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ABSTRACT

One hundred and forty years after his discovery, the Hall effect still deserves attention. If it is well-known that the Hall voltage measured in Hall bar devices is due to the electric charges accumulated at the edges in response to the magnetic field, the nature of the corresponding boundary conditions is still problematic. In order to study this out-of-equilibrium stationary state, the Onsager's least-dissipation principle is applied. It is shown that, beside the well-known expression of the charge accumulation and the corresponding Hall voltage, a longitudinal surface current proportional to the charge accumulation is generated. An expression of the surface current is given. The surface currents allow the Hall voltage to be stabilized at a stationary state, despite, e.g., the presence of leakage of charges at the edges.

I. INTRODUCTION

The classical Hall effect[1] is usually described by the local transport equations for the charge carriers that takes into account the effect of the Laplace–Lorentz force generated by a static magnetic field. Typically, in a planar Hall device, an electric generator imposes an electric current $\vec{J}$ along the x direction (see Fig. 1), and the Hall voltage is then measured transversally along the y direction at a stationary regime, as a function of the magnetic field. The physical mechanisms behind this effect and the corresponding transport equations are well-known and are described in all reference textbooks.

However, the non-equilibrium stationary state of the classical Hall effect still presents some important technical and conceptual difficulties while treating the boundary conditions. In contrast to usual non-equilibrium stationary sates, the presence of the static magnetic field leads to specific transverse boundary conditions—the charge accumulation at the edges—that are not imposed directly by external constraints but by the system itself, according to the Le Chatelier–Braun principle.[2,3]

This specificity of the boundary conditions for the Hall effect has been discussed within a large variety of theoretical models,[4–12] but the nature of the charge accumulation at the edges still remains rather mysterious. Indeed, the discovery of the extraordinary Hall effect at the turn of the millennium[13–15] and the recent developments about the Hall effect in ferromagnets[16,17] or about the spin-Hall effect[18,19] show that 140 years after his discovery, the classical Hall effect still deserves attention.

For the ideal Hall bar (i.e., with a translational invariance along x and symmetric edges), the local stationarity condition $\vec{\nabla} \cdot \vec{J} = 0$ is not sufficient to define a unique stationary state (see Appendix A). It is usually admitted that the stationary state corresponds to a vanishing transverse current $J_y = 0$ along the y axis,[20] as claimed in reference textbooks.[21–24] The argument invoked is that an accumulation of electric charges at the edges produces a transverse electric field E_y that balances the Lorentz force so that the system reaches "equilibrium" along the y axis.

However, the term "equilibrium" is misleading because the electric charges at the edges are not static but renewed permanently in order to maintain a stationary non-equilibrium charge distribution, typically in the case of charge leakage due to the Hall-voltage measurement.

It is shown in this paper that the non-equilibrium stationary state defined by the second law of thermodynamics—through the least-dissipation principle[25–28]—is indeed defined by $J_y = 0$, but with the generation of a surface current $J_x(y)$ flowing along the x axis (see Fig. 1) and proportional to the charge accumulation. This is valid as long as the total power dissipated by the charge leakage at the edges is negligible with respect to the total power dissipated in the device.[29]

The demonstration is presented in the following way: the system is first defined from a non-equilibrium thermodynamic

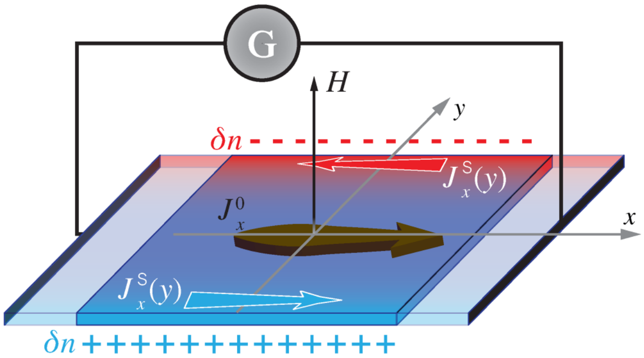

FIG. 1. Schematic representation of the Hall effect under a static magnetic field H applied along the z direction, with the electrostatic charge accumulation δn and surface currents $J_x^S(y) = J_x - J_x^0$ at the edges.

point of view in Sec. II. In Sec. III, the minimization of the Joule power is performed under the electrostatic screening and the galvanostatic constraint. A differential equation is obtained, where solutions would necessitate the knowledge of the local boundary conditions. The use of global constraints—instead of local—allows the stationary state to be uniquely defined. The expression of the surface currents is derived in Sec. IV. In Sec. V, the stability of the stationary state is investigated before concluding in Sec. VI.

II. JOULE DISSIPATION

The system under interest is defined in the context of non-equilibrium thermodynamics.[25–28,30–33] It is a thin homogeneous conducting layer of finite width contacted to an electric generator and submitted to a magnetic field (see Fig. 1). We assume that the device is planar, invariant by translation along the x axis (in particular, the region in contact to the generator is not under consideration here), and that the two lateral edges are symmetric. However, it is important to point out that we do not assume local boundary conditions for the current J_y. In particular, the model takes into account a possible charge leakage that can be present due to the Hall-voltage measurements (since any realistic voltmeter has a finite internal resistance). The power lost by this leakage of electric charges is assumed to be small with respect to the total power dissipated in the device (in other terms, the voltmeter has a large enough internal resistance).[29] Furthermore, we do not assume *a priori* the expression of the continuity equation $\vec{\nabla} \cdot \vec{J} = 0$.

Let us define the distribution of electric charge carriers by $n(y) = n_0 + \delta n(y)$, where $\delta n(y)$ is the charge accumulation and n_0 the homogeneous density of an electrically neutral system (e.g., density of carriers without the magnetic field). The charge accumulation is governed by the Poisson's equation $\nabla^2 V = \partial_y^2 V = -\frac{q}{\varepsilon}\delta n$, where V is the electrostatic potential, q is the electric charge, and ε is the electric permittivity. The local electrochemical potential $\mu(x, y)$, which takes into account not only the electrostatic potential V but also the energy (or the entropy) responsible for the diffusion, is given by the expression (local equilibrium is assumed everywhere),[31]

$$\mu = \frac{kT}{q} \ln\left(\frac{n}{n_0}\right) + V, \tag{1}$$

where k is the Boltzmann constant and T is the temperature of the heat bath in the case of a non-degenerate semiconductor or the Fermi temperature T_F in the case of a fully degenerated conductor.[34] Poisson's equation now reads

$$\nabla^2\mu - \lambda_D^2 \frac{q}{\varepsilon} n_0 \nabla^2 \ln\left(\frac{n}{n_0}\right) + \frac{q}{\varepsilon}\delta n = 0, \tag{2}$$

where $\lambda_D = \sqrt{\frac{kT\varepsilon}{q^2 n_0}}$ is the Debye–Fermi length. The invariance along x gives $\nabla^2 = \partial_y^2$.

On the other hand, the transport equation under a magnetic field is $\vec{J} = -\hat{\sigma}\vec{\nabla}\mu = -qn\hat{\eta}\vec{\nabla}\mu$, with the conductivity tensor $\hat{\sigma}$ and the mobility tensor $\hat{\eta}$. In two dimensions and for isotropic material, the mobility tensor is defined by Onsager relations,[25]

$$\hat{\eta} = \begin{pmatrix} \eta & \eta_H \\ -\eta_H & \eta \end{pmatrix} = \eta \begin{pmatrix} 1 & \theta_H \\ -\theta_H & 1 \end{pmatrix},$$

with

$$\theta_H = \frac{\eta_H}{\eta},$$

where η is the ohmic mobility, η_H the Hall mobility (usually proportional to the magnetic field $\vec{H} = H\vec{e}_z$), and θ_H the Hall angle. The electric current then reads

$\vec{J} = -qn\eta(\vec{\nabla}\mu - \theta_H\,\vec{e}_z \times \vec{\nabla}\mu)$ (where $\times$ denotes the cross product) or

$$-qn\eta(1 + \theta_H^2)\partial_x\mu = J_x - \theta_H J_y, \tag{3}$$

$$-qn\eta(1 + \theta_H^2)\partial_y\mu = J_y + \theta_H J_x. \tag{4}$$

The expression of the power dissipated by the system reads

$$P_J = \int_{\mathcal{D}} qn\eta \|\vec{\nabla}\mu\|^2 \,\mathrm{d}x\mathrm{d}y = \frac{1}{qn_0\eta(1 + \theta_H^2)} \int_{\mathcal{D}} \frac{n_0}{n} \|\vec{J}\|^2 \,\mathrm{d}x\mathrm{d}y.$$

III. LEAST-DISSIPATION PRINCIPLE

The least-dissipation principle states that the current distributes itself so as to minimize Joule heating P_J compatible with the constraints. After introducing the galvanostatic constraint $J_x = -qn(\eta\partial_x\mu + \eta_H\partial_y\mu)$, the screening equation [Eq. (2)], and their respective Lagrange multipliers $\beta(y)$ and $\gamma(y)$, the functional to be minimized reads

$$F[n, \vec{\nabla}\mu] = \int_{\mathcal{D}} q n\eta \|\vec{\nabla}\mu\|^2 \,\mathrm{d}y - \int_{\mathcal{D}} \beta(y)\left(-q\eta n\partial_x\mu - q\eta_H n\partial_y\mu\right)\mathrm{d}y - \int_{\mathcal{D}} \gamma(y)\left(\nabla^2\mu - \lambda_D^2 \frac{q}{\varepsilon} n_0 \nabla^2 \ln\left(\frac{n}{n_0}\right) + \frac{q}{\varepsilon}\delta n\right)\mathrm{d}y, \tag{5}$$

where $\mathcal{D}$ is the width of the device. It contains, on the right-hand side from left to right, the heat power, the galvanostatic constraint, and the electrostatic constraint. The minimization imposes the vanishing of the functional derivatives $\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta}(\partial_x\mu)} = 0$, $\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta}(\partial_y\mu)} = 0$, and $\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta}(n)} = 0$, from which we obtain the Euler–Lagrange equation corresponding to the stationary state (see Appendix B),

$$J_y - \lambda_D^2 \partial_y\left(\frac{n_0}{n}\partial_y J_y\right) = \frac{\varepsilon}{2q^2\eta(1+\theta_H^2)}\partial_y\left(\frac{(J_y)^2 + 2\theta_H J_x J_y - (J_x)^2}{n^2}\right). \tag{6}$$

This is a second order differential equation in J_y and first order in n, and its resolution—coupled with Poisson's equation and transport equations—would need the knowledge of four boundary conditions. However, a part of these conditions is not imposed externally but are fixed by the system itself in order to reach the state of minimum dissipation. Our approach does not consider them explicitly as the functional equation (5) does not include the treatment of the discontinuity between the conductor and its environment. Indeed, it is possible to find the minimum power dissipated without the aforementioned boundary conditions by taking into account the global constraints, which are known.

Let us define the width L of the conductor. Due to the symmetry of the device and that of the magnetic field (as an axial vector), we have $\int \delta n^{\mathrm{st}}\,\mathrm{d}y = 0$, and the total charge carrier density is constant $n_{\mathrm{tot}} = \frac{1}{L}\int n\,\mathrm{d}y$. For the sake of simplicity, we assume a global charge neutrality so that $n_{\mathrm{tot}} = n_0$. On the other hand, the global current flowing in the x direction throughout the device is also constant along x by definition of the galvanostatic condition. The two global constraints are

$$\int_{-L/2}^{L/2} n(y)\,dy = Ln_0 \quad \text{and} \quad \int_{-L/2}^{L/2} J_x(y)\,dy = LJ_x^0, \tag{7}$$

where J_x^0 is the uniform current density present at zero magnetic field. The inhomogeneity of the current density along the x axis (derived below) is then defined by the difference $J_x^S(y) = J_x(y) - J_x^0$.

We define for convenience the reduced power: $\tilde{P}_J = q\eta(1+\theta_H^2)\,P_J = \int \frac{J_x^2+J_y^2}{n}\,\mathrm{d}y$. Let us introduce the Lagrange multiplayers λ_J and λ_n corresponding to the constraints [Eq. (7)] so that the functional to be minimized reads now,

$$\tilde{P}_J^{\mathrm{Lag}}[J_x, J_y, n] = \int\left(\frac{J_x^2 + J_y^2}{n} - \lambda_J J_x - \lambda_n n\right)\mathrm{d}y. \tag{8}$$

The minimum corresponds to

$$\frac{\boldsymbol{\delta}\tilde{P}_J^{\mathrm{Lag}}}{\boldsymbol{\delta} J_x} = 0 \Longleftrightarrow 2J_x^{\mathrm{st}} = n^{\mathrm{st}}\lambda_J, \tag{9}$$

$$\frac{\boldsymbol{\delta}\tilde{P}_J^{\mathrm{Lag}}}{\boldsymbol{\delta} J_y} = 0 \Longleftrightarrow J_y^{\mathrm{st}} = 0, \tag{10}$$

$$\frac{\boldsymbol{\delta}\tilde{P}_J^{\mathrm{Lag}}}{\boldsymbol{\delta}(n)} = 0 \Longleftrightarrow (J_x^{\mathrm{st}})^2 + (J_y^{\mathrm{st}})^2 = -\lambda_n (n^{\mathrm{st}})^2, \tag{11}$$

where the superscript st stands for "stationary." Using Eqs. (7) and (9) leads to $\lambda_J = \frac{2J_x^0}{n_0}$ so that $J_x^{\mathrm{st}} = \frac{n}{n_0}J_x^0$ [and from Eq. (11), we have, furthermore, $\lambda_n = -(J_x^0 n^{\mathrm{st}}/n_0)$]. Hence, the minimum is reached for

$$J_x^{\mathrm{st}}(y) = J_x^0\frac{n(y)}{n_0} \quad \text{and} \quad J_y^{\mathrm{st}} = 0. \tag{12}$$

It is easy to verify that this state is a solution of the Euler–Lagrange equation (6), whatever the density distribution $n(y)$. As shown in Sec. V, this solution is stable. Note also that the usual stationarity condition $\vec{\nabla}\cdot\vec{J}^{\mathrm{st}} = 0$ is verified. Inserting the solution (12) into the relations (3) and(4), we deduce $\partial_x\mu^{\mathrm{st}} = \frac{-J_x^0}{qn_0\eta(1+\theta_H^2)}$ and $\partial_y\mu^{\mathrm{st}} = \frac{\theta_H J_x^0}{qn_0\eta(1+\theta_H^2)}$. These two terms are constant so that the electrochemical potential of the stationary state is harmonic: $\nabla^2\mu^{\mathrm{st}} = 0$. The corresponding current [Eq. (12)] is defined as a function of the charge density n. The solution is hence given by Poisson's equation for $\nabla^2\mu^{\mathrm{st}} = 0$,

$$\lambda_D^2\partial_y^2 \ln\left(1 + \frac{\delta n^{\mathrm{st}}}{n_0}\right) = \frac{\delta n^{\mathrm{st}}}{n_0}. \tag{13}$$

It is still necessary to know two boundary conditions on the density n in order to determine the solution. Once again, these boundary conditions are not explicitly given, but we can use global conditions instead. The first one is given by $\int n\,\mathrm{d}y = n_0 L$, and a second condition is imposed by the expression of the electric field E_y, given by Gauss's law $\vec{\nabla}\cdot\vec{E} = \partial_y E_y = \frac{q}{\varepsilon}\delta n$, at a point y_0 (see Appendix C),

$$E_y(y_0) = -\partial_y V(y_0) = -\frac{q}{2\varepsilon}\int_{-L/2}^{L/2}\delta n(y)\frac{y-y_0}{|y-y_0|}\,\mathrm{d}y + E(\infty) + E(-\infty), \tag{14}$$

whose derivative is nothing but the Poisson's equation. The constant $E(+\infty) + E(-\infty)$ accounts for the electromagnetic environment of the Hall device [$E(\pm\infty) = 0$ in vacuum]. Inserting the stationary solution (12) and the relation (4) for $\partial_y\mu$ gives the final

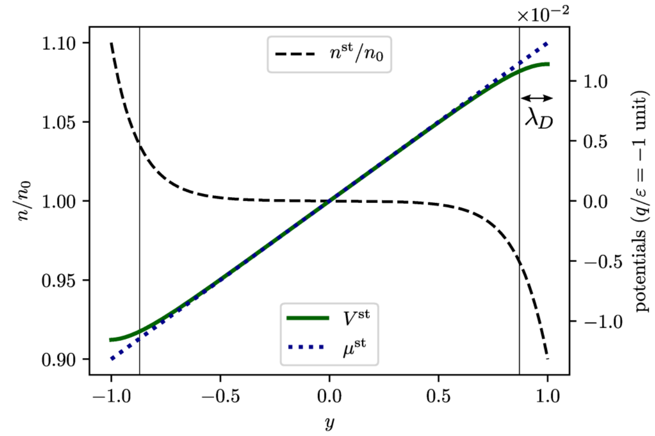

FIG. 2. Profile of the stationary density n^{st}/n_0, voltage V^{st}, and chemical potential μ^{st}. The sample is confined in the region $y \in [-1, 1]$, and the straight vertical lines represent the Debye–Fermi length λ_D. All quantities are dimensionless, for $q/\varepsilon = -1$. The density $n^{st}(y)$ is a solution of Eq. (13) for $\delta n^{st}(\pm L/2) = \pm 0.1$. The electric potential $V^{st}(y)$ is a solution of Eq. (14) for $V(0) = 0$, and the chemical potential μ^{st} is calculated from Eq. (1).

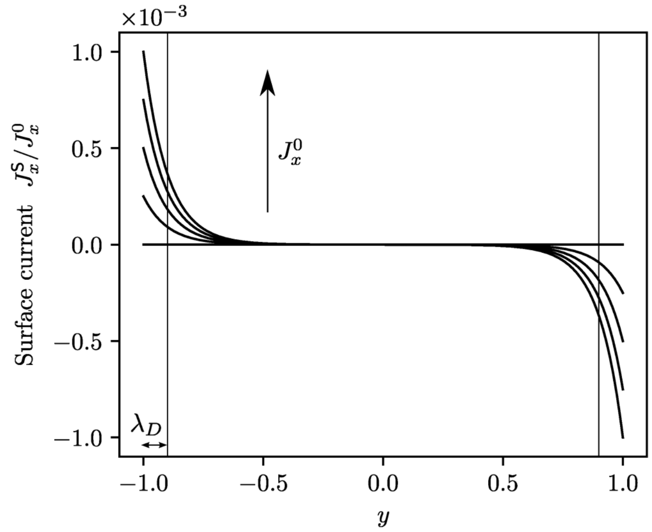

FIG. 3. Numerical solutions of Eq. (15) for the surface currents. The five profiles of the ratio $J_x^S(y)/J_x^0$ correspond to increasing values of J_x^0, or equivalently of the Hall angles θ_H, while C_0 and λ_D are fixed, and $C_E = 0$.

condition (see Appendix C),

$$\frac{2\theta_H J_x^0 C_0}{1+\theta_H^2} + 2\lambda_D^2 \partial_y \ln\left(\frac{n^{st}}{n_0}\right)(y_0) + 2C_E + \int_{-L/2}^{L/2} \delta n^{st}(y) \frac{y-y_0}{|y-y_0|}\, dy = 0, \tag{15}$$

where $C_0 = \frac{\varepsilon}{q^2 n_0 \eta}$ and $C_E = \frac{\varepsilon(E(\infty)+E(-\infty))}{q n_0}$. The sign of $(\delta n^{st}(y))(y-y_0)$ is fixed by the sign of $\theta_H J_x^0$ meaning that the side where $\delta n^{st} > 0$ is fixed by the direction of the current in x and by the magnetic field. Using this condition and fixing n_0 gives a unique solution for n^{st} and the surface currents [Eq. (12)] are now fully determined.

The typical characteristics of the stationary state are plotted in Figs. 2 and 3. The stationary charge accumulation, voltage, and chemical potential are shown in Fig. 2. Note that the minimum power is reached when the voltage $V^{st}(y)$ (plain line in Fig. 2) compensates the charge accumulation $n^{st}(y)$ (dashed line in Fig. 2) in order to maintain a linear chemical potential $\mu^{st}(y)$ (dotted line in Fig. 2) throughout the Hall bar. The surface currents associated with this state—deduced from Eqs. (15) and (12)—are plotted in Fig. 3. The calculation is performed for a disordered semiconductor (typically Ge) at room temperature. The ratio $J_x^S(y)/J_x^0 = 10^{-3}$ is obtained for a current density J_x^0 of the order of $10\,\mathrm{mA/m^2}$ and $\theta_H \approx 0.1$. The Debye length is typically $\lambda_D \approx 1\,\mu$m.

IV. EXPRESSION OF THE SURFACE CURRENTS

The linearization of Eqs. (13) and (15) when $\delta n^{st} \ll n_0$ gives an analytical solution of the problem. Indeed, the solution of the linearized Poisson equation reads $\frac{\delta n^{st}}{n_0} = A e^{\frac{y}{\lambda_D}} + B e^{-\frac{y}{\lambda_D}}$, where A and B are constants. The condition of neutrality ($\int \delta n^{st} dy = 0$) imposes that $A = -B$, and we can use the condition [Eq. (15)] to determine B at $y_0 = 0$,

$$\frac{2\theta_H J_x^0 C_0}{n_0(1+\theta_H^2)} + 2B\lambda_D + 2C_E + B\int_{-L/2}^{+L/2} \left(e^{\frac{y}{\lambda_D}} - e^{\frac{-y}{\lambda_D}}\right)\frac{y}{|y|}\, dy = 0, \tag{16}$$

after linearizing the $\ln(n/n_0)$ term. Thus, after integration and simplification, we have

$$B = -\left(\frac{\theta_H J_x^0 C_0}{(1+\theta_H^2) n_0 \lambda_D} + \frac{C_E}{\lambda_D}\right)\left(\frac{1}{e^{\frac{L}{2\lambda_D}} + e^{\frac{-L}{2\lambda_D}}}\right), \tag{17}$$

and the charge accumulation reads

$$\frac{\delta n^{st}}{n_0} = \frac{1}{\lambda_D}\left(\frac{J_x^0\, C_0 \theta_H}{n_0(1+\theta_H^2)} + C_E\right)\frac{e^{\frac{-y}{\lambda_D}} - e^{\frac{y}{\lambda_D}}}{e^{\frac{-L}{2\lambda_D}} + e^{\frac{L}{2\lambda_D}}}, \tag{18}$$

which leads, through the stationary solution [Eq. (12)] $J_x^{st} = J_x^0 \frac{n}{n_0}$, to the expression of a *surface current* $J_x^S(y) = J_x^{st} - J_x^0$ superimposed to the galvanostatic current J_x^0,

$$J_x^S(y) = -J_x^0 \frac{1}{\lambda_D}\left(\frac{J_x^0\, C_0 \theta_H}{n_0(1+\theta_H^2)} + C_E\right)\frac{\sinh\left(\frac{y}{\lambda_D}\right)}{\cosh\left(\frac{L}{2\lambda_D}\right)}. \tag{19}$$

For a vanishing screening length $\lambda_D \to 0$ and $E(\pm\infty) = 0$, the charge accumulation Eq. (18) reduces to Dirac distributions at the edges of the Hall bar,

$$q\,\delta n^{st} = Q^S\left\{\delta\left(y - \frac{L}{2}\right) - \delta\left(y + \frac{L}{2}\right)\right\}, \tag{20}$$

where Q^S is the surface charge,

$$Q^S = \frac{qJ_x^0 C_0 \theta_H}{1+\theta_H^2}. \tag{21}$$

At this point, all happens as if the system were at "equilibrium" along the y axis, like in a capacitor. When $\theta_H \ll 1$, this surface charge $\pm Q^S$ at $y = \mp \frac{L}{2}$ creates a Hall voltage $V_H = \frac{Q^S L}{\varepsilon} = \frac{\theta_H L J_x^0}{q n_0 \eta}$, and the usual formula for the Hall voltage is recovered by taking $\theta_H = \arctan(\eta H) \simeq \eta H$,

$$V_H = \frac{H J_x^0 L}{q n_0}. \tag{22}$$

Equations (20)–(22) show that the well-known results presented in the textbooks are recovered. However, Eq. (19) also reduces to Dirac distributions at the edges of the Hall bar,

$$J_x^S(y) = J_x^0\left(\frac{J_x^0 C_0\theta_H}{n_0(1+\theta_H^2)} + C_E\right)\left\{\delta\left(y-\frac{L}{2}\right) - \delta\left(y+\frac{L}{2}\right)\right\}. \tag{23}$$

Accordingly, even for vanishing screening length $\lambda_D \longrightarrow 0$ with a charge accumulation δn^{st} confined at the surface of the edges, the corresponding current is still present in the Hall device. The accumulation of electric charges is permanently renewed, maintaining a distribution δn^{st}, whatever the details of the local boundary conditions.

Note that for $C_E = 0$, the surface currents are proportional to the square of the injected current $J_x^S(y) \propto \left(J_x^0\right)^2$.

V. STABILITY

Finally, it is important to verify that the solution defined by Eq. (12) is regular or stable enough so that the stationary state does not depend on variations of the local boundary conditions. This is the case if the system described by the Poisson's equation and the solution of Eq. (6) (obtained without fixing local boundary conditions) can converge uniformly to the minimum defined by Eq. (12). This is indeed the case, as shown below: With a given J_y, J_x, and n, let us define $\epsilon_y = J_y/n$, $\epsilon_x = J_x/n - J_x^0/n_0$, $n_+ = n - n^{st}$ [with n^{st} the solution of Eq. (13)], and $f(y) = \frac{\delta n}{n_0} - \lambda_D^2 \partial_y^2 \ln\left(\frac{n}{n_0}\right)$. It is clear that $f(y)$ tends to 0 when n tends to n^{st}. Furthermore, Poisson's equation now reads

$$\partial_y \epsilon_y + \theta_H \partial_y \epsilon_x = f(y). \tag{24}$$

This equation shows that if both $f(y)$ and ϵ_y tend to 0, then ϵ_x tends to a constant. The galvanostatic constraint shows that this constant is 0. In the same way, if both ϵ_x and ϵ_y tend to 0, then $f(y)$ tends to 0 and n tends to n^{st}. Finally, Eq. (6) now reads

$$\left(n - \frac{\varepsilon}{q^2\eta} f(y)\right)\epsilon_y = \frac{\varepsilon}{2q^2\eta}\left((1+\theta_H^2)\,\partial_y\left(\frac{J_x^0}{n_0} + \epsilon_x\right)^2 + 2\theta_H \frac{J_x^0}{n_0}\left(f(y) - \theta_H\,\partial_y \epsilon_x\right)\right), \tag{25}$$

which shows that if both $f(y)$ and ϵ_x tend to 0, then ϵ_y tends to 0. As a conclusion, despite the fact that the stationary state is out-of-equilibrium (the charge accumulation δn being not static) and that it is described per Eq. (6) (where the solution would necessitate the knowledge of four boundary conditions), this stationary state is fully determined by Eq. (12), i.e., by the solution of the Poisson equation. Once again, apart from the surface currents, all happens as if the system were at equilibrium along the y axis.

VI. CONCLUSION

We have shown that the stationary state of the ideal Hall bar (planar device, translational invariance along x, and symmetric edges) with small charge leakage at the edges can be derived from the principle of least dissipation (i.e., on the second law of thermodynamics) and the global boundary conditions. The stationary state is defined by a vanishing transverse current $J_y = 0$, a harmonic chemical potential $\nabla^2\mu = 0$, and a longitudinal current $J_x(y)$ proportional to the charge accumulation $\delta n(y)$ (see Fig. 3). The usual expression of the continuity equation $\vec{\nabla}\cdot\vec{J} = 0$ is recovered.

Accordingly, the stationary state is now characterized not only by the accumulation of electric charges at the edges—that generates the Hall voltage like in a simple capacitor at equilibrium—but also by surface currents confined at the edges.

These surface currents allow the charge accumulation to be stable so that the Hall voltage is robust to perturbations of the boundary conditions, like that occurring while measuring the Hall voltage. These currents describe the fact that the accumulation of electric charges δn is not at equilibrium but is renewed permanently by the generator. A simple expression of the surface current is given for weak charge accumulation $\delta n/n_0 \ll 1$ in Eq. (19). The surface current is predicted to follow a power two of the injected current and is proportional to the magnetic field at low field. The method developed here can straightforwardly be generalized to the case of sizable charge leakage at the edges, i.e., to the hybrid case between the Hall bar geometry and the Corbino disk geometry. Beyond, the method allows describing devices comprising active interfaces in order to exploit the surface current (instead of the charge accumulation) for new magnetic sensors, actuators, or energy convertors at the nanoscale.

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APPENDIX A: TWOFOLD STATIONARY STATE

The goal of the following calculation is to make explicit the ensemble of states corresponding to the standard local stationarity condition $\vec{\nabla}\cdot\vec{J}=0$. Taking the divergence of the transport equation $\vec{J}=-qn\eta(\vec{\nabla}\mu-\theta_H\,\vec{e_z}\times\vec{\nabla}\mu)$, we have

$$\vec{\nabla}\cdot\vec{J}=-q\eta\,\vec{\nabla}.(n\vec{\nabla}\mu)+q\eta\theta_H\,\vec{\nabla}.(\vec{e_z}\times\vec{\nabla}\mu). \tag{A1}$$

Developing the two divergence terms, we obtain

$$\begin{aligned}\vec{\nabla}\cdot\vec{J}&=-q\eta\,\vec{\nabla}n.\vec{\nabla}\mu-q\eta n\,\nabla^2\mu\\&+q\eta\theta_H\left(n\vec{\nabla}\mu.(\vec{\nabla}\times\vec{e_z})-\vec{e_z}.(\vec{\nabla}\times\vec{\nabla}\mu)\right)\\&+q\eta\theta_H\,\vec{\nabla}n.(\vec{e_z}\times\vec{\nabla}\mu).\end{aligned} \tag{A2}$$

Note that the two curls in the second line of Eq. (A2) vanish. Using the symmetry of the Hall bar, the local stationary condition reduces to

$$\vec{\nabla}\cdot\vec{J}=-q\eta\left(n\partial_y^2\mu+\partial_y n(\partial_y\mu-\theta_H\partial_x\mu)\right)=0. \tag{A3}$$

The solution $\partial_y^2\mu=0$, $\partial_y n\neq 0$, and $\partial_y\mu=\theta_H\partial_x\mu$ holds. This state corresponds to $J_y=0$ and the stationary state discussed in the article. However, we can also have $\partial_y n\neq 0$, $J_y\neq 0$, $\partial_y^2\mu\neq 0$, and

$$n\partial_y^2\mu=-\partial_y n(\partial_y\mu-\theta_H\partial_x\mu). \tag{A4}$$

It is shown in the article that the condition [Eq. (A4)] does not correspond to the minimum of the dissipated power compatible with the constraints applied to the system.

APPENDIX B: DIFFERENTIAL EQUATION

In order to investigate the minimum dissipated power compatible with the constraints, we have to study the functional equation (5), which includes the corresponding Lagrange multipliers β and γ,

$$F[n,\partial_x\mu,\partial_y\mu]=\int\left(qn\eta\|\vec{\nabla}\mu\|^2-\beta qn(-\eta\partial_x\mu-\eta_H\partial_y\mu)-\gamma\left(\partial_y^2\mu-n_0\lambda_D^2\frac{q}{\epsilon}\partial_y^2\ln\left(\frac{n}{n_0}\right)+\frac{q\delta n}{\epsilon}\right)\right)\mathrm{d}y.$$

In what follows, we will use several forms of the transport equation $\vec{J}=-qn\hat{\eta}\vec{\nabla}\mu$,

$$J_x=-qn(\eta\partial_x\mu+\eta_H\partial_y\mu), \tag{B1}$$

$$J_y+\theta_H J_x=-qn\eta(1+\theta_H^2)\partial_y\mu, \tag{B2}$$

$$J_x-\theta_H J_y=-qn\eta(1+\theta_H^2)\partial_x\mu, \tag{B3}$$

$$J_x^2+J_y^2=\|\vec{J}\|^2=(qn\eta)^2(1+\theta_H^2)\|\vec{\nabla}\mu\|^2. \tag{B4}$$

Carrying out the extremization ($\boldsymbol{\delta}$ denoting the functional derivative), we have

$$\begin{aligned}\frac{\boldsymbol{\delta}F}{\boldsymbol{\delta}(\partial_x\mu)}&=0\Longleftrightarrow qn\eta 2\partial_x\mu+\beta qn\eta=0\quad\Longleftrightarrow\quad\beta=-2\partial_x\mu,\\\frac{\boldsymbol{\delta}F}{\boldsymbol{\delta}(\partial_y\mu)}&=0\Longleftrightarrow qn\eta 2\partial_y\mu+\beta qn\eta_H-\partial_y\lambda=0,\end{aligned} \tag{B5}$$

which, combined with Eq. (B5), gives

$$-J_y=qn(\eta\partial_y\mu-\eta_H\partial_x\mu)=\frac{1}{2}\partial_y\gamma. \tag{B6}$$

Finally,

$$\begin{aligned}\frac{\boldsymbol{\delta}F}{\boldsymbol{\delta}(n)}=0&\Longleftrightarrow q\eta\|\vec{\nabla}\mu\|^2+\beta q(\eta\partial_x\mu+\eta_H\partial_y\mu)=\lambda_D^2\frac{q}{\epsilon}\frac{n_0}{n}\partial_y^2\gamma-\frac{q}{\epsilon}\gamma\\&\Longleftrightarrow\eta\|\vec{\nabla}\mu\|^2+2(\partial_x\mu)\frac{J_x}{qn}=\frac{1}{\epsilon}\left(\lambda_D^2\frac{n_0}{n}\partial_y^2\gamma-\gamma\right)\end{aligned}$$

after using Eqs. (B5) and (B1).

Now, inserting Eq. (B6) in $\partial_y^2\gamma$, this last equation becomes

$$-\gamma=\epsilon\eta\|\vec{\nabla}\mu\|^2+\frac{2\epsilon}{qn}J_x\partial_x\mu+2\lambda_D^2\frac{n_0}{n}\partial_y J_y, \tag{B7}$$

and Eq. (B6) then becomes, with the help of relations (B3) and (B4),

$$\begin{aligned}J_y&=\partial_y\left(\frac{\epsilon\eta}{2}\|\vec{\nabla}\mu\|^2+\frac{\epsilon}{qn}J_x\partial_x\mu+\lambda_D^2\frac{n_0}{n}\partial_y J_y\right)\\&=\partial_y\left(\frac{\epsilon\eta}{2}\frac{J_x^2+J_y^2}{q^2n^2\eta^2(1+\theta_H^2)}-\frac{\epsilon}{qn}J_x\frac{J_x-\theta_H J_y}{qn\eta(1+\theta_H^2)}+\lambda_D^2\frac{n_0}{n}\partial_y J_y\right),\end{aligned}$$

which can be simplified to get the equation used in the main text,

$$\lambda_D^2\partial_y\left(\frac{n_0}{n}\partial_y J_y\right)+\frac{\epsilon}{2q^2\eta(1+\theta_H^2)}\partial_y\left(\frac{J_y^2+2\theta_H J_xJ_y-J_x^2}{n^2}\right)-J_y=0. \tag{B8}$$

APPENDIX C: ELECTRIC FIELD

The goal of this Appendix is to derive the condition [Eq. (15)] obtained in Sec. II.

The electric field is determined by Maxwell–Gauss' law from the charge density,

$$\vec{\nabla}\cdot\vec{E} = \underbrace{\partial_x E_x}_{=0} + \partial_y E_y = \frac{q}{\epsilon}\delta n,$$

which can be integrated, relative to a fixed y_0, to obtain

$$E_y(y_0) = E_y(-\infty) + \frac{q}{\epsilon}\int_{-\infty}^{y_0}\delta n(y)\,\mathrm{d}y \quad \text{and}$$
$$E_y(y_0) = E_y(+\infty) - \frac{q}{\epsilon}\int_{y_0}^{+\infty}\delta n(y)\,\mathrm{d}y.$$

By adding these two expressions [and because $\delta n(y) = 0$ outside $[-\ell, +\ell]$], we have

$$E_y(y_0) = E_y(-\infty) + E_y(+\infty) - \frac{q}{2\epsilon}\int_{-\ell}^{+\ell}\delta n(y)\mathrm{sgn}(y-y_0)\,\mathrm{d}y,$$

where $\ell = L/2$ and $\mathrm{sgn}(y-y_0) \equiv \frac{y-y_0}{|y-y_0|}$. The constant $E_y(-\infty) + E_y(+\infty)$ may account for the electromagnetic environment of the Hall device, like a voltmeter. For an isolated conductor under a galvanostatic constraint, this term vanishes. Using the definitions $\mu = n_0\lambda_D^2\frac{q}{\epsilon}\ln\left(\frac{n}{n_0}\right) + V$ and $E_y = -\partial_y V$, as well as the transport equation [Eq. (B2)], we then have

$$-\frac{J_y + \theta_H J_x}{qn\eta(1+\theta_H^2)} - n_0\lambda_D^2\frac{q}{\epsilon}\partial_y\ln\left(\frac{n}{n_0}\right)(y_0) = E_y(-\infty) + E_y(+\infty) + \frac{q}{2\epsilon}\int_{-\ell}^{+\ell}\delta n(y)\mathrm{sgn}(y-y_0)\,\mathrm{d}y. \tag{C1}$$

Thus, under the main result ($J_y = 0$, $J_x = J_x^0\frac{n}{n_0}$), we obtain the condition [Eq. (15)] used in the main text,

$$\frac{2\theta_H J_x^0 C_0}{(1+\theta_H^2)n_0} + 2\lambda_D^2\partial_y\ln\left(\frac{n}{n_0}\right)(y_0) + 2C_E + \int_{-\ell}^{+\ell}\frac{\delta n(y)}{n_0}\mathrm{sgn}(y-y_0)\,\mathrm{d}y = 0, \tag{C2}$$

with $C_0 = \frac{\epsilon}{q^2\eta n_0}$ and $C_E = \frac{\epsilon}{qn_0}(E_y(-\infty) + E_y(+\infty))$.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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