium delays do not necessarily become negligible.

3.4 Discounting factors

A very special case of the time preferences covered by Theorem 3.1 occurs when $\phi_1(s) = \phi_2(s) = s$, $0 \leq s \leq 1$. Suppose that we have an arbitrary $\tau > 0$, we have that $u_1^* = (1-\delta_2^\tau)/(1-\delta_1^\tau\delta_2^\tau) \to \rho_2/(\rho_1+\rho_2)$ as $\tau \to 0^+$.[4] When $\delta_1 = \delta_2$, players share the available surplus of 1 equally. If δ_1 decreases, then player 1 also share it with player 2. Then, we have a general result: it always pays to be more patient.

[4]The computation of the limit is a direct application of the L'Hôpital rule. The expression $\rho_i = -\ln\delta_i$ for $i = 1, 2$ is the continuous interest equivalent.

Chapter 4

Extension of games with incomplete information

So far we have dealt with complete information games. In this chapter we are going to focus our study in incomplete information games. So, the players do not have common knowledge of the game being played.

The game explained above can be more complex and complete by adding several factors that will be exposed throughout this chapter. First we will add an attrition factor over time and then we will add a deadline effect.

4.1 Bayesian Nash Equilibrium

We have seen that the Nash equilibrium appears when we study a static game of complete information and a subgame perfect equilibrium appears when we study a dynamic game of complete information. Now, we will introduce the equilibrium which appears when we study a static game of incomplete information: *the Bayesian Nash equilibrium.*

Recall that in a game of complete information the players' payoff functions are common knowledge. In a game of incomplete information, in contrast, at least one player is uncertain about another player's payoff function.

Let player i's possible payoff functions be represented by $u_i(a_1, \ldots, a_n; t_i)$, where $u_i(a_1, \ldots, a_n)$ is player i's payoff when the players choose the actions $(a_1, \ldots, a_n)$ and t_i is called player i's *type* and belongs to a set of possible types T_i. Each type t_i corresponds to a different payoff function that player i might have. Then, saying that player i knows his or her own payoff function is equivalent to say that player i knows his or her type. Likewise, saying that player i may be uncertain about the other players' payoff functions is equivalent to say that player i may be uncertain about the types of the others players, denoted by $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$. We use T_{-i} to denote the set of all possible values of t_{-i} and we use the probability distribution $p_i(t_{-i}|t_i)$ to denote player i's belief about the other player's types, t_{-i}, given player i's knowledge of his or her own type, t_i. This game is called Bayesian because players update their beliefs about the types of other players, once they know one's own type, using Bayes' formula.

Now, we can proceed to describe the game in normal form.

Definition 4.1. The *normal form representation* of an n-player static Bayesian game specifies the players' action spaces $A_1, \ldots, A_n$, their type spaces $T_1, \ldots, T_n$, their beliefs $p_1, \ldots, p_n$, and their payoffs functions $u_1, \ldots, u_n$. Player i's type, t_i, is privately known by player i, determines player i's payoff function, $u_i(a_1, \ldots, a_n; t_i)$, and is a member of the set of possible types, T_i. Player i's belief $p_i(t_{-i}|t_i)$ describes i 's uncertainty about the $n-1$ other players' possible types, t_{-i}, given i's own type, t_i. We denote this game by $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$.

Once we have the previous definition we need to define an equilibrium concept and, to do it, we must first define the players' strategy spaces in such a game. Recall from Chapter 1 that a player's strategy is a complete plan of action.

Definition 4.2. In the static Bayesian game $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$, a strategy for player i is a function $s_i(t_i)$, where for each type t_i in T_i, $s_i(t_i)$ specifies the action from the feasible set A_i that type t_i would choose if drawn by nature.

Unlike games of complete information, in a Bayesian game the strategy spaces are not given in the normal form representation of the game. Instead, in a static Bayesian game the strategy spaces are constructed from the type and action spaces: player i's set of possible strategies, S_i, is the set of all possible functions with domain T_i and range A_i.

Now, we can proceed to define the Bayesian Nash Equilibrium. The main idea of this concept is that each player's strategy must be a best response to the other players' strategies.

Definition 4.3. In the static Bayesian game $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$, the strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a *Bayesian Nash equilibrium* if for each player i and for each of i's types t_i in T_i, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n); t) p_i(t_{-i}|t_i).$$

That is, no player wants to change his or her strategy, even if the change involves only one action by one type.

4.2 The war of attrition

Ponsati and Sákovics [20], 1995, presented an analysis of the war of attrition with exponential discounting in continuous time with two-sided incomplete information about reservation values.

Let's explain the situation that we will study in this chapter. Consider a simple game where there are two players, 1 and 2, and two different alternatives. Each player has a different preferred alternative and they must choose a time at which to concede in case the other has not done so yet. We have to take into account that the payoff from any player, which we will suppose that is individually rational, decreases over time. Furthermore, at any time a player prefers that her opponent concedes rather than conceding herself.

John Maynard Smith [10], in 1974, was the one who originally proposed this type of game to study patterns of animal behavior. This game is useful in the study of a wide variety of conflict situations as price wars and exit in oligopolistic markets (Fudenberg and Tirole [6], 1986; Ghemawat and Nalebuff [7], 1985; Kreps and Wilson [9], 1982), patent races (Fudenberg [5], 1983), public good provision (Bliss and Nalebuff [2], 1984) and bargaining (Ordover and Rubinstein [14], 1982; Osborne [15], 1985; Chatterjee and Samuelson [3], 1987).

It can also be expanded with a new model in which the impatience is modeled by an additive linear cost of delay, using an exponential discounting. With this approach is it possible to incorporate a new behavior of the players: they may prefer disagreement to concession. That's because the disagreement payoff is zero as opposed to minus infinity. This type of situations have a unique equilibrium and are more appropriate to use them in generalizations of the war of attrition: to more than two players (Ponsati and Sákovics [19], 1996), to bargaining over many issues (Ponsati [16], 1992) or to bargaining over a finite number of alternatives (Ponsati and Sákovics [18], 1992).

In the model explained in this chapter, we have two main classes of types of players:

1. Weak types. This players prefer concession to disagreement.
2. Tough types. This players would rather never agree than concede.

So, we have different situations depending on the type of players:

a) If both players are though types, the unique equilibrium is to never concede.

b) If both of them are weak types, they distribute their concessions across time. These strategies are characterized by a pair of differential equations.

c) When it is known that only one of the two players is weak with probability one (and so, the other player is tough), reaching an agreement takes time, possibly infinite. That is because a strategy in which the player with weak type concede by some time cannot be supported as an equilibrium. Then, in this case, waiting for an instant to convince the opponent of their toughness always increases the expected payoff of the weak player.

Usually, a player more likely to be tough receives a higher expected payoff.

4.3 The model

Consider two players, 1 and 2, disputing over two alternatives, A and B. Player 1 favours A and player 2 favours B. Let us denote a generic alternative by X and the generic player associated with X by x. So, $X \in \{A, B\}$ and $x \in \{1, 2\}$. Let s be the least favorable point at which one player will accept a negotiated agreement and is called *reservation value*. The reservation value is a privately known parameter, which means that every player knows his/her reservation value but they don't know the reservation value of the other player.

Let $u_i(X, s)$ be the utility that receives player i when the alternative reached is X. This utility depends on the reservation value s of player i.

If we assume that we deal with player 1, the restrictions on preferences that we need to be considered are the ones that follow:

1. $u(B, s) \geq u$, where u is the disagreement payoff, if and only if s is non-positive (i.e., if player 1 is weak),

2. $u(A, s) > u(B, s)$ for all s,[1]

3. $\partial u(X, s)/\partial s < 0$,

4. $u(B, s') - u(B, s) \geq u(A, s') - u(A, s)$ for $s' \geq s$.

To simplify it, we will suppose that player i's, $i = 1, 2$, static preferences are described in the following way:

$$u_i(X, s) = \begin{cases} 1 - s, & if \quad i = x \\ \\ -s, & if \quad i \neq x \end{cases}$$

These preferences satisfy the above restrictions.

The game is played in continuous time, starting at $t = 0$ and consists of that each player proposes his/her preferred alternative. The players have a unique action available (player 1 has only the alternative A to propose and player 2 has only the alternative B to propose), though they can choose the time to propose it. They can also yield at any time if they think is necessary. So, two different situations can happen:

1. Player 1 starts out proposing alternative A and this situation persists until he finally yields. When player 1 yields, alternative B is implemented.

2. Player 2 starts out proposing alternative B and this situation persists until he finally yields. When player 2 yields, alternative A is implemented.

Observations. *In the case that both players concede at the same time, they use a lottery to decide the outcome of player i ($i = 1, 2$) with probability π^i. The lottery cannot assign a probability one to any of the two alternatives. Then,* $0 < \pi^i < 1$.

Thus, a strategy of one of the players, σ_i, is a function that goes from his type to the time of his concession, which can be infinite.

We also have to take into account that players are impatient. Their impatience is modeled by a discount function and it is common for both of them. This discount factor is normalized to be e^{-1} per unit of time. Therefore, player i receives the utility $U_i(X, s, t) = u_i(X, s) \cdot e^{-t}$ when the alternative X is reached at time t.[2] Perpetual disagreement gives utility 0 to both players.

The players make beliefs about the type of the other player and these beliefs are known for both of them. The beliefs are represented by a probability distribution function F_j, with positive density f_j, over the interval $[j_L, j_H]$, and $f_j \in C^\infty$. This interval must accomplish:

- $j_L < 0$.
- $j_H \leq 1$.

[1] Notice that if s is positive then $u(A, s) > u > u(B, s)$ and if s is non-positive $u(A, s) > u(B, s) > u$.

[2] The assumption on time preferences would be enough to have $U_i(X, s, t) = u_i(X, s) \cdot \phi(t)$, where ϕ is continuously differentiable, strictly decreasing and $lim_{t\to\infty}\phi(t) = 0$.

We can have some conclusions with the above assumptions: when the opponent of one player is weak type, with positive probability both players are of a type that derives a positive utility, even if the opponent's alternative predominates; when the opponent of one player is tough type, with positive probability both players are of a type that derives a negative utility if the opponent's alternative predominates; and there are no types of either player which derive negative utility, even when the agreed alternative is the preferred alternative of the player.

We will assume that the type of player 1 is independent from the type of player 2.

Let denote the type of player 1 with a and the type of player 2 with b. Given a strategy profile σ, let $(X(\sigma(a,b)), t(\sigma(a,b)))$ be the outcome generated by σ when the type of player 1 is a and the type of player 2 is b. Let

$$V_1^a(\sigma) = \int_{[j_L, j_H]} U_i(X(\sigma(a,b)), a, t(\sigma(a,b)))dF_j(b)$$

denote the expected payoff to player 1 of type a given σ. Similarly, the expected payoff to player 2 of type b given σ is:

$$V_2^b(\sigma) = \int_{[j_L, j_H]} U_i(X(\sigma(a,b)), a, t(\sigma(a,b)))dF_j(a).$$

Then, σ is a *Bayesian Equilibrium* (BE) if and only if for all (a,b) and $i = 1, 2:$

$$V_i^a(\sigma) \geq V_i^a(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i'.$$

4.4 The deadline effect

Ponsati (1995) [17] presents a simple game of concession in which the combination of a deadline with two-sided incomplete information leads to a unique Bayesian equilibrium (B.E.) with a deadline effect. This type of situations are very common in real life. An example could be the last minute agreements. An effective deadline yields a discontinuity in the payoffs that agents can enjoy over time. Some examples of the deadline could be the date at which the contract expires in labour negotiations, the date at which the firm expects to run out of inventories,etc.

Ponsati and Sákovics (1995) [20] characterize B.E. for concession games with incomplete information without a deadline, when $T = \infty$. That is what we have seen in the previous section. Now, we consider the effect of introducing a deadline. The distribution of dates of agreement along the unique B.E. is continuous in $(0, T)$, where $T < \infty$. Moreover, there is some date t, $0 < t < T$, such that the probability of concession is null in the interval (t, T).

The results of this game suggest that in very polarized negotiations a credible deadline has a very positive effect. In this case, since conceding means giving up almost all the surplus, most types do not concede, and the probability that the opponent concedes is very small. Thus, the average gains from trade without a deadline are very close to 0. An early deadline would yield a positive probability of a compromise at the deadline, which yields positive average gains.

The model of this game with the deadline effect is exactly the same as one in the previous section, without the deadline effect, but with some added notes. The main difference is that in this case the game is played in continuous time, starting at $t = 0$, but ending at the deadline T, $T < \infty$. We also have to take into account that if no player yields at $t \leq T$, both players receive a zero payoff. The results obtained provide the B.E. strategies in all the possible cases that can appear during the bargaining. To do it they use increasing and differentiable functions that uniquely solve a system of differential equations.

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