Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

ON THE CONSTRUCTION AND ALGEBRAIC SEMANTICS OF RELEVANCE LOGIC

Autor: Andrea Gastón Codony

Director: Dr. Joan Gispert Braso
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 21 de juny de 2020

Contents

Abstract

The truth-functional interpretation of classical implication gives rise to relevance paradoxes, since it doesn't adequately model our usual understanding of a valid implication, which assumes the antecedent is relevant to the truth of the consequent. This work gives an overview of the system **R** of relevance logic, which aims to avoid said paradoxes. We present the logic **R** with a Hilbert calculus and then prove the Variable-sharing Theorem. We also give an equivalent algebraic semantics for **R** and a semantics for its first-degree entailment fragment.

Resum

La implicació de la lògica clàssica dóna lloc a paradoxes de rellevància, ja que no modela adequadament el que intuïtivament entenem com a implicació vàlida, que assumeix que l'antecedent és rellevant per la veritat del conseqüent. Aquest treball pretén donar una visió general del sistema **R** de lògica de la rellevància, que té com a objectiu evitar aquestes paradoxes. Presentarem la lògica **R** amb un càlcul Hilbert i demostrarem el Teorema de la Variable Compartida. També donarem una semàntica algebraica equivalent per **R** i una semàntica pel seu fragment d'implicacions de primer grau.

2010 Mathematics Subject Classification. 03B47, 03G27

Introduction

We all know about *material* or *classical implication* for its role in mathematics. When we are writing a proof, we use expressions like "imply" or "if ... then", which assume a certain understanding of classical implication. We know that an implication is false only when the antecedent is true and the consequent is not, and we use this in our mathematical endeavours. But not only do we use implication in a mathematical framework, conversation is full of examples of its use in our day to day, as in a parent that warns his son "If it's raining, then you should take an umbrella" or a teacher lecturing his students "If you only study the day before the exam, then you will most probably fail". But if someone told you "If π is transcendental, then $3 \times 3 = 9$", you would respond that this doesn't make an ounce of sense, although it is in fact a true sentence, since the consequent is true. The problem is that "π is transcendental" and "$3 \times 3 = 9$" are sentences with no common meaning, so to us one implying the other is not *relevant*. This irrelevance is why there are no theorems which state that $3 \times 3 = 5$ implies the Bolzano theorem, since we don't consider all true classical implications as valid implications. Thus, we find that classical implication doesn't adequately model our understanding of implication, and that is the starting point of Anderson and Belnap's investigation in [2].

Anderson and Belnap's work on relevance logics centers on their system **E** of entailment, and they develop their logic **E** with an implication that they call *entailment* in which the antecedent is relevant to the truth of the consequent and which is concerned about matters of necessity or possibility (it is a *modal logic*). The book also comments on a system **R** of relevance logic, which doesn't take necessity in regard. The implication in **R** is called *relevant implication*. Our work will be to overview the system **R** of logic following its construction in [2] and giving it adequate semantics. The formal motivation of this work is the aim to avoid the so-called *paradoxes of material implication*:

$$\begin{array}{lr} p \to (q \to p) & \textit{Positive paradox} \\ p \to (\neg p \to q) & \textit{Explosion} \end{array}$$

We are considering these paradoxes arise from relevance concerns, since the Positive paradox would allow us to infer any truth from an aleatory sentence, and Explosion infers any sentence from a contradiction.

We start with an introduction into logic and the tools needed for understanding this work. It also features classical logic as an example of what we're about to present with our system **R**.

The second chapter concerns the construction of our relevance logic **R**. We will work parallelly to the construction of **E** in [2], and since **R** is usually only mentioned in comparison to **E**, we will develop all the proofs and properties which are sometimes not made explicit in Anderson and Belnap's book. The construction will start by presenting the implicational fragment of **R**, to which we will add negation afterwards. With this, we will turn to conjunction and disjunction, but instead of adding them on top of negation, we will construct the first-degree entailment fragment of **R** (implicational formulas with no other implication than the outermost one). The first-degree entailment fragment of **R** and the first-degree entailment fragment of **E** coincide. Finally, in the last section, we will join our negation-implication and first-degree entailment fragments adequately to present our relevance logic **R**. Along the way we will prove some interesting properties, such as the Variable-sharing Theorem, which are important results concerning the adequacy of **R** into the topic of relevance.

The third and last chapter will give semantics to **R** and its first-degree entailment fragment. We have chosen to present semantics for both these logics since the first-degree entailment fragment of **R** has an interesting semantics on its own. For this, we will follow Chapter III in [2], and we will present some required preliminaries on lattices and filters. The semantics we will give to **R** were presented in [10] and are purely algebraic, so we will turn to the work on algebraic semantics in [4] when necessary. Anderson and Belnap's book also give semantics for **R**, so a discussion on their adequacy will finish the chapter.

[2], [10] and [4] have been widely used in this work, although some other references will promptly appear which introduce different tools needed. The proofs which are not fully developed in [2] will be completed and the others referenced. We will add an annex of logical deductions from axioms and rules in order to make the reading of this work more accessible.

Acknowledgements

I want to thank Dr. Joan Gispert for introducing me to this interesting topic which I wasn't acquainted with and for assessing me any time I needed it.

Chapter 1

Preliminary concepts

1.1 Propositional logic

Given a language $\mathcal{L}$ (determining the connectives) we denote the set of formulas of $\mathcal{L}$ as $Fm_{\mathcal{L}}$. The set of formulas is constructed in the usual way: a variable is a formula, a connective of rank 0 (a constant) is a formula, and for every connective $c \in \mathcal{L}$ of rank $k \geqslant 1$ if $\varphi_1, ..., \varphi_k \in Fm_{\mathcal{L}}$, then $c(\varphi_1, ..., \varphi_k)$ is a formula. (The priority in connectives will be the usual: first $\neg$, then $\wedge, \vee$ and then $\rightarrow$, so $p \rightarrow q \vee p = p \rightarrow (q \vee p)$ for example). From now on, we will denote the language of a given logic $\mathscr{L}$ as $\mathcal{L}_{\mathscr{L}}$, the set of variables will be denoted by Var. Also, any assignment $s : Var \longrightarrow Fm_{\mathcal{L}}$ extends into a *substitution* $s : Fm_{\mathcal{L}} \longrightarrow Fm_{\mathcal{L}}$ where for any formula $\varphi(p_1, ..., p_n)$ with variables $p_1, ..., p_n$, $s(\varphi(p_1, ..., p_n)) = \varphi(s(p_1), ..., s(p_n))$.

Definition 1.1.1. *A* ***propositional logic*** *is a pair $\mathscr{L} =< \mathcal{L}, \vdash_{\mathscr{L}}>$ such that $\vdash_{\mathscr{L}}$ is a* ***structural consequence relation*** *i.e. a relation between sets of formulas and formulas which satisfies for every $\Sigma \cup \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:*

Reflexivity $\varphi \vdash_{\mathscr{L}} \varphi$

Monotony *If $\Sigma \vdash_{\mathscr{L}} \varphi$ and $\Sigma \subseteq \Gamma$ then $\Gamma \vdash_{\mathscr{L}} \varphi$*

Cut *If $\Sigma \vdash_{\mathscr{L}} \varphi$ and $\Gamma \vdash_{\mathscr{L}} \psi$ for all $\psi \in \Sigma$ then $\Gamma \vdash_{\mathscr{L}} \varphi$*

With these conditions we have a consequence relation. Moreover, it is ***structural****, so that for every substitution s, if $\Sigma \vdash_{\mathscr{L}} \varphi$ then $s[\Sigma] \vdash_{\mathscr{L}} s(\varphi)$.*

Since we won't work with higher-order logics than propositional logic, any time we say logic we are referring to a propositional logic. Propositional logics can be defined from a syntactical point of view or semantically. We can define them syntactically as follows:

Definition 1.1.2 (Syntactic consequence)**.** *Given a language $\mathcal{L}$, a* ***calculus*** *is a set of inference rules and/or axioms in $Fm_{\mathcal{L}}$. Let $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, a* ***deduction*** *of φ from Σ is a finite sequence of formulas $< \varphi_1, ..., \varphi_{n-1}, \varphi >$ where each φ_i is in Σ, is an axiom or is obtained from previous formulas using an inference rule. We can thus define a* ***deductive system*** *$\mathcal{DS} =< \mathcal{L}, \vdash_{\mathcal{DS}}>$*

*where given $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Sigma \vdash_{\mathcal{DS}} \varphi$ if and only if there is a deduction of φ from Σ. We then say φ is a **syntactic consequence of** Σ. If $\Sigma = \varnothing$, we say φ is a **theorem** of $\mathcal{DS}$.*

Clearly, a deductive system defined this way is a logic. Moreover, since deductions are finite it is also *finitary*, i.e. if $\Sigma \vdash_{\mathscr{L}} \varphi$ then $\Gamma \vdash_{\mathscr{L}} \varphi$ for some finite $\Gamma \subseteq \Sigma$. There are different types of calculi used in the field of logic, and we will work with two: the *Hilbert calculus*, a deductive system with a large number of axioms and small number of inference rules (most times only one), and *natural deduction calculus*, which has no axioms (or sometimes very few).

We can also define a logic from a semantic point of view, behind and idea of what "truth" should be. The basis of semantics is that a reasoning must preserve truth, that is, if the premises are true, then the conclusion is true, just as in classical logic and the truth tables. We now give a possible generalisation of this idea, but first we need to introduce some algebraic concepts:

Definition 1.1.3. *Let $\mathcal{L}$ be the propositional language of a given logic and ω the set of its connectives. A structure $\boldsymbol{A} =< A, \omega^{\boldsymbol{A}} >$ where $\omega^{\boldsymbol{A}} = \{c^{\boldsymbol{A}} : c \in \omega\}$ is called an **$\mathcal{L}$-algebra** if A is a non-empty set (we call it the **universe**) and each $c^{\boldsymbol{A}} \in \omega^{\boldsymbol{A}}$ is an operation on A with the same rank as $c \in \omega$ (a connective of rank 0 is a constant). Let D be a subset of A. We say $\mathcal{A} =< \boldsymbol{A}, D >$ is an **$\mathcal{L}$-matrix**, and we call the elements of D **designated elements**.*

Definition 1.1.4. *Given a mapping $s : Var_{\mathcal{L}} \longrightarrow A$ of the variables of the language into A, we define an **interpretation** $I : Fm_{\mathcal{L}} \longrightarrow A$ by recursion so that:*

- *If $\varphi \in Var_{\mathcal{L}}$, then $I(\varphi) = s(\varphi)$*
- *If $\varphi = c(\varphi_1, ..., \varphi_k)$ for some connective $c \in \omega$ of rank k, then $I(\varphi) = c^{\boldsymbol{A}}(I(\varphi_1), ..., I(\varphi_k))$*

We will define semantic consequence the following way:

Definition 1.1.5 (Semantic consequence)**.** *Let $\mathcal{A}$ be an **$\mathcal{L}$-matrix**, given $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Sigma \vDash_{\mathcal{A}} \varphi$ if for every interpretation $I : Fm_{\mathcal{L}} \longrightarrow A$:*

$$I[\Sigma] \subseteq D \implies I(\varphi) \in D$$

We can generalise this into the class of $\mathcal{L}$-matrices $\mathbb{M}$ the following way: write $\Sigma \vDash_{\mathbb{M}} \varphi$ if for every $\mathcal{A} \in \mathbb{M}$, $\Sigma \vDash_{\mathcal{A}} \varphi$. $< Fm_{\mathcal{L}}, \vDash_{\mathcal{A}} >$ and $< Fm_{\mathcal{L}}, \vDash_{\mathbb{M}} >$ are logics.

1.2 Completeness and consistency

It is interesting to study the relationship between a semantics and a calculus.

Definition 1.2.1. *We say that an **$\mathcal{L}$-matrix** $\mathcal{A}$ (or a class of **$\mathcal{L}$-matrices** $\mathbb{M}$) is a **matrix model** (**class model**) of a given deductive system $\mathcal{DS}$ if for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*

$$\Sigma \vdash_{\mathcal{DS}} \varphi \implies \Sigma \vDash_{\mathcal{A}} \varphi \quad (\Sigma \vDash_{\mathbb{M}} \varphi)$$

*Looking from the other point of view, if this is satisfied we can say $\mathcal{DS}$ is **consistent** with respect to $\mathcal{A}$ ($\mathbb{M}$). If $\Longleftarrow$ is satisfied we can say $\mathcal{DS}$ is **complete** with respect to $\mathcal{A}$ (with respect to $\mathbb{M}$).*

*If $\mathcal{DS}$ is complete and consistent with respect to $\mathcal{A}$ (with respect to $\mathbb{M}$) we say that $\mathcal{A}$ (the class $\mathbb{M}$) is a **matrix semantics** of $\mathcal{DS}$ i.e. if for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*

$$\Sigma \vdash_{\mathcal{DS}} \varphi \iff \Sigma \vDash_{\mathcal{A}} \varphi \quad (\Sigma \vDash_{\mathbb{M}} \varphi)$$

Usually, the matrix semantics of a logic are found by abstracting its algebraic structure. This is done via the Lindembaum algebra of the logic:

Definition 1.2.2. *Let $Fm_{\mathcal{L}}$ be the set of formulas of a given language $\mathcal{L}$ with a set of connectives ω (where $\rightarrow \in \omega$). Given a logic $< \mathcal{L}, \vdash_{\mathscr{L}} >$ we define the relation $\equiv$ on $Fm_{\mathcal{L}}$ to be:*

$$\varphi \equiv \psi \stackrel{def}{\iff} \vdash_{\mathscr{L}} \varphi \rightarrow \psi \text{ and } \vdash_{\mathscr{L}} \psi \rightarrow \varphi$$

*We consider the algebra of sentential formulas of $\mathscr{L}$, $\mathcal{F}_{\mathscr{L}} =< Fm_{\mathscr{L}}, \omega >$. If $\equiv$ is a congruence over $\mathcal{F}_{\mathscr{L}}$, we say ${}^{\mathcal{F}_{\mathscr{L}}}/_{\equiv}$ is the **Lindenbaum algebra** of $\mathscr{L}$.*

Definition 1.2.3. *We will say an algebra $\mathcal{A}$ is **free** in its class of algebras if any identification of the generators of $\mathcal{A}$ into another algebra of the same class can be extended to an **homomorphism** between the two algebras, a mapping which preserves the operations of the class.*

Proving that the Lindenbaum algebra of a logic is free in its variety of algebras is important in the sense that it assures an adequate representation of the Lindenbaum algebra into the variety, because it will satisfy the defining identities.

1.3 Algebraic semantics

But we can go further than matrix semantics in our study of the semantics of a given deductive system, since nothing assures us that a matrix semantics is unique, or that it is the most adequate to represent our deductive system. That is why in [4] a general theory on *algebraizability* is developed.

An $\mathcal{L}$-*equation* (or simply an *equation*) is a formal expression $\varphi \approx \psi$ where $\varphi, \psi \in Fm_{\mathcal{L}}$; the set of all equations is denoted by $Eq_{\mathcal{L}}$. A *quasi-equation* is a formal expression $(\varphi_1 \approx \psi_1)\&...\&(\varphi_n \approx \varphi_n) \implies \varphi \approx \psi$. Let $\varphi(p_1, ..., p_m) \in Fm_{\mathcal{L}}$ and $p_i \in Var$, if $\mathbf{A}$ is an $\mathcal{L}$-algebra, then $\varphi^{\mathbf{A}}(a_1, ..., a_m)$ is the interpretation of φ in $\mathbf{A}$ when the variables and connectives are interpreted as elements and operations of $\mathbf{A}$.

Definition 1.3.1. *Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras, $\Sigma \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$, we will say $\varphi \approx \psi$ is a $\mathbb{K}$-**consequence** of Σ and write $\Sigma \vDash_{\mathbb{K}} \varphi \approx \psi$ if for every $\mathbf{A} \in \mathbb{K}$ and every assignment $s : Var \longrightarrow A$, we have:*

$$\text{If } \xi^{\mathbf{A}}(s(\bar{p}_{\xi})) = \eta^{\mathbf{A}}(s(\bar{p}_{\eta})) \text{ for every } \xi \approx \eta \in \Sigma \text{ then } \varphi^{\mathbf{A}}(s(\bar{p}_{\varphi})) = \psi^{\mathbf{A}}(s(\bar{p}_{\psi}))$$

where we write $\bar{p}$ for the sequence $(p_1, ..., p_m)$ of variables of a given formula, so that $s(\bar{p}) = (s(p_1), ..., s(p_m))$.

Clearly, $\vDash_{\mathbb{K}}$ satisfies reflexivity, monotony and cut. From Chapter 2 in [4], $\vDash_{\mathbb{K}}$ is also structural and therefore it determines a logic. Also, if $\vDash_{\{\mathbf{A}\}} \varphi \approx \psi$, we will say $\varphi \approx \psi$ is an *identity* of $\mathbf{A}$, and if $\{\varphi_1 \approx \psi_1, ..., \varphi_n \approx \varphi_n\} \vDash_{\{\mathbf{A}\}} \varphi \approx \psi$, we will say $(\varphi_1 \approx \psi_1)\&...\&(\varphi_n \approx \varphi_n) \implies \varphi \approx \psi$ is a *quasi-identity* of $\mathbf{A}$. The class of algebras defined as satisfying a given set of identities is a *variety*, and if it satisfies a given set of quasi-identities it is a *quasi-variety*.

Definition 1.3.2. *We say that a class of $\mathcal{L}$-**algebras** $\mathbb{K}$ is an **algebraic semantics** of a given deductive system $\mathcal{DS}$ if there is a system of equations of one variable $\delta(p) \approx \epsilon(p)$ (called the **defining equations**) such that for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*

$$\Sigma \vdash_{\mathcal{DS}} \varphi \iff \{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Sigma\} \vDash_{\mathbb{K}} \delta(\varphi) \approx \epsilon(\varphi)$$

Theorem 1.3.3. [4] *Let $\mathcal{DS}$ be a deductive system, $\mathbb{K}$ a quasi-variety and $\delta(p) \approx \epsilon(p)$ a system of equations, the following are equivalent:*

(i) *$\mathbb{K}$ is an algebraic semantics for $\mathcal{DS}$ with defining equations $\delta(p) \approx \epsilon(p)$.*

(ii) *The class of $\mathcal{L}$-matrices $\mathbb{M} = \{< \mathbf{A}, F_{\mathbf{A}}^{\delta\approx\epsilon} >: \mathbf{A} \in \mathbb{K}\}$ where $F_{\mathbf{A}}^{\delta\approx\epsilon} = \{a \in \mathbf{A} : \delta^{\mathbf{A}}(a) = \epsilon^{\mathbf{A}}(a)\}$ is a matrix semantics for $\mathcal{DS}$.*

Definition 1.3.4. *Let $\mathcal{DS}$ be a deductive system, $\mathbb{K}$ an algebraic semantics for $\mathcal{DS}$ with defining equations $\delta \approx \epsilon$. We say $\mathbb{K}$ is equivalent to $\mathcal{DS}$ if there is a finite system of formulas with two variables Δ such that for every $\varphi \approx \psi \in Eq_{\mathcal{L}}$*

$$\varphi \approx \psi \vDash_{\mathbb{K}}\dashv \delta(\varphi\Delta\psi) \approx \epsilon(\varphi\Delta\psi)$$

*where $\delta(\varphi\Delta\psi) \approx \epsilon(\varphi\Delta\psi) = \{\delta_i(\varphi\Delta_j\psi) \approx \epsilon_i(\varphi\Delta_j\psi) : \delta_i \approx \epsilon_i$ is an equation of the system $\delta \approx \epsilon$ and Δ_j is a formula of the system Δ, for all possible $i, j\}$. The system Δ are the **equivalence formulas**.*

Corollary 1.3.5. [4] *$\mathbb{K}$ is equivalent to $\mathcal{DS}$ with equivalence formulas Δ if and only if for every $\Sigma \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$*

$$\Sigma \vDash_{\mathbb{K}} \varphi \approx \psi \iff \{\xi\Delta\eta : \xi \approx \eta \in \Sigma\} \vdash_{\mathcal{DS}} \varphi\Delta\psi$$

and for every $\xi \in Fm_{\mathcal{L}}$, $\xi \vdash_{\mathcal{DS}}\dashv \delta(\xi)\Delta\epsilon(\xi)$

This indicates that we can interpret $\vdash_{\mathcal{DS}}$ from $\vDash_{\mathbb{K}}$ and conversely, and that these interpretations are essentially inverse, so that $\mathbb{K}$ is the algebraic counterpart of $\mathcal{DS}$. Thus, Blok and Pigozzi [4] propose the following definition:

Definition 1.3.6. *A deductive system is **algebraizable** if it has an equivalent algebraic semantics.*

Moreover, we have the following very desirable property for algebraizable logics:

Theorem 1.3.7. [4] *Every algebraizable logic has a unique equivalent semantics on a quasi-variety.*

Therefore, the equivalent algebraic semantics on a quasi-variety is the algebraic semantics which characterises best the algebraic structure of our deductive system, and by Theorem 1.3.3 we can obtain the most adequate matrix semantics via the analogous algebraic semantics.

1.4 Classical logic as an example

The usual language $\mathcal{L}_{\text{CPL}}$ of classical propositional logic contains connectives $\{\neg, \to, \wedge, \vee\}$. The habitual semantic approach to CPL is given by the interpretations into $\{0,1\}$ and truth-tables. We can see this in the algebraic point of view of the previous sections defining the $\mathcal{L}_{\text{CPL}}$-matrix $\mathcal{B} :=<< \{0,1\}, \neg^*, \wedge^*, \vee^*, \to^*>, \{1\} >$ where the operations are the usual boolean operators. We can then present classical propositional logic as CPL $=< \mathcal{L}_{\text{CPL}}, \vDash_{\mathcal{B}}>$. Whenever two formulas $\varphi, \psi \in Fm_{\mathcal{L}_{\text{CPL}}}$ satisfy that $\varphi \vDash_{\mathcal{B}} \psi$ and $\psi \vDash_{\mathcal{B}} \varphi$ we will say they are *classically equivalent*.

We present now some calculi which determine classical propositional logic. There are many Hilbert calculi for classical propositional logic, but we will use a Russell axiomatisation of the classical propositional calculus (CPC), where only $\to, \neg$ are taken as primitive and the only rule is *modus ponens*, i.e. from $\varphi, \varphi \to \psi$ we infer ψ:

$$\begin{array}{ll}
\text{CPC 1} & \varphi \to (\psi \to \varphi) \\
\text{CPC 2} & (\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi)) \\
\text{CPC 3} & (\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi)) \\
\text{CPC 4} & (\varphi \to \neg\psi) \to (\psi \to \neg\varphi) \\
\text{CPC 5} & (\varphi \to \neg\varphi) \to \neg\varphi \\
\text{CPC 6} & \neg\neg\varphi \to \varphi
\end{array}$$

If needed, we can then define $\varphi \vee \psi := \neg\varphi \to \psi$ and $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$. In [11] [12] we can find its equivalence to an axiomatic system by Łukasiewicz, for which completeness and consistency with respect to $\mathcal{B}$ can be found in [13].

A Fitch-style natural deduction calculus for CPL was developed in [9]. Fitch-style natural deduction calculi consist on opening hypothesis which start an inner subformula and then using rules that can introduce new connectives or eliminate them. Some of these rules can close hypotheses, so that we return to an outer subformula, and a deduction is finished when all hypotheses are closed. Some rules are:

$$\begin{array}{l|ll}
n & \varphi & \\
\vdots & \vdots & \\
m & \varphi \to \psi & \\
m+1 & \psi & \to\text{E}, n, m
\end{array}
\qquad
\begin{array}{l|l|l}
n & \quad \begin{array}{|l} \varphi \\ \hline \end{array} & \text{hyp} \\
\vdots & \quad \vdots & \\
m & \quad \psi & \\
m+1 & \varphi \to \psi & \to\text{I}, n, m
\end{array}$$

Rules of Implication: Elimination ($\to E$) and *Introduction* ($\to I$)
($\to I$ closes hypotheses)

With this,

Theorem 1.4.1. *Let $\Sigma \cup \varphi \subseteq Fm_{\mathcal{L}_{CPL}}$, then $\Sigma \vDash_{\mathcal{B}} \varphi$ if and only if $\Sigma \vdash_{CPC} \varphi$ if and only if $\Sigma \vdash_{FCPL} \varphi$.*

We can also study the algebraic semantics for CPL, which is given by *Boolean algebras*. Blok and Pigozzi proved the algebraizability of classical logic in [5], so we give the definitions necessary to present its equivalent algebraic semantics.

Definition 1.4.2. *A **lattice** is a pair* $< L, \leqslant >$ *such that* L *is a partially ordered set with order* $\leqslant$ *satisfying the following for any* $a, b \in L$*:*

(i) There is a least upper bound (lub) of $\{a, b\}$*, we denote it by* $a \vee b$ *and call it the **join**.*

(ii) There is a greatest lower bound (glb) of $\{a, b\}$*, we denote it by* $a \wedge b$ *and call it the **meet**.*

An equational presentation of lattices can be given by taking $\wedge$ and $\vee$ as primitive operations satisfying idempotence, associativity and commutativity each one and absorption, i.e. $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$. And so a lattice presented this way is an algebra and the class of lattices is a variety. In this case $a \leqslant b \iff a \wedge b = a$. [6]

Definition 1.4.3. *A **distributive lattice** is a lattice* $< L, \leqslant >$ *satisfying that:*

D1 *: For all* $a, b, c \in L$*,* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$*.*

D2 *: For all* $a, b, c \in L$*,* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$*.*

Definition 1.4.4. *A **Boolean algebra** (or **complemented distributive lattice**) is a distributive lattice* $< L, \leqslant >$ *which has a least element (we denote it by* 0*) and a greatest element (we denote it by* 1*) satisfying that for every* $a \in L$*, there is* $\neg a \in L$ *such that* $a \wedge \neg a = 0$ *and* $a \vee \neg a = 1$*. We say* $\neg a$ *is the complement of* a*.*

In [17] Tarski proved that the Lindenbaum algebra of classical propositional logic is a Boolean algebra, and that all the theorems of classical logic belonged to the same class, $1 = [\top] = [p \vee \neg p]$. Also, the Lindenbaum algebra of CPL is the free Boolean algebra.[7] In [5] it is shown that the class of Boolean algebras is an equivalent algebraic semantics for classical propositional logic in the style of Blok and Pigozzi.

Finally, we give some properties which characterise classical propositional logic:

Theorem 1.4.5. [8] *Let* $\Sigma \cup \{\varphi, \psi, \xi\}$ *be a set of formulas in the language of CPL, then:*

Deduction Theorem $\Sigma \cup \{\varphi\} \vdash_{CPC} \psi$ *if and only if* $\Sigma \vdash_{CPC} \varphi \to \psi$

Reductio ad Absurdum $\Sigma \cup \{\varphi\} \vdash_{CPC} \varphi$ *if and only if* $\Sigma \cup \{\neg\varphi\}$ *is inconsistent, i.e. for every formula* ψ $\Sigma \cup \{\neg\varphi\} \vdash_{CPC} \psi$*.*

Left conjunction property $\Sigma \cup \{\varphi, \psi\} \vdash_{CPC} \xi$ *if and only if* $\Sigma \cup \{\varphi \wedge \psi\} \vdash_{CPC} \xi$*.*

Right conjuction property $\Sigma \vdash_{CPC} \varphi \wedge \psi$ *if and only if* $\Sigma \vdash_{CPC} \varphi$ *and* $\Sigma \vdash_{CPC} \psi$*.*

Left disjunction property $\Sigma \cup \{\varphi \vee \psi\} \vdash_{CPC} \xi$ *if and only if* $\Sigma \cup \{\varphi\} \vdash_{CPC} \xi$ *and* $\Sigma \cup \{\psi\} \vdash_{CPC} \xi$*.*

Right disjunction property *If* $\Sigma \vdash_{CPC} \varphi$ *then* $\Sigma \vdash_{CPC} \varphi \vee \psi$ *and* $\Sigma \vdash_{CPC} \psi \vee \varphi$*.*

Chapter 2

Construction of R

As discussed in the introduction, material implication doesn't adequately represent our use of implication, since relevance concerns arise between antecedent and consequent. Our task will be to construct a logic that avoids fallacies of relevance in the sense that for an entailment to hold the antecedent must be relevant to the truth of the consequent. We will call this logic **R**. As aforementioned, **R** must avoid the fallacies of material implication:

$$\begin{array}{lr} p \to (q \to p) & \textit{Positive paradox} \\ p \to (\neg p \to q) & \textit{Explosion} \end{array}$$

Other classical theorems which we don't consider relevant are $p \wedge \neg p \to q$, $p \to q \vee \neg q$ and $p \to (q \to q)$, the first one representing that if the antecedent is false then the classical implication is always true, and the second and third ones representing that if the consequent is true then the classical implication is true no matter the antecedent. In fact, $p \to (\neg p \to q)$ and $p \wedge \neg p \to q$ are classically equivalent and closely related, as we will see in the third section of this work, but they may not be equivalent in **R** (as it happens, they aren't), and so must be treated separately.

2.1 Axiomatising relevant implication

We start by presenting the implicational fragment, since the problem with relevance arises in implication. We will give the natural deduction and Hilbert presentations in [2], for which equivalence is not proven in the book, since it centers on the system **E** of entailment.

2.1.1 Natural deduction

We define the implicational fragment of our relevance logic, $\mathbf{R}_{\to}$, by giving a Fitch-style natural deduction calculus. What interests us about natural deduction is that it has the rule of *implication introduction* ($\to$I, see page 5), in which you can deduce an implication from its antecedent and consequent. This is done by previously taking the antecedent as a hypothesis and then deducing the consequent under it. With this rule, it is easy to control in what cases an entailment is valid, but this still doesn't guarantee that the antecedent will be relevant to the consequent, since using the classical rule of *repetition* (repeating a formula under the same hypothesis) we could prove the following:

$$
\begin{array}{l|l|l}
1 & \quad \varphi & \text{hyp} \\
2 & \qquad \psi & \text{hyp} \\
3 & \qquad \psi & \text{rep, 2} \\
4 & \quad \psi \rightarrow \psi & \rightarrow\text{I, 2, 3} \\
5 & \varphi \rightarrow (\psi \rightarrow \psi) & \rightarrow\text{I, 1, 4}
\end{array}
$$

But we don't want $\varphi \rightarrow (\psi \rightarrow \psi)$ to be a theorem of this logic, because φ and ψ may have no relation. In classical logic, since $\psi \rightarrow \psi$ is always true, it can be inferred from φ (or in fact any formula), but φ needn't be relevant to the actual deduction of $\psi \rightarrow \psi$. As we see in the example above, even though $\psi \rightarrow \psi$ is deduced under the hypothesis of φ, it is deduced independently of φ. So our aim will be to modify these rules and add others so as to get a deductive system in which only relevant entailments can be deduced. We will add the classical rules of *implication elimination* (→ *E* or *modus ponens*, see 5) and *reiteration* (under a certain hypothesis, repeating a formula that is under an outer hypothesis).

Now, the main issue found is that when the rule of implication introduction is used, the consequent may not have been deduced actually using the hypothesis. Therefore it is proposed to affix some kind of mark to the hypothesis opened, making the rules → E, repetition and reiteration pass on the mark to the formulas deduced using them. This way, we know whether the hypothesis has been used in deducing a given formula. Restricting the use of →I only when the consequent is marked would assure that the hypothesis was actually used in the deduction of the consequent, fulfilling our requirement of relevance. But we might have different hypotheses opened during a deduction, so we need different marks for all of them. Thus, in order to simplify the notation, we will subscript the formulas in a deduction with sets of numbers in the way we detail now. When opening a hypothesis we subscript it with a new singleton like so: $\varphi_{\{k\}}$. In the case of repetition and reiteration, we keep the same subscript. This way we keep the property of "having been deduced using certain hypothesis" when repeating a formula (see Figure 1 in the next page).

The same idea works for implication elimination, where we take the union of the two subscripts as below. Therefore, we are indicating that ψ carries all the hypoteses used in deducing φ and $\varphi \rightarrow \psi$:

$$
\begin{array}{l|ll}
n & \varphi_a & \\
\vdots & \vdots & \\
m & \varphi \rightarrow \psi_b & \\
m+1 & \psi_{a\cup b} & \rightarrow\text{E}, n, m
\end{array}
$$

The case of implication introduction is more complex. First, we restrict its use to the

$$
\begin{array}{l|l|l}
n & \varphi_{\{k\}} & \text{hyp} \\
\vdots & \vdots & \\
m & \psi_a & \\
\vdots & \vdots & \\
r & \psi_a & \text{rep}, m
\end{array}
\qquad
\begin{array}{l|l|l}
n & \varphi_{\{k\}} & \text{hyp} \\
\vdots & \vdots & \\
m & \psi_a & \\
\vdots & \ddots & \\
r & \quad \varphi'_{\{k'\}} & \text{hyp} \\
\vdots & \quad \vdots & \\
s & \quad \psi_a & \text{reit}, m
\end{array}
$$

Figure 1: repetition and reiteration

case where the subscript of the consequent contains the number in the subscript of the hypotheses (in the diagram below, $k \in a$). Then, in the resulting implication, we take out said number, since we have closed the hypothesis.

$$
\begin{array}{l|l|l}
n & \quad \varphi_{\{k\}} & \text{hyp} \\
\vdots & \quad \vdots & \\
m & \quad \psi_a & \\
m+1 & \varphi \to \psi_{a-\{k\}} & \to\text{I}, n, m
\end{array}
$$

With this, we have a Fitch-style natural deduction calculus $F\mathbf{R}_{\to}$ with rules

hyp	opening hypothesis
rep	repetition
reit	reiteration
$\to$ E	implication elimination
$\to$ I	implication introduction

Definition 2.1.1. *Let Σ be a set of formulas. We will say a formula φ* ***follows from*** *Σ in $F\mathbf{R}_{\to}$ and we will write $\Sigma \vdash_{F\mathbf{R}_{\to}} \varphi$ if there is a deduction of φ in $F\mathbf{R}_{\to}$ where all the hypotheses have been closed and the formulas of Σ were taken as premises at the start of the deduction (with an empty subscript).*

2.1.2 Equivalence to Hilbert calculus

We now give a Hilbert-style presentation of this logic $\mathbf{R}_{\to}$, with *modus ponens* as the only rule, and the following axioms:

$\mathbf{R}_{\to}1$	$\varphi \to \varphi$	*identity*
$\mathbf{R}_{\to}2$	$(\varphi \to \psi) \to ((\xi \to \varphi) \to (\xi \to \psi))$	*transitivity*
$\mathbf{R}_{\to}3$	$(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$	*permutation*
$\mathbf{R}_{\to}4$	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	*self-distribution*

To see that the natural deduction and Hilbert-style calculi defined previously determine the same logic we need to prove that $\vdash_{\mathbf{R}_{\to}}$ and $\vdash_{F\mathbf{R}_{\to}}$ are *equivalent*, that is, for every $\Sigma \cup \{\varphi\} \subset Prop(X)$ $\Sigma \vdash_{\mathbf{R}_{\to}} \varphi$ if and only if $\Sigma \vdash_{F\mathbf{R}_{\to}} \varphi$, written $\vdash_{\mathbf{R}_{\to}} = \vdash_{F\mathbf{R}_{\to}}$.

Theorem 2.1.2. *$\vdash_{R_{\to}}$ is equivalent to $\vdash_{FR_{\to}}$.*

Proof. $\boxed{\Longrightarrow}$ To prove this implication we find a deduction of the axioms of $\mathbf{R}_{\to}$ in $\vdash_{F\mathbf{R}_{\to}}$ and we see that $\vdash_{F\mathbf{R}_{\to}}$ satisfies *modus ponens*. Therefore, every deduction in $\vdash_{\mathbf{R}_{\to}}$ will have a valid counterpart in $\vdash_{F\mathbf{R}_{\to}}$ by substituting the axioms by its deduction. We give only deductions for $\mathbf{R}_{\to}1$ and $\mathbf{R}_{\to}4$ so that the reading is more accessible, the others may be found in Appendix A.

1	$\quad \varphi_{\{1\}}$	hyp
2	$\quad \varphi_{\{1\}}$	rep, 1
3	$\varphi \to \varphi$	$\to$I, 1, 2

1	$\quad \varphi \to (\varphi \to \psi)_{\{1\}}$	hyp
2	$\quad\quad \varphi_{\{2\}}$	hyp
3	$\quad\quad \varphi \to (\varphi \to \psi)_{\{1\}}$	reit, 1
4	$\quad\quad \varphi \to \psi_{\{1,2\}}$	$\to$E, 2, 3
5	$\quad\quad \psi_{\{1,2\}}$	$\to$E, 2, 4
6	$\quad \varphi \to \psi_{\{1\}}$	$\to$I, 2, 5
7	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	$\to$I, 1, 6

Finally, *modus ponens* is a special case of the rule $\to$E where both a and b (as in the initial definition) are the empty set. Thus, we conclude $\vdash_{\mathbf{R}_{\to}} \leqslant \vdash_{F\mathbf{R}_{\to}}$ ($\Longrightarrow$ is satisfied).

$\boxed{\Longleftarrow}$ For the proof of the other implication, we define a new calculus $F\mathbf{R}^{*}_{\to}$ whose deductions will be called *quasi-deductions*. We define the concept of *quasi-deduction* as a deduction in $F\mathbf{R}_{\to}$ where an axiom of $\vdash_{\mathbf{R}_{\to}}$ may be introduced, with the empty set as subscript (we remark that said axiom can appear in any subdeduction without restrictions). That is, $F\mathbf{R}^{*}_{\to}$ has the same rules as $F\mathbf{R}_{\to}$ and a rule of axiom introduction. Clearly, $F\mathbf{R}^{*}_{\to}$ and $F\mathbf{R}_{\to}$ are equivalent, since any deduction in $F\mathbf{R}_{\to}$ is a quasi-deduction and given a quasi-deduction, if we substitute every axiom introduced by its deduction in $F\mathbf{R}_{\to}$, this is still a quasi-deduction, but also a deduction in $F\mathbf{R}_{\to}$. Also, we define the *degree* of a subdeduction as the number of hypothesis under which it is included. We will say that a quasi-deduction has *maximum degree n* if its innermost subdeductions are of degree n. For example, the deduction of $\mathbf{R}_{\to}1$ in $\boxed{\Longrightarrow}$ has maximum degree 1, whereas the deduction of $\mathbf{R}_{\to}4$ has maximum degree 2.

Now, we need to prove that $\mathbf{R}_{\to}$ is equivalent to $F\mathbf{R}^{*}_{\to}$. Using induction over the natural numbers we prove the following statement for $n \geqslant 0$:

> (*) Any quasi-deduction with maximum degree n can be expressed as a quasi-deduction without subdeductions (i.e. with maximum degree 0).

Base case ($n = 0$): A quasi-deduction with maximum degree 0 has no subdeductions.

Inductive step: We suppose that if a quasi-deduction has maximum degree $n-1$ it can be expressed as a quasi-deduction without subdeductions (HI1). We take a quasi-deduction with maximum degree n. This quasi-deduction has m subdeductions of degree n for a certain $m \geqslant 1$. By induction over m, we prove that we can reduce the m subdeductions to equivalent ones without opening hypotheses so that the overall deduction has maximum degree $n-1$.

Base case ($m = 1$): We have a subdeduction similar to this one, since a subdeduction can only be closed by the rule $\to I$:

$$
\begin{array}{l|l}
 & \ddots \\
s+1 & \quad \left| \; \underline{\varphi_{1\,\{k\}}} \qquad \text{hyp} \right. \\
\vdots & \quad \left| \; \vdots \right. \\
s+i & \quad \left| \; \varphi_{i\ a_i} \right. \\
\vdots & \quad \left| \; \vdots \right. \\
s+r & \quad \left| \; \varphi_{r\ a_r} \right. \\
s+r+1 & \varphi_1 \to \psi_{a_r - \{k\}}
\end{array}
$$

where $k \in a_r$. We replace every φ_i for φ'_i ($i \geqslant 1$) this way:

- If $k \in a_i$, then $\varphi'_i := \varphi_1 \to \varphi_{i_{a_i - \{k\}}}$
- If $k \notin a_i$, then $\varphi'_i := \varphi_{i_{a_i}}$

We use a third nested induction over r to prove that with this substitution each step in the deduction $< \varphi_1{}'_{\{k\}}, ..., \varphi_r{}'_{a_r} >$ (where we can only use axioms, repetition, reiteration and $\to$E) can be justified without using a hypotheses (in the sense that they are deduced from preceding ones and axioms using the rules available).

Base case ($r = 1$): Since $k \in \{k\}$ we have $\varphi'_1 = \varphi_1 \to \varphi_1$, justified by $\mathbf{R}_{\to}1$.

Inductive step: We suppose the subdeduction has i steps and we suppose if a deduction has $i-1$ steps all of them are justified (HI3). In particular, $< \varphi'_{1\{k\}}, ..., \varphi'_{i-1\,a_{i-1}}, \widehat{\varphi'_{i\,a_i}} >$ is a subdeduction with $i-1$ steps, so using HI3 we can proceed supposing all steps in our deduction but the last one are justified. We take φ'_i and, depending on the rule used originally to deduce φ_i, we have four cases:

- *Reiteration*: Then $k \notin a_i$, so $\varphi'_i = \varphi_{i\ a_i}$, which can still be deduced from reiteration (or repetition) from the same formula as before.
- *Repetition*: Then $\varphi_i = \varphi_j$, $a_i = a_j$, for some $j < i$, so $\varphi'_i = \varphi'_j$, therefore φ'_i is deduced from φ'_j by repetition.
- *Axiom introduction*: Since φ_i is an axiom, then $a_i = \varnothing$, so $\varphi'_i = \varphi_{i\varnothing}$, which is still justified because it is an instance of an axiom.
- *Implication Elimination*: $\varphi_{i\ a_i}$ is deduced from $\varphi_{j\ b}$ and $\varphi_{j'\ c} = \varphi_j \to \varphi_{i\ c}$, where $j, j' < i$ and $a_i = b \cup c$. We must distinguish 4 cases:
  1. $\underline{k \notin b \text{ and } k \notin c}$: then $\varphi'_j = \varphi_{j\ b}$ and $\varphi'_{j'} = \varphi_{j'\ c}$, and using →E we obtain $\varphi_{i\ b\cup c} = \varphi_{i\ a_i} = \varphi'_i$ since $k \notin a_i = b \cup c$.
  2. $\underline{k \notin b \text{ and } k \in c}$: then $\varphi'_j = \varphi_{j\ b}$ and $\varphi'_{j'} = \varphi_1 \to (\varphi_j \to \varphi_i)_{c-\{k\}}$. We add axiom $\mathbf{R}_{\to}3$
  $$(\varphi_1 \to (\varphi_j \to \varphi_i)) \to (\varphi_j \to (\varphi_1 \to \varphi_j))\ \varnothing$$
  so that using →E we obtain $\varphi_j \to (\varphi_1 \to \varphi_i)_{c-\{k\}}$. And finally using again →E we get $\varphi_1 \to \varphi_{i\ b\cup(c-\{k\})} = \varphi_1 \to \varphi_{i\ a_i-\{k\}} = \varphi'_i$ since $k \in a_i = b \cup c$.
  3. $\underline{k \in b \text{ and } k \notin c}$: then $\varphi'_j = \varphi_1 \to \varphi_{j\ b-\{k\}}$ and $\varphi'_{j'} = \varphi_j \to \varphi_{i\ c}$. We add axiom $\mathbf{R}_{\to}2$
  $$(\varphi_j \to \varphi_i) \to ((\varphi_1 \to \varphi_j) \to (\varphi_1 \to \varphi_i))\ \varnothing$$
  so that using →E twice first with $\varphi'_{j'}$ and then with φ'_j we obtain $\varphi_1 \to \varphi_{i\ (b-\{k\})\cup d} = \varphi_1 \to \varphi_{i\ a_i-\{k\}} = \varphi'_i$ since $k \in a_i = b \cup c$.
  4. $\underline{k \in b \text{ and } k \in c}$: we have that $\varphi'_j = \varphi_1 \to \varphi_{j\ b-\{k\}}$ and that $\varphi'_{j'} = \varphi_1 \to (\varphi_j \to \varphi_i)_{c-\{k\}}$. We add axiom $\mathbf{R}_{\to}4$
  $$(\varphi_1 \to (\varphi_j \to \varphi_i)) \to ((\varphi_1 \to \varphi_j) \to (\varphi_1 \to \varphi_i))\ \varnothing$$
  so that using $\to E$ twice first with $\varphi'_{j'}$ and then with φ'_j we obtain $\varphi_1 \to \varphi_{i\ (b-\{k\})\cup(c-\{k\})} = \varphi_1 \to \varphi_{i\ a_i-\{k\}} = \varphi'_i$ since $k \in a_i = b \cup c$.

We have substituted our initial subdeduction for an equivalent one, since φ'_r coincides with the formula for step $s + r + 1$, thus justifying it by repetition.

Inductive step: As hypotesis of induction, we suppose that if we had $m - 1$ subdeductions of degree n, we could find equivalent deductions without opening a hypothesis, integrating them into the subdeduction of degree $n - 1$ (HI2). We suppose we have m subdeductions of degree n, and we consider the first one. We can use the substitution of the basis case in the same way as before so that we have $m - 1$ subdeductions of degree n, which by HI2 have equivalent deductions of degree $n - 1$. Now, our overall *quasi-deduction* has maximum degree $n - 1$.

Since we have proven that we can reduce our *quasi-deduction* to a *quasi-deduction* of maximum degree $n - 1$, by HI1 we can express it as a *quasi-deduction* without subdeductions, ending the proof of the initial statement.

We have proven that if we can deduce a formula in $F\mathbf{R}^*_{\to}$, it can be deduced with a quasi-deduction without subdeductions. We note that without opening any hypothesis, the only rules which can be used are $\to$E, repetition and introduction of axioms and premises. Since $\to$E is *modus ponens*, a quasi-deduction without subdeductions is a valid deduction in $\mathbf{R}_{\to}$. Therefore, $F\mathbf{R}^*_{\to}$ and $\mathbf{R}_{\to}$ are equivalent deduction systems. We can conclude if $\Sigma \vdash_{F\mathbf{R}_{\to}} \varphi$ then $\Sigma \vdash_{\mathbf{R}_{\to}} \varphi$. □

2.1.3 Variable-sharing and fallacies

The first result we get that reassures us that we have taken the right direction in the study of relevance is the *variable-sharing theorem*, which states that for an entailment to be a theorem of $\mathbf{R}_{\to}$ the antecedent and consequent must share a variable. This guarantees that the antecedent and consequent aren't completely unrealated, they must share some kind of meaning. To prove this theorem, we will use some tools of matrix semantics which we introduced in the first chapter.

Theorem 2.1.3 (Variable-sharing in $\mathbf{R}_{\to}$). *Let φ, ψ be implicational formulas. If $\vdash_{\mathbf{R}_{\to}} \varphi \to \psi$, then φ and ψ share a variable.*

Proof. This proof is similar to the one in [2] for $\mathbf{E}_{\to}$; the matrix presented here is the one used in the book, since it is also a model for $\mathbf{E}_{\to}$. We consider the matrix $\mathcal{M}$ with universe $\{0, 1, 2, 3\}$, designated elements $\{2, 3\}$ and a single operation $\to$ defined as in the following table:

$\to$	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2	0	1	2	3
3	0	0	0	3

To prove that $\mathcal{M}$ is a matrix model of $\mathbf{R}_{\to}$ we need only prove that for every interpretation $I : Fm \longrightarrow \{0, 1, 2, 3\}$ (where Fm refers to the set of implicational formulas) and every instance of an axiom ξ, we have $I(\xi) \in \{2, 3\}$, and also that for every $\xi_1, \xi_1 \to \xi_2 \in Fm_{\mathcal{L}}$ if $I(\xi_1) \in \{2, 3\}$ and $I(\xi_1 \to \xi_2) \in \{2, 3\}$ then $I(\xi_2) \in \{2, 3\}$. Therefore if the premises of a deduction are interpreted as 2 or 3, the consequence will be too. The second statement is clear from the previous table, the first one is proven by checking all possible cases, something done using the program I coded in Appendix B.

Now, we suppose that φ and ψ share no variables. Therefore, we can assign the value 3 to all variables of φ and the value 2 to all the variables of ψ, so that for the associated interpretation I, $I(\varphi) = 3$ (since $3 \to 3 = 3$) and $I(\psi) = 2$ (since $2 \to 2 = 2$). But $I(\varphi \to \psi) = I(\psi) \to I(\varphi) = 0$, so $\not\models_{\mathcal{M}} \varphi \to \psi$. Finally, $\mathcal{M}$ is a matrix model of $\mathbf{R}_{\to}$, which makes $\nvdash_{\mathbf{R}_{\to}} \varphi \to \psi$ unprovable in $\mathbf{R}_{\to}$. □

This theorem assures us that $\varphi \to (\psi \to \psi)$, a formula we discussed earlier, isn't a theorem of $R_{\to}$, since φ and $\psi \to \psi$ may not share a variable (as is the case in $p \to (q \to q)$). With this, we can already reject the Positive Paradox $\varphi \to (\psi \to \varphi)$ as being a theorem of $\mathbf{R}_{\to}$, because if it were, using *permutation* we would conclude $\varphi \to (\psi \to \psi)$ is too. The Positive Paradox is the epitome of the fallacy of relevance, since it would enable us to infer something true from anything we wanted. This is the complete opposite of what we are considering, and we are thus satisfied to have rejected the Positive Paradox.

2.2 The negation-implication fragment

The objective of this section is to add negation into our relevant implicational calculus in a way that keeps the classical properties of negation but altogether avoiding paradoxes of implication. We will do so by extending our Hilbert calculus and proving afterwards that we avoid the fallacies. This chapter follows the development of Chapter II in [2] for their calculus of entailment, applying it to our system $\mathbf{R}_{\to}$. In the book, the initial motivation is an axiomatisation due to Ackermann [1], but we have found more straightforward to use an axiomatisation of Russell instead.

2.2.1 Adding negation axioms

We take Russell's axiomatisation in section 1.4 (with the rule of *modus ponens*). This particular axiomatisation is convenient to us since only implication and negation are taken as primitive connectives and, as we have stated in the introduction of this chapter, we want to keep the classical properties of negation. Therefore, we add the negational axioms of this axiomatisation to $\mathbf{R}_{\to}$ (note that CPC1 is the Positive Paradox, so we are avoiding it):

$$\begin{array}{ll} \text{Ax 1} & (\varphi \to \neg\psi) \to (\psi \to \neg\varphi) \\ \text{Ax 2} & (\varphi \to \neg\varphi) \to \neg\varphi \\ \text{Ax 3} & \neg\neg\varphi \to \varphi \end{array}$$

But this axiomatisation is not *independent*, that is, there are axioms we can deduce from others. We are going to prove we can take out Ax 2 or $\mathbf{R}_{\to}4$ and the resulting axiomatisations are equivalent.

Lemma 2.2.1. *$\mathbf{R}_{\to}1-4+$ Ax 1-3, $\mathbf{R}_{\to}1-4+$ Ax 1,3 and $\mathbf{R}_{\to}1-3+$ Ax 1-3 with the rule modus ponens are equivalent axiomatisations.*

Proof. Firstly, we prove with $\mathbf{R}_{\to}1-4$, Ax 1,3 we can deduce Ax 2:

$$\begin{array}{rll}
1 & (\varphi \to \neg\varphi) \to (\varphi \to \neg\varphi) & \mathbf{R}_{\to}1 \\
2 & ((\varphi \to \neg\varphi) \to (\varphi \to \neg\varphi)) \to (\varphi \to ((\varphi \to \neg\varphi) \to \neg\varphi)) & \mathbf{R}_{\to}3 \\
3 & \varphi \to ((\varphi \to \neg\varphi) \to \neg\varphi) & 1,2-\text{MP} \\
4 & ((\varphi \to \neg\varphi) \to \neg\varphi) \to (\varphi \to \neg(\varphi \to \neg\varphi)) & \text{Ax 1} \\
5 & ((\varphi \to \neg\varphi) \to \neg\varphi) \to (\varphi \to \neg(\varphi \to \neg\varphi)) \to & \mathbf{R}_{\to}2 \\
 & \quad ((\varphi \to ((\varphi \to \neg\varphi) \to \neg\varphi)) \to (\varphi \to (\varphi \to \neg(\varphi \to \neg\varphi)))) & \\
6 & \varphi \to (\varphi \to \neg(\varphi \to \neg\varphi)) & 3,4,5-\text{MP}(\times 2) \\
7 & (\varphi \to (\varphi \to \neg(\varphi \to \neg\varphi))) \to (\varphi \to \neg(\varphi \to \neg\varphi)) & \mathbf{R}_{\to}4 \\
8 & \varphi \to \neg(\varphi \to \neg\varphi) & 6,7-\text{MP} \\
9 & (\varphi \to \neg(\varphi \to \neg\varphi)) \to ((\varphi \to \neg\varphi) \to \neg\varphi) & \text{Ax 1} \\
10 & (\varphi \to \neg\varphi) \to \neg\varphi & 8,9-\text{MP}
\end{array}$$

Secondly, we need to prove with $\mathbf{R}_{\to}1-3$, Ax 1-3 we can deduce $\mathbf{R}_{\to}4$. A proof that $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi), (\psi \to \varphi) \to (\neg\varphi \to \neg\psi)$ and $\varphi \to \neg\neg\varphi$ can be deduced from

$\mathbf{R}_{\to}1-3$, Ax 1-3 can be found in Appendix A. Using this, we give a deduction of $\mathbf{R}_{\to}4$ with the axioms chosen:

1	$((\varphi \to \neg\varphi) \to \neg\varphi) \to ((\neg\psi \to (\varphi \to \neg\varphi)) \to (\neg\psi \to \neg\varphi))$	$\mathbf{R}_{\to}2$
2	$(\varphi \to \neg\varphi) \to \neg\varphi$	Ax 2
3	$(\neg\psi \to (\varphi \to \neg\varphi)) \to (\neg\psi \to \neg\varphi)$	$1,2-\mathrm{MP}$
4	$(\varphi \to (\neg\psi \to \neg\varphi)) \to (\neg\psi \to (\varphi \to \neg\varphi))$	$\mathbf{R}_{\to}3$
5	$(\varphi \to (\neg\psi \to \neg\varphi)) \to (\neg\psi \to \neg\varphi)$	$3,4,\mathbf{R}_{\to}2-\mathrm{MP}(\times 2)$
6	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	*Thm*
7	$(\varphi \to (\neg\psi \to \neg\varphi)) \to (\psi \to \varphi)$	$5,6,\mathbf{R}_{\to}2-\mathrm{MP}(\times 2)$
8	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	*Thm*
9	$((\varphi \to \psi) \to (\neg\psi \to \neg\varphi)) \to$ $((\varphi \to (\varphi \to \psi)) \to (\varphi \to (\neg\psi \to \neg\varphi)))$	$\mathbf{R}_{\to}2$
10	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to (\neg\psi \to \neg\varphi))$	$8,9-\mathrm{MP}$
11	$(\varphi \to (\varphi \to \psi)) \to (\psi \to \varphi)$	$7,10-\mathrm{MP}$

where in 5 and 7 the adequate instance of $\mathbf{R}2$ must be taken. Since all the axioms from one of the axiomatisations can be deduced from the others, and the only rule is *modus ponens*, all axiomatisations are equivalent. □

For ease in the proofs of the next section, we take the following axiomatisation of $\mathbf{R}_{\underset{\to}{\neg}}$, with the rule of *modus ponens* together with the following axioms:

$\mathbf{R}_{\underset{\to}{\neg}}1$	$\varphi \to \varphi$	*identity*
$\mathbf{R}_{\underset{\to}{\neg}}2$	$(\varphi \to \psi) \to ((\xi \to \varphi) \to (\xi \to \psi))$	*transitivity*
$\mathbf{R}_{\underset{\to}{\neg}}3$	$(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$	*permutation*
$\mathbf{R}_{\underset{\to}{\neg}}4$	$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$	*contraposition*
$\mathbf{R}_{\underset{\to}{\neg}}5$	$(\varphi \to \neg\varphi) \to \neg\varphi$	
$\mathbf{R}_{\underset{\to}{\neg}}6$	$\neg\neg\varphi \to \varphi$	

2.2.2 Equivalence to natural deduction

In this section, we will add natural deduction rules for negation to F$\mathbf{R}$. We do so by substituting each axiom we have added by a rule. For example, in the case of axiom $\mathbf{R}_{\underset{\to}{\neg}}6$ we need to be able to infer φ from $\neg\neg\varphi$, we thus get the rule of *negation elimination*:

$$\begin{array}{l|ll} s & \neg\neg\varphi_a & \\ s+1 & \varphi_a & \neg\neg\mathrm{E},\ s \end{array}$$

in which the subindex is kept since φ carries all hypotheses used in deducing $\neg\neg\varphi$. $\mathbf{R}_{\underset{\to}{\neg}}5$ allows us to infer $\neg\varphi$ from $\varphi \to \neg\varphi$, which translates into natural deduction by deducing $\neg\varphi$ in the case that $\neg\varphi$ can be deduced *from* φ (in the relevant sense of *from*). The resulting rule is called *negation introduction*:

$$
\begin{array}{l|l|l}
r & \quad \begin{array}{|l}\varphi_{\{k\}} \\ \hline \end{array} & \text{hyp} \\
\vdots & \quad \begin{array}{|l}\vdots\end{array} & \\
s & \quad \begin{array}{|l}\neg\varphi_a\end{array} & \\
s+1 & \neg\varphi_{a-\{k\}} & \neg\text{I}, r, s
\end{array}
$$

We observe that for the rule to be used, k must be in a, since otherwise the deduction of $\neg\varphi$ under hypothesis φ is not relevant. At last, and in the same way as before, the axiom for contraposition is translated into a natural deduction rule this way:

$$
\begin{array}{l|l|l}
t & \psi_b & \\
\vdots & \quad \vdots & \\
r & \quad \begin{array}{|l}\varphi_{\{k\}} \\ \hline \end{array} & \text{hyp} \\
\vdots & \quad \begin{array}{|l}\vdots\end{array} & \\
s & \quad \begin{array}{|l}\neg\psi_a\end{array} & \\
s+1 & \neg\varphi_{a\cup b-\{k\}} & \text{contrap}, t, r, s
\end{array}
$$

In the end, we have added three new rules to the ones we already had: negation introduction ($\neg$I), contraposition (contrap) and double negation elimination ($\neg\neg E$). Now, we must prove the equivalence between natural deduction and Hilbert calculus:

Lemma 2.2.2. *$\vdash_{\mathbf{R}^{\neg}_{\rightarrow}}$ is equivalent to $\vdash_{F\mathbf{R}^{\neg}_{\rightarrow}}$.*

Proof. First, we prove $F\mathbf{R}^{*\neg}_{\rightarrow}$ (a deduction system with the rules of $F\mathbf{R}_{\rightarrow}$ in which the axioms of $\mathbf{R}^{\neg}_{\rightarrow}$ can be introduced with subscript $\varnothing$) is equivalent to $F\mathbf{R}^{\neg}_{\rightarrow}$:

$\boxed{\Longleftarrow}$ $\neg I$, $\neg\neg E$ and contraposition are valid in $F\mathbf{R}^{*\neg}_{\rightarrow}$:

$$
\begin{array}{l|l|l}
1 & \quad \begin{array}{|l}\varphi_{\{k\}} \\ \hline \end{array} & \text{hyp} \\
\vdots & \quad \begin{array}{|l}\vdots\end{array} & \\
s & \quad \begin{array}{|l}\neg\varphi_a\end{array} & \\
s+1 & \varphi \rightarrow \neg\varphi_{a-\{k\}} & \rightarrow\text{I}, 1, s \\
s+2 & (\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi & \text{axiom} \\
s+3 & \neg\varphi_{a-\{k\}} & \rightarrow\text{E}, s+1, s+2
\end{array}
$$

We proved negation introduction as an example, the proofs of the other rules may be found in Appendix A. Therefore if $\Sigma \vdash_{F\mathbf{R}^{\neg}_{\rightarrow}} \varphi$ we can find a deduction of $\Sigma \vdash_{F\mathbf{R}^{*\neg}_{\rightarrow}} \varphi$ by substituting the occurrences of these three rules.

$\boxed{\Longrightarrow}$ $\mathbf{R}^{\neg}_{\rightarrow}4-6$ can be proven in $F\mathbf{R}^{\neg}_{\rightarrow}$. We give $\mathbf{R}^{\neg}_{\rightarrow}4$ as example (for the others, see Appendix A).

1	$\varphi \rightarrow \neg\psi_{\{1\}}$	hyp
2	$\quad \psi_{\{2\}}$	hyp
3	$\quad\quad \varphi_{\{3\}}$	hyp
4	$\quad\quad \neg\psi_{\{1,3\}}$	$\rightarrow$E, 1, 3
5	$\quad \neg\varphi_{\{1,2\}}$	contrap, 2, 3, 4
6	$\psi \rightarrow \neg\varphi_{\{1\}}$	$\rightarrow$I, 2, 5
7	$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$	$\rightarrow$I, 1, 6

Since $F\mathbf{R}^{\neg}_{\rightarrow}$ has every rule in $F\mathbf{R}_{\rightarrow}$, then $\mathbf{R}^{\neg}_{\rightarrow}1-3$ can also be proven in $F\mathbf{R}^{\neg}_{\rightarrow}$. Therefore, if $\Sigma \vdash_{F\mathbf{R}^{*\neg}_{\rightarrow}} \varphi$, by substituting each axiom introduction by its deduction in $F\mathbf{R}^{\neg}_{\rightarrow}$, we get a deduction of φ in $F\mathbf{R}^{\neg}_{\rightarrow}$, thus proving the other direction.

Since all new axioms are implicational, one may use the same proof as in 2.1.2 to show that $F\mathbf{R}^{*\neg}_{\rightarrow}$ is equivalent to $\mathbf{R}^{\neg}_{\rightarrow}$, hence concluding this proof. □

2.2.3 Fallacies avoided

Definition 2.2.3. *Let $\mathcal{L}, \mathcal{L}^*$ be propositional languages, and $< Fm_{\mathcal{L}}, \vdash >$ and $< Fm_{\mathcal{L}^*}, \vdash^* >$ be two logics, we say $\vdash$ is an* ***expansion*** *of $\vdash^*$ if $\vdash$ is an extension of $\vdash^*$ (i.e. if $\Sigma \vdash^* \varphi$ then $\Sigma \vdash \varphi$) and $\mathcal{L}^* \subsetneq \mathcal{L}$, that is, if the language is broadened. Moreover, we say the expansion is* ***conservative*** *if $\vdash$ proves no new theorems in the language of $\mathcal{L}^*$, that is, if $\varphi \in Fm_{\mathcal{L}^*}$, then $\vdash \varphi$ implies $\vdash^* \varphi$.*

Checking that every expansion in the steps to constructing our relevance logic is conservative is enormously important, since conservativity assures we are not adding any unknown theorems or metalogical properties which would trump the properties of our initial logic. We are constructing our relevance logic starting from implication and working our way up, and so being able to ensure conservativity is necessary.

Theorem 2.2.4. *$\mathbf{R}^{\neg}_{\rightarrow}$ is a conservative expansion of $\mathbf{R}_{\rightarrow}$.*

Proof. That it is an expansion is clear from the presentation of both logics, but that it is conservative falls out of the scope of this work, so we won't reproduce any proof of it. A proof can be found in [2], (Theorem 2 in page 146). □

What we have stated assures us that the Positive Paradox isn't a theorem of $\mathbf{R}^{\neg}_{\rightarrow}$ and, more importantly, since we don't obtain new implicational theorems, only the properties of implicational formulas that we already had are present in $\mathbf{R}^{\neg}_{\rightarrow}$. For example, Theorem 2.1.3 still holds for implicational formulas of $\mathbf{R}^{\neg}_{\rightarrow}$.

Moreover, conservativity assures us that the paradox of Explosion is not a theorem of $\mathbf{R}_{\overrightarrow{\neg}}$, since if it were we could deduce the Positive Paradox, as in the deduction at the right.

1	$p \to (\neg p \to \neg q)$	Explosion
2	$(\neg p \to \neg q) \to (q \to p)$	Thm
3	$p \to (q \to p)$	1,2, $\mathbf{R}_{\overrightarrow{\neg}}$2-MP

2.3 First-degree entailments

A *first-degree entailment* (fde) is an implicational formula $\varphi \to \psi$ where neither φ nor ψ have any $\to$. In this section we are going to construct the fragment of **R** that contains the first degree entailments, we will denote it by $\mathbf{R}_{fde}$. With this, we avoid the nesting of implications while adding disjunction and conjunction, allowing us to treat $\to$ more like a consequence relation than a connective, the same way the Deduction Theorem (Theorem 1.4.5) allows us to do so in classical logic. We will start by characterising the first degree entailments and deciding which ones satisfy the relevance criteria by comparing them to their classical counterparts. Then, we will find the equivalent Hilbert-style calculus for $\mathbf{R}_{fde}$. Since the first-degree entailment fragment of **R** is the same as the one for the calculus of entailment developed by Anderson and Belnap, we will follow Chapter III in [2].

2.3.1 Motivation and first characterisation

Definition 2.3.1. *We give the necessary preliminary definitions:*

- *We will say a formula consisting of a variable or its negate is a* ***literal****.*
- *A formula $\varphi_1 \wedge ... \wedge \varphi_n$ where each φ_i is a literal is called a* ***primitive conjunction****.*
- *A formula $\varphi_1 \vee ... \vee \varphi_n$ where each φ_i is a literal is called a* ***primitive disjunction****.*
- *A formula $\varphi \to \psi$ where φ is a primitive conjunction and ψ is a primitive disjunction is called a* ***primitive implication****.*

Remark 2.3.2. We use the notation $\varphi_1 \wedge ... \wedge \varphi_n$ as $(...(\varphi_1 \wedge \varphi_2) \wedge ...\varphi_{n-1}) \wedge \varphi_n$, and the same for disjunction. We will see further into this work it is not necessary to incur in this distinction.

We now pause to determine the validity of a primitive implication. First, we study the case of a formula $\varphi \to \psi$ where both φ and ψ are literals. φ and ψ have each one variable, so we consider two cases: if these variables are the same or not. If the variables are different, we don't want the formula to be valid since the variable-sharing theorem wouldn't hold. If the variables are the same we have four possible formulas:

1	$p \to p$
2	$p \to \neg p$
3	$\neg p \to p$
4	$\neg p \to \neg p$

In order to keep axiom $\mathbf{R}_{\overrightarrow{\neg}}2$ we need 1 and 4 to be valid formulas. If 3 was a theorem of $\mathbf{R}_{\overrightarrow{\neg}}$, by means of axiom $\mathbf{R}_{\overrightarrow{\neg}}5$, 2 would be too. And if 2 were a theorem, since we know Ax 2 (in the previous section) is too, then $\neg p$ would be a theorem of $\mathbf{R}_{\overrightarrow{\neg}}$. Therefore, we discard 2 and 3. Hence, we will have $\varphi \to \psi$ valid if and only if $\varphi = \psi$. In keeping with this condition, we give the first characterisation of validity in $\mathbf{R}_{fde}$:

Definition 2.3.3. *We will say a primitive implication $\varphi \to \psi$ is **explicitly tautological** if one of the literals of φ is identical to one of the literals of ψ.*

In classical logic, a primitive implication $\varphi_1 \wedge ... \wedge \varphi_n \to \psi_1 \vee ... \vee \psi_m$ would have to satisfy one of the following criteria in order to be a tautology:

(i) φ has a contradiction, i.e. for some i, j and a variable p, $\varphi_i = p$ and $\varphi_j = \neg p$.

(ii) ψ has an excluded middle, i.e. for some i, j and a variable p, $\psi_i = p$ and $\psi_j = \neg p$.

(iii) φ and ψ share an literal.

We observe that, by only taking 3, we have avoided the relevance issues that arise in 1 and 2. Therefore, formulas as $p \wedge \neg p \to q$ or $p \to q \vee \neg q$ aren't valid in our system, although they are classical tautologies.

We now go on to define a more general form of first-degree entailment:

Definition 2.3.4. *We will say a first degree entailment is in **normal form** if it is a formula $\varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m$ where $n, m \geqslant 1$, every φ_i is a primitive conjunction and every ψ_j is a primitive disjunction.*

Remark 2.3.5. If $n = m = 1$, the formula is a primitive implication. Also, we say a formula like $\varphi_1 \vee ... \vee \varphi_n$ (a disjunction of conjunctions of literals) is in *disjunctive normal form* and a formula as $\psi_1 \wedge ... \wedge \psi_m$ (a conjunction of disjunctions of literals) is in *conjunctive normal form*.

To extend validity to all fde's in normal form, we turn to the classical conjunction and disjunction properties (in Theorem 1.4.5) which together with the Deduction Theorem (in Theorem 1.4.5) give:

$$\vdash_{\mathrm{CPC}} \varphi \to \psi \wedge \xi \iff \vdash_{\mathrm{CPC}} \varphi \to \psi \text{ and } \vdash_{\mathrm{CPC}} \varphi \to \xi$$
$$\vdash_{\mathrm{CPC}} \varphi \vee \psi \to \xi \iff \vdash_{\mathrm{CPC}} \varphi \to \xi \text{ and } \vdash_{\mathrm{CPC}} \psi \to \xi$$

Since these are sound in regards to relevance, we will take the following properties to be true for fde's:

$\varphi \to \psi \wedge \xi$ is explicitly tautological if and only if $\varphi \to \psi$ and $\varphi \to \xi$ are too.
$\varphi \vee \psi \to \xi$ is explicitly tautological if and only if $\varphi \to \psi$ and $\varphi \to \xi$ are too.

The previous properties extend to fde's in normal form naturally:

Definition 2.3.6. *We will say an fde $\varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m$ in normal form is **explicitly tautological** if for every $i, j \leqslant m$ $\varphi_i \to \psi_j$ is explicitly tautological, i.e. φ_i and ψ_j share a literal.*

Clearly, the case of primitive implication still holds. The last step is to extend explicit tautologyhood to all first-degree entailments. We have chosen this particular characterisation of fde's (the normal form) because in classical logic any formula can be expressed as a classically equivalent one in conjunctive or disjunctive normal form. This is achieved by using certain classical equivalences; we give a list of the ones referring to disjunction, conjunction and negation:

Commutation: $\varphi \wedge \psi \vDash\dashv \psi \wedge \varphi$, $\varphi \vee \psi \vDash\dashv \psi \vee \varphi$

Association: $(\varphi \wedge \psi) \wedge \xi \vDash\dashv \varphi \wedge (\psi \wedge \xi)$, $(\varphi \vee \psi) \vee \xi \vDash\dashv \varphi \vee (\psi \vee \xi)$

Distribution: $\varphi \wedge (\psi \vee \xi) \vDash\dashv (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$, $\varphi \vee (\psi \wedge \xi) \vDash\dashv (\varphi \vee \psi) \wedge (\varphi \vee \xi)$

Double negation: $\neg\neg\varphi \vDash\dashv \varphi$

De Morgan laws: $\neg(\varphi \wedge \psi) \vDash\dashv \neg\varphi \vee \neg\psi$, $\neg(\varphi \vee \psi) \vDash\dashv \neg\varphi \wedge \neg\psi$

A short examination of these assures us they are relevantly sound, in the sense that for us a deduction of $\varphi \vee \psi$ from $\psi \vee \varphi$, for example, would be relevant. We can now consider these classical equivalences as substitution rules taking $\vDash\dashv$ to mean "we can substitute ... by ... or viceversa and the resulting formula is *equivalent*". We define these rules for non-implicational formulas. For example, the rules of association would be:

$$\frac{(\varphi \wedge \psi) \wedge \xi}{\varphi \wedge (\psi \wedge \xi)} \qquad \frac{\varphi \wedge (\psi \wedge \xi)}{(\varphi \wedge \psi) \wedge \xi} \qquad \frac{(\varphi \vee \psi) \vee \xi}{\varphi \vee (\psi \vee \xi)} \qquad \frac{\varphi \vee (\psi \vee \xi)}{(\varphi \vee \psi) \vee \xi}$$

Using these rules, any non-implicational formula can be expressed as an equivalent formula in disjunctive normal form or conjunctive normal form. Therefore, we will say any fde has an *equivalent normal form*, found substituting the antecedent and consequent by equivalent disjunctive and conjunctive normal forms respectively. For example, an equivalent normal form for $\neg(p \vee \neg q) \to q \vee (p \wedge r)$ would be $\neg p \wedge q \to (q \vee p) \wedge (q \vee r)$.

Remark 2.3.7. We note that we are using the term "equivalence" in two different contexts: when referring to equivalence in classical logic, we mean classically equivalent in the sense of the first chapter; when referring to equivalence in fde's (in the context of defining validity in $\mathbf{R}_{\text{fde}}$), we mean two formulas where one is generated by the other by means of the rules available and conversely.

Remark 2.3.8. We remit to Remark 2.3.2 to note that because of the association rule the distinction we made is avoided in the context of this section, since any correct positioning of parentheses would determine an equivalent formula.

Definition 2.3.9. *We will say an fde is a* ***tautological implication*** *if it has an equivalent normal form which is explicitly tautological.*

An fde may have more than one equivalent normal form, although its normal forms $\varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m$ only differ in the order of φ's and ψ's and the literals in them – for example, $p \vee q \to r$ and $q \vee p \to r$ are equivalent. Therefore, a formula is explicitly tautological if and only if all its equivalent normal forms are explicitly tautological. In conclusion, we only need to find one non-explicitly tautological normal form equivalent to an *fde* to disprove its validity.

2.3.2 An equivalent Hilbert calculus

We present now a Hilbert-style calculus for $\mathbf{R}_{\text{fde}}$:

	Axioms	Rules
Implication		IR: From $\varphi \to \psi$ and $\psi \to \xi$ we infer $\varphi \to \xi$
Conjunction	$\mathbf{R}_{fde}1$: $\varphi \wedge \psi \to \varphi$ $\mathbf{R}_{fde}2$: $\varphi \wedge \psi \to \psi$	CR: From $\varphi \to \psi$ and $\varphi \to \xi$ we infer $\varphi \to \psi \wedge \xi$
Disjunction	$\mathbf{R}_{fde}3$: $\varphi \to \varphi \vee \psi$ $\mathbf{R}_{fde}4$: $\varphi \to \psi \vee \varphi$	DR: From $\varphi \to \xi$ and $\psi \to \xi$ we infer $\varphi \vee \psi \to \xi$
Distribution	$\mathbf{R}_{fde}5$: $\varphi \wedge (\psi \vee \xi)$ $\to (\varphi \wedge \psi) \vee \xi$	
Negation	$\mathbf{R}_{fde}6$: $\varphi \to \neg\neg\varphi$ $\mathbf{R}_{fde}7$: $\neg\neg\varphi \to \varphi$	NR: From $\varphi \to \psi$ we infer $\neg\psi \to \neg\varphi$

The following result states the equivalence of this axiomatisation with the system in the previous section:

Proposition 2.3.10. *Let φ be first-degree entailment, then $\vdash_{\mathbf{R}_{fde}} \varphi$ if and only if φ is a tautological implication.*

Proof. $\boxed{\Longleftarrow}$ First, we prove that every explicitly tautological primitive implication is a theorem of $\mathbf{R}_{fde}$. Let $\varphi_1 \wedge ... \wedge \varphi_n \to \psi_1 \vee ... \vee \psi_m$ be an explicitly tautological primitive implication. Therefore, there exist $i \leqslant n$, $j \leqslant m$ such that $\varphi_i = \psi_j$. Since $\vdash_{\mathbf{R}_{fde}} \varphi \to \neg\neg\varphi$ and $\vdash_{\mathbf{R}_{fde}} \neg\neg\varphi \to \varphi$, using IR we get $\vdash_{\mathbf{R}_{fde}} \varphi \to \varphi$. With this, we obtain $\vdash_{\mathbf{R}_{fde}} \varphi_i \to \psi_j$, so:

1	$\varphi_i \to \psi_j$	Thm
2	$(\varphi_1 \wedge ... \wedge \varphi_{i-1}) \wedge \varphi_i \to \varphi_i$	$\mathbf{R}_{fde}2$
3	$(\varphi_1 \wedge ... \wedge \varphi_{i-1}) \wedge \varphi_i \to \psi_j$	$1, 2 -$ IR
4	$(\varphi_1 \wedge ... \wedge \varphi_i) \wedge (\varphi_{i+1} \wedge ... \wedge \varphi_n) \to \varphi_1 \wedge ... \wedge \varphi_i$	$\mathbf{R}_{fde}1$
5	$\varphi_1 \wedge ... \wedge \varphi_n \to \psi_j$	$3, 4 -$ IR
6	$\psi_j \to (\psi_1 \vee ... \vee \psi_{j-1}) \vee \psi_j$	$\mathbf{R}_{fde}4$
7	$\varphi_1 \wedge ... \wedge \varphi_n \to \psi_1 \vee ... \vee \psi_j$	$5, 6 -$ IR
8	$\psi_1 \vee ... \vee \psi_j \to (\psi_1 \vee ... \vee \psi_j) \vee (\psi_{j+1} \vee ... \vee \psi_m)$	$\mathbf{R}_{fde}3$
9	$\varphi_1 \wedge ... \wedge \varphi_n \to \psi_1 \vee ... \vee \psi_m$	$7, 8 -$ IR

Secondly, we prove that every explicitly tautological fde in normal form is a theorem of $\mathbf{R}_{fde}$. Let $\varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m$ in normal form be explicitly tautological. Therefore, for every $i \leqslant n$, $j \leqslant m$ we have $\vdash_{\mathbf{R}_{fde}} \varphi_i \to \psi_j$, since the formula is explicitly tautological. By using DR, we get $\vdash_{\mathbf{R}_{fde}} \varphi_1 \vee ... \vee \varphi_n \to \psi_j$ for every $j \leqslant m$ and through CR we conclude $\vdash_{\mathbf{R}_{fde}} \varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m$, in a similar way as for primitive implications.

Finally, we prove that the substitution rules we use to find equivalent normal forms are theorems of $\mathbf{R}_{\mathrm{fde}}$. As an example, we prove the case of one of the De Morgan laws:

1	$\neg\varphi \to \neg\varphi \vee \neg\psi$	$\mathbf{R}_{\mathrm{fde}}3$
2	$\neg(\neg\varphi \vee \neg\psi) \to \neg\neg\varphi$	$1 - \mathrm{NR}$
3	$\neg\neg\varphi \to \varphi$	$\mathbf{R}_{\mathrm{fde}}7$
4	$\neg(\neg\varphi \vee \neg\psi) \to \varphi$	$2,3 - \mathrm{IR}$
5	$\neg\psi \to \neg\varphi \vee \neg\psi$	$\mathbf{R}_{\mathrm{fde}}3$
6	$\neg(\neg\varphi \vee \neg\psi) \to \neg\neg\psi$	$5 - \mathrm{NR}$
7	$\neg\neg\psi \to \psi$	$\mathbf{R}_{\mathrm{fde}}7$
8	$\neg(\neg\varphi \vee \neg\psi) \to \psi$	$6,7 - \mathrm{IR}$
9	$\neg(\neg\varphi \vee \neg\psi) \to \varphi \wedge \psi$	$4,8 - \mathrm{CR}$
10	$\neg(\varphi \wedge \psi) \to \neg\neg(\neg\varphi \vee \neg\psi)$	$9 - \mathrm{NR}$
11	$\neg\neg(\neg\varphi \vee \neg\psi) \to \neg\varphi \vee \neg\psi$	$\mathbf{R}_{\mathrm{fde}}7$
12	$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$	$10,11 - \mathrm{IR}$

1	$\varphi \wedge \psi \to \psi$	$\mathbf{R}_{\mathrm{fde}}2$
2	$\neg\psi \to \neg(\varphi \wedge \psi)$	$1 - \mathrm{NR}$
3	$\varphi \wedge \psi \to \varphi$	$\mathbf{R}_{\mathrm{fde}}1$
4	$\neg\varphi \to \neg(\varphi \wedge \psi)$	$3 - \mathrm{NR}$
5	$\neg\varphi \vee \neg\psi \to \neg(\varphi \wedge \psi)$	$2,4 - \mathrm{DR}$

To end this proof, we show:

Lemma 2.3.11 (Replacement)**.** *Let φ, ψ_1, ψ_2 be non-implicational formulas and p be a variable of φ. If $\vdash_{\mathbf{R}_{fde}} \psi_1 \to \psi_2$ and $\vdash_{\mathbf{R}_{fde}} \psi_2 \to \psi_1$, then*

$$\vdash_{\mathbf{R}_{fde}} \varphi\left(\frac{\psi_1}{p}\right) \to \varphi\left(\frac{\psi_2}{p}\right) \text{ and } \vdash_{\mathbf{R}_{fde}} \varphi\left(\frac{\psi_2}{p}\right) \to \varphi\left(\frac{\psi_1}{p}\right)$$

(where $\varphi\left(\frac{\psi_1}{p}\right)$ and $\varphi\left(\frac{\psi_2}{p}\right)$ are obtained by substituting the same instance of p in φ by ψ_1 and ψ_2 respectively)

Proof. (*of lemma*) (For ease in the notation, we denote "$\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi_1 \to \xi_2$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi_2 \to \xi_1$" by "$\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi_1 \leftrightarrows \xi_2$"). Let φ_1, φ_2 be such that $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi_1 \leftrightarrows \psi_2$. By induction over φ:

* If φ is a variable, then $\varphi = p$, and therefore $\varphi\left(\frac{\psi_1}{p}\right) = \psi_1$ and $\varphi\left(\frac{\psi_2}{p}\right) = \psi_2$, by hypothesis, we obtain what we wanted.

* If $\varphi = \neg\varphi'$ and suppose $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi'\left(\frac{\psi_1}{p}\right) \leftrightarrows \varphi'\left(\frac{\psi_2}{p}\right)$ (HI). Using the rule of negation (NR),
$$\vdash_{\mathbf{R}_{\mathrm{fde}}} \neg\varphi'\left(\frac{\psi_1}{p}\right) \leftrightarrows \neg\varphi'\left(\frac{\psi_2}{p}\right)$$

* Let $\varphi = \varphi_1 \wedge \varphi_2$ and suppose $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi_1\left(\frac{\psi_1}{p}\right) \leftrightarrows \varphi_1\left(\frac{\psi_2}{p}\right)$ if p appears in φ_1 and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi_2\left(\frac{\psi_1}{p}\right) \leftrightarrows \varphi_2\left(\frac{\psi_2}{p}\right)$ if p appears in φ_2 (HI). Then since the lemma states we only substitute one instance of p in the formula:
$$(\varphi_1 \wedge \varphi_2)\left(\frac{\psi_1}{p}\right) = \begin{cases} \varphi_1\left(\frac{\psi_1}{p}\right) \wedge \varphi_2 & \text{if } p \text{ is substituted in } \varphi_1 \\ \varphi_1 \wedge \varphi_2\left(\frac{\psi_1}{p}\right) & \text{if } p \text{ is substituted in } \varphi_2 \end{cases}$$
and
$$(\varphi_1 \wedge \varphi_2)\left(\frac{\psi_2}{p}\right) = \begin{cases} \varphi_1\left(\frac{\psi_2}{p}\right) \wedge \varphi_2 & \text{if } p \text{ is substituted in } \varphi_1 \\ \varphi_1 \wedge \varphi_2\left(\frac{\psi_2}{p}\right) & \text{if } p \text{ is substituted in } \varphi_2 \end{cases}$$

We prove the equivalence in the first case, the other is proven analogously:

1	$\varphi_1\left(\frac{\psi_1}{p}\right) \wedge \varphi_2 \to \varphi_1\left(\frac{\psi_1}{p}\right)$	$\mathbf{R}_{\text{fde}}1$	1	$\varphi_1\left(\frac{\psi_2}{p}\right) \wedge \varphi_2 \to \varphi_1\left(\frac{\psi_2}{p}\right)$	$\mathbf{R}_{\text{fde}}1$
2	$\varphi_1\left(\frac{\psi_1}{p}\right) \to \varphi_1\left(\frac{\psi_2}{p}\right)$	HI	2	$\varphi_1\left(\frac{\psi_2}{p}\right) \to \varphi_1\left(\frac{\psi_1}{p}\right)$	HI
3	$\varphi_1\left(\frac{\psi_1}{p}\right) \wedge \varphi_2 \to \varphi_1\left(\frac{\psi_2}{p}\right)$	$1,2-$ IR	3	$\varphi_1\left(\frac{\psi_2}{p}\right) \wedge \varphi_2 \to \varphi_1\left(\frac{\psi_1}{p}\right)$	$1,2-$ IR
4	$\varphi_1\left(\frac{\psi_1}{p}\right) \wedge \varphi_2 \to \varphi_2$	$\mathbf{R}_{\text{fde}}1$	4	$\varphi_1\left(\frac{\psi_2}{p}\right) \wedge \varphi_2 \to \varphi_2$	$\mathbf{R}_{\text{fde}}1$
5	$\varphi_1\left(\frac{\psi_1}{p}\right) \wedge \varphi_2 \to \varphi_1\left(\frac{\psi_2}{p}\right) \wedge \varphi_2$	$3,4-$ CR	5	$\varphi_1\left(\frac{\psi_2}{p}\right) \wedge \varphi_2 \to \varphi_1\left(\frac{\psi_1}{p}\right) \wedge \varphi_2$	$3,4-$ CR

* The case where $\varphi = \varphi_1 \vee \varphi_2$ is analogous to the previous one, using DR instead of CR.

□

Remark 2.3.12. With this lemma and the fact that $\vdash_{\mathbf{R}_{\text{fde}}} \varphi \wedge (\psi \wedge \xi) \leftrightarrows (\varphi \wedge \psi) \wedge \xi$ and the same for disjunction, the distinction in Remark 2.3.2 is avoided in $\mathbf{R}_{\text{fde}}$.

This lemma and rule IR, conclude that if $\vdash_{\mathbf{R}_{\text{fde}}} \psi_1 \leftrightarrows \psi_2$, then

$$\vdash_{\mathbf{R}_{\text{fde}}} \varphi_1\left(\frac{\psi_1}{p}\right) \to \varphi_2\left(\frac{\psi_1}{p}\right) \iff \vdash_{\mathbf{R}_{\text{fde}}} \varphi_1\left(\frac{\psi_2}{p}\right) \to \varphi_2\left(\frac{\psi_2}{p}\right)$$

therefore the use of the substitution rules preserves the theoremhood of formulas, proving that if φ is a tautological implication, then $\vdash_{\mathbf{R}_{fde}} \varphi$, since we know every explicitly tautological implication in normal form is a theorem of $\mathbf{R}_{fde}$.

$\boxed{\Longrightarrow}$ We need to prove that the axioms are tautological entailments and that the rules preserve this property.

□ *Rule of conjunction*

Let $\varphi \to \psi$ and $\varphi \to \xi$ be tautological entailments, we need to prove $\varphi \to \psi \wedge \xi$ is too. We express all formulas in equivalent normal forms:

$$\begin{aligned}
\varphi \to \psi &= \varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m \\
\varphi \to \xi &= \varphi_1 \vee ... \vee \varphi_n \to \xi_1 \wedge ... \wedge \xi_k \\
\varphi \to \psi \wedge \xi &= \varphi_1 \vee ... \vee \varphi_n \to \psi_1 \wedge ... \wedge \psi_m \wedge \xi_1 \wedge ... \wedge \xi_k
\end{aligned}$$

Clearly $\varphi \to \psi \wedge \xi$ is a tautological entailment bacause for all $i = 1, .., n$ φ_i shares a literal with every ψ_j and $\xi_{j'}$.

□ *Rule of disjunction*

Let $\varphi \to \xi$ and $\psi \to \xi$ be tautological entailments, we need to prove $\varphi \vee \psi \to \xi$ is too. We express all formulas in equivalent normal forms:

$$\begin{array}{rcccl}
\varphi \to \xi & = & \varphi_1 \vee ... \vee \varphi_n & \to \xi_1 \wedge ... \wedge \xi_k \\
\psi \to \xi & = & \psi_1 \vee ... \vee \psi_m & \to \xi_1 \wedge ... \wedge \xi_k \\
\varphi \vee \psi \to \xi & = & \varphi_1 \vee ... \vee \varphi_n \vee \psi_1 \vee ... \vee \psi_m & \to \xi_1 \wedge ... \wedge \xi_k
\end{array}$$

Since any ξ_i shares a literal with φ_j or $\psi_{j'}$ for all possible j, j', clearly $\varphi \to \psi \wedge \xi$ is a tautological entailment.

▫ *Rule of negation*

Let $\varphi \to \psi$ be a tautological entailment, we need to prove $\neg\psi \to \neg\varphi$ is too. This relies on the fact that if φ is in disjunctive normal form (or conjuntive normal form), then by changing $\wedge$'s into $\vee$'s and viceversa and the literals into their negates or taking out negation, we obtain a conjuntive normal form (disjunctive normal form) for $\neg\varphi$. We show the first case, but first we prove by induction over $n \geqslant 1$ that

> (*) if $\psi_1 \wedge ... \wedge \psi_n$ is a primitive conjunction, by changing $\wedge$ into $\vee$ and $\neg p$ into p and conversely we obtain a primitive disjunction equivalent to $\neg(\psi_1 \wedge ... \wedge \psi_n)$ (in the sense of being able to obtain it from the rules)

- **Base case ($n = 1$):** We have the literal ψ_1. Therefore, $\psi_1 = p$ or $\psi_1 = \neg p$ for some variable p in the language of $\mathbf{R}_{\text{fde}}$. In the first case, it is clear that $\neg p$ is an equivalent primitive disjunction for $\neg\psi_1$, and in the second case p is equivalent to $\neg\psi_1 = \neg\neg p$ by using double negation.
- **Inductive step:** We take the primitive conjunction $\psi_1 \wedge ... \wedge \psi_n \wedge \psi_{n+1}$ and we suppose that $\neg(\psi_1 \wedge ... \wedge \psi_n)$ has an equivalent primitive disjunction ψ by changing $\wedge$ into $\vee$ and $\neg p$ into p and conversely (HI). $\neg(\psi_1 \wedge ... \wedge \psi_n \wedge \psi_{n+1})$ is equivalent to $\neg((\psi_1 \wedge ... \wedge \psi_n) \wedge \psi_{n+1})$, which by using De Morgan gives $\neg(\psi_1 \wedge ... \wedge \psi_n) \vee \neg\psi_{n+1}$. By using HI, this formula is in turn equivalent to $\psi \vee \neg\psi_{n+1}$. Now, ψ_{n+1} is a literal, therefore as in the base case either $\neg p$ or p is equivalent to $\neg\psi_{n+1}$, giving us the equivalent formula we wanted.

Lastly, we show by induction over $n \geqslant 1$ that if $\varphi_1 \vee ... \vee \varphi_n$ is in disjunctive normal form, making the transformation described previously we obtain a conjuntive normal form for $\neg(\varphi_1 \vee ... \vee \varphi_n)$:

- **Base case ($n = 1$):** We have φ_1 which is a primitive conjunction, that is, $\varphi_1 = \psi_1 \wedge ... \wedge \psi_m$ where each ψ_i is a literal. Therefore, the base case is reduced to (*).
- **Inductive step:** We take $\varphi_1 \vee ... \vee \varphi_n \vee \varphi_{n+1}$ in disjunctive normal form and suppose $\neg(\varphi_1 \vee ... \vee \varphi_n)$ has an equivalent conjunctive normal form φ using the indicated transformation (HI). Since φ_{n+1} is a primitive conjunction, because of (*) there is a primitive disjunction φ^* equivalent to $\neg\varphi_{n+1}$. Therefore, $\varphi \wedge \varphi^*$ is equivalent to $\neg(\varphi_1 \vee ... \vee \varphi_n)$ and satisfies that it is obtained from $\varphi_1 \vee ... \vee \varphi_n$ by using the indicated transformation.

The same result for conjunctive normal forms can be obtained analogously. With this, it is clear that if $\varphi \to \psi$ is a tautological entailment, $\neg\psi \to \neg\varphi$ is too, since if the normal form for $\varphi \to \psi$ is explicitly tautological, the normal form for $\neg\psi \to \neg\varphi$ found through the changes would be too.

▫ *Rule of implication*

First, we observe that $\varphi \to \varphi$ is a tautological implication, since we can easily prove by induction over n and using Distribution that if we have the conjunctive normal form of φ below then we get its disjunctive normal form like so:

$$\wedge_{i=1}^{n}\varphi_i = \wedge_{i=1}^{n}(\varphi_1^i \vee ... \vee \varphi_{n_i}^i) \qquad \vee_{j=1}^{i_1 \cdot ... \cdot i_n} \varphi_j' = \vee_{j_i \leqslant n_i}(\varphi_{j_1}^1 \wedge ... \wedge \varphi_{j_n}^n)$$

(taking all possible conjunctions of elements each from one of the terms in the conjunctive normal form). So for every possible i and j, $\varphi_{j_i}^i$ in $\varphi_{j_1}^1 \vee ... \vee \varphi_{j_n}^n$ will be in $\varphi_1^i \wedge ... \wedge \varphi_{n_i}^i$, making $\varphi \to \varphi$ a tautological entailment.

Let $\varphi \to \psi$ and $\psi \to \xi$ be tautological entailments, we need to prove $\varphi \to \xi$ is too. First, we express all the formulas in normal form:

$$\begin{array}{rcccl} \varphi \to \psi & = & \varphi_1 \vee ... \vee \varphi_n & \to & \psi_1 \wedge ... \wedge \psi_m \\ \psi \to \xi & = & \psi_1^* \vee ... \vee \psi_{m*}^* & \to & \xi_1 \wedge ... \wedge \xi_k \\ \varphi \to \xi & = & \varphi_1 \vee ... \vee \varphi_n & \to & \xi_1 \wedge ... \wedge \xi_k \end{array}$$

Each φ_i must share a literal with each ξ_j for $\varphi \to \xi$ to be a tautological entailment. Let i be such that $1 \leqslant i \leqslant n$, then φ_i shares a literal ψ_j^i with ψ_j for every $j = 1, ..., m$, so $\psi_1^i \wedge ... \wedge \psi_m^i$ is a subconjunction of φ_i. We see that it is also one of the terms ψ_l^* for some $l \leqslant m^*$, since the terms of $\psi_1^* \vee ... \vee \psi_{m*}^*$ are generated taking conjunctions of elements each from one of the terms in $\psi_1 \wedge ... \wedge \psi_m$ (in all possible combinations). As $\psi \to \xi$ is a tautological entailment, each ξ_j for $j = 1, ..., k$ shares a literal with ψ_l^*, and in turn with φ_i, which was what we wanted.

▫ *Axioms*

With the initial observation in the proof for the rule of implication, we can prove that the axioms are tautological entailments. $\mathbf{R}_{\text{fde}}1 - 4$ are very similar, so we only prove $\mathbf{R}1$. We do it by induction over the number of terms of the disjunctive (conjunctive if it is $\mathbf{R}_{\text{fde}}3, 4$) form of ψ. The base case is that ψ is equivalent to a primitive conjunction and so (following the same notation as in the observation) $(\varphi_1 \vee .. \vee \varphi_n) \wedge \psi$ is equivalent by Distribution to $(\varphi_1 \wedge \psi) \vee .. \vee (\varphi_n \wedge \psi)$ which is in disjunctive normal form. Since for all possible i and j, φ_i and φ_j' share a literal, $\varphi_i \wedge \psi$ and φ_j' do too. For the inductive step we suppose that if ψ' is equivalent to $\psi_1 \vee .. \vee \psi_{m-1}$ then $\varphi \wedge \varphi' \to \varphi$ is a tautological implication. Now, if ψ is equivalent to $\psi_1 \vee ... \vee \psi_m$, then taking $\psi' = \psi_1 \vee .. \vee \psi_{m-1}$ we have that $\varphi \wedge \psi$ is equivalent to $(\varphi \wedge \psi') \vee (\varphi \wedge \psi_m)$ by Distribution. By hypothesis of induction and the base case, and since we have proven the rule of disjunction (conjunction if it is $\mathbf{R}_{\text{fde}}3, 4$) works for tautological entailments, $(\varphi \wedge \psi') \vee (\varphi \wedge \psi_m) \to \psi$ is a tautological implication, and as it has an explicitly tautological normal form and it also is equivalent to $\varphi \wedge \psi \to \varphi$ then $\varphi \wedge \psi \to \varphi$ is a tautological implication.

To prove $\mathbf{R}_{\text{fde}}5$ is a tautological implication we first observe that it is equivalent by Distribution to $\varphi \wedge (\psi \vee \xi) \to (\varphi \vee \xi) \wedge (\psi \vee \xi)$. We have proven that $\varphi \wedge (\psi \vee \xi) \to \varphi$, $\varphi \wedge (\psi \vee \xi) \to \psi \vee \xi$ and $\varphi \to \varphi \vee \xi$ are tautological implications, and since we already have that the rules preserve this property by IR $\varphi \wedge (\psi \vee \xi) \to \varphi \vee \xi$ is a tautological implication and finally by CR $\varphi \wedge (\psi \vee \xi) \to (\varphi \vee \xi) \wedge (\psi \vee \xi)$ is a tautological implication, so $\mathbf{R}5$ is too.

$\mathbf{R}_{\text{fde}}6, 7$ are easy to see, since by Double Negation $\neg\neg\varphi$ is equivalent to φ, so with the initial observation in the proof of IR we are done. ▫

2.4 Syntax of R

2.4.1 Presentation of the axioms

We first present the axiomatisation of the logic we have finally constructed, and afterwards we will discuss the choice of axioms and rules:

	AXIOMS
R1	$\varphi \to \varphi$
R2	$(\varphi \to \psi) \to ((\xi \to \varphi) \to (\xi \to \psi))$
R3	$(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$
R4	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
R5	$\varphi \wedge \psi \to \varphi$
R6	$\varphi \wedge \psi \to \psi$
R7	$(\varphi \to \psi) \wedge (\varphi \to \xi) \to (\varphi \to \psi \wedge \xi)$
R8	$\varphi \to \varphi \vee \psi$
R9	$\psi \to \varphi \vee \psi$
R10	$(\varphi \to \xi) \wedge (\psi \to \xi) \to (\varphi \vee \psi \to \xi)$
R11	$\varphi \wedge (\psi \vee \xi) \to (\varphi \wedge \psi) \vee \xi$
R12	$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$
R13	$\neg\neg\varphi \to \varphi$

RULES

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{MP, } \textit{modus ponens}$$

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ \text{\&I, adjunction}$$

If **R** didn't extend the logics which we have discussed until now, we wouldn't have taken adequate axioms, so we need to check that:

Proposition 2.4.1. *$\mathbf{R}$ is an expansion of $\mathbf{R}_{\underset{\to}{\neg}}$ and $\mathbf{R}_{\to}$*

Proof. This result is derived from Lemma 2.2.1, since $\mathbf{R}1-4+\mathbf{R}12,13+\text{MP}$ give $\mathbf{R}_{\underset{\to}{\neg}}$. Since $\mathbf{R}_{\underset{\to}{\neg}}$ is an expansion of $\mathbf{R}_{\to}$(Proposition 2.2.4), **R** is also an expansion of $\mathbf{R}_{\to}$. □

Proposition 2.4.2. *$\mathbf{R}$ is an expansion of $\mathbf{R}_{fde}$*

Proof. $\mathbf{R}_{fde}1-5$ and $\mathbf{R}_{\text{fde}}7$ are axioms of **R**, and since we know $\mathbf{R}_{\text{fde}}6$ is a theorem of $\mathbf{R}_{\underset{\to}{\neg}}$ from the proof of Lemma 2.2.1, we need only prove that the inference rules of $\mathbf{R}_{\text{fde}}$ are valid in **R**:

□	*Rule of disjunction*		□	*Rule of conjunction*	
1	$\varphi \to \xi$	Premise	1	$\varphi \to \psi$	Premise
2	$\psi \to \xi$	Premise	2	$\varphi \to \xi$	Premise
3	$(\varphi \to \xi) \wedge (\psi \to \xi)$	$1,2-$&I	3	$(\varphi \to \psi) \wedge (\varphi \to \xi)$	$1,2-$&I
4	$(\varphi \to \xi) \wedge (\psi \to \xi) \to (\varphi \vee \psi \to \xi)$	**R**10	4	$(\varphi \to \psi) \wedge (\varphi \to \xi) \to (\varphi \to \psi \wedge \xi)$	**R**7
5	$\varphi \vee \psi \to \xi$	$3,4-$MP	5	$\varphi \to \psi \wedge \xi$	$3,4-$MP

$\square$	*Rule of implication*		$\square$	*Rule of negation*	
1	$\varphi \to \psi$	Premise	1	$\varphi \to \psi$	Premise
2	$\psi \to \xi$	Premise	2	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	Thm
3	$(\psi \to \xi) \to$ $((\varphi \to \psi) \to (\varphi \to \xi))$	**R**2	3	$\neg\psi \to \neg\varphi$	$1,2-$MP
4	$(\varphi \to \psi) \to (\varphi \to \xi)$	$2,3-$MP			
5	$\varphi \to \xi$	$1,4-$MP			

By the proof of Lemma 2.2.1 $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ is a theorem of $\mathbf{R}_{\overrightarrow{\neg}}$, so of **R** too. $\square$

With this discussion we have given reason to all axioms except **R**$7, 10$ and rule &I. It is clear they are useful in deriving the rules of disjunction and conjunction, but couldn't we have avoided having a new rule? In classical logic, when we want to infer ξ from φ and ψ, we prove $\varphi \to (\psi \to \xi)$ and apply *modus ponens* twice. Therefore, couldn't we have an axiom $\varphi \to (\psi \to \varphi \wedge \psi)$ and avoid &I altogether? In fact, $\varphi \to (\psi \to \varphi \wedge \psi)$ is a tautology in classical propositional logic, since given any interpretation into $\{0, 1\}$ we have:

p	q	$p \to (q \to p \wedge q)$
0	0	$0 \to^{\mathcal{B}} (0 \to^{\mathcal{B}} 0 \wedge^{\mathcal{B}} 0) = 0 \to^{\mathcal{B}} (0 \to^{\mathcal{B}} 0) = 0 \to^{\mathcal{B}} 1 = 1$
0	1	$0 \to^{\mathcal{B}} (1 \to^{\mathcal{B}} 0 \wedge^{\mathcal{B}} 1) = 0 \to^{\mathcal{B}} (1 \to^{\mathcal{B}} 0) = 0 \to^{\mathcal{B}} 0 = 1$
1	0	$1 \to^{\mathcal{B}} (0 \to^{\mathcal{B}} 1 \wedge^{\mathcal{B}} 0) = 1 \to^{\mathcal{B}} (0 \to^{\mathcal{B}} 0) = 1 \to^{\mathcal{B}} 1 = 1$
1	1	$1 \to^{\mathcal{B}} (1 \to^{\mathcal{B}} 1 \wedge^{\mathcal{B}} 1) = 1 \to^{\mathcal{B}} (1 \to^{\mathcal{B}} 1) = 1 \to^{\mathcal{B}} 1 = 1$

where the operations are the ones in the first chapter. But adding this axiom to **R** gives:

1	$(\varphi \wedge \psi \to \varphi) \to ((\psi \to \varphi \wedge \psi) \to (\psi \to \varphi))$	**R**2
2	$\varphi \wedge \psi \to \varphi$	**R**5
3	$(\psi \to \varphi \wedge \psi) \to (\psi \to \varphi)$	$1,2-$MP
4	$\varphi \to (\psi \to \varphi \wedge \psi)$	New Axiom
5	$(\varphi \to (\psi \to \varphi \wedge \psi)) \to (((\psi \to \varphi \wedge \psi) \to (\psi \to \varphi)) \to (\varphi \to (\psi \to \varphi)))$	**R**2
6	$((\psi \to \varphi \wedge \psi) \to (\psi \to \varphi)) \to (\varphi \to (\psi \to \varphi))$	$4,5-$MP
7	$\varphi \to (\psi \to \varphi)$	$3,6-$MP

This generates a problem for two reasons. First and foremost, $\varphi \to (\psi \to \varphi)$ is the Positive Paradox, and the object of this logic is to avoid it, therefore, this is enormously problematic on its own. In addition, we have previously shown that the Positive Paradox isn't a theorem of $\mathbf{R}_{\to}$, therefore, by being one of **R**, it would signify the extension between $\mathbf{R}_{\to}$ and **R** isn't conservative. So, this course of action doesn't comply with our requirements. A second idea would be to avoid &I not by substituting the rule, but by substituting axioms **R**$7, 10$ with

$$\mathbf{R}7^* \quad (\varphi \to \psi) \to ((\varphi \to \xi) \to (\varphi \to \psi \wedge \xi))$$
$$\mathbf{R}10^* \quad (\varphi \to \xi) \to ((\psi \to \xi) \to (\varphi \vee \psi \to \xi))$$

Hence, the need for &I in the proof that the rules of disjunction and conjunction are valid in **R** is lost. But this doesn't satisfy us either. We see in this case $(\varphi \to \psi) \to (\varphi \to \varphi)$

would be a theorem of **R**:

1	$(\varphi \to \varphi) \to ((\varphi \to \psi) \to (\varphi \to \varphi \wedge \psi))$	**R**7*
2	$\varphi \to \varphi$	**R**1
3	$(\varphi \to \varphi) \to (\varphi \to \varphi \wedge \psi)$	1, 2 – MP
4	$\varphi \wedge \psi \to \varphi$	**R**5
5	$(\varphi \wedge \psi \to \varphi) \to ((\varphi \to \varphi \wedge \psi) \to (\varphi \to \varphi))$	**R**2
6	$(\varphi \to \varphi \wedge \psi) \to (\varphi \to \varphi)$	4, 5 – MP
7	$((\varphi \to \psi) \to (\varphi \to \varphi \wedge \psi)) \to ((\varphi \to \psi) \to (\varphi \to \varphi))$	6, **R**2 – MP
8	$(\varphi \to \psi) \to (\varphi \to \varphi)$	3, 7 – MP

The matrix in Theorem 2.1.3 (Matrix 1 below) disproves that $(\varphi \to \psi) \to (\varphi \to \varphi)$ is a theorem of $\mathbf{R}_{\to}$ for if we take the interpretation I such that $I(p) = 1$ and $I(q) = 3$, then $I((p \to q) \to (p \to p)) = (1 \to 3) \to (1 \to 1) = 3 \to 2 = 0$. This implies that this manner of avoiding &I would also establish **R** to be non-conservative with respect to $\mathbf{R}_{\to}$. We are thus satisfied by having the rule &I.

$\to$	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2*	0	1	2	3
3*	0	0	0	3

$\to$	0	1	2
0	2	2	2
1	0	2	0
2*	0	0	2

Matrices 1 and 2 respectively

Remark 2.4.3. In [2] the fact that $(\varphi \to \psi) \to (\varphi \to \varphi)$ is not a theorem of $\mathbf{E}_{\to}$ is indicated to be refuted by Matrix 2 above. For any interpretation I in this matrix, $I(\varphi \to \varphi)$ would be 2, so for $(\varphi \to \psi) \to (\varphi \to \varphi)$ to be unsatisfied (i.e. $I((\varphi \to \psi) \to (\varphi \to \varphi)) \neq 2$), $I(\varphi \to \psi)$ would need to be 1. That is impossible since an implication can only be interpreted as 0 or 2 in this matrix. This error in [2] is easily avoided by using the matrix in Theorem 2.1.3 as previously, which is also a matrix model for $\mathbf{E}_{\to}$.

Remark 2.4.4. By Proposition 2.4.1 and Proposition 2.4.2, the following formulas are theorems of **R**, since we've proven they are theorems either of $\mathbf{R}_{\text{fde}}$ or of $\mathbf{R}_{\to}^{\neg}$:

$$(\varphi \to \neg\varphi) \to \neg\varphi \qquad \varphi \to \neg\neg\varphi$$
$$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi) \qquad (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$
$$(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$$

where the last theorem is deduced from **R**2 and **R**3.

2.4.2 Variable-sharing Theorem and its consequences

Definition 2.4.5. *We define the concepts of* ***antecedent*** *and* ***consequent*** *parts of a formula* φ *recursively:*

- φ *is a consequent part of* φ*.*

- *If $\neg\psi$ is a consequent part of φ, then ψ is an antecedent part of φ.*
- *If $\neg\psi$ is an antecedent part of φ, then ψ is a consequent part of φ.*
- *If $\psi \wedge \xi$ or $\psi \vee \xi$ is a consequent part of φ, then ψ and ξ are consequent parts of φ.*
- *If $\psi \wedge \xi$ or $\psi \vee \xi$ is an antecedent part of φ, then ψ and ξ are antecedent parts of φ.*
- *If $\psi \to \xi$ is a consequent part of φ, then ψ is an antecedent part of φ and ξ is a consequent part of φ.*
- *If $\psi \to \xi$ is an antecedent part of φ, then ψ is a consequent part of φ and ξ is an antecedent part of φ.*

Theorem 2.4.6 (Variable-sharing in **R**). *If $\vdash_{\mathbf{R}} \varphi \to \psi$ then there is a variable p occurring as a consequent in both φ and ψ or as an antecedent in both φ and ψ.*

Proof. The matrix in Appendix C is a matrix model for **R** (we don't reproduce it here since it's very large). It has elements $\{\pm 0, \pm 1 \pm 2, \pm 3\}$ and the positives are the designated elements. We have proven the axioms take designated values in it using the program in Appendix B and a simple look at the table defining implication ratifies that if for some interpretation I, $I(\varphi)$ and $I(\varphi \to \psi)$ are positive, then $I(\psi)$ is too and if $I(\varphi)$ and $I(\psi)$ are positive, then $I(\varphi \wedge \psi)$ is.

The proof of this theorem is by contrapositive. Let $\varphi \to \psi$ be a formula in which there doesn't exist any variable occurring neither as a consequent in both φ and ψ nor as an antecedent in both φ and ψ. Then, for every variable occurring in $\varphi \to \psi$ there are six cases depending on whether it appears in φ or ψ and if it does so as a consequent or an antecedent. Depending on this, we give to the variables the following assignment in the universe our matrix:

φ	ψ	$v(p)$
consequent	˘˘	-1
antecedent	˘˘	$+1$
˘˘	consequent	-2
˘˘	antecedent	$+2$
consequent	antecedent	$+3$
antecedent	consequent	-3

This assignment is extended to an interpretation I for the matrix model in Appendix C in the usual way. If we prove

(i) If ξ is an antecedent of φ then $I(\xi) \in \{\pm 1, -3\}$ and if it is a consequent of φ then $I(\xi) \in \{\pm 1, +3\}$.

(ii) If ξ is an antecedent of ψ then $I(\xi) \in \{\pm 2, +3\}$ and if it is a consequent of ψ then $I(\xi) \in \{\pm 2, -3\}$.

we are finished, because since φ is a consequent part of φ and ψ is a consequent part of ψ, then $I(\varphi) \in \{\pm 1, +3\}$ and $I(\psi) \in \{\pm 2, -3\}$, and restricting implication to these values:

$\to$	-3	-2	$+2$
-1	-3	-3	-3
$+1$	-3	-3	-3
$+3$	-3	-3	-3

we obtain that $I(\varphi \to \psi) = -3 \notin \{+0, +1, +2, +3\}$. We consequently conclude $\nvdash_{\mathbf{R}} \varphi \to \psi$. We need only prove (i) and (ii), but we will only explicitly prove (i) since the proof of (ii) is analogous. We prove (i) by induction over the construction of ξ:

- If $\xi = p$, a variable, then

  Case 1 : If ξ is an antecedent part of φ, then $I(p) = v(p) = +1, -3$.

  Case 2 : If ξ is a consequent part of φ, then $I(p) = v(p) = -1, +3$.

- If $\xi = \neg\xi'$ and ξ' complies with the conditions of (i), then

  Case 1 : If ξ is an antecedent part of φ, then ξ' is a consequent part of φ, therefore $I(\xi) = \neg I(\xi') \in \{\mp 1, -3\}$ since $I(\xi') \in \{\pm 1, +3\}$.

  Case 2 : If ξ is a consequent part of φ, then ξ' is an antecedent part of φ, therefore $I(\xi) = \neg I(\xi') \in \{\mp 1, +3\}$ since $I(\xi') \in \{\pm 1, -3\}$.

- If $\xi = \xi_1 \vee \xi_2$ and ξ_1, ξ_2 comply with the conditions of (i), then

  Case 1 : If ξ is an antecedent part of φ, then ξ_1, ξ_2 are antecedent parts of φ and $I(\xi_1), I(\xi_2) \in \{\pm 1, -3\}$, so if we restrict disjunction to these values (see table below), $I(\xi) \in \{\pm 1, -3\}$.

  Case 2 : If ξ is a consequent part of φ, then ξ_1, ξ_2 are consequent parts of φ and $I(\xi_1), I(\xi_2) \in \{\pm 1, +3\}$, so if we restrict disjunction to these values (see table below), $I(\xi) \in \{\pm 1, +3\}$.

$\vee$	-3	-1	$+1$
-3	-3	-1	$+1$
-1	-1	-1	$+1$
$+1$	$+1$	$+1$	$+1$

$\vee$	-1	$+1$	$+3$
-1	-1	$+1$	$+3$
$+1$	$+1$	$+1$	$+3$
$+3$	$+3$	$+3$	$+3$

- The cases where $\xi = \xi_1 \wedge \xi_2$ and $\xi = \xi_1 \to \xi_2$ are analogous, and we will simply state the matrices with restricted values:

$\wedge$	-3	-1	$+1$
-3	-3	-3	-3
-1	-3	-1	-1
$+1$	-3	-1	$+1$

Case 1 $\wedge$

$\wedge$	-1	$+1$	$+3$
-1	-1	-1	-1
$+1$	-1	$+1$	$+1$
$+3$	-1	$+1$	$+3$

Case 2 $\wedge$

$\to$	-3	-1	$+1$
-1	-3	$+1$	$+1$
$+1$	-3	-1	$+1$
$+3$	-3	-3	-3

Case 1 $\to$

$\to$	-1	$+1$	$+3$
-3	$+3$	$+3$	$+3$
-1	$+1$	$+1$	$+3$
$+1$	-1	$+1$	$+3$

Case 2 $\to$ $\square$

Corollary 2.4.7. *If $\vdash_{\mathbf{R}} \varphi \to \psi$ then φ and ψ share a variable.*

This theorem is an extension of the one we saw when constructing **R**. Again, the fact that antecedent and consequent share common meaning in the form of sharing a variable is a remarkably desirable property, since if you stated that X implied something with no relation to X, most probably the reaction of your interlocutor would be to ask "But how is that relevant?". But Theorem 2.4.6 not only allows us to discard formulas with no intensional meaning: depending on the position of the shared variable, the formula can be discarded too. For example, the following are not theorems of **R**, since an instance of them doesn't comply with the necessary conditions:

$$\varphi \to (\varphi \to \psi) \quad \Big| \quad \underbrace{p}_{\text{conseq.}} \quad \underbrace{\underbrace{p}_{\text{ante.}} \to \underbrace{q}_{\text{conseq.}}}_{\text{conseq.}}$$

$$\varphi \to (\psi \to \neg\varphi) \quad \Big| \quad \underbrace{p}_{\text{conseq.}} \quad \underbrace{\underbrace{q}_{\text{ante.}} \to \underbrace{\neg \underbrace{p}_{\text{ante.}}}_{\text{conseq.}}}_{\text{conseq.}}$$

$$\varphi \to \neg\varphi \quad \Big| \quad \underbrace{p}_{\text{conseq.}} \quad \underbrace{\neg \underbrace{p}_{\text{ante.}}}_{\text{conseq.}}$$

$$\neg\varphi \to \varphi \quad \Big| \quad \underbrace{\neg \underbrace{p}_{\text{ante.}}}_{\text{conseq.}} \quad \underbrace{p}_{\text{conseq.}}$$

Another consequence of Theorem 2.4.6 is the conservativity in $\vdash_{\mathbf{R}_{\text{fde}}} \leqslant \vdash_{\mathbf{R}}$:

Theorem 2.4.8. *$\mathbf{R}$ is a conservative expansion of $\mathbf{R}_{fde}$.*

Proof. That it is an expansion is clear from Proposition 2.4.2, so we only need to prove conservativity. First, we observe that since $\vdash_{\mathbf{R}_{\text{fde}}} \leqslant \vdash_{\mathbf{R}}$ and through the use of &I, the following are theorems of **R** (with $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$):

$$\begin{array}{ll}
\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi & \varphi \vee \psi \leftrightarrow \psi \vee \varphi \\
(\varphi \wedge \psi) \wedge \xi \leftrightarrow \varphi \wedge (\psi \wedge \xi) & (\varphi \vee \psi) \vee \xi \leftrightarrow \varphi \vee (\psi \vee \xi) \\
\varphi \wedge (\psi \vee \xi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi) & \varphi \vee (\psi \wedge \xi) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \xi) \\
\neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi & \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi \\
\neg\neg\varphi \leftrightarrow \varphi &
\end{array}$$

Let $\varphi \to \psi$ be a first-degree entailment such that $\vdash_{\mathbf{R}} \varphi \to \psi$. As a result of the previous observation, if $\varphi' \to \psi'$ is a normal form of $\varphi \to \psi$ then $\vdash_{\mathbf{R}} \varphi \leftrightarrow \varphi'$, $\vdash_{\mathbf{R}} \psi \leftrightarrow \psi'$ and through the use of *modus ponens* and **R**2, $\vdash_{\mathbf{R}} \varphi' \to \psi'$ (note that $\vdash_{\mathbf{R}} \varphi \leftrightarrow \varphi' \iff \vdash_{\mathbf{R}} \varphi \leftrightarrows \varphi'$). Now, if $\varphi' = \varphi_1 \vee ... \vee \varphi_n$ and $\psi' = \psi_1 \wedge ... \wedge \psi_m$, for $1 \leqslant j \leqslant m$, we have $\vdash_{\mathbf{R}} \psi_1 \wedge ... \wedge \psi_n \to \psi_j$ because of **R**5 and **R**6. Similarly, for $1 \leqslant i \leqslant n$ $\vdash_{\mathbf{R}} \varphi_i \to \varphi_1 \vee ... \vee \varphi_n$. Then, by the use of *modus ponens*, we conclude that for every $i = 1, ..., n$ $j = 1, ..., m$ $\vdash_R \varphi_i \to \psi_j$. Applying Theorem 2.4.6, we obtain that for each i, j available there is a variable $p_{i,j}$ such that it appears as an antecedent in φ_i and ψ_j, or as a consequent in φ_i and ψ_j. But we need to prove that φ_i and ψ_j share a literal, not only a variable. We study the literal in which $p_{i,j}$ is contained. Since φ_i is a primitive conjunction, $\varphi_i = \varphi^i_1 \wedge ... \wedge \varphi^i_{n_i}$ and since ψ_j is a primitive disjunction, $\psi_j = \psi^j_1 \vee ... \vee \varphi^j_{m_j}$ where all φ^i_r, ψ^j_s are literals. There must be $1 \leqslant r \leqslant n_i$, $1 \leqslant s \leqslant m_j$ such that $p_{i,j}$ is in φ^i_r and ψ^j_s, but these literals are consequent parts of φ_i and ψ_j respectively, and since $p_{i,j}$ can only appear as an antecedent or a consequent in both at the same time, we get $\varphi^i_r = \psi^j_s = p_{i,j}$ or $\varphi^i_r = \psi^j_s = \neg p_{i,j}$. This implies that $\varphi \to \psi$ is a tautological entailment, concluding our proof. □

As remarked in section 2.2.3, it is very important to see that **R** is a conservative expansion of all the subsystems we have defined before. Otherwise, all the effort we put into constructing them so they are relevantly sound would have been unnecessary and we wouldn't be able to ensure the properties we proved for the formulas of $\mathbf{R}_{\to}$, $\mathbf{R}_{\overset{\neg}{\to}}$, $\mathbf{R}_{\text{fde}}$ still apply to **R**, because we may be unknowingly adding theorems or properties to our logic which it didn't have. That is why we state the following theorem, although we cannot give a full proof of it in this work:

Theorem 2.4.9. *$\mathbf{R}$ is a conservative expansion of $\mathbf{R}_{\substack{\neg\\\rightarrow}}$ and $\mathbf{R}_{\rightarrow}$.*

Proof. They are both expansions because of Proposition 2.4.1, but as in Theorem 2.2.4, we don't have the necessary tools to prove **R** is a a conservative expansion of $\mathbf{R}_{\substack{\neg\\\rightarrow}}$ and $\mathbf{R}_{\rightarrow}$. A proof of this may be found in [14] (Corollary 2 and Corollary 4*). □

Both these theorems assure that all the paradoxes we mentioned at the start of this chapter (Positive Paradox, Explosion,...) and which we avoided through the careful construction of $\mathbf{R}_{\text{fde}}$, $\mathbf{R}_{\substack{\neg\\\rightarrow}}$ and $\mathbf{R}_{\rightarrow}$ are not theorems of **R**.

2.4.3 Expanding the language

As in [2], new connectives may be introduced in **R** which will aid us in the following chapters. We present them here.

Co-tenability

We begin by discussing the role of conjunction and disjunction in this logic. Since $\nvdash_{\mathbf{R}} \varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$, it is clear that the relevant conjunction lacks some of the properties classical conjunction has, so we can ask ourselves if some other connective can assimilate them. In the first chapter, we presented an axiomatisation for classical logic which defined $\vee$ as $\varphi \vee \psi := \neg\varphi \rightarrow \psi$ so we define a new connective:

Definition 2.4.10. *Let φ, ψ be formulas in the language of $\mathbf{R}$, we define **intensional disjunction** as the connective $+$ such that $\varphi + \psi := \neg\varphi \rightarrow \psi$.*

Now, the classical way to extend $+$ into a conjunction is to use the De Morgan laws, which would give us the definition:

Definition 2.4.11. *Let φ, ψ be formulas in the language of $\mathbf{R}$, we define **intensional conjunction** as the connective $\circ$ such that $\varphi \circ \psi := \neg(\varphi \rightarrow \neg\psi)$.*

Clearly, $\vdash_{\mathbf{R}} \varphi \circ \psi \leftrightarrow \neg(\neg\varphi + \neg\psi)$.

This definition is equivalent to adding the following axioms to **R**:

$$\begin{array}{ll} \circ 1 & \varphi \rightarrow (\psi \rightarrow \varphi \circ \psi) \\ \circ 2 & (\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow (\varphi \circ \psi \rightarrow \xi) \end{array}$$

That is, when these axioms are added $\vdash_{\mathbf{R}} \varphi \circ \psi \leftrightarrow \neg(\varphi \rightarrow \neg\psi)$ (these deductions are long and have been added to Appendix A). Therefore, we can take $\circ$ as primitive in **R**. We can also see that $\vdash_{\mathbf{R}} (\varphi \circ \psi \rightarrow \xi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \xi))$, making $\circ 2$ an equivalence, this way: $\vdash_{\mathbf{R}} (\psi \rightarrow \varphi \circ \psi) \rightarrow ((\varphi \circ \psi \rightarrow \xi) \rightarrow (\psi \rightarrow \xi))$ because $\vdash_{\mathbf{R}} (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \xi) \rightarrow (\varphi \rightarrow \xi))$

from Remark 2.4.4, from **R**2 and $\circ 1$ we get $\vdash_{\mathbf{R}} \varphi \to ((\varphi \circ \psi \to \xi) \to (\psi \to \xi))$, and **R**3 gives us the converse of $\circ 2$.

Consequently, $\circ$ keeps the classical properties that we avoided in $\wedge$ when constructing $\mathbf{R}_{\mathrm{fde}}$, but it lacks one of the fundamental properties needed to be considered a conjunction, since $\nvdash_{\mathbf{R}} \varphi \circ \psi \to \varphi$. This is true because otherwise the Positive Paradox would be a theorem of **R**, which we are certain it is not. This is shown through the use of the converse of $\circ 2$, that gives $\vdash_{\mathbf{R}} (\varphi \circ \psi \to \varphi) \to (\varphi \to (\psi \to \varphi))$. Thus, we change the nomenclature "intensional conjunction" in favour of "co-tenability", finding it more adequate.

Adding t

A natural neighbor of **R** can be constructed from the addition of a simple constant: **t**, the conjunction of all logical truths. The idea behind this extension is being able to define truth from a constant. As we saw in the preliminaries, in classical logic a formula is a tautology if and only if it is classically equivalent to $\top$ (in the Lindenbaum algebra of CPC the tautologies are elements of $1 = [\top]$), and therefore all tautologies are equivalent. In **R**, $p \to p$ and $q \to q$ are not equivalent (they don't imply each other) by the Variable-Sharing theorem, although in CPC they are. Therefore, instead of having a constant equivalent to all tautologies, we will have a constant implying all tautologies, since $\vdash_{\mathbf{R}} \varphi \wedge \psi \to \varphi$. This constant is included by adding the following axioms to **R**:

$$\begin{array}{ll} \mathbf{t}1 & \mathbf{t} \\ \mathbf{t}2 & \mathbf{t} \to (\varphi \to \varphi) \end{array}$$

The resulting logic is called $\mathbf{R}^t$, and it clearly is an expansion of **R**. We prove the following property, which will be useful to us in the future:

$$\vdash_{\mathbf{R}^{\mathbf{t}}} \varphi \leftrightarrow (\mathbf{t} \to \varphi) \tag{2.1}$$

$\to$ is obtained through the use of **R**3 in **t**2 and the other implication is obtained from **R**4, $\vdash_{\mathbf{R}^{\mathbf{t}}} \mathbf{t} \to ((\mathbf{t} \to \varphi) \to \varphi)$, by using *modus ponens* with **t**1.

Chapter 3

Algebraic semantics for relevance logic

In this final chapter we present semantics for $\mathbf{R}_{\mathrm{fde}}$ and $\mathbf{R}$. As we saw in the first chapter, classical logic can be given a semantics with the class of Boolean algebras, but also with only one Boolean algebra, $\mathcal{B}$. Parallelly, $\mathbf{R}_{\mathrm{fde}}$ can also be given a semantics in the class of intensional lattices, and then reduce them to only one element of the class. The first semantics is not actually stated in [2], though it is hinted at; the second one is what the book centers on with respect to a first-degree entailment semantics, since $\mathbf{E}_{\mathrm{fde}}$ is equivalent to $\mathbf{R}_{\mathrm{fde}}$. For $\mathbf{R}$, we will present an equivalent algebraic semantics from [10], and then give an equivalent algebraic semantics for $\mathbf{R}^{\mathbf{t}}$ which is found in [2], but for which equivalence is not proven in the book.

3.1 Semantics for $\mathbf{R}_{\mathrm{fde}}$

3.1.1 Preliminaries on filters

Definition 3.1.1. *A **sublattice** of a lattice $< L, \leqslant >$ is a non-empty subset of L which is closed under the operations $\wedge$ and $\vee$ of $< L, \leqslant >$.*

We present two types of sublattices which are important to our work:

Definition 3.1.2. *Let $\mathbf{L} =< L, \leqslant >$ be a lattice, a **filter** (or **ideal**) of $\mathbf{L}$ is a subset $F \subseteq L$, $F \neq \varnothing$ ($I \subseteq L$, $I \neq \varnothing$) such that:*

F1(I1) *: If $a, b \in F$ then $a \wedge b \in F$ (if $a, b \in I$ then $a \vee b \in I$)*

F2(I2) *: If $a \in F$ then $a \vee b \in F$ (if $a \in I$ then $a \wedge b \in I$) for every $b \in L$*

Lemma 3.1.3. *Condition **F2**(**I2**) is equivalent to:*

F2′(I2′) : *If $a \in F$ and $a \leqslant b$, then $b \in F$ (if $a \in I$ and $b \leqslant a$, then $b \in F$)*

Proof. We prove both implications for filters; for ideals the proof is analogous. For **F2** $\Longrightarrow$ **F2′**, let a be an element of F and let $a \leqslant b$, then the lub of $\{a, b\}$ is b, so that $a \vee b = b$, which by **F2** implies $b \in F$. For the converse, let a be an element of F, then for any b we have $a \vee b \geqslant a$ since $a \vee b$ is the lub of $\{a, b\}$, which through **F2′** means $a \vee b \in F$. □

Clearly, filters and ideals are closed under $\wedge$, $\vee$ and therefore are sublattices.

Definition 3.1.4. *Let $\mathbf{L} =< L, \leqslant>$ be a lattice, a filter F of $\mathbf{L}$ is **maximal** if $F \neq L$ and there is no ideal F' of $\mathbf{L}$ such that $F \subseteq F'$. An ideal I of $\mathbf{L}$ is **maximal** if $I \neq L$ and there is no ideal I' of $\mathbf{L}$ such that $I \subseteq I'$.*

Definition 3.1.5. *A **prime filter** (**prime ideal**) is a filter F (ideal I) which satisfies*

PF (PI) : *If $a \vee b \in F$, then $a \in F$ or $b \in F$ (if $a \wedge b \in I$, then $a \in I$ or $b \in F$)*

We can give a more elegant characterisation of prime filters and ideals as follows:

Lemma 3.1.6. *Let $\mathbf{L} =< L, \leqslant>$ be a lattice, a set $F \subseteq L$, $F \neq \varnothing$, ($I \subseteq L$, $I \neq \varnothing$) is a **prime filter** (**prime ideal**) of $\mathbf{L}$ if and only if:*

F1′(I1′) $a, b \in F \iff a \wedge b \in F$ ($a, b \in I \iff a \vee b \in I$)

PF′[PI′] $a \in F$ *or* $b \in F \iff a \vee b \in F$ ($a \in I$ *or* $b \in I \iff a \wedge b \in I$)

Proof. **F1′**($\Longrightarrow$) is **F1**, **PF′**($\Longrightarrow$) is **F2** and **PF′**($\Longleftarrow$) is **PF**, so we only need to prove that every prime filter satisfies **F1′**($\Longleftarrow$). If $a \wedge b \in F$, then since $a \wedge b \leqslant a, b$ using **F2′** we get that $a, b \in F$, concluding our proof. The case for ideals is analogous. □

Definition 3.1.7. *Let $\mathbf{L} =< L, \leqslant>$ be a lattice and let $A \subseteq L$, we define the **filter (ideal) generated by** A as the least filter (ideal) including A. If $A = \{a\}$ for some $a \in L$, we say it is the **principal filter (ideal) generated by** a.*

Remark 3.1.8. The filter or ideal generated by A always exists, since trivially the intersection of filters is a filter and the intersection of ideals is an ideal.

We can characterise filters generated by sets more explicitly:

Lemma 3.1.9. *Let $\mathbf{L} =< L, \leqslant>$ be a lattice and let $A \subseteq L$, the filter generated by A is $F(A) = \{x \in L :$ there are $a_i \in A$ such that $a_1 \wedge ... \wedge a_n \leqslant x\}$.*

Proof. $F(A)$ is a filter because if $a_1 \wedge ... \wedge a_n \leqslant x$ and $a'_1 \wedge ... \wedge a'_n \leqslant x'$, then $a_1 \wedge ... \wedge a_n \wedge a'_1 \wedge ... \wedge a'_n \leqslant x \wedge x'$ for any $y \in L$ $a_1 \wedge ... \wedge a_n \leqslant x \leqslant x \vee y$. And since $a \leqslant a$ for every $a \in A$ then $A \subseteq F(A)$. To finish, we prove that if F is a filter that contains A $F(A) \subseteq F$. We take $x \in F(A)$, therefore $x \geqslant a_1 \wedge a_n$ for some $a_i \in A$. Since F is a filter that contains A $a_1 \wedge a_n \in F$, and by **F2′** $x \in F$. □

Lemma 3.1.10. *Let* $\mathbf{L} =< L, \leqslant >$ *be a lattice and let* $a \in L$*, then the principal filter generated by* a *is* $\{x \in L : a \leqslant x\}$ *and the principal ideal generated by* a *is* $\{x \in L : x \leqslant a\}$*.*

Proof. We denote as F_a the principal filter generated by a. First, we prove the conditions to show $A = \{x \in L : a \leqslant x\}$ is a filter. Since $a \leqslant a$, $a \in A$, so $A \neq \varnothing$. Now, if $b, c \in A$, since $\leqslant_A$ is total, $b \wedge c$ is b or c, and therefore $b \wedge c \in A$. Finally, if $b \in A$, then $a \leqslant b$, so if we take $c \in L$, $a \leqslant b \leqslant b \vee c$, therefore $b \vee c \in A$. Since A is a filter that contains a, $F_a \subseteq A$. Now we prove the other inclusion. Let $b \in A$, then $b \geqslant a$ and since $a \in F_a$, **F2** implies that $a \vee b = b \in F_a$. In conclusion, $F_a = A$. The case of ideals has an analogous proof. □

From Theorem 6 in [16] (where an α-ideal is a filter and a μ-ideal is an ideal) we have:

Proposition 3.1.11. *Let* $\mathbf{L} =< L, \leqslant >$ *be a lattice, if* $a, b \in L$ *are such that* $a \nleqslant b$*, then there is a prime filter* P *satisfying that* $a \in P$ *and* $b \notin P$*.*

3.1.2 The Lindenbaum algebra of $\mathbf{R}_{\mathrm{fde}}$

This section will characterise algebraically the Lindenbaum algebra of $\mathbf{R}_{fde}$, we proceed as follows:

Definition 3.1.12. *We define* $\leftrightarrows$ *as the following relation over non-implicational formulas:*

$$\varphi \leftrightarrows \psi \overset{def}{\Longleftrightarrow} \vdash_{\mathbf{R}_{fde}} \varphi \to \psi \textit{ and } \vdash_{\mathbf{R}_{fde}} \psi \to \varphi$$

Proposition 3.1.13. $\leftrightarrows$ *is a congruence over non-implicational formulas.*

Proof. Let $\varphi, \psi, \xi, \varphi', \psi'$ be non-implicational formulas. Reflexivity is satisfied since $\vdash_{\mathbf{R}} \varphi \to \varphi$, by conservativity (Theorem 2.4.8), $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \varphi$, therefore $\varphi \leftrightarrows \varphi$. For symmetry, if $\varphi \leftrightarrows \psi$, then $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \psi$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \varphi$, which clearly implies $\psi \leftrightarrows \varphi$. Proving transitivity we will have an equivalence relation. Suppose $\varphi \leftrightarrows \psi$ and $\psi \leftrightarrows \xi$. Then, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \psi$, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \varphi$, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \xi$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi \to \psi$. Using IR, we get $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \xi$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi \to \varphi$, so $\varphi \leftrightarrows \psi$. With this, $\leftrightarrows$ is an equivalence relation. Now we only lack distribution over $\neg, \wedge, \vee$:

- $\neg$ Suppose $\varphi \leftrightarrows \psi$, then $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \psi$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \varphi$, from NR we obtain $\vdash_{\mathbf{R}_{\mathrm{fde}}} \neg\psi \to \neg\varphi$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \neg\varphi \to \neg\psi$. Therefore $\neg\varphi \leftrightarrows \neg\psi$.

- $\wedge$ Suppose $\varphi \leftrightarrows \psi$ and $\varphi' \leftrightarrows \psi'$. Then $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \psi$, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \varphi$, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi' \to \psi'$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi' \to \varphi'$. Through the axioms of $\mathbf{R}_{\mathrm{fde}}$, we have:

$$\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \wedge \varphi' \to \varphi \text{ and } \vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \wedge \varphi' \to \varphi'$$

  and also for the other implication. So using IR, we obtain:

$$\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \wedge \varphi' \to \psi \text{ and } \vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \wedge \varphi' \to \psi'$$

  and conversely. Finally, by CR, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \wedge \varphi' \to \psi \wedge \psi'$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \wedge \psi' \to \varphi \wedge \varphi'$, therefore, $\varphi \wedge \varphi' \leftrightarrows \psi \wedge \psi'$.

$\vee$ Suppose $\varphi \leftrightarrows \psi$ and $\varphi' \leftrightarrows \psi'$. Then $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \psi$, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \varphi$, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi' \to \psi'$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi' \to \varphi'$. Through the axioms of $\mathbf{R}_{\mathrm{fde}}$, we have:

$$\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \to \psi \vee \psi' \text{ and } \vdash_{\mathbf{R}_{\mathrm{fde}}} \psi' \to \psi \vee \psi'$$

and vice-versa. So using IR, we obtain:

$$\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \psi \vee \psi' \text{ and } \vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi' \to \psi \vee \psi'$$

and the same for the other implication. Finally, by DR, $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \vee \varphi' \to \psi \vee \psi'$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \psi \vee \psi' \to \varphi \vee \varphi'$, therefore, $\varphi \vee \varphi' \leftrightarrows \psi \vee \psi'$.

In conclusion, $\leftrightarrows$ is a congruence. □

We denote by $\mathbf{R}_{fde} / \leftrightarrows$ the *Lindenbaum algebra of $\mathbf{R}_{fde}$* (doing the quotient over non-implicational formulas). Note that the Lindembaum algebra of $\mathbf{R}_{\mathrm{fde}}$ is actually not an algebra, since its equivalence classes contain only non-implicational formulas, so there is no operation for $\to$. We only have three operations, mirroring the tree connectives in non-implicational formulas. We now want to characterise algebraically the structure of $\mathbf{R}_{\mathrm{fde}}$ and then finde the role of $\to$, so we introduce some new algebraic concepts:

Definition 3.1.14. *An **De Morgan lattice** is a tuple $< L, \leqslant, \neg >$ where $\leqslant$ is a relation and $\neg$ is a unary function called **intensional complementation** satisfying:*

DL *: $< L, \leqslant >$ is a distributive lattice*

N1 *: For every $a \in A$, $\neg\neg a = a$*

N2 *: For every $a, b \in L$ if $a \leqslant b$ then $\neg b \leqslant \neg a$*

Lemma 3.1.15. *Intensional complementation satisfies the De Morgan rules, i.e. for every $a, b \in L$ $\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg(a \vee b) = \neg a \wedge \neg b$*

Proof. We prove $\neg(a \wedge b) = \neg a \vee \neg b$ showing the two inequatilites. $\leqslant$ is true since $\neg a, \neg b \leqslant \neg a \vee \neg b$, by **N2,N1** $\neg(\neg a \vee \neg b) \leqslant a, b$ which implies $\neg(\neg a \vee \neg b) \leqslant a \wedge b$ and using **N2,N1** again we get what we wanted. $\geqslant$ is since $a, b \geqslant a \wedge b$, then $\neg(a \wedge b) \geqslant \neg a \geqslant \neg a \vee \neg b$. The proof of $\neg(a \vee b) = \neg a \wedge \neg b$ can be done analogously. □

Now we can state our first algebraic characterisation of the Lindenbaum algebra of $\mathbf{R}_{\mathrm{fde}}$:

Proposition 3.1.16. *$< \mathbf{R}_{fde} / \leftrightarrows, \leqslant, \neg >$, where if φ, ψ are non-implicational formulas,*

$$[\varphi] \leqslant [\psi] \overset{def}{\iff} \vdash_{\mathbf{R}_{fde}} \varphi \to \psi$$

is a De Morgan lattice, and the join and meet operations coincide with $\vee$ and $\wedge$.

Proof. First we prove it is a **lattice**. $\leqslant$ is a partial order of $\mathbf{R}_{\mathrm{fde}}$, since $\vdash_{\mathbf{R}_{\mathrm{fde}}} \varphi \to \varphi$ (reflexivity), if $\vdash_{\mathbf{R}_{fde}} \varphi \to \psi$ and $\vdash_{\mathbf{R}_{fde}} \psi \to \varphi$ then $[\varphi] = [\psi]$ (antisymmetry) and we have rule IR (transitivity). We need to see that every two elements $[\varphi], [\psi]$ of $\mathbf{R}_{fde} / \leftrightarrows$ have a glb which is $[\varphi] \wedge [\psi]$ and a lub which is $[\varphi] \vee [\psi]$. Because of axioms $\mathbf{R}_{\mathrm{fde}}1-4$, $[\varphi] \wedge [\psi] \leqslant [\varphi], [\psi]$ and $[\varphi], [\psi] \leqslant [\varphi] \vee [\psi]$. If there is $[\xi] \in \mathbf{R}_{fde} / \leftrightarrows$ such that $[\xi] \leqslant [\varphi], [\psi]$, then $\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi \to \varphi$ and $\vdash_{\mathbf{R}_{\mathrm{fde}}} \xi \to \psi$ which by CR would give $[\xi] \leqslant [\varphi \wedge \psi] = [\varphi] \wedge [\psi]$. Similarly, using DR if $[\xi] \in \mathbf{R}_{fde} / \leftrightarrows$ satisfies $[\varphi], [\psi] \leqslant [\xi]$ then $[\varphi] \vee [\psi] \leqslant [\xi]$. **Distributivity** is easy to prove, since we know from the proof of Proposition 2.3.10 that the rules used to find equivalent normal forms translate into theorems of $\mathbf{R}_{\mathrm{fde}}$, so distributivity does too. **N1** is a direct consequence of axioms $\mathbf{R}_{\mathrm{fde}}6, 7$ and **N2** of rule NR. ◻

But we can go further with this, and introduce some set that acts as the "designated elements":

Definition 3.1.17. *Let $< L, \leqslant, \neg >$ be a De Morgan lattice, then $T \subseteq L$ is a* ***truth filter*** *if it is a filter and it is*

- ***Consistent****: There is no $a \in L$ such that $a \in T$ and $\neg a \in T$.*
- ***Exhaustive****: For every $a \in L$ $a \in T$ or $\neg a \in T$.*

Definition 3.1.18. *An* ***intensional lattice*** *is a tuple $< L, \leqslant, \neg, T >$ where $< L, \leqslant, \neg >$ is a De Morgan lattice (i.e. $< L, \leqslant, \neg >$ satisfies* **DL, N1, N2***) and $T \subseteq L$ is a truth filter.*

An intensional lattice is a De Morgan lattice with a truth filter, therefore, it is useful to characterise the existence of truth filters in De Morgan lattices, the first result with respect to this follows:

Definition 3.1.19. *Let $< L, \leqslant, \neg >$ be a De Morgan lattice, $a \in L$ is a* ***fixed point*** *if $a = \neg a$.*

Theorem 3.1.20. *Let* **L***$=< L, \leqslant, \neg >$ be a De Morgan lattice, then* **L** *has a truth-filter if and only if it has no fixed points.*

Proof. $\boxed{\Longrightarrow}$ By contradiction. Suppose $T \subseteq L$ is a truth filter and there is $a \in L$ such that $a = \neg a$. We have two cases:

Case 1: If $a \in T$, then $\neg a \in T$, so T isn't consistent, which is a contradiction.

Case 2: If $a \notin T$, then since T is exhaustive $\neg a \in T$, which is a contradiction.

Therefore, there isn't any $a \in L$ such that $a = \neg a$.

$\boxed{\Longleftarrow}$ Complete in [2] (Theorem 1 in page 194). ◻

Finally, since φ is an antecedent part of $\neg\varphi$ and a consequent part of φ, by Variable-sharing (Theorem 2.4.6) $\nvdash_{\mathbf{R}} \varphi \to \neg\varphi$ and $\nvdash_{\mathbf{R}} \neg\varphi \to \varphi$. Since $\mathbf{R}$ is an expansion of $\mathbf{R}_{\mathrm{fde}}$ $< \mathbf{R}_{fde} / \leftrightarrows, \leqslant, \neg >$ has no fixed points, and by Theorem 3.1.20 we obtain:

Theorem 3.1.21. *There is $T \subseteq \mathbf{R}_{fde} / \leftrightarrows$ such that $< \mathbf{R}_{fde} / \leftrightarrows, \leqslant, \neg, T >$ is an intensional lattice.*

3.1.3 Two semantics for $\mathbf{R_{fde}}$

We observe how in intensional lattices we don't have an operation for $\rightarrow$, since implication is treated as an order relation. Therefore, our semantics can't be the usual matrix semantics as defined in the first section of this work. Therefore, we take a language $\mathcal{L}_{\not\rightarrow}$ with connectives $\{\wedge, \vee, \neg\}$.

Definition 3.1.22. *A **model** Q is a pair $< L, s >$ where L is an intensional lattice and s is an assignment. We extend s into an interpretation $I_Q : Fm_{\mathcal{L}_{\not\rightarrow}} \longrightarrow L$ as usual. Let $\varphi \rightarrow \psi$ be a first-degree entailment, we say it is*

(i) ***true in Q*** *if $I_Q(\varphi) \leqslant I_Q(\psi)$, otherwise it is **false in Q**.*

(ii) ***valid in L*** *if for every possible assignment s of the variables into L it is true in $< L, s >$, otherwise it is **falsifiable in L**.*

(iii) ***valid in the class of intensional lattices*** *if for every intensional lattice L it is valid in L, otherwise it is **falsifiable in the class of intensional lattices**.*

Note that here truth is not defined from belonging to a given set (which would be the truth-filter). Also, we are going to prove completeness and consistency only for theorems of $\mathbf{R}_{\text{fde}}$, this is why semantic consequence is not defined.

We present $< \mathbf{R}_{fde}/_{\leftrightarrows}, c >$ where $c(p) = [p]$, called the *canonical model* where c is the *canonical assignment function*. The *canonical interpretation* that extends c is $I_C(\varphi) = [\varphi]$, and this can be proven easily by induction over non-implicational formulas:

- If φ is a variable, then $I_C(\varphi) = c(\varphi) = [\varphi]$.
- If $\varphi = \neg\psi$ and $I_C(\psi) = [\psi]$ then $I_C(\varphi) = \neg I_C(\psi) = \neg[\psi] = [\neg\psi]$
- If $\varphi = \psi \hat{\vee} \xi$ and $I_C(\psi) = [\psi]$, $I_C(\xi) = [\xi]$, then $I_C(\varphi) = I_C(\psi) \hat{\vee} I_C(\xi) = [\psi] \hat{\vee} [\xi] = [\varphi]$

With the aid of this, we can prove:

Theorem 3.1.23 (Completeness)**.** *If an fde is valid in the class of intensional lattices, then it is provable in $\mathbf{R}_{fde}$.*

Proof. This is a consequence of the fact that if an fde is not provable in $\mathbf{R}_{\text{fde}}$ then it is falsifiable in $\mathbf{R}_{fde}/_{\leftrightarrows}$, since if $\vdash_{\mathbf{R}_{\text{fde}}} \varphi \rightarrow \psi$, then $[\varphi] \nleqslant [\psi]$ and hence the canonical interpretation falsifies $\varphi \rightarrow \psi$. $\square$

Conversely, we have:

Theorem 3.1.24 (Consistency)**.** *If $\vdash_{\mathbf{R}_{fde}} \varphi \rightarrow \psi$, then $\varphi \rightarrow \psi$ is valid in in the class of intensional lattices.*

Proof. Let $\mathbf{L}$ be an intensional lattice. First, we prove the axioms are valid in $\mathbf{L}$. Let I be an interpretation from non-implicational formulas into $\mathbf{L}$, since all the theorems of R_{fde} are implications $\varphi \to \psi$, we want to prove for each of them that $I(\varphi) \leqslant I(\psi)$, using the properties of the operations in intensional lattices:

$\mathbf{R_{fde}}1, 2$ $I(\varphi), I(\psi) \geqslant I(\varphi) \wedge I(\psi) = I(\varphi \wedge \psi)$

$\mathbf{R_{fde}}3, 4$ $I(\varphi), I(\psi) \leqslant I(\varphi) \vee I(\psi) = I(\varphi \vee \psi)$

$\mathbf{R_{fde}}6, 7$ Since $I(\varphi) = \neg\neg I(\varphi) = I(\neg\neg\varphi)$, $I(\varphi) \leqslant I(\neg\neg\varphi)$ and $I(\neg\neg\varphi) \leqslant I(\varphi)$

Now, we check $\mathbf{R}_{\text{fde}}5$. $I(\varphi \wedge (\psi \vee \xi)) = I(\varphi) \wedge (I(\psi) \vee I(\xi)) \leqslant I(\varphi), I(\psi) \vee I(\xi)$, therefore $I(\varphi \wedge (\psi \vee \xi)) \leqslant I(\varphi), I(\psi)$ (which implies $I(\varphi \wedge (\psi \vee \xi)) \leqslant I(\varphi) \wedge I(\psi)$) or $I(\varphi \wedge (\psi \vee \xi)) \leqslant I(\varphi), I(\xi)$. Either way, $I(\varphi \wedge (\psi \vee \xi)) \leqslant I(\varphi \wedge \psi) \vee I(\xi) = I((\varphi \wedge \psi) \vee \xi)$. The following step is to prove that the rules maintain validity in $\mathbf{L}$.

IR : We suppose $\varphi \to \psi$ and $\psi \to \xi$ are valid. Let I be an interpretation from non-implicational formulas into $\mathbf{L}$, then $I(\varphi) \leqslant I(\psi) \leqslant I(\xi)$ so $\varphi \to \xi$ is valid in $\mathbf{L}$.

CR : We suppose $\varphi \to \psi$ and $\varphi \to \xi$ are valid. Let I be an interpretation from non-implicational formulas into $\mathbf{L}$, then since $I(\varphi) \leqslant I(\psi), I(\xi)$ we have $I(\varphi) \leqslant I(\psi) \wedge I(\xi) = I(\psi \wedge \xi)$, making $\varphi \to \psi \wedge \xi$ valid in $\mathbf{L}$.

DR : We suppose $\varphi \to \xi$ and $\psi \to \xi$ are valid. Let I be an interpretation from non-implicational formulas into $\mathbf{L}$, then since $I(\varphi), I(\psi) \leqslant I(\xi)$ we get $I(\varphi \vee \psi) = I(\varphi) \vee I(\psi) \leqslant I(\xi)$, so $\varphi \vee \psi \to \xi$ is valid in $\mathbf{L}$.

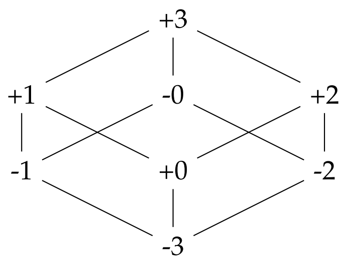
NR : We suppose $\varphi \to \psi$ is valid in $\mathbf{L}$. Let I be an interpretation from non-implicational formulas into $\mathbf{L}$, since $I(\varphi) \to I(\psi)$ by **N2** $I(\neg\psi) = \neg I(\psi) \to \neg I(\varphi) = I(\neg\varphi)$. Then, $\neg\psi \to \neg\varphi$ is valid in $\mathbf{L}$.

With all this, the theorems of $\mathbf{R}_{\text{fde}}$ are valid in the class of intensional lattices. □

These two theorems together give:

Theorem 3.1.25. *An fde is provable in $\mathbf{R}_{fde}$ if and only if it is valid in the class of intensional lattices.*

Therefore, the class of intensional lattices induces a semantics for $\mathbf{R}_{\text{fde}}$, but, as in classical logic, we can reduce completeness and consistency to only one intensional lattice. We now present $\mathbf{M}_0 =< \{-3, -2, -1, -0, +0, +1, +2, +3\}, \leqslant, \neg, \{+0, ... + 3\} >$ where $\neg(\pm a) = \mp a$ and $\leqslant$ is defined as in the following diagram:

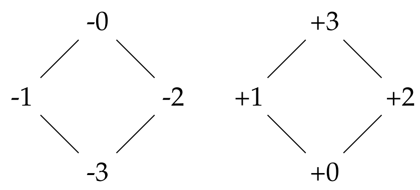


$\mathbf{M}_0$ is an intensional lattice, since it is lattice and all necessary properties can be proven easily by looking at the diagram. **D1**, **D2** are checked case by case, for example,

$$+0 \wedge (-1 \vee +2) = +0 \wedge -3 = +0$$
$$(+0 \wedge -1) \vee (+0 \wedge +2) = +1 \vee +2 = +0$$

We prove **N1:** let $a \in \{0, ..., 3\}$, then $\neg\neg \pm a = \neg \mp a = \pm a$. If $a, b \in \{-3, ..., +3\}$ and $a \leqslant b$ it is clear from the diagram of order restricted to negative and positive elements that **N2** holds if both a, b are positive or both are negative (see the figure below).

If one is negative and the other one is positive, then a is negative and b is positive, because we can observe by looking at the original diagram that no negative element is grater than a positive element. Now, we can see their complements are related case by case. Finally, $\{+0, ... +3\}$ is a truth filter because it is consistent and exhaustive.



The final result of this section will be the completeness and consistency of $\mathbf{R}_{\text{fde}}$ with respect to $\mathbf{M}_0$, but first we will present some useful tools.

Definition 3.1.26. *Let $h : L \longrightarrow L'$ be an homomorphism between intensional lattices $\mathbf{L}$ and $\mathbf{L'}$, and let T, T' be the truth filters of $\mathbf{L}$ and $\mathbf{L'}$ respectively. We say h is a **T-homomorphism** (or a T-**preserving homomorphism**) if it satisfies $h(a) \in T'$ for every $a \in T$. If h is also a one-to-one function, we call it a **T-isomorphism**.*

Theorem 3.1.27. *Let $\mathbf{L}=< L, \leqslant, \neg, T >$ be an intensional lattice and $P \subseteq L$ a prime filter of $\mathbf{L}$. P determines a T-homomorphism $h : L \longrightarrow \mathbf{M}_0$ satisfying for every $a \in L$:*

(i) $h(a) \in F_{-1}$ *if and only if* $a \in P$

(ii) $h(a) \in F_{-2}$ *if and only if* $\neg a \in -P(= L - P)$

(iii) $h(a) \in F_{+0}$ *if and only if* $a \in T$

where $F_{\pm i}$ is the principal filter of $\mathbf{M}_0$ generated by $\pm i$

Proof. First, we prove h is a function (i.e. for each $a \in L$ a has only one image). we consider $\mu F_{\pm i}$ to be $F_{\pm i}$ or $-F_{\pm i}(= L - F_{\pm i})$. Then depending on wheter a is in P, T and $\neg a$ is in $-P$, $h(a) \in \mu_1 F_{-1} \cap \mu_2 F_{-2} \cap \mu_0 F_{+0}$ and looking at the diagram for $\leqslant$ we can see this intersection uniquely determines one element of $\mathbf{M}_0$, the following table gives all possible cases:

μ_1		$-$			$-$	$-$		$-$
μ_2			$-$		$-$		$-$	$-$
μ_0				$-$		$-$	$-$	$-$
$h(a)$	$+3$	$+2$	$+1$	-0	$+0$	-2	-1	-3

Secondly, we prove h is an homomorphism. To show that $h(a \wedge b) = h(a) \wedge h(b)$ we only need to see that $h(a \wedge b) \in F_i$ if and only if $h(a) \wedge h(b) \in F_i$ where $i = -1, -2, +0$ because of our observation about intersection determining only one element. -1 and $+0$ are similar:

$$h(a \wedge b) \in F_{-1} \iff a \wedge b \in P \overset{\mathbf{F1'}}{\iff} a, b \in P \iff h(a), h(b) \in F_{-1}$$

$$h(a \wedge b) \in F_{+0} \iff a \wedge b \in T \overset{\mathbf{F1'}}{\iff} a, b \in T \iff h(a), h(b) \in F_{+0}$$

Now, for $i = -2$, we first observe that $-P$ is a prime ideal because the complement of a prime filter is a prime ideal (the negation of **F1′** is **PI′** and the negation of **PF′** is **I1′**). Using Lemma 3.1.15 in the first equivalence,

$$h(a \wedge b) \in F_{-2} \overset{\text{DeM}}{\iff} \neg a \vee \neg b \in -P \overset{\mathbf{I1'}}{\iff} \neg a, \neg b \in -P \iff h(a), h(b) \in F_{-2}$$

We show $h(\neg a) = \neg h(a)$, but first define $\overline{F_i} := \{\neg b : b \in F_i\}$. We prove the following:

$$\overline{-F_{-1}} = \overline{\{+0, -2, +2, -3\}} = \{-0, +2, -2, +3\} = F_{-2}$$
$$\overline{-F_{-2}} = \overline{\{+0, -1, +1, -3\}} = \{-0, +1, -1, +3\} = F_{-1}$$
$$\overline{-F_{+0}} = \overline{\{-0, -1, -2, -3\}} = \{+0, +1, +2, +3\} = F_{-2}$$

Finally,

$$h(\neg a) \in F_{-1} \iff \neg a \in P \iff h(a) \in -F_{-2} \iff \neg h(a) \in \overline{-F_{-2}} = F_{-1}$$

$$h(\neg a) \in F_{-2} \iff \neg a \in -P \iff h(a) \in -F_{-1} \iff \neg h(a) \in \overline{-F_{-1}} = F_{-2}$$

$$h(\neg a) \in F_{+0} \iff \neg a \in T \iff a \in -T \iff h(a) \in -F_{+0} \iff \neg h(a) \in \overline{-F_{+0}} = F_{+0}$$

The only thing left to see is h is T-preserving, which is deduced from (iii) since $F_{+0} = \{+0, +1, +2, +3\}$, the truth-filter of $\mathbf{M}_0$. □

From Proposition 3.1.11, we have a prime filter P where $a \in P$ and $b \notin P$, therefore with this theorem we can conclude:

Theorem 3.1.28. *Let $L <, \leqslant, \neg, T >$ be an intensional lattice and $a, b \in L$ such that $a \nleqslant b$, then there exists a T-preserving homomorphism $h : L \longrightarrow \mathbf{M}_0$ satisfying that $h(a) \in F_{-1}$ and $h(b) \notin F_{-1}$.*

Finally, we return to our logic to conclude the section.

Theorem 3.1.29 (Completeness and consistency with respect to $\mathbf{M}_0$)**.** *$\varphi \to \psi$ is valid in $\mathbf{M}_0$ if and only if $\vdash_{\mathbf{R}_{fde}} \varphi \to \psi$.*

Proof. If $\vdash_{\mathbf{R}_{\text{fde}}} \varphi \to \psi$, then by Theorem 3.1.24 $\varphi \to \psi$ is valid in every intensional lattice, in particular, it is valid in $\mathbf{M}_0$. This gives consistency.

Completeness is by contrapositive. We suppose $\nvdash_{\mathbf{R}_{\text{fde}}} \varphi \to \psi$, so $[\varphi] \nleqslant [\psi]$. By Theorem 3.1.28 there is an homomorphism $h : {}^{\mathbf{R}_{fde}}\!/_{\leftrightarrows} \longrightarrow \mathbf{M}_0$ such that $h([\varphi]) \in F_{-1}$ and $h([\psi]) \notin F_{-1}$. Therefore, since $F_{-1} = \{x | -1 \leqslant x\} = \{-1, -0, +1, +3\}$ it can't be that $h([\varphi]) \leqslant h([\psi])$. Now, $h \circ c$ is an assignment function for $\mathbf{M}_0$, so we consider the model $M =< \mathbf{M}_0, h \circ c >$, by induction we prove that for every non-implicational formula ξ $I_M(\xi) = h([\xi])$:

- If ξ is a variable p, $I_M(\xi) = h \circ c(p) = h([p])$.
- If $\xi = \neg\xi'$ and $I_M(\xi') = h([\xi'])$, $I_M(\xi) =_M (\xi') = \neg h([\xi']) = h([\neg\xi'])$.
- If $\xi = \xi_1 \hat{\vee} \xi_2$ and $I_M(\xi_1) = h([\xi_1])$, $I_M(\xi_2) = h([\xi_2])$, then

$$I_M(\varphi) = I_M(\psi) \hat{\vee} I_M(\xi) = h([\psi]) \hat{\vee} h([\xi]) = h([\varphi])$$

Finally, since $h([\varphi]) \nleqslant h([\psi])$, $I_M(\varphi) \nleqslant I_M(\psi)$, so $\varphi \to \psi$ is falsifiable in $\mathbf{M}_0$. □

Because of its finite number of designated elements, or *truth-values*, this type of semantics is useful since it allows us to easily prove if a proposition is a theorem. And actually, Belnap gave it a more manageable, four-valued semantics in [3]. In fact, this is the usual presentation this logic, which has been independently researched for 40 years now (a overview of this field can be found in [15]).

3.2 Semantics for R

3.2.1 An equivalent algebraic semantics for R

First we give the algebraic definitions and properties necessary:

Definition 3.2.1. *A **relevant algebra** (or **R-algebra**) is a tuple* $\mathbf{A} =< A, \wedge, \vee, \to, \neg >$ *where* $< A, \wedge, \vee, \neg >$ *is a De Morgan lattice, and for all* $a, b, c \in A$*:*

RA1 $a \to (b \to c) \leqslant b \to (a \to c)$

RA2 $a \leqslant ((a \to b) \wedge c) \to b$

RA3 $a \to \neg b \leqslant b \to \neg a$

RA4 $a \to \neg a \leqslant \neg a$

RA5 $((a \to a) \wedge (b \to b)) \to c \leqslant c$

*The class of all **R**-algebras will be denoted by* $\mathcal{R}$*.*

RA1,3 are actually equalities, by symmetry. Also, note that we are taking the algebraic presentation of lattices, with $\wedge$ and $\vee$ as primitive.

Proposition 3.2.2. *We now list some properties of **R**-algebras. Let* $\mathbf{A} =< A, \wedge, \vee, \to, \neg >\in \mathcal{R}$*, then for all* $a, b, c \in A$*:*

P1 *If* $a \leqslant b \to c$ *then* $b \leqslant a \to c$*.*

P2 *If* $a \leqslant b$ *then* $b \to c \leqslant a \to c$*.*

P3 *If* $a \leqslant b$ *then* $c \to a \leqslant c \to b$*.*

P4 $a \to b \leqslant (c \to a) \to (c \to b)$

P5 $a \to (a \to b) \leqslant a \to b$

P6 $a \to b \wedge c = (a \to b) \wedge (a \to c)$

P7 $a \vee b \to c = (a \to c) \wedge (b \to c)$

Proof. Probably because the proof of these properties is short, it isn't in the article we are following, but we reproduce it nonetheless:

P1 $b \overset{\text{RA2}}{\leqslant} ((b \to c) \wedge a) \to c = a \to c$.

P2 $a \leqslant b \overset{\text{RA2}}{\leqslant} (b \to c) \wedge (b \to c) \to c = (b \to c) \to c$ so by P1, $b \to c \leqslant a \to c$.

P3 If $a \leqslant b$, $\neg b \leqslant \neg a$, by P2 $\neg a \to \neg c \leqslant \neg b \to \neg c$ and since RA3 is an equality, $c \to a \leqslant c \to b$.

P4 By RA2, $a \leqslant (a \to b) \to b$ and $c \leqslant (c \to a) \to a$. The first identity implies by P3 that $c \leqslant (c \to a) \to a \leqslant (c \to a) \to ((a \to b) \to b) \overset{\text{RA1}}{=} (a \to b) \to ((c \to a) \to b)$. Now, P1 implies that $a \to b \leqslant c \to ((c \to a) \to b) \overset{\text{RA1}}{=} (c \to a) \to (c \to b)$.

P5 $a \to (a \to b) \overset{\text{RA3}}{=} a \to (\neg b \to \neg a) \overset{\text{RA1}}{=} \neg b \to (a \to \neg a) \overset{\text{RA3}}{=} \neg(a \to \neg a) \to b \leqslant a \to b$ where the last inequality is obtained from the fact that $a \leqslant \neg(a \to \neg a)$ (by RA4) using P2.

P6 is directly from P3 and **P7** directly from P2. □

Definition 3.2.3. *For any* $\mathbf{A} \in \mathcal{R}$*, we denote by* $E(\mathbf{A})$ *the filter generated by* $\{a \to a : a \in A\}$*. We will write* $\mathbb{R} = \{< \mathbf{A}, E(\mathbf{A}) >: \mathbf{A} \in \mathcal{R}\}$.

Lemma 3.2.4. *If* $\mathbf{A} \in \mathcal{R}$*, then for any* $a, b \in A$*,* $a \leqslant b$ *if and only if* $a \to b \in E(\mathbf{A})$.

Proof. The first implication is from **P3** putting $c = a$. Conversely, supposing $a \to b \in E(\mathbf{A})$, this implies by Lemma 3.1.9 that there are $c_1, ..., c_n \in A$ such that $c = (c_1 \to c_1) \wedge ... \wedge (c_n \to c_n) \leqslant a \to b$. Now, we prove by induction over n that for $a_i, ..., a_n \in A$ if $a_i \to a_i \leqslant a_i$ for all $i \leqslant n$ then $a_1 \wedge ... \wedge a_n \to a_1 \wedge ... \wedge a_n \leqslant a_1 \wedge ... \wedge a_n$:

Base case($n = 1$): if $a \to a \leqslant a$ then it is clear this holds.

Inductive step: We suppose the statement is true for $n - 1$ and $a_i \to a_i \leqslant a_i$ for all $i \leqslant n$. By hypothesis of induction, $a_1 \wedge ... \wedge a_{n-1} \to a_1 \wedge ... \wedge a_{n-1} \leqslant a_1 \wedge ... \wedge a_{n-1}$ and since $a_n \to a_n \leqslant a_n$, $(a_1 \wedge ... \wedge a_{n-1} \to a_1 \wedge ... \wedge a_{n-1}) \wedge (a_n \to a_n) \leqslant (a_1 \wedge ... \wedge a_{n-1}) \wedge a_n$. Therefore, using **P2**, $a_1 \wedge ... \wedge a_n \to a_1 \wedge ... \wedge a_n \leqslant (a_1 \wedge ... \wedge a_{n-1} \to a_1 \wedge ... \wedge a_{n-1}) \wedge (a_n \to a_n) \to a_1 \wedge ... \wedge a_n$. Finally, since by **RA5**, $(a_1 \wedge ... \wedge a_{n-1} \to a_1 \wedge ... \wedge a_{n-1}) \wedge (a_n \to a_n) \to a_1 \wedge ... \wedge a_n \leqslant a_1 \wedge ... \wedge a_n$ we get what we wanted.

With this, since from **RA5** (putting $a = b = z_i$ and $c = c_i \to c_i$) we obtain that $(c_i \to c_i) \to (c_i \to c_i) \leqslant (c_i \to c_i)$, we can conclude $c \to c \leqslant c$. This implies $c \to c \leqslant a \to b$, so that using **P1** $a \leqslant (c \to c) \to b \overset{\text{RA5}}{\leqslant} b$. □

Definition 3.2.5. *We define* $\leftrightarrows_\Sigma$ *as the following relation over formulas in the language of* $\boldsymbol{R}$*: let* $\Sigma \cup \{\varphi, \psi\} \subseteq Fm_{\boldsymbol{R}}$*, then* $\varphi \leftrightarrows_\Sigma \psi \overset{def}{\iff} \Sigma \vdash_{\boldsymbol{R}_{fde}} \varphi \to \psi$ *and* $\Sigma \vdash_{\boldsymbol{R}_{fde}} \psi \to \varphi$.

Proposition 3.2.6. $\leftrightarrows_\Sigma$ *is a congruence over the algebra of formulas of* $\boldsymbol{R}$.

Proof. The proof is analogous to the case of fde's, and the case of distributivity with respect to $\to$ is done using only $\mathbf{R}2$ and IR (which we have proven is valid in $\mathbf{R}$ in Lemma 2.2.4). Supposing $\varphi \leftrightarrows_\Sigma \psi$, $\varphi' \leftrightarrows_\Sigma \psi'$ we give the proof of one direction, the other one is symmetrical. Taking Σ as premises, from the hypothesis we have $\varphi \to \psi$ and $\psi' \to \varphi'$. From the first formula, by $\mathbf{R}2$ and using *modus ponens* $\Sigma \vdash_{\mathbf{R}} (\psi' \to \varphi) \to (\psi' \to \psi)$. From the second, and using the equivalent form of $\mathbf{R}2$ (by contraction) $\Sigma \vdash_{\mathbf{R}} (\varphi' \to \varphi) \to (\psi' \to \varphi)$. Finally, IR gives $\Sigma \vdash_{\mathbf{R}} (\varphi' \to \varphi) \to (\psi' \to \psi)$. □

Theorem 3.2.7. *Let* $\Sigma \subseteq Fm_{\mathcal{L}_R}$, *then* ${}^{\mathcal{F}_{\mathbf{R}}}\!\big/_{\leftrightarrows_\Sigma} \in \mathcal{R}$ *and the order induced by the underlying lattice is* $\leqslant_\Sigma$ *where* $[\varphi] \leqslant_\Sigma [\psi]$ *if and only if* $\Sigma \vdash_{\mathbf{R}} \varphi \to \psi$.

Proof. To find the order induced by the lattice we use the characterisation of the first chapter, so $[\varphi] \leqslant_\Sigma [\psi] \iff [\varphi] = [\varphi] \wedge [\psi] = [\varphi \wedge \psi] \iff \Sigma \vdash_{\mathbf{R}} \varphi \leftrightarrows \varphi \wedge \psi$. This is equivalent to $\Sigma \vdash_{\mathbf{R}} \varphi \to \psi$, the first direction using axioms $\mathbf{R}5, 2$ and the converse is by $\mathbf{R}7$ and the rule of adjunction. Now we can use the non-algebraic presentation of lattice since it will make the proof easier. $\leqslant_\Sigma$ is reflexive ($\mathbf{R}1$), transitive (IR) and anti-symmetric by the definition of $\leftrightarrows_\Sigma$, and therefore is a partial order. As in the case of fde's, there always are lub and glb $[\varphi] \vee [\psi] = [\varphi \vee \psi]$ and $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$. Distributivity is proven from the fact that $\vdash_{\mathbf{R}_{\text{fde}}} \varphi \wedge (\psi \vee \xi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$ and $\vdash_{\mathbf{R}_{\text{fde}}} \varphi \vee (\psi \wedge \xi) \to (\varphi \vee \psi) \wedge (\varphi \vee \xi)$. Also, $\mathbf{R}13$ and its converse give $[\varphi] = [\neg\neg\varphi] = \neg\neg[\varphi]$, and since $\vdash_{\mathbf{R}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$, if $\Sigma \vdash_{\mathbf{R}} \varphi \to \psi$, then $\Sigma \vdash_{\mathbf{R}} \neg\psi \to \neg\varphi$, so $[\neg\psi] \leqslant_\Sigma [\neg\varphi]$. This proves that $< {}^{Fm_{\mathbf{R}}}\!\big/_{\leftrightarrows_\Sigma}, \leqslant_\Sigma, \neg >$ is a De Morgan lattice.

Finally, we need to prove **RA1-5**. First, $\mathbf{R}3$ proves **RA1** and $\mathbf{R}12$ proves **RA3**. From $\mathbf{R}5$ and monotony we get $\Sigma \vdash_{\mathbf{R}} ((\varphi \to \psi) \wedge \xi) \to (\varphi \to \psi)$, so $\mathbf{R}3$ gives $\Sigma \vdash_{\mathbf{R}} \varphi \to (((\varphi \to \psi) \wedge \xi) \to \psi)$, which concludes **RA2**. From $\mathbf{R}3, 12$ we obtain $\Sigma \vdash_{\mathbf{R}} \varphi \to ((\varphi \to \neg\varphi) \to \neg\varphi)$, and with contraposition $\Sigma \vdash_{\mathbf{R}} \varphi \to (\neg\neg\varphi \to (\varphi \to \neg\varphi))$. The double negation is eliminated by the Replacement Theorem, and using $\mathbf{R}4$ concludes $\Sigma \vdash_{\mathbf{R}} \varphi \to (\varphi \to \neg\varphi)$, which implies **RA4**. To show **RA5** we need $\Sigma \vdash_{\mathbf{R}} ((\varphi \to \varphi) \wedge (\psi \to \psi) \to \xi) \to \xi$. From $\mathbf{R}1$ and $\mathbf{R}3$, $\vdash_{\mathbf{R}} (\varphi \to \varphi) \wedge (\psi \to \psi) \to (((\varphi \to \varphi) \wedge (\psi \to \psi) \to \xi) \to \xi)$. $\mathbf{R}1$ again and &I give $\vdash_{\mathbf{R}} (\varphi \to \psi) \wedge (\psi \to \psi)$, so using *modus ponens* and by monotony of $\vdash_{\mathbf{R}}$, we get what we wanted. □

Theorem 3.2.8 (Consistency)**.** *For every* $\mathbf{A} \in \mathcal{R}$, $< \mathbf{A}, E(\mathbf{A}) >$ *is a matrix model for* $\boldsymbol{R}$.

Proof. We need to prove the interpretation of the axioms is in $E(\mathbf{A})$ and the rules preserve belonging. The properties of filters assure us that rules preserve belonging to $E(\mathbf{A})$. Now, since all axioms are implicational i.e. of the form $\varphi \to \psi$, Lemma 3.2.4 tells us that proving $I(\varphi) \leqslant_\Sigma I(\psi)$ we would be done. $\mathbf{R}1$ is verified because orders are reflexive. $\mathbf{R}2$ is from **P4**, $\mathbf{R}3$ from **RA1**, $\mathbf{R}4$ from **P5**. The fact that $a \wedge b, b$ verifies $\mathbf{R}5, 6$, and **P6** checks $\mathbf{R}7$. Since $a \vee b \geqslant a, b$ $\mathbf{R}8, 9$ are verified, and **P7** checks $\mathbf{R}10$. By the properties of distributive lattices, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \overset{a \wedge c \leqslant c}{\leqslant} (a \wedge b) \vee c$, so $\mathbf{R}11$ is checked. **RA3** verifies $\mathbf{R}12$ and finally the negation properties of De Morgan lattices verify $\mathbf{R}13$. □

Theorem 3.2.9 (Completeness)**.** *Let* $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_{\mathbf{R}}}$*, if* $\Sigma \vDash_{\mathbb{R}} \varphi$*, then* $\Sigma \vdash_{\mathbf{R}} \varphi$*.*

Proof. By contrapositive. Suppose $\Sigma \nvdash_{\mathbf{R}} \varphi$, we want to see that $\Sigma \nvDash_{\mathbb{R}} \varphi$, that is, there is some relevant algebra for which φ is not a consequence of Σ. We take ${}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma} \in \mathcal{R}$ and the canonical interpretation $I_c : Fm_{\mathcal{L}_{\mathbf{R}}}$ where $I_c(\psi) = [\psi]$ (in the same way as with $\mathbf{R}_{\text{fde}}$). $I_c[\Sigma] \subseteq E\left({}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma}\right)$ since if $\psi \in \Sigma$ by the reflexivity of logics $\Sigma \vdash_{\mathbf{R}} \psi$. And since ${}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma} \in \mathbb{R}$, by Theorem 3.2.8 $[\psi] \in E\left({}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma}\right)$. The only thing left is to see that $[\varphi] \notin E\left({}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma}\right)$. We suppose it is, and arrive at a contradiction. From **RA2**, we obtain that for all a in a relevant algebra $a \leqslant (a \to a) \to a$. Therefore, since $[\varphi] \in E\left({}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma}\right)$, by the properties of filters $[(\varphi \to \varphi) \to \varphi] \in E\left({}^{\mathcal{F}_{\mathbf{R}}}\!/_{\leftrightarrows_\Sigma}\right)$, which by Lemma 3.2.4 implies $[\varphi \to \varphi] \leqslant_\Sigma [\varphi]$. Now, this means by the definition of order we gave that $\Sigma \vdash_{\mathbf{R}} (\varphi \to \varphi) \to \varphi$, which in turn by axiom **R**1 gives $\Sigma \vdash_{\mathbf{R}} \varphi$, in contradiction with our hypothesis. ◻

Corollary 3.2.10. $\mathbb{R}$ *is a matrix semantics for* ***R****.*

Proving that ${}^{\mathbf{R}}\!/_{\leftrightarrows}$ is the free **R**-algebra will assure **R**-algebras are a good representation of our relevance logic.

Theorem 3.2.11. ${}^{\boldsymbol{R}}\!/_{\leftrightarrows}$ *is the free* ***R****-algebra.*

Proof. The generators of ${}^{\mathbf{R}}\!/_{\leftrightarrows}$ are ${}^{Var}\!/_{\leftrightarrows}$. Suppose $\mathbf{A} \in \mathcal{R}$ and we have the identification $f : {}^{Var}\!/_{\leftrightarrows} \longrightarrow \mathbf{A}$. We need to prove it can be extended to an homorphism from ${}^{\mathbf{R}}\!/_{\leftrightarrows}$ into $\mathbf{A}$. We consider the assignment

$$\begin{aligned} s : Var &\longrightarrow \mathbf{A} \\ p &\longmapsto f([p]) \end{aligned}$$

s extends naturally into an interpretation $I_s : Fm_{\mathcal{L}_{\mathbf{R}}} \longrightarrow \mathbf{A}$.Now, we define

$$\begin{aligned} h : {}^{Var}\!/_{\leftrightarrows} &\longrightarrow \mathbf{A} \\ [\varphi] &\longmapsto I_s(\varphi) \end{aligned}$$

h is well defined, since if $[\varphi] = [\psi]$, then $\vdash_{\mathbf{R}} \varphi \to \psi$ and $\vdash_{\mathbf{R}} \psi \to \varphi$, and by Theorem 3.2.8 $I_s(\varphi \to \psi), I_s(\psi \to \varphi) \in E({}^{\mathbf{R}}\!/_{\leftrightarrows})$, so $I_s(\varphi) = I_s(\psi)$ via Lemma 3.2.4. Since clearly h is an homomorphism (from the definition of interpretation), we have proven that ${}^{\mathbf{R}}\!/_{\leftrightarrows}$ is the free **R**-algebra. ◻

Now, in order to prove this matrix semantics ($\mathbb{R}$) is the more adequate for **R**, we need to find the defining equations for its equivalent algebraic semantics, so we need to characterise the set of designated elements for relevant algebras:

Lemma 3.2.12. *If* $\mathbf{A} \in \mathcal{R}$ *then* $E(\mathbf{A}) = \{a \in A : a \to a \leqslant a\}$.

Proof. If $a \in \mathbf{A}$, then $a \to ((a \to a) \to a) \overset{\mathbf{RA1}}{=} (a \to a) \to (a \to a) \in E(\mathbf{A})$, so if moreover $a \in E(\mathbf{A})$ then since $E(\mathbf{A})$ is a filter $(a \to a) \to a \in E(\mathbf{A})$, and by Lemma 3.2.4 $a \to a \leqslant a$ □

As is noted in [10], in Chapter 5 of [4] it is proven that **R** is algebraizable with defining equation $\delta \approx \epsilon$, where $\delta(p) = p \wedge (p \to p)$ and $\epsilon(p) = p \to p$, and equivalence formula $\Delta(p,q) = (p \to q) \wedge (q \to p) =: p \leftrightarrow q$, but its equivalent algebraic semantics is not stated. We can now determine the algebraizability of **R** with respect to the class of relevant algebras. But first we observe this class is a variety, since changing any $a \leqslant b$ into the equivalent $a \wedge b = a$ all the postulates for **R**-algebras are identities.

Theorem 3.2.13. $\mathcal{R}$ *is the equivalent algebraic semantics for* **R**.

Proof. Since for $a, b \in \mathbf{A}$, $\mathbf{A} \in \mathcal{R}$, $a \leqslant b$ if and only if $a \wedge b = a$, $E(\mathbf{A}) = \{a \in A : a \wedge (a \to a) = a \to a\}$. Since $\mathbb{R}$ is a matrix semantics for **R** (Corollary 3.2.10) and $\mathcal{R}$ is a variety (in particular, a quasi-variety), from Theorem 1.3.3 we obtain that $\mathcal{R}$ is an algebraic semantics for **R** with defining equation $p \wedge (p \to p) = p \to p$. To prove that $\mathcal{R}$ is equivalent to **R**, we need to see that given $\varphi, \psi \in Fm$ $\varphi \approx \psi \vDash_{\mathcal{R}} \dashv \delta(\varphi\Delta\psi) \approx \epsilon(\varphi\Delta\psi)$. Let $\mathbf{A} \in \mathcal{R}$ and $I : Fm_{\mathcal{L}_{\mathbf{R}}} \to \mathbf{A}$ be an interpretation, and put $a = I(\varphi)$, $b = I(\psi)$. We prove that $a = b$ if and only if $\delta^{\mathbf{A}}(a\Delta^{\mathbf{A}}b) = \epsilon^{\mathbf{A}}(a\Delta^{\mathbf{A}}b)$. If $a = b$, then

$$\begin{aligned}(a \leftrightarrow b) \wedge ((a \leftrightarrow b) \to (a \leftrightarrow b)) = \\ &= (a \leftrightarrow b) \wedge ((a \leftrightarrow b) \to (a \leftrightarrow b)) = (a \to a) \wedge ((a \to a) \to (a \to a)) \overset{\mathbf{P4}}{=} \\ &= (a \to a) \to (a \to a) = (a \leftrightarrow a) \to (a \leftrightarrow a) = \\ &= (a \leftrightarrow b) \to (a \leftrightarrow b)\end{aligned}$$

Conversely, if $(a \leftrightarrow b) \wedge ((a \leftrightarrow b) \to (a \leftrightarrow b)) = (a \leftrightarrow b) \to (a \leftrightarrow b)$, then $a \leftrightarrow b \in E(\mathbf{A})$ and by Lemma 3.2.4 $a = b$. □

3.2.2 The least truth in R

The previous section on the semantics of **R** followed [10], but Anderson and Belnap also gave a semantics for **R**. They started by seeing that the Lindenbaum algebra of **R** is a De Morgan semi-group, a De Morgan lattice which satisfies **RA1-4**. But **R** is not free in the class of De Morgan semi-groups, since any interpretation of the formulas of **R** will satisfy **RA5**, but a De Morgan semi-group won't necessarily do so. Therefore, De Morgan semi-groups are not an adequate representation of **R**, it's too large a class, and so Anderson and Belnap find a matrix semantics for $\mathbf{R^t}$ in the class of De Morgan monoids (smaller than De Morgan semi-groups) and from there prove completeness and consistency of **R** with respect to De Morgan monoids (which would mean that $\mathbf{R^t}$ is a conservative expansion of **R**). But De Morgan monoids have a constant **t** which **R** doesn't have, and so this class of algebras doesn't adequately represent **R**, which is why we chose to present the semantics in [10].

We now turn to the proof that De Morgan monoids are a matrix semantics for $\mathbf{R^t}$ with the adequate set of designated elements in the style of [10], and we will end by enriching this semantics using the tools of algebraic semantics in [4], which were developed after the publishing of [2]. What interests us about this is that the algebraic structure of $\mathbf{R^t}$, being close to $\mathbf{R}$, gives us more information about our relevance logic.

Definition 3.2.14. *A* ***De Morgan monoid*** *is a tuple* $\mathbf{A} =< A, \leqslant, \to, \neg, \mathbf{t} >$ *where* $< A, \leqslant, \neg >$ *is a De Morgan lattice, and for all* $a, b, c \in A$*:*

- **D1** (RA1) $a \to (b \to c) \leqslant b \to (a \to c)$
- **D2** (RA2) $a \leqslant ((a \to b) \wedge c) \to b$
- **D3** (RA3) $a \to \neg b \leqslant b \to \neg a$
- **D4** (RA4) $a \to \neg a \leqslant \neg a$
- **D5** $\mathbf{t} \to a = a$

The class of all De Morgan monoids will be denoted by $\mathcal{R}^{\mathbf{t}}$.

Lemma 3.2.15. *To the properties we had for* ***R****-algebras, which are valid for De Morgan monoids since its proof did not use* **RA5***, we add the following:*

P8 *For all* $a \in A$, $\mathbf{t} \leqslant a \to a$

Proof. From D2, $\mathbf{t} \leqslant (\mathbf{t} \to a) \wedge (\mathbf{t} \to a) \to a = (\mathbf{t} \to a) \to a = a \overset{D5}{=} a \to a$ □

Definition 3.2.16. *For any* $\mathbf{A} \in \mathcal{R}^{\mathbf{t}}$*, we denote by* $E^{\mathbf{t}}(\mathbf{A})$ *the principal filter generated by* $\mathbf{t}$*. We denote* $\mathbb{R}^{\mathbf{t}} = \{< \mathbf{A}, E^t(\mathbf{A}) >: \mathbf{A} \in \mathcal{R}^t\}$.

We note that this is relevant because the filters defined in the previous section had no lower bound, since for any $a \in E(\mathbf{A})$, $a \to a \leqslant a$ and $a \to a \in E(\mathbf{A})$. No constant may be defined in $\mathbf{R}$ that takes the role of least truth, and the reason that $\mathbf{t}$ is not naturally in $\mathbf{R}$ as $\top$ is in CPC is precisely that. Therefore, $\mathbf{t}$ acts as the smallest truth in $\mathbf{R}$, the truth from which all other truths arise, but is not in $\mathbf{R}$.

We return to our logic to find the Lindembaun algebra of $\mathbf{R^t}$. In the same way as $\mathbf{R}$, the relation $\leftrightarrows^t_\Sigma$ defined analogously as $\leftrightarrows_\Sigma$ is a congruence in the algebra $\mathcal{F}_{\mathbf{R^t}}$ of formulas of $\mathbf{R^t}$, and we can take the relation $\leqslant^{\mathbf{t}}_\Sigma$ as primitive in $\mathcal{F}_{\mathbf{R^t}} / _{\leftrightarrows^{\mathbf{t}}_\Sigma}$. We denote by $\mathbf{R^t} / _{\leftrightarrows}$ the **Lindenbaum algebra of $\mathbf{R^t}$**.

Theorem 3.2.17. *Let* $\Sigma \subseteq Fm_{\mathcal{R}_{\mathbf{R^t}}}$*, then* $\mathcal{F}_{\mathbf{R^t}} / _{\leftrightarrows^{\mathbf{t}}_\Sigma} \in \mathcal{R}$.

Proof. The fact that it satisfies D1-4 can be seen with the same proof as for $\mathbf{R}$. We only need to show there is an element acting as $\mathbf{t}$ in D5, and that is easy since we have proven Equation 2.1 for the constant $\mathbf{t}$ in $\mathbf{R^t}$, therefore for every $\varphi \in Fm_{\mathcal{R}_{\mathbf{R^t}}}$ $[\mathbf{t}] \to [\varphi] = [\mathbf{t} \to \varphi] = [\varphi]$ making $[\mathbf{t}] = \mathbf{t}$ in the algebra. □

An in the previous chapter, we characterise order in our class of matrices:

Lemma 3.2.18. *If $\mathbf{A} \in \mathcal{R}^{\mathbf{t}}$, then for every $a, b \in A$, $a \leqslant b$ if and only if $a \to b \in E^{\mathbf{t}}(\mathbf{A})$.*

Proof. $a \leqslant b = t \to b$ is equivalent by P1 to $t \leqslant a \to b$, and by the characterisation of principal filters in Lemma 3.1.10 this is equivalent to $a \to b$ being an element of $E(\mathbf{A})$. ▫

Theorem 3.2.19 (Completeness and consistency)**.** *Let $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_{\mathbf{R}^{\mathbf{t}}}}$, $\Sigma \vDash_{\mathbb{R}^{\mathbf{t}}} \varphi$ if and only if $\Sigma \vdash_{\mathbf{R}^t} \varphi$. Therefore, $\mathbb{R}^{\mathbf{t}}$ is a matrix semantics for $\mathbf{R}^{\mathbf{t}}$.*

Proof. With the same proof as in Theorem 3.2.8, and by Lemma 3.2.2, the rules and axioms of $\mathbf{R}^t$ except **t**1 and **t**2. **t**1 is verified by the fact that $t \in E^t(\mathbf{A})$ for every $\mathbf{A} \in \mathcal{R}^t$ and **t**2 by P8. This concludes our proof of consistency.

For the converse, we see that if $\Sigma \nvdash_{\mathbf{R}^t} \varphi$, then $\Sigma \nvDash_{\mathbb{R}^t} \varphi$. For that, we only need a matrix in $\mathbb{R}^t$ and an interpretation which falsify φ. We consider $\mathcal{F}_{\mathbf{R}^{\mathbf{t}}} / \underset{\Sigma}{\overset{\mathbf{t}}{\leftrightarrows}} \in \mathcal{R}$ and the canonical interpretation $I : Fm_{\mathcal{L}_{\mathbf{R}^{\mathbf{t}}}} \longrightarrow \mathcal{F}_{\mathbf{R}^{\mathbf{t}}} / \underset{\Sigma}{\overset{\mathbf{t}}{\leftrightarrows}}$ defined by $I(\psi) = [\psi]$, which can be seen to be the interpretation extending the assignment of variables $s(p) = [p]$ by induction. We suppose $\varphi \in E^{\mathbf{t}}(\mathcal{F}_{\mathbf{R}^{\mathbf{t}}} / \underset{\Sigma}{\overset{\mathbf{t}}{\leftrightarrows}})$, then $[t] \leqslant^{\mathbf{t}}_{\Sigma} [\varphi]$, so by the definition of the order, $\vdash_{\mathbf{R}^{\mathbf{t}}} t \to \varphi$. Axiom **t**1 gives $\vdash_{\mathbf{R}^{\mathbf{t}}}$, which is a contradiction with the supposition that $\Sigma \nvdash_{\mathbf{R}^t} \varphi$. Therefore $\varphi \notin E^{\mathbf{t}}(\mathcal{F}_{\mathbf{R}^{\mathbf{t}}} / \underset{\Sigma}{\overset{\mathbf{t}}{\leftrightarrows}})$, which finishes our proof of completeness. ▫

As in **R**, and with an analogous proof, the following is true:

Theorem 3.2.20. *$\mathbf{R}^{\mathbf{t}} / \leftrightarrows$ is the free De Morgan monoid.*

Finally, we can state:

Theorem 3.2.21. *$\mathcal{R}$ is the equivalent algebraic semantics for $\mathbf{R}^{\mathbf{t}}$ with equivalence formula $\Delta(p, q) = p \leftrightarrow q$ and defining equation $\delta \approx \epsilon$, where $\delta(p) = p \wedge t$ and $\epsilon(p) = \mathbf{t}$.*

Proof. From Theorem 3.2.19, $\mathcal{R}^{\mathbf{t}}$ is a matrix semantics for $\mathbf{R}^{\mathbf{t}}$, and it is also clearly a quasi-variety (in fact, a variety). Moreover by Lemma 3.1.10 given $\mathbf{A} \in \mathcal{R}$, $E(\mathbf{A}) = \{a \in A : \mathbf{t} \leqslant a\} = \{a \in A : a \wedge \mathbf{t} = \mathbf{t}\}$. By Theorem 1.3.3, $\mathcal{R}$ is an algebraic semantics for $\mathbf{R}^{\mathbf{t}}$ with defining equation $p \wedge t = t$. Now, given $\mathbf{A} \in \mathcal{R}$ and an interpretation $I : Fm_{\mathcal{L}_{\mathbf{R}^{\mathbf{t}}}} \longrightarrow \mathbf{A}$, writing $I(\varphi) = a$ and $I(\psi) = b$ we need to prove $a = b$ if and only if $(a \leftrightarrow b) \wedge \mathbf{t} = \mathbf{t}$: $a = b$ is equivalent by Lemma 3.2.2 to $a \leftrightarrow b \in E^{\mathbf{t}}(\mathbf{A})$ which implies $\mathbf{t} \wedge (a \leftrightarrow b) = \mathbf{t}$ and conversely. ▫

Conclusions

In the context of non-classical logics, **R** is an alternative to classical logic which avoids the *material implication paradoxes* by assessing their relevance. The motivation for it is very clear, and it fulfills its aim seeing as with this study, we have found that relevance in implication can be modelised by the logic **R**. A formal account of our understanding of implication is given in this system, which assures that the antecedent is relevant to the truth of the consequent.

R has some interesting properties revealing the depth of the topic of relevance. From a syntactical point of view, we have found that relevance is expressed in the sharing of variables. The fact that this is a necessary condition is a very important consequence of the study of **R**, since it tells us that antecedent and consequent must share meaning. But not only do we have this, the shared meaning must be in a determined position, something that allows us to easily discard false formulas. This is a powerful tool characteristic to **R** which has no parallel in classical logic, and reveals the strength of our logic.

But **R** also has other relevant properties which are unveiled in the study of its semantics. The equivalent algebraic semantics of **R** induces a matrix semantics where the designated elements form a filter with infinitely many generators. Therefore, truth in **R** is not defined by a determined constant, in contrast to the fact that all tautologies are equivalent to $\top$ in classical logic. A constant t may be added as the lower bound of the filter of designated elements, acting as the conjunction of all truths, the least truth. But this gives rise to a larger logic, $\mathbf{R^t}$.

Apart from $\mathbf{R^t}$, there are other neighbours of **R** which unveil the intricacies of relevance logic. **E**, $\mathbf{R}_{\mathrm{fde}}$ and many others which haven't been introduced in this work are relevance logics closely related to **R** which have their own importance in the field of relevance logic. In this work, we have highlighted the eight-valued semantics of $\mathbf{R}_{\mathrm{fde}}$, interesting in the sense that no operation is defined as analogous to $\to$, but instead an order relation acts as implication, giving rise to a non-algebraic semantics.

Finally, the importance of [2] in the development of the field of relevance logics is clear, and we have given an overview which unifies and clarifies the book's knowledge of the system of logic **R**, adding relevant information posterior to its publication. I recommend this book to anyone interested, since it gives a very complete account of many aspects of relevance logic and it does so in a very clear and at times even funny way. It has been a delight to read, and I am grateful to have been able to do so.

Bibliography

[1] W. Ackermann, *Begründung einer strengen implikation*, The Journal of Symbolic Logic, **21**(1956), 113–128.

[2] A.R. Anderson and N.D. Belnap, *Entailment - The logic of relevance and necessity I*, Princeton University Press (1975).

[3] N.D. Belnap, *A useful four-valued logic*, in J.M. Dunn, and G. Epstein, *Modern uses of multiple-valued logic*, D. Reidel Publishing Co. (1977), 8-–37.

[4] W. Blok and D. Pigozzi, *Algebraizable logics*, Series Memoirs of the American Mathematical Society, **396** Providence: American Mathematical Society (1989).

[5] W. Blok and D. Pigozzi, *Abstract algebraic logic and the deduction theorem*, Bulletin of Symbolic Logic (To appear) (2001)

[6] S. Burris and H.P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, (1981). Edition consulted available at: `http://www.math.uwaterloo.ca/~snburris/htdocs/UALG/univ-algebra2012.pdf`

[7] J.M. Dunn and G.Hardegree, *Algebraic methods in philosophical logic*, Oxford University Press (2001).

[8] H.B. Enderton, *A mathematical introduction to logic*, Harcourt/Academic Press (1972).

[9] F.B. Fitch, *Symbolic logic: an introduction*, Ronald Press Co. (1952).

[10] J.M. Font and G. Rodríguez, *Note on algebraic models for relevance logic*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, **36** (1990), 535–540.

[11] Y. Imai and K. Iséki, *Axiom systems of propositional calculi. I*, Proc. Japan Academy, **41** (1965), no. 6, 436–439.

[12] K. Iséki and S. Tanaka, *Axiom systems of propositional calculi. V*, Proc. Japan Academy, **41** (1965), no. 8, 661–662.

[13] J. Łukasiewicz, *Elements of mathematical logic*, Pergamon Press (1963).

[14] R.K. Meyer and R. Routley, *E is a conservative extension of* $E\bar{I}$, Philosophia **4** (1974), 223–249.

[15] H. Omori and H. Wansig, *40 years of FDE: an introductory overview*, Studia Logica (2017).

[16] M.H. Stone, *Topological representations of distributive lattices and Browerian logics*, Casopis pro pestovaní matematiky a fysiky, **67** (1937), 1–25.

[17] A. Tarski, *Grundzügue des systemenkallkül, erster teil*, Fund. Math. **25** (1935), 503–526. English translation *Foundations in the calculus of systems* in A. Tarski, *Logic, semantic, mathematics*, Oxford (1956), 342–364.

Appendices

A Deductions from different calculi

$\vdash_{\mathbf{FR}_{\rightarrow}} (\varphi \rightarrow \psi) \rightarrow ((\xi \rightarrow \varphi) \rightarrow (\xi \rightarrow \psi))$

1	$\varphi \rightarrow \psi_{\{1\}}$	hyp
2	$\xi \rightarrow \varphi_{\{2\}}$	hyp
3	$\xi_{\{3\}}$	hyp
4	$\xi \rightarrow \varphi_{\{2\}}$	reit, 2
5	$\varphi_{\{2,3\}}$	$\rightarrow$E, 3, 4
6	$\varphi \rightarrow \psi_{\{1\}}$	reit, 1
7	$\psi_{\{1,2,3\}}$	$\rightarrow$E, 5, 6
8	$\xi \rightarrow \psi_{\{1,2\}}$	$\rightarrow$I, 3, 7
9	$(\xi \rightarrow \varphi) \rightarrow (\xi \rightarrow \psi)_{\{2\}}$	$\rightarrow$I, 2, 8
10	$(\varphi \rightarrow \psi) \rightarrow ((\xi \rightarrow \varphi) \rightarrow (\xi \rightarrow \psi))$	$\rightarrow$I, 1, 9

$\vdash_{\mathbf{FR}_{\to}} (\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$

1	$\mid\ \mid\ \varphi \to (\psi \to \xi)_{\{1\}}$	hyp
2	$\mid\ \mid\ \mid\ \psi_{\{2\}}$	hyp
3	$\mid\ \mid\ \mid\ \mid\ \varphi_{\{3\}}$	hyp
4	$\mid\ \mid\ \mid\ \mid\ \varphi \to (\psi \to \xi)_{\{1\}}$	reit, 1
5	$\mid\ \mid\ \mid\ \mid\ \psi \to \xi_{\{1,3\}}$	$\to$E, 3, 4
6	$\mid\ \mid\ \mid\ \mid\ \psi_{\{2\}}$	reit, 2
7	$\mid\ \mid\ \mid\ \mid\ \xi_{\{1,2,3\}}$	$\to$E, 5, 6
8	$\mid\ \mid\ \mid\ \varphi \to \xi_{\{1,2\}}$	$\to$I, 3, 7
9	$\mid\ \mid\ \psi \to (\varphi \to \xi)_{\{1\}}$	$\to$I, 2, 8
10	$\mid\ (\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$	$\to$I, 1, 9

$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi), (\psi \to \varphi) \to (\neg\varphi \to \neg\psi), \varphi \to \neg\neg\varphi$ from $\mathbf{R}_{\to}1-3$, Ax 1-3

1	$(\neg\varphi \to \neg\psi) \to (\psi \to \neg\neg\varphi)$	Ax 1
2	$\neg\neg\varphi \to \varphi$	Ax 3
3	$(\neg\neg\varphi \to \varphi) \to ((\psi \to \neg\neg\varphi) \to (\psi \to \varphi))$	$\mathbf{R}_{\to}2$
4	$(\psi \to \neg\neg\varphi) \to (\psi \to \varphi)$	$2,3-\mathrm{MP}$
5	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	$1,4,\mathbf{R}_{\to}2-\mathrm{MP}$

1	$(\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\neg\varphi)$	Ax 1
2	$\neg\varphi \to \neg\varphi$	$\mathbf{R}_{\to}1$
3	$\varphi \to \neg\neg\varphi$	$1,2-\mathrm{MP}$

1	$(\varphi \to \neg\neg\psi) \to (\neg\psi \to \neg\varphi)$	Ax 1
2	$\varphi \to \neg\neg\varphi$	Ax 3
3	$(\varphi \to \neg\neg\varphi) \to ((\psi \to \varphi) \to (\psi \to \neg\neg\varphi))$	$\mathbf{R}_{\to}2$
4	$(\psi \to \varphi) \to (\psi \to \neg\neg\varphi)$	$2,3-\mathrm{MP}$
5	$(\psi \to \varphi) \to (\neg\varphi \to \neg\psi)$	$1,4,\mathbf{R}_{\to}2-\mathrm{MP}$

$\neg\neg$E and contraposition are valid in $FR^{*}_{\substack{\neg\\\to}}$

1	$\mid\ \neg\neg\varphi_a$	
2	$\mid\ \neg\neg\varphi \to \varphi$	axiom
3	$\mid\ \varphi_a$	$\to$E, 1, 2

1	ψ_b	
$\vdots$	$\quad\vdots$	
r	$\quad \varphi_{\{k\}}$	hyp
$\vdots$	$\quad\vdots$	
s	$\quad \neg\psi_a$	
$s+1$	$\varphi \to \neg\psi_{a-\{k\}}$	$\to$I, r, s
$s+2$	$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$	axiom
$s+3$	$\psi \to \neg\varphi_{a-\{k\}}$	$\to$E, $s+1$, $s+2$
$s+4$	$\neg\varphi_{a\cup b-\{k\}}$	$\to$E, 1, $s+3$

$\vdash_{F\mathbf{R}^*_{\neg\atop\to}} (\varphi \to \neg\varphi) \to \neg\varphi$

1	$\quad \varphi \to \neg\varphi_{\{k\}}$	hyp
2	$\qquad \varphi_{\{2\}}$	hyp
3	$\qquad \neg\varphi_{\{1,2\}}$	$\to$E, 1, 2
4	$\quad \neg\varphi_{\{1\}}$	$\neg$I, 2, 3
5	$(\varphi \to \neg\varphi) \to \neg\varphi$	$\to$I, 1, 4

$\vdash_{F\mathbf{R}^*_{\neg\atop\to}} \neg\neg\varphi \to \varphi$

1	$\quad \neg\neg\varphi_{\{1\}}$	hyp
2	$\quad \varphi_{\{1\}}$	$\neg\neg$E, 1
3	$\neg\neg\varphi \to \varphi$	$\to$I, 1, 2

$\vdash_{\mathbf{R}+\circ} (\varphi \circ \psi) \leftrightarrow \neg(\varphi \to \neg\psi)$

1	$(\varphi \to \neg\psi) \to (\varphi \to \neg\psi)$	$\mathbf{R}1$
2	$\varphi \to ((\varphi \to \neg\psi) \to \neg\psi)$	$1, \mathbf{R}3 - \text{MP}$
3	$((\varphi \to \neg\psi) \to \neg\psi) \to (\neg\neg\psi \to \neg(\varphi \to \neg\psi))$	$\mathbf{R}12$
4	$\varphi \to (\neg\neg\psi \to \neg(\varphi \to \neg\psi))$	$2, 3, \mathbf{R}2 - \text{MP}(\times 2)$
5	$\psi \to \neg\neg\psi$	$\mathbf{R}13$
6	$(\neg\neg\psi \to \neg(\varphi \to \neg\psi)) \to (\psi \to \neg(\varphi \to \neg\psi))$	$\mathbf{R}2, 5 - \text{MP}$
7	$\varphi \to (\psi \to \neg(\varphi \to \neg\psi))$	$4, 6, \mathbf{R}2 - \text{MP}$
8	$(\varphi \to (\psi \to \neg(\varphi \to \neg\psi))) \to (\varphi \circ \psi \to \neg(\varphi \to \neg\psi))$	$\circ 2$
9	$\varphi \circ \psi \to \neg(\varphi \to \neg\psi)$	$7, 8 - \text{MP}$

1	$(\psi \to \varphi \circ \psi) \to (\neg(\varphi \circ \psi) \to \neg\psi)$	Thm (Remark 2.4.4)
2	$(\varphi \to (\psi \to \varphi \circ \psi)) \to (\varphi \to (\neg(\varphi \circ \psi) \to \neg\psi))$	$1, \mathbf{R}2 - \text{MP}$
3	$\varphi \to (\psi \to \varphi \circ \psi)$	$\circ 1$
4	$\varphi \to (\neg(\varphi \circ \psi) \to \neg\psi)$	$2, 3 - \text{MP}$
5	$\neg(\varphi \circ \psi) \to (\varphi \to \neg\psi)$	$4, \mathbf{R}3 - \text{MP}$
6	$(\varphi \to \neg\psi) \to \neg\neg(\varphi \to \neg\psi)$	$\mathbf{R}13$
7	$\neg(\varphi \circ \psi) \to \neg\neg(\varphi \to \neg\psi)$	$5, 6, \mathbf{R}2 - \text{MP}$
8	$\neg(\varphi \to \neg\psi) \to \neg\neg(\varphi \circ \psi)$	$7, \mathbf{R}12 - \text{MP}$
9	$\neg\neg(\varphi \circ \psi) \to \varphi \circ \psi$	Thm (Remark 2.4.4)
10	$\neg(\varphi \to \neg\psi) \to \varphi \circ \psi$	$8, 9, \mathbf{R}2 - \text{MP}$

B Theorem checking program

The following program takes as input a file with the number n of elements (which assumes are $0, 1, 2, ..., n$), m of designated elements (which assumes are $n - m + 1, ..., n$) and operations of an $\mathcal{L}$-matrix $\mathbb{M}$, it also takes the tables defining the operations. A formula φ in polish notation (explained at the end of this annex) is introduced and the program checks it it is a valid formula and if all possible assignments to the variables (any symbol which is not for a connective is taken as a variable) to determine whether $\vDash_{\mathbb{M}} \varphi$ or not. In case $\nvDash_{\mathbb{M}} \varphi$ it gives an interpretation for which φ is not satisfied.

```
#include <math.h>
#include <stdio.h>
#include <string.h>

int isConnective (char);

struct formula {
 int nchar;
 char fwrite[30];
 int var;
 int con;
 int assoc[20][3];
};


int main (void){
 int choose, go, elem, desig, con, i, j, k, interp1, interp2;
 struct formula form;
 FILE *ent;
 char listcon, ent_name[25], var[4];
 int imp[10][10], conj[10][10], disj[10][10], neg[10], aux_form[30], interp
    [4];

 var[0]='p';
 var[1]='q';
 var[2]='r';
 var[3]='s';

 printf("\nEnter file name\n");
 scanf(" %s", ent_name);
 ent = fopen(ent_name, "r");
 if (ent == NULL) {
  printf("\nMemory error.\n");
  return 23;
 }
 fscanf(ent, "%d", &elem);
 if(elem > 10 || elem < 2){
  printf("\nInvalid number of elements\n");
  return 23;
 }
 fscanf(ent, "%d", &desig);
 if(desig >= elem || desig < 1){
  printf("\nInvalid number of designated elements\n");
  return 23;
 }
 fscanf(ent, "%d", &con);
 if(con < 1 || con > 4){
```

```
  printf("\nInvalid number of connectives\n");
  return 23;
 }
 for (k = 0; k < con; k++){
  fscanf(ent, " %c", &listcon);
  if(listcon == 'C'){
   for(i=0; i<elem; i++)
    for(j=0; j<elem; j++)
     fscanf(ent, "%d", &imp[i][j]);
  } else {
   if(listcon == 'K') {
    for(i=0; i<elem; i++)
     for(j=0; j<elem; j++)
      fscanf(ent, "%d", &conj[i][j]);
   } else {
    if(listcon == 'A') {
     for(i=0; i<elem; i++)
      for(j=0; j<elem; j++)
       fscanf(ent, "%d", &disj[i][j]);
    } else {
     if(listcon == 'N'){
      for(i=0; i<elem; i++)
       fscanf(ent, "%d", &neg[i]);
     }
    }
   }
  }
 }
 do {
  for (i=0; i<20; i++){
   aux_form[i]=0;
  }
  do{
   do{
    printf("\nEnter formula with less than 4 variables\n");
    scanf(" %s", form.fwrite);
    form.nchar = strlen(form.fwrite);
   }while(form.nchar>30 || form.nchar <2);
   form.var = 0;
   for(i = 0; i < form.nchar; i++){
    if(!isConnective(form.fwrite[i])){
     k = 0;
     for (j = 0; j < form.var && j < 4; j++)
      if (form.fwrite[i] != var[j]) ++k;
     if (k == form.var && k < 4) {
      var[k] = form.fwrite[i];
      ++form.var;
     }
     else if (k == 4)
      ++form.var;
    }
   }
   printf("The formula has %d variable(s)\n", form.var);
  }while(form.var > 4 || form.var == 0);

  // SAVING THE FORMULA //

  form.con = 0;
  for (i=0; i<form.nchar; i++) {
```

```
  if (isConnective(form.fwrite[i])) {
   form.assoc[form.con][0]=i;
   form.assoc[form.con][2]=-1;
   ++form.con;
  }
 }

 for (i=form.con-1; i>= 0; i--){
  if (isConnective(form.fwrite[form.assoc[i][0]])){
   j = form.assoc[i][0] + 1;

   while (aux_form[j] == 1 && j<form.nchar) ++j;
   if(j == form.nchar){
    printf("Error in formula. Ending process.\n");
    return 23;
   }
   form.assoc[i][1]=j;
   aux_form[j] = 1;

   if (form.fwrite[form.assoc[i][0]] !='N'){
    while (aux_form[j] == 1 && j<form.nchar) ++j;
    if(j == form.nchar){
     printf("Error in formula. Ending process.\n");
     return 23;
    }
    form.assoc[i][2]=j;
    aux_form[j] = 1;
   }
  }
 }
 k = 1;
 for(i=1; i<form.nchar; i++)
  if(aux_form[i]==0){
   k = 0;
   i = form.nchar;
  }
 if(!k) printf("Invalid formula.\n");

 // FOR EACH INTERPRETATION IS IT TRUE? //

 if (k){
 for(interp[0] = 0;  interp[0] < elem; interp[0]++)
  for(interp[1] = 0;  interp[1] < elem; interp[1]++)
   for(interp[2] = 0;  interp[2] < elem; interp[2]++)
    for(interp[3] = 0;  interp[3] < elem; interp[3]++) {
     for (i = form.var + 1; i < 4; i++)
      interp[form.var] += interp[i];
     if(interp[form.var] == 0 || form.var == 4) {
      for (i = form.con-1; i >= 0; i--) {
       if (isConnective(form.fwrite[form.assoc[i][1]])){
        interp1 = aux_form[form.assoc[i][1]];
       }
       else {
        for (j=0; j<form.var; j++)
          if (form.fwrite[form.assoc[i][1]]==var[j])
           interp1 = interp[j];
       }
       if (form.fwrite[form.assoc[i][0]] == 'N'){
        aux_form[form.assoc[i][0]] = neg[interp1];
```

```
        }
        else{
         if (isConnective(form.fwrite[form.assoc[i][2]])){
          interp2 = aux_form[form.assoc[i][2]];
         }
         else {
          for (j=0; j<form.var; j++)
           if (form.fwrite[form.assoc[i][2]]==var[j])
            interp2 = interp[j];
         }
         if (form.fwrite[form.assoc[i][0]] == ’C’)
          aux_form[form.assoc[i][0]] = imp[interp1][interp2];
         if (form.fwrite[form.assoc[i][0]] == ’K’)
          aux_form[form.assoc[i][0]] = conj[interp1][interp2];
         if (form.fwrite[form.assoc[i][0]] == ’A’)
          aux_form[form.assoc[i][0]] = disj[interp1][interp2];
        }
       }
       for (i = 0; i < elem-desig; i++)
        if (aux_form[0]==i){
         printf("\nInterpretation ");
         for(j=0; j < form.var; j++)
          printf("I(%c)=%d ", var[j], interp[j]);
         printf("fails to satisfy the formula\n");
         for (i=0; i<4; i++){
          interp[i] = 20;
         }
        }
      }
     }
  if (interp[0] == elem)
   printf("\nThe formula is valid\n");
  }
  printf("\n...Enter 1 for another formula, 0 to stop\n");
  scanf("%d", &go);
 } while (go);
 fclose(ent);
 return 1;
}

int isConnective(char a){
 if (a ==’C’||a ==’K’||a ==’A’||a ==’N’) return 1;
 return 0;
}
```

Here’s an example of an input file, the one for classical logic, where each operation is denoted by its letter in polish notation:

```
2 1 4

C
1 1
0 1

K
0 0
0 1

A
```

```
0 1
1 1

N
1 0
```

Polish notation is a way of writing formulas without delimiting symbols, since the connectives precede their "operands". The connectives are written as follows:

Usual notation	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
Polish notation	Np	Kpq	Apq	Cpq

Therefore, for example, the following formulas are written in polish notation like so:

Usual notation	Polish notation
$p \wedge q \to q$	$CKpqq$
$(p \to q) \to ((r \to p) \to (r \to q))$	$CCpqCCrpCrq$
$p \to (\neg p \to q)$	$CpCNpq$
$(p \to r) \wedge (q \to r) \to (p \vee q \to r)$	$CKCprCqrCApqr$

C A matrix model for R

The following tables define a matrix model for **R** as stated in the proof of Theorem 2.4.6 (designated elements have an asterisk). The elements in blue signal the cases in which an implication and its antecedent take a designated value (note that then the consequent also takes a designated value). The elements in red indicate that when both elements of a conjunction are positive the conjunction is too.

→	−3	−2	−1	−0	+0	+1	+2	+3
−3	+3	+3	+3	+3	+3	+3	+3	+3
−2	−3	+2	−3	+2	−3	−3	+2	+3
−1	−3	−3	+1	+1	−3	+1	−3	+3
−0	−3	−3	−3	+0	−3	−3	−3	+3
+0*	−3	−2	−1	−0	+0	+1	+2	+3
+1*	−3	−3	−1	−1	−3	+1	−3	+3
+2*	−3	−2	−3	−2	−3	−3	+2	+3
+3*	−3	−3	−3	−3	−3	−3	−3	+3

¬	
−3	+3
−2	+2
−1	+1
−0	+0
+0*	−0
+1*	−1
+2*	−2
+3*	−3

∧	−3	−2	−1	−0	+0	+1	+2	+3
−3	−3	−3	−3	−3	−3	−3	−3	−3
−2	−3	−2	−3	−2	−3	−3	−2	−2
−1	−3	−3	−1	−1	−3	−1	−3	−1
−0	−3	−2	−1	−0	−3	−1	−2	−0
+0*	−3	−3	−3	−3	+0	+0	+0	+0
+1*	−3	−3	−1	−1	+0	+1	+0	+1
+2*	−3	−2	−3	−2	+0	+0	+2	+2
+3*	−3	−2	−1	−0	+0	+1	+2	+3

∨	−3	−2	−1	−0	+0	+1	+2	+3
−3	−3	−2	−1	−0	+0	+1	+2	+3
−2	−2	−2	−0	−0	+2	+3	+2	+3
−1	−1	−0	−1	−0	+1	+1	+3	+3
−0	−0	−0	−0	−0	+3	+3	+3	+3
+0*	+0	+2	+1	+3	+0	+1	+2	+3
+1*	+1	+3	+1	+3	+1	+1	+3	+3
+2*	+2	+2	+3	+3	+2	+3	+2	+3
+3*	+3	+3	+3	+3	+3	+3	+3	+3