Equation of state for neutron star calculations

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Abstract: Neutron star matter is composed of neutrons, protons and electrons in a state of charge neutrality and β-equilibrium. In order to study the equation of state of this type of stellar matter, a zero-range type of nucleon-nucleon effective interactions, called Skyrme interactions, will be used within the Hartree-Fock approach. We also study the relation between the mass and the radius of neutron stars by solving the Tolman-Oppenheimer-Volkoff equations using the equation of state of neutron star matter.

I. INTRODUCTION

Neutron stars (NSs) are the remnants of supernova explosions of massive stars ($10-40M_{\odot}$). A NS has typically a mass of around $1-2M_{\odot}$ and radius $\simeq$ 10km, which makes them the densest objects of the observable Universe, with central densities in the range of $10^{14}-10^{15}$ g/cm^3 [1].
NSs have an onion shell structure. The external region is called crust and consists of a lattice of atomic nuclei permeated by a gas of free electrons (outer crust) and a gas of free electrons and neutrons (inner crust). The core of the star is formed of an homogeneous liquid compound of neutrons, protons, leptons and, eventually, other exotic particles.
The first pulsar was discovered in 1967 by Jocellyn Bell. From then on, many NS have been detected with quite well determined masses. The radii of these stars is much more uncertain than their mass.
A NS remains as a bound object because of the equilibrium between the gravitational attraction and the pressure due to its components and can be described theoretically by means of the Tolman-Oppenheimer-Volkoff equations in the case of static NSs. An essential ingredient to solve these equations, which provides de mass-radius relation in the NS, is the Equation of State (EoS) of its components. At first, only the pressure due to the electrons was considered and the maximum mass reached was around $0.7M_{\odot}$. When nuclear forces are also taken into account, the pressure increases due to the neutron and proton contributions and larger masses up to $2M_{\odot}$ can be achieved [2].
In this work, we construct the nuclear part of the EoS with the Hartree-Fock method using the effective SLy4 [3], SkM* [4], SkI2 and SkI5 [5] Skyrme forces, which reproduce successfully many properties of finite nuclei.

II. SKYRME INTERACTION

The Skyrme effective nucleon-nucleon interaction can be expressed in form of a potential of the type

$$\hat{v} = \sum_{i<j} v_{ij}^{(2)} + \sum_{i<j<k} v_{ijk}^{(3)}. \tag{1}$$

The matrix elements of this force in momentum space read

$$\left\langle \vec{k} \middle| v_{12} \middle| \vec{k}' \right\rangle = t_0 \left(1 + x_0 P^{\sigma}\right) + \frac{1}{2} t_1 \left(\vec{k}^2 + \vec{k}'^2\right) + t_2 \vec{k} \cdot \vec{k}', \tag{2}$$

which in coordinate space can be expressed as

$$\begin{aligned} v_{12} = & t_0 \left(1 + x_0 P^{\sigma}\right) \delta\left(\vec{r}_1 - \vec{r}_2\right) \\ & + \frac{1}{2} t_1 \left(1 + x_1 P^{\sigma}\right) \left[\delta\left(\vec{r}_1 - \vec{r}_2\right) \vec{k}^2 + \vec{k}'^2 \delta\left(\vec{r}_1 - \vec{r}_2\right)\right] \\ & + t_2 \left(1 + x_2 P^{\sigma}\right) \vec{k}' \delta\left(\vec{r}_1 - \vec{r}_2\right). \end{aligned} \tag{3}$$

In Eqs. (2) and (3), t_0, t_1, t_2, x_0, x_1 and x_2 are six of the free parameters that define the Skyrme force. P^{σ} is the spin exchange operator $P^{\sigma} = \frac{1}{2}\left(1 + \vec{\sigma}_1 \vec{\sigma}_2\right)$ and $\vec{k}$ and $\vec{k}'$ are relative momentum operators

$$\vec{k} = \frac{\overrightarrow{\nabla}_1 - \overrightarrow{\nabla}_2}{2i} \qquad \vec{k}' = -\frac{\overleftarrow{\nabla}_1 - \overleftarrow{\nabla}_2}{2i}.$$

A zero-range force is also assumed for the three-body part of the interaction which reads

$$v_{123}^{(3)} = \delta\left(\vec{r}_1 - \vec{r}_2\right) \delta\left(\vec{r}_2 - \vec{r}_3\right), \tag{4}$$

that is equivalent to the following density dependent two-body interaction.

$$v_{12} = \frac{1}{6} t_3 \left(1 + x_3 P^{\sigma}\right) \delta\left(\vec{r}_1 - \vec{r}_2\right) \rho \left(\frac{\vec{r}_1 - \vec{r}_2}{2}\right)^{\gamma}. \tag{5}$$

In Eq. (5), x_3, $t3$ and γ are the remaining parameters of the Skyrme force.

III. HARTREE-FOCK METHOD IN COORDINATE SPACE

In the Hartree-Fock approach, the total energy is given by

$$E = \sum_i \langle i|\hat{t}|i\rangle + \frac{1}{2}\sum_{i,j}\left(\langle ij|\hat{v}|ij\rangle - \langle ij|\hat{v}|ji\rangle\right), \quad (6)$$

where $\hat{t}$ and $\hat{v}$ are respectively the kinetic and potential operators.
In order to study nuclear matter, as it is a uniform system, the spatial part of the wave function is a plane wave normalized to a volume Ω. Taking into account the spin and isospin degrees of freedom, the total wave function reads

$$\phi_i(\vec{r}) = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\vec{r}} \xi_\sigma \xi_\tau, \quad (7)$$

where ξ_σ and ξ_τ are the spin and isospin spinors.
The particle density is given by

$$\rho(\vec{r}) = \sum_i \left|\phi_i(\vec{r})\right|^2. \quad (8)$$

In Eq. (6), the sum over i can be replaced as the sum over the spin and isospin configurations.

$$\sum_i \to \sum_\sigma \sum_\tau \frac{\Omega}{(2\pi)^3} \int_0^{k_F} d\vec{k}, \quad (9)$$

where k_F is the Fermi momentum.
Therefore, using Eqs. (7) and (9), the density in symmetric nuclear matter can be written as

$$\rho = \frac{2k_F^3}{3\pi^2}, \quad (10)$$

as it involves a degeneracy of 4 whereas for neutron matter the degeneracy is 2 as

$$\rho = \frac{k_F^3}{3\pi^2}. \quad (11)$$

IV. ASYMMETRIC NUCLEAR MATTER

To study infinite matter, we consider a volume Ω of homogeneous matter with N neutrons, Z protons and, consequently, $A = N + Z$ nucleons. In the case of asymmetric nuclear matter ($N \neq Z$), the asymmetry parameter is introduced as

$$\delta = \frac{N-Z}{A} = \frac{\rho_n - \rho_p}{\rho} = \frac{\rho_n - \rho_p}{\rho_n + \rho_p}. \quad (12)$$

In order to find the energy per particle of asymmetric nuclear matter (ANM), one needs to compute the matrix elements of Eq. (6) assuming the nuclear matter to be composed of neutrons and protons with an asymmetry δ.
If we define an asymmetry factor

$$F_m(\delta) = \frac{1}{2}\left[(1+\delta)^m + (1-\delta)^m\right], \quad (13)$$

the energy per particle reads as

$$\begin{aligned}\frac{E}{A}(\rho,\delta) &= \frac{\hbar^2}{2m}\frac{3}{5}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}} \rho^{\frac{2}{3}} F_{5/3} + \frac{t_0}{8}\rho\left[3 - \delta^2(2x_0+1)\right] \\ &+ \frac{t_3}{48}\rho^{\gamma+1}\left[3 - \delta^2(2x_3+1)\right] + \frac{3}{40}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}} \rho^{\frac{5}{3}} \\ &\times \Big(\left[t_1(x_1+2) + t_2(x_2+2)\right] F_{5/3} \\ &+ \frac{1}{2}\left[t_2(2x_2+1) - t_1(2x_1+1)\right] F_{8/3}\Big).\end{aligned} \quad (14)$$

A very important property of nuclear matter is the symmetry energy, defined as

$$E_{sym} = \frac{1}{2}\frac{\partial^2 \frac{E}{A}(\rho,\delta)}{\partial \delta^2}\Big|_{\delta=0}. \quad (15)$$

To a good approximation, the energy per particle in ANM is a quadratic function of the asymmetry δ. We can then compute the symmetry energy as the difference between neutron and symmetric nuclear matter energy per particle as

$$E_{sym} \simeq \frac{E}{A}(\rho,1) - \frac{E}{A}(\rho,0). \quad (16)$$

On the one hand, pure neutron matter is calculated with $\delta = 1$ as $\rho = \rho_n$. On the other hand, for symmetric nuclear matter, $\rho_n = \rho_p = \frac{\rho}{2}$ and, consequently, $\delta = 0$. Eq. (16) clearly shows that the symmetry energy is the energy cost for converting all the protons of symmetric nuclear matter into neutrons.
The energy per particle of symmetric nuclear matter and pure neutron matter are

$$\begin{aligned}\frac{E}{A}(\rho,0) &= \frac{3}{5}\left(\frac{3\pi^2}{2}\rho\right)^{\frac{2}{3}}\left(\frac{\hbar^2}{2m} + \frac{\rho}{16}\left[3t_1 + 5t_2 + 4t_2x_2\right]\right) \\ &+ \frac{3}{8}t_0\rho + \frac{t_3}{16}\rho^{\gamma+1},\end{aligned} \quad (17)$$

$$\frac{E}{A}(\rho,1)=\frac{3}{5}\left(3\pi^2\rho\right)^{\frac{2}{3}}\left(\frac{\hbar^2}{2m}+\frac{\rho}{8}\left[t_1(1-x_1)+3t_2(1+x_2)\right]\right) + \frac{t_0}{4}\rho(1-x_0)+\frac{t_3}{24}\rho^{\gamma+1}(1-x_3)\,. \tag{18}$$

By plugging Eqs. (17) and (18) into (16) one estimates the symmetry energy as

$$\begin{aligned}E_{sym}\simeq{}&\frac{\hbar^2}{2m}\frac{3}{5}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}}\left(2^{\frac{2}{3}}-1\right)\\&-\frac{t_0}{8}\rho(1+2x_0)-\frac{t_3}{48}\rho^{\gamma+1}(1+2x_3)\\&+\frac{3}{5}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}}\rho^{\frac{5}{3}}\left(\frac{2^{\frac{2}{3}}}{8}\left[t_1(1-x_1)\right.\right.\\&\left.\left.+3t_2(1+x_2)\right]-\frac{3t_1+5t_2+4t_2x_2}{16}\right).\end{aligned} \tag{19}$$

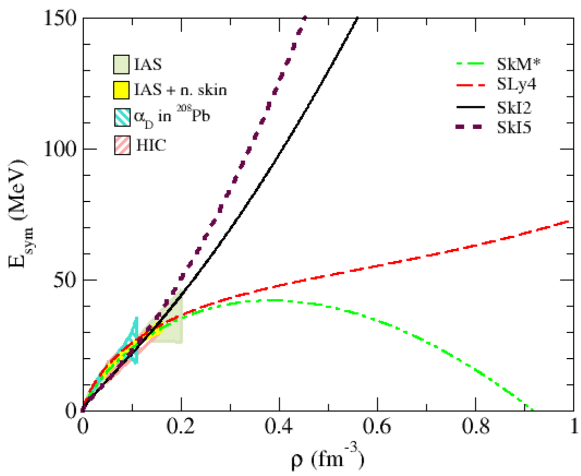

FIG. 1: Symmetry energy defined in Eq. (16) as a function of the density for SLy4, SkM*, SkI2 and SkI5 Skyrme parametrizations. Four constraints are also included [6].

At low densities, the symmetry energy predicted by all models is quite similar due to the fact that this region is constrained by experimental data of finite nuclei. It is also remarkable that for the Skyrme force SkM*, the symmetry energy becomes negative for large enough densities, implying isospin instabilities in such regions.

V. EQUATION OF STATE FOR NEUTRON STARS

The outer part of the NS core is formed of uniform matter made out of neutrons, protons and electrons in β-equilibrium. This β-stable matter is characterized by charge neutrality and equilibrium in the weak interaction processes:

$$\begin{aligned}&n\rightarrow p+e^-+\bar{\nu}_e\\&p+e^-\rightarrow n+\nu_e.\end{aligned} \tag{20}$$

This equilibrium implies that the chemical potentials should satisfy $\mu_n=\mu_p+\mu_e$ as well as $\rho_e=\rho_p$, where the chemical potential of each spice is defined as $\mu_q=\frac{\partial}{\partial\rho_q}\left(\rho\frac{E}{A}\right)$.

Considering nuclear matter at zero temperature, the pressure as a function of the density is

$$P(\rho)=\rho^2\frac{\partial}{\partial\rho}\left(\frac{E}{A}(\rho,\delta)\right)\Big|_\delta. \tag{21}$$

As we have only considered neutrons and protons, one needs also to take into account the electron contribution which we consider to be ultra-relativistic:

$$P_e(\rho_e)=\frac{\hbar c}{4}\left(3\pi^2\right)^{\frac{1}{3}}\rho_e^{\frac{4}{3}}. \tag{22}$$

Adding (22) to the ANM pressure having in mind that $\rho_e=\rho_p$, the EoS for β-stable matter reads as follows

$$\begin{aligned}P(\rho,\delta)={}&\frac{\hbar^2}{2m}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}}\frac{2}{5}\rho^{\frac{5}{3}}F_{5/3}+\frac{t_0}{8}\rho^2\left[3-\delta^2(2x_0+1)\right]\\&+\frac{t_3}{48}(\gamma+1)\rho^{\gamma+2}\left[3-\delta^2(2x_3+1)\right]\\&+\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}}\frac{\rho^{\frac{8}{3}}}{8}\Big(\left[t_1(x_1+2)+t_2(x_2+2)\right]F_{5/3}\\&+\frac{1}{2}\left[t_2(2x_2+1)-t_1(2x_1+1)\right]F_{8/3}\Big)\\&+\frac{\hbar c}{4}\left(3\pi^2\right)^{\frac{1}{3}}\left[\frac{\rho(1-\delta)}{2}\right]^{\frac{4}{3}}.\end{aligned} \tag{23}$$

Therefore, after some algebra,

$$\begin{aligned}\mu_n-\mu_p-\mu_e={}&\left[(1+\delta)^{\frac{2}{3}}-(1-\delta)^{\frac{2}{3}}\right]k_F^2\\&\times\left[\frac{\hbar^2}{2m}+\frac{\rho}{8}\left[t_1(x_1+2)+t_2(x_2+2)\right]\right]\\&+\left[(1+\delta)^{\frac{5}{3}}-(1-\delta)^{\frac{5}{3}}\right]k_F^2\rho\\&\times\left[t_2(2x_2+1)-t_1(2x_1+1)\right]\\&-\frac{t_0}{2}\rho\delta(2x_0+1)-\frac{t_3}{12}\rho^{\gamma+1}\delta(2x_3+1)\\&-\hbar c\cdot k_F(1-\delta)^{\frac{1}{3}}=0.\end{aligned} \tag{24}$$

For a given value of the total density ρ, the corresponding δ value that fulfills the β-stability condition can be obtained by solving Eq. (24). We have computed this equation using the Newton-Rhapson method of finding the roots of a function. The results obtained are shown in Figure 2.

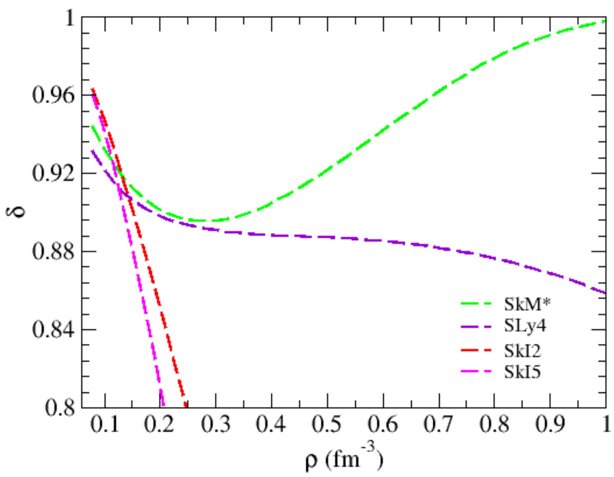

FIG. 2: Isospin asymmetry as a function of the density for β-stable nuclear matter for SLy4, SkM*, SkI2 and SkI5 Skyrme parametrizations.

These values of the asymmetry as a function of the density, agree with the results from Figure 1. For instance, for the SkM* force, the asymmetry tends to 1 (neutron matter) when the symmetry energy tends to 0.

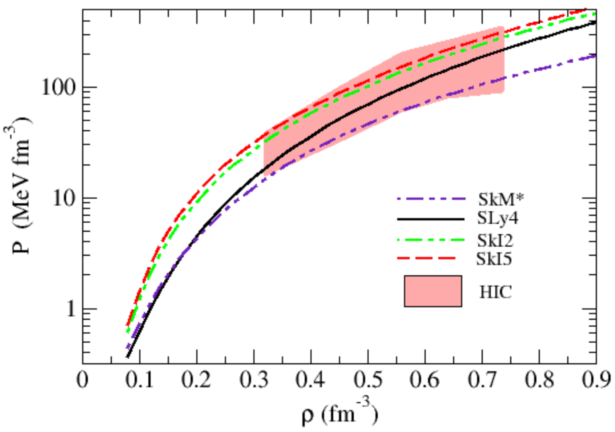

FIG. 3: Pressure in logarithmic scale as a function of the density for β-stable nuclear matter obtained for SLy4, SkM*, SkI2 and SkI5 Skyrme parametrizations. A constraint coming from heavy ion collisions is also added [6].

Once we have the values of the asymmetry and density obtained from Eq. (24), we can introduce them into our EoS of β-stable matter in Eq. (23) and plot the pressure of NS matter as shown in Figure 3.

All models considered in this work are compatible at high densities (from 2 to 4 times the critical density $\rho_0 \simeq 0.16$ fm^{-3}) with the constraints provided by the analysis of HIC.

VI. TOLMAN-OPPENHEIMER-VOLKOFF EQUATIONS

The most important property of cold non-rotational NSs is the relation between its gravitational mass and its radius. This relation is obtained by solving the Tolman-Oppenheimer-Volkoff (TOV) equations

$$\frac{dP(r)}{dr} = -\frac{[\epsilon(r)+P(r)]\left[m(r)+4\pi r^3 P(r)\right]}{r^2\left[1-\frac{2m(r)}{r}\right]} \tag{25}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \epsilon(r), \tag{26}$$

where $\epsilon(r)$, $P(r)$ and $m(r)$ are respectively the energy density, the pressure and the mass inside the NS. The solution is constrained to the following conditions: the mass at the center of the star is zero ($m(0) = 0$), the central pressure is a given value ($P(0) = P_c$) and the pressure at the surface is zero ($P(R) = 0$). Integrating from the center to the surface of the star, where $m(R) = M$, we obtain the mass-radius profile of the star.
We have computed this equations by introducing our values of the pressure from Eq. (23) and the energy density of neutron star matter that is expressed as

$$\begin{aligned}\epsilon(\rho,\delta) = &\frac{\hbar^2}{2m}\frac{3}{5}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}}\rho^{\frac{5}{3}}F_{5/3} + \frac{t_0}{8}\rho^2\left[3-\delta^2\left(2x_0+1\right)\right]\\ &+\frac{t3}{48}\rho^{\gamma+2}\left[3-\delta^2\left(2x_3+1\right)\right] + \frac{3}{40}\left(\frac{3\pi^2}{2}\right)^{\frac{2}{3}}\rho^{\frac{8}{3}}\\ &\cdot\Bigg[\left[t_1(2+x_1)+t_2(2+x_2)\right]F_{5/3} + \Big[t_2(2x_2+1)\\ &- t_1(2x_1+1)\Big]\frac{1}{4}\left[(1+\delta)^{\frac{8}{3}}-(1-\delta)^{\frac{8}{3}}\right]\Bigg]\\ &+\frac{3}{4}\hbar c\left(3\pi^2\right)^{\frac{1}{3}}\left[\frac{\rho}{2}(1-\delta)\right]^{\frac{4}{3}} + M_{nuc}c^2\rho.\end{aligned} \tag{27}$$

In this last equation, we have added to the energy density of ANM contributions of the electrons and the rest mass.

In our calculation of the EoS, the crust region of the star has been taken into account using the tabulated values of [7].

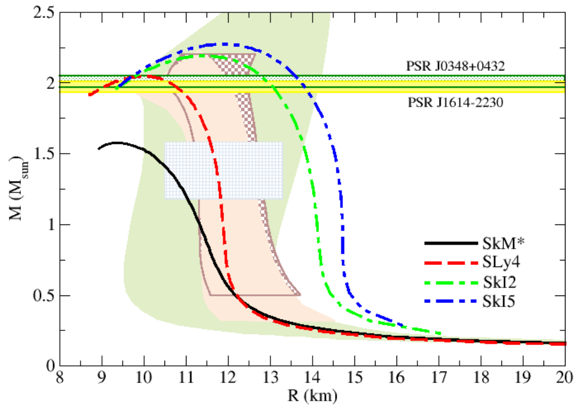

FIG. 4: Mass-radius relation of neutron star matter for SLy4, SkM*, SkI2 and SkI5 Skyrme parametrizations with several experimental constraints [6].

Skyrme Force	$M_{max}(M_{\odot})$	R_{max}(km)
SkM*	1.61	9.49
SLy4	2.04	10.00
SkI2	2.19	11.47
SkI5	2.27	11.85

TABLE I: Maximum values of mass for the four Skyrme forces studied and the radius at which the maximum mass is reached.

From Figure 4 and Table I, we can observe that the parametrization that fits best the description of neutron star profile is SLy4 out of the four parametrizations studied. It matches the constraints and does correlate with the maximum mass of NSs observed, which is around $2M_{\odot}$ [2] as is shown in Figure 4 with the horizontal bands. In the case of SkM*, the curve does not reach the estimated $2M_{\odot}$ and for SkI2 and SkI5 this limit is widely surpassed.

VII. CONCLUSIONS

- A successful representation of the relation between the mass and the radius of NSs has been achieved by the implementation of the equation of state and integrating the TOV equations.
- As Skyrme interactions have been of use to our study, one can say that this type of zero-range effective interactions can be used to study neutron star matter apart from its usual use in finite nuclei.
- Having studied different Skyrme forces, from the parametrizations chosen, SLy4 is the one that fits the best the description of the behaviour of the gravitational mass as a function of its radius.

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