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i Informàtica

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Treball final de grau

BROWNIAN MOTION

Autora: Ingrid Mases Solé

Director: Dr. Josep Vives

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Contents

Abstract

The aim of this work is to study the Brownian motion from a theoretical approach. Brownian motion (also named Wiener process) is one of the best known stochastic processes and plays an important role in both pure and applied Mathematics.

In the first chapter, we present the basic concepts of the theory of stochastic processes such as filtrations, stopping times and martingales which are needed to develop further sections of the project.

In the second chapter, we define the Brownian motion itself. Furthermore, two different constructions of Brownian motion are provided. The first one presents theorems of existence and continuity of stochastic processes from which we end up building the Brownian motion. The second construction provides another proof for the existence of Brownian motion based on the idea of the weak limit of a sequence of random walks.

In the third chapter, we present a discussion of some properties of Brownian motion paths, also called sample path properties. These include characterizations of bad behaviour such as the nondifferentiability, as well as characterizations of good behaviour like the law of the iterated logarithm. Moreover, we study the zero sets, the quadratic variation and the lack of monotonicity of the Brownian paths.

Finally, we show some Python simulations of one dimensional Brownian paths.

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Resum

L'objectiu d'aquest treball és l'estudi del moviment Brownià des d'un punt de vista teòric. El moviment Brownià (també anomenat procés de Wiener) és un dels processos estocàstics més coneguts i té un paper important tant en les Matemàtiques pures com aplicades.

En el primer capítol es presenten els conceptes bàsics de la teoria de processos estocàstics, així com les filtracions, els instants d'aturada i les martingales, que seran necessaris per a desenvolupar seccions posteriors en el treball.

En el segon capítol es defineix de manera rigurosa el moviment Brownià. A més a més, es demostren dues construccions diferents per al moviment Brownià. La primera presenta teoremes d'existència i de continuitat de processos estocàstics a partir dels quals podem construir el moviment Brownià. La segona construcció proporciona una altra demostració d' existència del moviment Brownià basada en la idea de la convergència feble d'una seqüència de camins aleatoris.

En el tercer capítol presentem les propietats més representatives dels camins del moviment Brownià. Aquestes inclouen caracteritzacions de mal comportament com per exemple la no-diferenciabilitat, i també caracteritzacions de bon comportament com ara la llei del logaritme iterat. Adicionalment, estudiem el conjunt de zeros, la variació quadràtica i la manca de monotonia dels camins Brownians.

Finalment, es mostren simulacions unidimensionals dels camins del moviment Brownià realitzades en Python.

Motivation of the project

I studied the double degree in Physics and Mathematics and my interest in Brownian motion began in a class of the subject called Statistical Physics. In 1827, the Scottish botanist Robert Brown made microscopic observations of the irregular movement that pollen grains describe when they are suspended in water. Many scientists attempted to interpret this strange phenomenon. This erratic motion, named Brownian motion, comes from the extremely large number of collisions of the suspended pollen particles with the molecules of the liquid. Albert Einstein also studied this phenomenon, giving a theoretical and quantitative approach to Brownian motion, such as the diffusion coefficient, the diffusion equation or Einstein's equation. In addition to this physical analysis, I wanted to go in depth in the mathematical treatment of Brownian motion, seen as a stochastic process.

Moreover, students of the double degree don't take the subject named *Modelització* where stochastic processes and martingales are introduced. So, this is another reason why I decided tho study Brownian motion.

Note that the mathematical study of Brownian motion was highly developed by Bachelier, Lévy and Wiener during the twentieth century. The first quantitative work on Brownian motion is due to L. Bachelier (1900) who was interested in stock prices fluctuations. A rigorous mathematical treatment began with N. Wiener (1923) who provided the first existence proof. For this reason, Brownian motion is also named Wiener process. Furthermore, the most profound work is that of P. Lévy (1939, 1948) who described in detail the properties of Brownian sample paths.

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Chapter 1

Introduction to stochastic processes

In this chapter we present the basic concepts of probability theory and stochastic processes which will be needed in the next chapters to study the Brownian motion. The references that have been followed are [1], [2] and [4].

1.1 Stochastic processes and filtrations

A stochastic process is a mathematical model of a random phenomenon fluctuating in time. In other words, it is a mathematical model that aims to describe a random phenomenon at each moment after the initial time. Let (Ω, $\mathcal{F}$) be a measurable space, called the *sample space*, where probability measures can be placed. Remember that a measurable space is an ordered pair where Ω is a non-empty set and $\mathcal{F}$ is a σ-algebra of subsets of Ω.

Definition 1.1. A *stochastic process* is a collection of random variables $X = \{X_t; t \in \mathcal{T}\}$ defined on the probability space (Ω, $\mathcal{F}$, P), where t is to be thought as the time parameter.

$\mathcal{T}$ is the set of parameters. If $\mathcal{T}$ is a finite or infinite interval subset of $\mathbb{R}^+$, then the process X is said to be a *continuous time process*. Otherwise, if $\mathcal{T}$ is $\mathbb{Z}^+$ or a subset of $\mathbb{Z}^+$, then X is a *discrete time process*. In this work, only continuous time processes are taken into account. So, X can also be written as $X = \{X_t; 0 \leq t < \infty\}$.

The collection of random variables take values in another measurable space (S, $\mathcal{S}$) called *state space*. In this project we will have $S = \mathbb{R}^d$ and $\mathcal{S}$=$\mathcal{B}(\mathbb{R}^d)$. Note that $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ-field containing all open sets of the space $\mathbb{R}^d$.

Definition 1.2. For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is the *sample path* or *trajectory* of the process X associated with ω.

A stochastic process can be expressed as the following way,

$$X : (t, \omega) \in \mathcal{T} \times \Omega \longrightarrow X_t(\omega) \in \mathbb{R}^d \tag{1.1}$$

Definition 1.3. A stochastic process X is *measurable* if for every $U \in \mathcal{B}(\mathbb{R}^d)$, the set $\{(t, \omega); X_t(\omega) \in U\}$ belongs to $\mathcal{B}([0, \infty)) \bigotimes \mathcal{F}$.

Definition 1.4. A *filtration* on $(\Omega, \mathcal{F})$ is a non-decreasing family $\{\mathcal{F}_t; 0 \leq t < \infty\}$ of sub-σ-fields of $\mathcal{F}$ such that, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$, and $\mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq t < \infty$.

Definition 1.5. A stochastic process X is *adapted* to the filtration $\{\mathcal{F}_t; 0 \leq t < \infty\}$ if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$-measurable random variable.

Definition 1.6. Let X be a stochastic process. Its *natural filtration* is defined as the succession of σ-fields

$$\mathcal{F}_t^X := \sigma\{X_s; 0 \leq s \leq t\}$$

which are the σ-fields generated by the process variables themselves.

Remark 1.7. Every stochastic process is adapted to its natural filtration.

Notice that the σ-field $\mathcal{F}_t$ can be interpreted as the accumulated information of the process up to time t. σ-fields are included in the study of stochastic processes to keep track of information. At every moment $t \geq 0$ we can think about *past*, *present* and *future*, and we can ask how much an observer of the process knows about it at the present, compared to how much she knew at some point in the past or how much she will know at some point in the future. That is the reason why our sample space $(\Omega, \mathcal{F})$ is equipped with a filtration (as defined in Definition 1.4). If $A \in \mathcal{F}_t^X$, it means that by time t an observer of the process X knows whether A has occurred or not.

Definition 1.8. Let $\{\mathcal{F}_t; t \geq 0\}$ be a filtration. We define $\mathcal{F}_{t^-} \triangleq \sigma(\bigcup_{s<t} \mathcal{F}_s)$ as the *σ-field of events strictly prior to t* > 0 and $\mathcal{F}_{t^+} \triangleq \sigma(\bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon})$ as the *σ-field of events immediately after t* ≥ 0. We establish $\mathcal{F}_{0-} \triangleq \mathcal{F}_0$ and we say that the filtration $\{\mathcal{F}_t\}$ is *right-continuous* (respectively *left-continuous*) if $\mathcal{F} = \mathcal{F}_{t^+}$ (respectively $\mathcal{F} = \mathcal{F}_{t^-}$) holds for every $t \geq 0$.

Definition 1.9. A filtration $\{\mathcal{F}_t\}$ satisfies the *usual conditions* if it is right-continuous and $\mathcal{F}_0$ contains all the P-negligible events in $\mathcal{F}$. Recall that a subset of Ω is called P-negligible if it belongs to a subset of $\mathcal{F}$ of probability zero.

1.2 Stopping times

In this section, the useful concept of stopping time is introduced. Let's suppose that we are interested in the instant at which a given stochastic process exhibits a certain behavior of interest, for example, the moment at which the price of a stock exceeds a certain value. We name $T(\omega)$ the instant at which the phenomenon manifests itself for the first time.

Definition 1.10. A *random time* is an $\mathcal{F}$-measurable random variable which takes values in [0,∞]. We can write

$$T : \omega \in \Omega \longrightarrow T(\omega) \in \mathcal{T} \cup \{\infty\} \tag{1.2}$$

Definition 1.11. Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $\{\mathcal{F}_t\}$. A random time T is a *stopping time* with respect to the filtration $\{\mathcal{F}_t\}$ if the event $\{T \leq t\}$ belongs to the σ-field $\mathcal{F}_t$, for every $t \geq 0$.

Intuitively, this condition means that the "decision" of whether to stop at time t must be based only on the information present at time t, not on any future information.

Example 1.12. One dimensional simple random walk. Let $\{Y_i\}_{i\geq 0}$, $i \in \mathbb{N}$, be a collection of independent identically distributed random variables that can only take two values, each one with probability $1/2$:

$$Y_i = \begin{cases} +1, & p = \frac{1}{2} \\ -1, & p = \frac{1}{2} \end{cases}$$

Now we define for each integer t, $t \geq 0$,

$$X_0 = 0, \;\; X_t = \sum_{i=0}^{t} Y_i$$

and then, the collection of random variables $\{X_0, X_1, X_2, \dots\}$ is called *simple random walk*. This is a discrete time stochastic process. Imagine the coin toss game, and that at each turn the balance goes up \$1 or goes down \$1 (starting with \$0). The total balance at each turn can be described as a simple random walk. To provide an example of stopping time, imagine that a player plays until he wins \$100, and let T be the first time at which the balance becomes \$100. Then T is a stopping time. On the contrary, let τ be the time at which the balance reaches the maximum amount of money; then τ is not a stopping time because it requires information about the future as well as the present and past.

Example 1.13. Hitting time. Let X be a stochastic process adapted to a filtration $\{\mathcal{F}_t\}$. Consider a subset $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ of the state space of the process. Then the *hitting time* is defined as

$$H_\Gamma(\omega) = \inf\{t \geq 0; X_t(\omega) \in \Gamma\}.$$

Furthermore, we can ask how to measure the information accumulated up to a stopping time T. Suppose that an event A is part of this information, that is, by time T the occurrence or non occurrence of A is known. If by time t we observe the value of T (which is possible if $t \geq T$) then we will also be able to tell whether A has occurred or not. In other words, $A \cap \{T \leq t\}$ and $A^c \cap \{T \leq t\}$ must both be measurable, for every $t \geq 0$.

Definition 1.14. Let T be a stopping time of the filtration $\{\mathcal{F}_t\}$. The *σ-algebra $\mathcal{F}_T$ of events determined prior to the stopping time T* consists of those events $A \in \mathcal{F}$ for which $A \cap \{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

1.3 Martingales

Now, let's define a martingale in continuous time. The standard example of a continuous-time martingale is the one-dimensional Brownian motion, as we will see in next chapter. Consider a real-valued process $X = \{X_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$, which is adapted to a given filtration $\{\mathcal{F}\}$.

Definition 1.15. The process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a *submartingale* (respectively a *supermartingale*) with respect to the filtration $\{\mathcal{F}\}$ if,

1. $X_t \in \mathcal{F}_t$ for every $t \geq 0$,
2. $E(|X_t|) < \infty$ for every $t \geq 0$,
3. for every $0 \leq s < t < \infty$, we have that, a.s.P -almost surely P-, $E(X_t|\mathcal{F}_s) \geq X_s$ (respectively a $E(X_t|\mathcal{F}_s) \leq X_s$).

Then, $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a *martingale* if it is both a submartingale and a supermartingale.

If we have a discrete time process $M = \{M_n, \mathcal{F}_n; n \in \mathbb{Z}^+\}$, condition 3 in Definition 1.15 can be written as $E(M_n|\mathcal{F}_{n-1}) \geq M_{n-1}$ for a submartingale, and $E(M_n|\mathcal{F}_{n-1}) \leq M_{n-1}$ for a supermartingale.

If M is a martingale, then $E(M_n|\mathcal{F}_{n-1}) = M_{n-1}$ and it can also be written as $E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0$, a.s. for every $n \in \mathbb{N}$.

Definition 1.16. Let $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ be a right-continuous martingale. We say that X is *square-integrable* if $E(X_t^2) < \infty$ for every $t \geq 0$. If, in addition $X_0 = 0$ a.s., we write $X \in \mathcal{M}_2$ (or $X \in \mathcal{M}_2^c$, if X is also continuous). $\mathcal{M}_2$ ($\mathcal{M}_2^c$) is the space of (continuous) square-integrable martingales.

1.4 Markov Processes

Definition 1.17. The stochastic process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F}, P)$ is a *Markov process* if, for all $B \in \mathbb{R}$, and $\{t_1, \ldots, t_n\} \subset [0, \infty)$ we have

$$P(X_{t_n} \in B | \mathcal{F}_{t_{n-1}}) = P(X_{t_n} \in B | X_{t_{n-1}}), \;\; a.s.$$

In other words, the probability of each event depends only on the state attained in the previous event.

The past $\sigma(X_s, s \leq t)$ and the future $\sigma(X_s, s \geq t)$ play symmetric roles, and the intuitive meaning is that the past and the future are independent given the present. Note that if the probability measure P is changed, there is no reason why X should remain a Markov process.

A similar statement, only when the deterministic time t_{n-1} is changed by a stopping time T, is typically referred to as the Strong Markov Property [4], [12].

Chapter 2

Brownian motion. Constructions of Brownian motion

The range of application of Brownian motion as defined here goes far beyond a study of microscopic particles in suspension and includes modeling of stock prices, of thermal noise in electrical circuits and of random perturbations in physical, biological and economic systems, among others. In this chapter, we define in a rigorous way the Brownian motion, and furthermore we prove its existence and construction in two different ways. We mainly follow references [1] and [5].

2.1 Definition of Brownian motion

Definition 2.1. A standard, one-dimensional *Brownian motion* is a continuous, adapted process $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ defined in a probability space $(\Omega, \mathcal{F}, P)$, accomplishing that $B_0 = 0$, a.s. and for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and it is normally distributed with mean zero and variance $t - s$.

If B is a Brownian motion and $0 = t_0 < t_1 < ... < t_n < \infty$, the increments $\{B_{t_i} - B_{t_{i-1}}\}_{i=1}^{n}$ are independent random variables and the distribution of $B_{t_i} - B_{t_{i-1}}$ depends only on the difference $t_i - t_{i-1}$. So, it is a normal distribution with mean zero and variance $t_i - t_{i-1}$. Then we say that the process B has *stationary, independent increments*. The distribution function for $(B_{t_1}, \dots, B_{t_n})$ is,

$$F_{(t_1,\dots,t_n)}(x_1, \dots, x_n) =$$

$$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} p(t_1; 0, y_1)p(t_2 - t_1; y_1, y_2)\dots p(t_n - t_{n-1}; y_{n-1}, y_n)\, dy_n \dots dy_2\, dy_1, \tag{2.1}$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$ and p is the Gaussian distribution,

$$p(t; x, y) \triangleq \frac{1}{\sqrt{2\pi t}} e^{\frac{-(x-y)^2}{2t}}, t > 0; x, y \in \mathbb{R}. \tag{2.2}$$

Definition 2.2. Let d be a positive integer. A *standard d-dimensional Brownian motion* is a vector-valued stochastic process $B_t = (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)})$, $t \geq 0$, whose components $B_t^{(i)}$, $i = 1, \dots, d$, are independent, standard one-dimensional Brownian motions.

We present now the Dynkin system theorem which is used when we need to establish that a certain property (which holds for a collection of sets closed under finite intersection) also holds for the σ-field generated by this collection.

Definition 2.3. *A collection $\mathcal{D}$ of subsets of a set Ω is a Dynkin system if*

1. $\Omega \in \mathcal{D}$
2. *$A, B \in \mathcal{D}$ and $B \subseteq A$ imply that $A \setminus B \in \mathcal{D}$*
3. *$\{A_n\}_{n=1}^{\infty} \in \mathcal{D}$ and $A_1 \subseteq A_2 \subseteq \dots$ imply that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.*

Theorem 2.4. *Let $\mathcal{C}$ be a collection of subsets of Ω which is closed under pairwise intersection. If $\mathcal{D}$ is a Dynkin system containing $\mathcal{C}$, then $\mathcal{D}$ also contains the σ-field $\sigma(\mathcal{C})$ generated by $\mathcal{C}$.*

Remark 2.5. Remember that a collection C of subsets of Ω is *closed under intersection* if $A \cap B$ belongs to C whenever A and B belongs to C.

Remark 2.6. Let $X = \{X_t; 0 \leq t < \infty\}$ be a stochastic process which accomplishes that $X_0, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent random variables, for every integer $n \geq 1$ and indices $0 = t_0 < t_1 < \dots < t_n < \infty$. Then for any fixed $0 \leq s < t < \infty$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s^X$.

Indeed, for fixed $0 \leq s < t < \infty$, $n \geq 1$ and indices $0 = s_0 < s_1 < \dots < s_n = s$, the σ-algebra $\sigma(X_0, X_{s_1}, \dots, X_{s_n}) = \sigma(X_0, X_{s_1} - X_{s_0}, \dots, X_{s_n} - X_{s_{n-1}})$ is independent of $X_t - X_s$. The union of all σ-algebras of this form constitutes a collection $\mathcal{C}$ of sets independent of $X_t - X_s$ which is closed under finite intersections. Now $\mathcal{D}$, the collection of all sets in $\mathcal{F}_s^X$ which are independent of $X_t - X_s$, is a Dynkin system containing $\mathcal{C}$. From Theorem 2.4 we can conclude that $\mathcal{F}_s^X = \sigma(\mathcal{C})$ is contained in $\mathcal{D}$.

The filtration $\{\mathcal{F}_t\}$ is part of the definition of Brownian motion. However, if we have $B = \{B_t; 0 \leq t < \infty\}$ with no filtration, and we know that B has stationary, independent increments and that $B_t = B_t - B_0$ is normal with mean zero and variance t, then $\{B_t, \mathcal{F}_t^B; 0 \leq t < \infty\}$ is a Brownian motion (which follows from the discussion on Remark 2.6 above).

When we study the Brownian motion, one of the first problems that we come across is its *existence*. In this work two different constructions of Brownian motion are provided. First in Section 2.2, given the finite-dimensional distributions of the process we can construct a probability measure and a process on an appropriate measurable space in order to obtain the original finite-dimensional distributions. In Section 2.3 a proof based on the weak convergence of random walks is provided. Moreover, if the reader is interested, another construction of Brownian motion can be found in Chapter 2 in [1]. It is an approach for Brownian motion which exploits the Gaussian property of this process and it is also based on Hilbert space theory. It is very similar to Wiener's original construction (1923), which was later modified by Lévy (1948).

2.2 First construction of Brownian motion

In this section we present the Kolmogorov consisteny theorem (or Daniell-Kolmogorov theorem) that guarantees that a suitable consistent collection of finite-dimensional distributions can define a stochastic process. With this proposal in mind, we first define the concept of a family of finite-dimensional distributions, and then we proof the Kolmogorov consisteny theorem. In what follows, in order to avoid heavy notations we restrict to the one dimensional case, $d = 1$.

The Kolmogorov-Čentsov theorem (or Kolmogorov continuity theorem) is stated and proved in the second part of this section. As in the definition of Brownian motion it is required that the sample paths are continuous almost surely, we use the Kolmogorov continuity theorem to construct a continuous modification of the process obtained by the Kolmogorov consistency theorem.

Definition 2.7. Let $\widetilde{T}$ be the set of finite sequences $\underset{\sim}{t} = (t_1, \dots, t_n)$ of distinct non-negative numbers, where $n \in \mathbb{N}$. Suppose that for every $\underset{\sim}{t}$ of length n there exists a probability measure $Q_{\underset{\sim}{t}}$ on the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. The collection $\{Q_{\underset{\sim}{t}}\}_{\{\underset{\sim}{t} \in \widetilde{T}\}}$ is called a *family of finite-dimensional distributions*. This family is *consistent* if the following two conditions are satisfied:

i) If $\underset{\sim}{s} = (t_{i_1}, \dots, t_{i_n})$ is a permutation of $\underset{\sim}{t} = (t_1, \dots, t_n)$, then for any $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \dots, n$, we have that $Q_{\underset{\sim}{t}}(A_1 \times A_2 \times \cdots \times A_n) = Q_{\underset{\sim}{s}}(A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n})$.

ii) If $\underset{\sim}{t} = (t_1, t_2, \ldots, t_n)$ with $n \geq 1$, $\underset{\sim}{s} = (t_1, t_2, \ldots, t_{n-1})$, and $A \in \mathcal{B}(\mathbb{R}^{n-1})$ then we have that $Q_{\underset{\sim}{t}}(A \times \mathbb{R}) = Q_{\underset{\sim}{s}}(A)$.

We consider $\mathbb{R}^{[0,\infty)}$ the set of all real-valued functions in [0,∞).

Definition 2.8. We define an *n-dimensional cylinder set* in $\mathbb{R}^{[0,\infty)}$ which is a set of the form

$$C \triangleq \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in U\}, \tag{2.3}$$

taking into account that $U \in \mathcal{B}(\mathbb{R}^n)$ and $t_i \in [0, \infty)$, $i = 1, \ldots, n$. Let $\mathcal{C}$ be the field of all cylinder sets of all finite dimensions in $\mathbb{R}^{[0,\infty)}$.

If we have a probability measure P on the space $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, it is possible to define a family of finite-dimensional distributions by,

$$Q_{\underset{\sim}{t}}(U) = P[\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in U], \tag{2.4}$$

where $U \in \mathcal{B}(\mathbb{R}^n)$ and $\underset{\sim}{t} = (t_1, \ldots, t_n) \in \widetilde{T}$. The inverse fact is of great interest because it allows the construction of the probability measure P using the finite-dimensional distributions of Brownian motion.

2.2.1 Daniell-Kolmogorov Theorem

The Daniell-Kolmogorov theorem (or Daniell-Kolmogorov extension theorem, or Kolmogorov consistency theorem) is a very important theorem of the theory of stochastic processes because it provides existence results for probability measures.

Theorem 2.9. ***(Daniell-Kolmogorov Theorem).*** *Let $\{Q_{\underset{\sim}{t}}\}$ be a consistent family of finite-dimensional distributions. Then, there exists a probability measure P on the space $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ such that Equation* (2.4) *holds for each $\underset{\sim}{t} \in \widetilde{T}$.*

Before starting with the proof of Theorem 2.9 we present the Carathéodory Extension Theorem and a useful lemma that will be used in the demonstration. The proof of Lemma 2.11 can be found in [5].

Theorem 2.10. ***(Carathéodory Extension Theorem)*** *Let Ω be a non-empty set and let $\mathcal{G}$ be a family of subsets that satisfy:*

i) $\Omega \in \mathcal{G}$,

ii) *If* $A, B \in \mathcal{G} \Rightarrow A \cup B \in \mathcal{G}$, *iii)* *If* $A \in \mathcal{G} \Rightarrow \Omega \setminus A \in \mathcal{G}$.

Let $\sigma(\mathcal{G})$ *be the* σ*-algebra generated by* $\mathcal{G}$. *If* Q_0 *is a* σ*-additive measure on the space* $(\Omega, \mathcal{G})$ *which is* σ*-finite, then there exists a unique* σ*-additive measure* Q *on* $(\Omega, \sigma(\mathcal{G}))$ *such that for* $A \in \mathcal{G}$, $Q_0(A) = Q(A)$.

Lemma 2.11. *Let* $B_n \subset \mathbb{R}^n$, $n \in \mathbb{N}$ *be a sequence of Borel sets that satisfy* $B_{n+1} \subset B_n \times \mathbb{R}$. *Let us assume that for every* $n \in \mathbb{N}$ *a probability measure* μ_n *is given on the space* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ *and that these probability measures are compatible in the sense that,* $\mu_n(A_1 \times A_2 \times \cdots \times A_{n-1} \times \mathbb{R}) = \mu_{n-1}(A_1 \times A_2 \times \cdots \times A_{n-1})$, *where* $A_i \in \mathcal{B}(\mathbb{R}^n)$ *and satisfy that* $\mu_n(B_n) > \epsilon$, *with* $0 < \epsilon < 1$. *Then, there exists a sequence of compact sets* $K_n \subset \mathbb{R}^n$, *such that:*

i) $K_n \subset B_n$,

ii) $K_{n+1} \subset K_n \times \mathbb{R}$,

iii) $\mu_n(K_n) \geq \epsilon/2$.

Proof. **(Theorem 2.9 Daniell-Kolmogorov)** We begin by defining a set function Q on the field of cylinders $\mathcal{C}$. If C is given by Definition 2.8 and $\underset{\sim}{t} = (t_1, t_2, \ldots, t_n) \in \widetilde{T}$, we set

$$Q(C) = Q_{\underset{\sim}{t}}(U), \;\; C \in \mathcal{C}. \tag{2.5}$$

Thanks to the assumptions on $\{Q_{\underset{\sim}{t}}\}$, the set function Q is well defined and it is finitely additive on $\mathcal{C}$, with $Q(\mathbb{R}^{[0,\infty)}) = 1$.

The set of all possible cylinders $\mathcal{C}$ satisfies the assumptions of Carathéodory's theorem. So, in order to conclude, we only have to prove the countable additivity of Q on $\mathcal{C}$, and we can then use the Carathéodory's theorem to assert the existence of the desired extension P of Q to $\mathcal{B}(\mathbb{R}^{[0,\infty)})$. Therefore, we have to show that, if $\{C_n\}_{n=0}^{\infty}$ is a sequence of pairwise disjoint cylinders in $\mathcal{C}$ and $C = \sum_{n=0}^{\infty} C_n$ is a cylinder also in $\mathcal{C}$, then

$$Q(C) = \sum_{n=0}^{\infty} Q(C_n). \tag{2.6}$$

This is the most difficult part of the proof. For $M \in \mathbb{N}$ we have

$$Q(C) = Q(C \setminus \bigcup_{n=0}^{M} C_n) + Q(\bigcup_{n=0}^{M} C_n) = Q(C \setminus \bigcup_{n=0}^{M} C_n) + \sum_{n=0}^{M} Q(C_n), \tag{2.7}$$

due to the finite additivity of Q. So, countable additivity will follow from

$$\lim_{M \to \infty} Q(C \setminus \bigcup_{n=0}^{M} C_n) = 0. \tag{2.8}$$

We name $D_M = C \setminus \bigcup_{n=0}^{M} C_n$. Observe that, $Q(D_M) = Q(D_{M+1}) + Q(C_{M+1}) \geq Q(D_{M+1})$, so the limit in equation (2.8) exists and $\{Q(D_M)\}_{M\in\mathbb{N}}$ is a positive decreasing sequence. At this point, let's assume that this sequence converges toward $\epsilon > 0$, and we shall see that in that case,

$$\bigcap_{M\in\mathbb{N}} D_M \neq \emptyset \tag{2.9}$$

which is clearly absurd.

D_M is a cylinder, then $\bigcup_{M\in\mathbb{N}} D_M$ only involves a countable sequence of times $t_1 < t_2 < \ldots$ and every D_M can be described as follows (as in Definition 2.8)

$$D_M = \{f \in \mathbb{R}^{[0,\infty)}; (f(t_1), \ldots, f(t_M)) \in B_M\} \tag{2.10}$$

where $B_n \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a sequence of Borel sets satisfying that $B_{n+1} \subset B_n \times \mathbb{R}$. Since we assumed that $Q(D_M) \geq \epsilon$, we can use the previous lemma to construct a sequence of compact sets $K_n \subset \mathbb{R}^n$, $n \in \mathbb{N}$, accomplishing that

i) $K_n \subset B_n$,

ii) $K_{n+1} \subset K_n \times \mathbb{R}$,

iii) $Q_{\underset{\sim}{t}}(K_n) \geq \epsilon/2$.

Since K_n is a non-empty set, we can pick $(x_1^n, \ldots, x_n^n) \in K_n$. We know that the sequence $(x_1^n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_1^{j_1(n)})_{n\in\mathbb{N}}$ that converges to $x_1 \in K_1$. The same way, the sequence $(x_1^n, x_2^n)_{n\in\mathbb{N}}$ has a convergent subsequence which converges to $(x_1, x_2) \in K_2$. By repeating this process, we obtain a sequence $(x_n)_{n\in\mathbb{R}}$ such that for every $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \in K_n$.

Then, the event

$$\{f \in \mathbb{R}^{[0,\infty)}, (f(t_1), \ldots, f(t_M)) = (x_1, \ldots, x_M)\}$$

is in D_M, which contradicts the fact that $\bigcap_{M\in\mathbb{N}} D_M = \emptyset$. Therefore, the sequence $\{Q(D_M)\}_{M\in\mathbb{N}}$ converges towards 0, which implies the σ-additivity of Q. □

Our goal is to build a probability measure P on $(\Omega, \mathcal{F}) = (\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ in a way that the process $B = \{B_t, \mathcal{F}_t^B; 0 \leq t < \infty\}$ defined by the *coordinate mapping process*, $B_t(\omega) \triangleq \omega(t)$, is a standard one-dimensional Brownian motion under the probability P. Remember that, $\mathcal{F}_t^B \triangleq \sigma(B_s; 0 \leq s \leq t)$.

Now, let $\underset{\sim}{t} = (t_1, t_2, \ldots, t_n)$ with t_i's distinct $(i = 1, \ldots, n)$, and let the random vector $(B_{t_1}, \ldots, B_{t_n})$ have the distribution determined by the expression in (2.1) (the t_i must be ordered from smallest to largest). For $U \in \mathcal{B}(\mathbb{R}^n)$, let $Q_{\underset{\sim}{t}}(U)$ be the probability under this distribution that $(B_{t_1}, \dots, B_{t_n})$ is in U. This defines a family of finite-dimensional distributions $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in \widetilde{T}}$, which is consistent. Clearly, given $\underset{\sim}{t} = (t_1, t_2, \dots, t_n)$ and $\underset{\sim}{s} = (t_{i_1}, t_{i_2}, \dots, t_{i_n})$ a permutation of $\underset{\sim}{t}$, we have constructed a distribution for the random vector $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ under which

$$\begin{aligned} Q_{\underset{\sim}{t}}(U_1 \times U_2 \times \cdots \times U_n) &= P[(B_{t_1}, B_{t_2}, \dots, B_{t_n}) \in U_1 \times U_2 \times \cdots \times U_n] \\ &= P[(B_{t_{i_1}}, B_{t_{i_2}}, \dots, B_{t_{i_n}}) \in U_{i_1} \times U_{i_2} \times \cdots \times U_{i_n}] \\ &= Q_{\underset{\sim}{s}}(U_{i_1} \times U_{i_2} \times \cdots \times U_{i_n}). \end{aligned}$$

Moreover, for $U \in \mathcal{B}(\mathbb{R}^{n-1})$ and $\underset{\sim}{s}' = (t_1, t_2, \dots, t_{n-1})$ we have that,

$$Q_{\underset{\sim}{t}}(U \times \mathbb{R}) = P[(B_{t_1}, B_{t_2}, \dots, B_{t_{n-1}}) \in U] = Q_{\underset{\sim}{s}'}(U).$$

From Theorem 2.9 we get the following Corollary.

Corollary 2.12. *There exists a probability measure P on the space $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ under which the coordinate mapping process $B_t(\omega) = \omega(t)$, $\omega \in \mathbb{R}^{[0,\infty)}$, $t \geq 0$, has stationary, independent increments. That is, an increment $B_t - B_s$, $0 \leq s < t$, is normally distributed with mean zero and variance $t - s$.*

Remark 2.13. Note that the Brownian motion that we have built is on the sample space $\mathbb{R}^{[0,\infty)}$ of all real-valued functions on $[0, \infty)$, not on the space of all continuous functions on $[0, \infty)$, denoted as $C[0, \infty)$.

2.2.2 Kolmogorov's Continuity Theorem

Definition 2.14. A stochastic process X with values in $(\Omega, \mathcal{F})$ is a.s. *continuous* if, for almost all $\omega \in \Omega$, the function $t \longrightarrow X_t(\omega)$ is continuous.

We would like our process B to have this property, but note that there is no reason why the set $\{\omega : t \longrightarrow X_t(\omega) \text{ continuous}\}$ should be measurable. In fact, the only $\mathcal{B}(\mathbb{R}^{[0,\infty)})$-measurable set contained in $C[0, \infty)$ is the empty set.

Since we want to construct the Brownian motion with state space $\mathbb{R}$ it is tempting, as we did, to use as sample space $\mathbb{R}^{[0,\infty)}$ of all possible paths, and as random variables X_t the coordinate mapping over t, namely $X_t(\omega) = \omega(t)$. Each set in $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ depends only in a countable set of coordinates and therefore the set of continuous ω's is not in $\mathcal{B}(\mathbb{R}^{[0,\infty)})$. That is, $C[0, \infty)$ is not in the σ-algebra $\mathcal{B}(\mathbb{R}^{[0,\infty)})$, and $P(C[0, \infty))$ is not defined. To overcome this issue we will construct a continuous modification of the coordinate mapping process seen in Corollary 2.12. Next, some useful concepts are defined to continue with the construction of the continuous Brownian motion process.

Definition 2.15. Let X and Y be two stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. We say that X and Y are *modifications* of each other if, for every $t \geq 0$, we have that,

$$P\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} = 1.$$

Note that two processes can be modifications of one another and have completely different sample paths.

Definition 2.16. Let X and Y be two stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. We say that X and Y have the *same finite-dimensional distributions* if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, and $U \in \mathcal{B}(\mathbb{R}^n)$, we have that,

$$P[(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \in U] = P[(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \in U].$$

Definition 2.17. Let f be a function $f : [0, \infty) \longrightarrow \mathbb{R}^d$. We say that f is *Hölder-continuous* if there exist nonnegative real constants $C, \alpha > 0$ such that,

$$|f(t) - f(s)| \leq C\|t - s\|^{\alpha} \tag{2.11}$$

for every $s, t \in [0, \infty)$.

Remark 2.18. As we will see in Theorem 2.23 we are also interested in functions which are locally Hölder-continuous.

Now, three useful lemmas are sketched out.

Lemma 2.19. *(The Borel-Cantelli Lemma). Let $\{A_n\}$ be a sequence of events.*

1. *If $\sum_n P(A_n) < \infty$, then*
$$P\left[\lim_{n\to\infty} \sup A_n\right] = 0, \tag{2.12}$$
2. *If $\sum_n P(A_n) = \infty$ and, in addition, the events of the sequence $\{A_n\}$ are pairwise independent, then*
$$P\left[\lim_{n\to\infty} \sup A_n\right] = 1. \tag{2.13}$$

Lemma 2.20. *(Markov's inequality). Let X be a positive random variable, then for every $a > 0$,*

$$P(X \geq a) \leq \frac{E(X)}{a}. \tag{2.14}$$

Lemma 2.21. *(Tchebychev's inequality). Suppose that X is a random variable with finite variance $V(X)$. Then we have the following inequality:*

$$P\left(|X - E(X)| \geq t\right) \leq \frac{V(X)}{t^2}, \ \ t > 0. \tag{2.15}$$

Lemma 2.22. *(Kolmogorov's inequality). Suppose that $\{X_n\}$ is an independent sequence of random variables with finite variance and $S_n = \sum_{k=1}^{n} X_k$. Then we have the inequality,*

$$P\left(\max_{1<k<n} |S_k - E(S_k)| \geq t\right) \leq \frac{V(S_n)}{t^2}, \ \ t > 0. \tag{2.16}$$

Theorem 2.23. ***(Kolmogorov's Continuity Theorem).*** *Suppose that a real-valued process $X = \{X_t; 0 \leq t \leq T\}$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the following condition,*

$$E(|X_t - X_s|^{\alpha}) \leq C|t-s|^{1+\beta} \tag{2.17}$$

for $0 \leq s,t \leq T$ and for some positive constants α, β and C. Then there exists a continuous modification of X, $\widetilde{X} = \{\widetilde{X}_t; 0 \leq t \leq T\}$ which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, it means that,

$$P\left[\omega; \sup_{\substack{0<t-s<h(\omega)\\ s,t\in[0,T]}} \frac{|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)|}{|t-s|^{\gamma}} \leq \delta\right] = 1, \tag{2.18}$$

where $h(\omega)$ is a positive random variable and $\delta > 0$ is an appropiate constant.

Proof. In order to simplify the notation we take $T = 1$, without loss of generality. As a consequence of Lemma 2.20 and using the hypothesis of the theorem we have that, for any $\epsilon > 0$,

$$P\left[|X_t - X_s| \geq \epsilon\right] \leq \frac{E|X_t - X_s|^{\alpha}}{\epsilon^{\alpha}} \leq \frac{C}{\epsilon^{\alpha}}|t-s|^{1+\beta}, \tag{2.19}$$

so when $s \to t$ then $X_s \to X_t$ in probability. Moreover, setting $t = \frac{k}{2^n}$, $s = \frac{k-1}{2^n}$ and $\epsilon = 2^{-n\gamma}$, with $0 < \gamma < \frac{\beta}{\alpha}$, we obtain the following inequality:

$$P\left[|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geq 2^{-n\gamma}\right] \leq C2^{-n(1+\beta-\alpha\gamma)}, \tag{2.20}$$

and consequently,

$$\begin{aligned} P\left[\max_{1\leq k\leq 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geq 2^{-n\gamma}\right] &= P\left[\bigcup_{k=1}^{2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geq 2^{-n\gamma}\right] \leq \\ &\leq \sum_{K=1}^{2^n} P\left[|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geq 2^{-n\gamma}\right] \leq C2^{-n(\beta-\alpha\gamma)} \end{aligned} \tag{2.21}$$

where the last inequality comes from the above inequality (2.20). Observe that

$$\sum_{n=1}^{\infty} P\left[\max_{1\le k\le 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \ge 2^{-n\gamma}\right] < +\infty, \tag{2.22}$$

due to the fact that $\gamma \in (0, \beta/\alpha)$. So, we can use the Lemma 2.19 (Borel-Cantelli) and we obtain that

$$P\left[\limsup_{n\to\infty}\left(\max_{1\le k\le 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \ge 2^{-n\gamma}\right)\right] = P\left[\Omega_0\right] = 0.$$

So that, there exists a set $\Omega' = \Omega \setminus \Omega_0 \in \mathcal{F}$ with the property $P(\Omega') = 1$ such that for each $\omega \in \Omega'$,

$$\max_{1\le k\le 2^n} |X_{\frac{k}{2^n}}(\omega) - X_{\frac{k-1}{2^n}}(\omega)| < 2^{-n\gamma}, \tag{2.23}$$

for every $n \ge n'(\omega)$, where $n'(\omega)$ is a positive, integer-valued random variable. Now, let us consider, for each integer $n \ge 1$, the partition $D_n = \{(k/2^n); k = 0, 1, \dots, 2^n\}$ of the considered interval $[0,1]$, and then, $D = \bigcup_{n=1}^{\infty} D_n$ is the set of dyadic rationals in $[0,1]$. We now claim that the paths of the restricted process $X_{|\Omega'}$ are γ-Hölder continuous on D. We shall fix $\omega \in \Omega'$ and $n \ge n'(\omega)$ and show that for every $m > n$, we have

$$|X_t(\omega) - X_s(\omega)| \le 2\sum_{j=n+1}^{m} 2^{-j\gamma}, \;\; \forall t, s \in D_m, \;\; 0 < t - s < 2^{-n}. \tag{2.24}$$

We prove it inductively. For the initial case $m = n+1$, we only have $t = k/2^m$ and $s = (k-1)/2^m$ and the expression in (2.24) comes from the expression in (2.23). Let's suppose that (2.24) is valid for $m = n+1, \dots, M-1$. Now take $s < t$, $s, t \in D_M$ and consider the numbers $t^1 = \max\{u \in D_{M-1}; u \le t\}$ and $s^1 = \min\{u \in D_{M-1}; u \ge s\}$, which follow these relationships: $s \le s^1 \le t^1 \le t$, $\;s^1 - s \le 2^{-M}$ and $t - t^1 \le 2^{-M}$. From the expression (2.23) we have that $|X_{s^1}(\omega) - X_s(\omega)| \le 2^{-M\gamma}$ and $|X_t(\omega) - X_{t^1}(\omega)| \le 2^{-M\gamma}$. Using these two inequalities together with inequality (2.24) in the case $m = M-1$, we get

$$|X_{t^1}(\omega) - X_{s^1}(\omega)| \le 2\sum_{j=n+1}^{M-1} 2^{-j\gamma},$$

Consequently we obtain (2.24) for $m = M$.

At this point, we can show that $\{X_t(\omega); t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega'$. Given $s, t \in D$ with $0 < t - s < h(\omega) \triangleq 2^{-n'(\omega)}$, we select $n \ge n'(\omega)$ such that $2^{-(n+1)} \le t - s < 2^{-n}$. Then, from (2.24) we have

$$|X_t(\omega) - X_s(\omega)| \le 2\sum_{j=n+1}^{\infty} 2^{-j\gamma} \le \delta|t-s|^{\gamma}, 0 < t - s < h(\omega), \tag{2.25}$$

where $\delta = 2/(1-2^{-\gamma})$. With this we have proved the desired γ-Hölder continuity.

Now, let's construct the continuous modification of X. We start by defining $\widetilde{X}$ as follows.

1. For $\omega \notin \Omega'$, we set $\widetilde{X}_t(\omega) = 0$, $0 \leq t \leq 1$.

2. For $\omega \in \Omega'$ and $t \in D$, we set $\widetilde{X}_t(\omega) = X_t(\omega)$.

3. For $\omega \in \Omega'$ and $t \in D^c \cap [0,1]$, we choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$. Then, uniform continuity and Cauchy criterion imply that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not in the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ that we have chosen to converge to t. We set $\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$.

Then the resulting process $\widetilde{X}$ is continuous; indeed, $\widetilde{X}$ satisfies (2.25), so (2.18) is established. Finally, we see that $\widetilde{X}$ is a modification of X. For $t \in D$ we have that $\widetilde{X}_t = X_t$ a.s.. For $t \in [0,1] \cap D^c$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$ we have that $X_{s_n} \to X_t$ in probability and $X_{s_n} \to \widetilde{X}_t$ a.s., so $\widetilde{X}_t = X_t$ a.s. $\square$

In the case of our process B, which satisfies that $B_t - B_s$, $0 \leq s < t$, is normally distributed with mean zero and variance $t - s$, then for each positive integer n, there exists a positive constant C_n for which we have

$$E(|B_t - B_s|)^{2n} = C_n |t-s|^n. \tag{2.26}$$

Equation (2.26) comes from the calculation of the moments of the normal distribution, whose expressions can be found in [6]. In particular,

$$E(|B_t - B_s|)^4 = 3 \cdot |t-s|^2, \tag{2.27}$$

so the Kolmogorov's continuity theorem can be applied and the following corollary can be stated.

Corollary 2.24. *There exists a probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, and a stochastic process $W = \{W_t, \mathcal{F}^W; t \geq 0\}$ on the same space, such that under P, the process W is a Brownian motion.*

Proof. According to Kolmogorov continuity theorem (Theorem 2.23) and expression in equation (2.27), there exists for each $T > 0$ a modification W^T of the process B in Corollary 2.12 such that W^T is continuous in $[0,T]$. Let

$$\Omega_T = \{\omega; W_t^T(\omega) = B_t(\omega) \text{ for every rational } t \in [0,T]\},$$

so $P(\Omega_T) = 1$. On $\bigcap_{T=1}^{\infty} \Omega_T$ we have that for positive integers T_1 and T_2, $W_t^{T_1}(\omega) = W_t^{T_2}(\omega)$, for every rational $t \in [0, T_1 \wedge T_2]$. Since both processes are continuous on $[0, T_1 \wedge T_2]$, we must have $W_t^{T_1}(\omega) = W_t^{T_2}(\omega)$ for every $t \in [0, T_1 \wedge T_2]$, $\omega \in \bigcap_{T=1}^{\infty} \Omega_T$. And we define $W_t(\omega)$ to be this common value. For $\omega \notin \bigcap_{T=1}^{\infty} \Omega_T$, we set $W_t(\omega) = 0$ for all $t \geq 0$.

$\square$

2.3 Second construction of Brownian motion

The most convenient space for Brownian motion is the space of all continuous real-valued functions on $[0, \infty)$, $C[0, \infty)$, with the following metric,

$$\rho(\omega_1, \omega_2) \triangleq \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1). \tag{2.28}$$

In this section, we show how to construct a measure, named Wiener measure, on the space $C[0, \infty)$ in a way that the coordinate mapping process is a Brownian motion. With this aim, the notion of weak convergence of random walks to Brownian motion will be needed.

If X is a random variable in a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(S, \mathcal{B}(S))$, that is, the function $X : \Omega \longmapsto S$ is $\mathcal{F}/\mathcal{B}(S)$-measurable, then X induces a probability measure PX^{-1} on $(S, \mathcal{B}(S))$ by the following expression:

$$PX^{-1}(B) = P\{\omega \in \Omega; X(\omega) \in B\}, \;\; B \in \mathcal{B}(S). \tag{2.29}$$

Definition 2.25. Particularly, when $X = \{X_t; t \geq 0\}$ is a continuous stochastic process on $(\Omega, \mathcal{F}, P)$ and X can be regarded as a random variable with values in $(C[0, \infty), \mathcal{B}(C[0, \infty)))$, then PX^{-1} is called *the law of* X.

Remark 2.26. The law of a continuous process is determined by its finite dimensional distributions.

Definition 2.27. Recalling the cylinders defined in Definition 2.8, at this point we name $\mathcal{C}$ the collection of finite-dimensional cylinder sets of the form

$$C = \{\omega \in C[0, \infty); (\omega(t_1), \ldots, \omega(t_n)) \in U\}; \;\; n \geq 1, \; U \in \mathcal{B}(\mathbb{R}^n) \tag{2.30}$$

where for all $i = 1, \ldots, n$, $t_i \in [0, \infty)$. And we denote $\mathfrak{C}$ the smallest σ-field containing $\mathcal{C}$.

Remark 2.28. $\mathcal{B}(C[0, \infty))$ is generated by the one-dimensional cylinder sets.

2.3.1 Weak Convergence

Definition 2.29. Let (S,ρ) be a metric space with Borel σ-field $\mathcal{B}(S)$. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(S,\mathcal{B}(S))$, and let P be another measure on this space. We say that $\{P_n\}_{n=1}^{\infty}$ *converges weakly* to P (and we write $P_n \xrightarrow{W} P$), if and only if,

$$\lim_{n\to\infty}\int_S f(s)dP_n(s) = \int_S f(s)dP(s)$$

for every bounded, real-valued continuous function f on S.

It follows, in particular, that the weak limit P is a probability measure and that it is unique.

Definition 2.30. Let $\{(\Omega_n,\mathcal{F}_n,P_n)\}_{n=1}^{\infty}$ be a sequence of probability spaces, and consider on each of them a random variable X_n with values in the metric space (S,ρ). Let $(\Omega,\mathcal{F},P)$ be another probability space and a random variable X on it, with values in (S,ρ). We say that $\{X_n\}_{n=1}^{\infty}$ *converges to X in distribution* (and we write $X_n \xrightarrow{\mathcal{D}} X$), if the sequence of measures $\{P_nX_n^{-1}\}_{n=1}^{\infty}$ converges weakly to the measure PX^{-1}.

Equivalently, $X_n \xrightarrow{\mathcal{D}} X$ if and only if

$$\lim_{n\to\infty} E_nf(X_n) = Ef(X)$$

for every bounded, real-valued continuous function f on S, and where E_n and E denote the expectations with respect to P_n and P, respectively.

A very important example of convergence in distribution is that provided by the Central Limit Theorem, which asserts that if $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of independent, identically distributed random variables with mean zero and variance σ^2, then $\{S_n\}$ defined by

$$S_n = \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n}\xi_k,$$

converges in distribution to a standard normal random variable. As we will see in next sections, it is this fact which dictates that a properly normalized sequence of random walks will converge in distribution to a Brownian motion (Donsker's Invariance Principle).

2.3.2 Tightness Property

We introduce here the concept of *tightness* which will be used in next sections.

Definition 2.31. Let (S, ρ) be a metric space and let Π be a family of probability measures on $(S, \mathcal{B}(S))$. We say that Π is *relatively compact* if every sequence of elements of Π has a weakly convergent subsequence.

We say that Π is *tight* if for every $\epsilon > 0$, there exists a compact set $K \subseteq S$ such that $P(K) \geq 1 - \epsilon$, for every $P \in \Pi$.

If $\{X_\alpha\}_{\alpha \in A}$ is a family of random variables, each of them defined in a probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ and take values in S, we say that this family is *relatively compact* or *tight* if the measures $\{P_\alpha X_\alpha^{-1}\}$ is relatively compact or tight, respectively.

We state two theorems which will be used in further sections to prove the Invariance Principle of Donsker (Theorem 2.43).

Theorem 2.32. ***(Prohorov)*** *Let Π be a family of probability measures on a complete, separable metric space S. This family is relatively compact if and only if it is tight.*

Theorem 2.33. *Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$. It is tight if and only if*

$$\lim_{\lambda \to \infty} \sup_{n \geq 1} P_n[\omega; |\omega(0)| > \lambda] = 0,$$

$$\lim_{\delta \to 0^+} \sup_{n \geq 1} P_n[\omega; m^T(\omega, \delta) > \epsilon] = 0; \quad \forall T > 0, \;\; \epsilon > 0,$$

where $m^T(\omega, \delta)$ is the modulus of continuity on [0,T] *defined as*

$$m^T(\omega, \delta) \triangleq \max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |\omega(s) - \omega(t)|$$

with $\omega \in C[0,\infty)$ and $T, \delta > 0$.

The proofs of these theorems can be found in [1] (pages 62-64).

2.3.3 Convergence of finite-dimensional distributions

We begin by considering that X is a continuous process on $(\Omega, \mathcal{F}, P)$. For each $\omega \in \Omega$, the function $t \longmapsto X_t(\omega)$ (which we denote by $X(\omega)$) is on $C[0,\infty)$. Furthermore, since $\mathcal{B}(C[0,\infty))$ is generated by the one-cylinder sets and $X_t(\cdot)$ is $\mathcal{F}$-measurable for each t, the random function $X : \Omega \longmapsto C[0,\infty)$ is $\mathcal{F}/\mathcal{B}(C[0,\infty))$-measurable. Thereby, if $\{X^{(n)}\}_{n=1}^{\infty}$ is a sequence of continuous processes which can be defined on different probability spaces, we can ask whether $X^{(n)} \xrightarrow{\mathcal{D}} X$ in the sense defined in Definition 2.30.

Definition 2.34. Let $\{t_1, \dots, t_d\}$ be a finite subset of $[0, \infty)$. We define the *projection mapping* $\pi_{t_1,\dots,t_d} : C[0,\infty) \longrightarrow \mathbb{R}^d$ as follows,

$$\pi_{t_1,\dots,t_d}(\omega) = (\omega(t_1), \dots, \omega(t_d)).$$

If a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous and bounded, so it is $f \circ \pi_{t_1,\dots,t_d}$. Thus, if $X^{(n)} \xrightarrow[n\to\infty]{\mathcal{D}} X$, then

$$\begin{aligned}\lim_{n\to\infty} E_n f(X^{(n)}_{t_1}, \dots, X^{(n)}_{t_d}) &= \lim_{n\to\infty} E_n (f \circ \pi_{t_1,\dots,t_d})(X^{(n)}) \\ &= E(f \circ \pi_{t_1,\dots,t_d})(X) = Ef(X_{t_1}, \dots, X_{t_d}).\end{aligned} \tag{2.31}$$

So that, if the sequence of processes $\{X^{(n)}\}_{n=1}^{\infty}$ converges in distribution to the process X, then all finite-dimensional distributions converge as well.

Remark 2.35. The converse holds in the presence of tightness, but not in general, as we see in next theorem.

Theorem 2.36. *Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a **tight** sequence of continuous processes accomplishing that, with $0 \le t_1 < \cdots < t_d < \infty$, the sequence of random vectors $\{(X^{(n)}_{t_1}, \dots, X^{(n)}_{t_d})\}_{n=1}^{\infty}$ converges in distribution. Let P_n be the measure induced by $X^{(n)}$ on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$. Then, $\{P_n\}_{n=1}^{\infty}$ converges weakly to a measure P, under which the coordinate mapping process $W_t(\omega) \triangleq \omega(t)$ on $C[0,\infty)$ satisfies that*

$$(X^{(n)}_{t_1}, \dots, X^{(n)}_{t_d}) \xrightarrow{\mathcal{D}} (W_{t_1}, \dots, W_{t_d}),\ \ 0 \le t_1 < \cdots < t_d < \infty,\ d \ge 1.$$

Proof. Every subsequence $\{\widetilde{X}^{(n)}\}$ of $\{X^{(n)}\}$ is tight, and therefore has a further subsequence $\{\widehat{X}^{(n)}\}$ such that the measures induced on $C[0,\infty)$ by $\{\widehat{X}^{(n)}\}$ converge weakly to a probability measure P, by Theorem 2.32 (Prohorov Theorem): If another subsequence $\{\check{X}^{(n)}\}$ induces measures on $C[0,\infty)$ converging to a probability measure Q, then P and Q must have the same finite-dimensional distributions, that is,

$$P[\omega \in C[0,\infty); (\omega(t_1), \dots, \omega(t_d)) \in A] = Q[\omega \in C[0,\infty); (\omega(t_1), \dots, \omega(t_d)) \in A],$$

where $0 \le t_1 < t_2 < \cdots < t_d < \infty$, $A \in \mathcal{B}(\mathbb{R}^d)$ and $d \ge 1$. This means $P = Q$. Suppose the sequence of measures $\{P_n\}_{n=1}^{\infty}$ induced by $\{X_n\}_{n=1}^{\infty}$ did not converge weakly to P. Then, there must be a bounded, continuous function $f : C[0,\infty) \longrightarrow \mathbb{R}$ such that $\lim_{n\to\infty} \int f(\omega) P_n(d\omega)$ does not exist, or it does exist but it is different from $\int f(\omega) P(d\omega)$. In both cases, we can chose a subsequence $\{\widetilde{P}_n\}_{n=1}^{\infty}$ for which $\lim_{n\to\infty} \int f(\omega) \widetilde{P}_n(d\omega)$ exists, but it is different from $\int f(\omega) P(d\omega)$. Consequently, this subsequence can have no further subsequence $\{\widehat{P}_n\}_{n=1}^{\infty}$ with $\widehat{P}_n \xrightarrow{W} P$, and this fact contradicts the conclusion of the previous paragraph. $\square$

Proposition 2.37. *Let $\{X_n\}_{n=1}^{\infty}$, $\{Y_n\}_{n=1}^{\infty}$ and X be random variables with values in the metric space (S, ρ), and we assume that for each $n \geq 1$, $X^{(n)}$ and $Y^{(n)}$ are defined on the same probability space. If $X^{(n)} \xrightarrow{\mathcal{D}} X$ and $\rho(X^{(n)}, Y^{(n)}) \to 0$ in probability, when $n \to \infty$, then $Y^{(n)} \xrightarrow{\mathcal{D}} X$ as $n \to \infty$.*

Proof. Let $(\Omega_n, \mathcal{F}_n, P_n)$ denote the space on which X_n and Y_n are defined, and let E_n denote the expectation with respect to P_n. Let X be defined on $(\Omega, \mathcal{F}, P)$. Due to the hypothesis, we know that $\lim_{n\to\infty} E_n f(X^{(n)}) = E(f(X))$ for every bounded, continuous function $f : S \longrightarrow \mathbb{R}$. To prove that $Y^{(n)} \xrightarrow{\mathcal{D}} X$ it suffices to show that $\lim_{n\to\infty} E_n[f(X^{(n)}) - f(Y^{(n)})] = 0$, whenever f is bounded and continuous. Suppose that such and f is given, and set $M = sup_{x\in S}|f(x)| < \infty$. Since $\{X_n\}_{n=1}^{\infty}$ is relatively compact it is tight. So, for each $\epsilon > 0$ there exists a compact set $K \subset S$, such that $P_n[X^{(n)} \in K] \geq 1 - \epsilon/6M$, $\forall n \geq 1$. We now choose $1 > \delta > 0$ so $|f(x) - f(y)| < \epsilon/3$ whenever $x \in K$ and $\rho(x, y) < \delta$. Finally, we choose a positive integer N such that $P_n[\rho(X^{(n)}, Y^{(n)}) \geq \delta] \leq \epsilon/6M$, $\forall n \geq N$. We have

$$\left| \int_{\Omega_n} [f(X^{(n)}) - f(Y^{(n)})] dP_n \right| \leq \frac{\epsilon}{3} P_n[X^{(n)} \in K, \rho(X^{(n)}, Y^{(n)}) < \delta] +$$
$$+ 2M \cdot P_n[X^{(n)} \notin K] + 2M \cdot P_n[\rho(X^{(n)}, Y^{(n)}) \geq \delta] \leq \epsilon.$$

□

We state a proposition which will be used in next section.

Proposition 2.38. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables taking values in a metric space (S_1, ρ_1) and converging in distribution to X. Suppose that (S_2, ρ_2) is another metric space, and $\phi : S_1 \longrightarrow S_2$ is continuous. Then, $Y_n \triangleq \phi(X_n)$ converges in distribution to $Y \triangleq \phi(X)$.*

2.3.4 The Wiener Measure and the Invariance Principle

We are going to consider a sequence of independent, dentically distributed andom variables $\{\xi_j\}_{j=1}^{\infty}$ with mean zero and variance σ^2, $0 < \sigma^2 < \infty$. We consider as well the sequence of partial sums $S_0 = 0$, $S_k = \sum_{j=1}^{k} \xi_j$, $k \geq 1$. A continuous-time process $Y = \{Y_t, t \geq 0\}$ can be obtained from the sequence $\{S_k\}_{k=1}^{\infty}$ by the following way (linear interpolation):

$$Y_t = S_{\lfloor t \rfloor} + (t + \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1}, \ \ t \geq 0, \tag{2.32}$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. From Y we can obtain another sequence of processes $\{X^{(n)}\}$,

$$X_t^{(n)} = \frac{1}{\sigma\sqrt{n}} Y_{nt}, \ \ t \geq 0. \tag{2.33}$$

Remark 2.39. Note that using $s = k/n$ and $s = (k+1)/n$, then the increment $X_t^{(n)} - X_s^{(n)} = (1/\sigma\sqrt{n})\xi_{(k+1)}$ is independent of $\mathcal{F}_s^{X^{(n)}} = \sigma(\xi_1, \ldots, \xi_k)$. Moreover, $X_t^{(n)} - X_s^{(n)}$ has mean zero and variance $t - s$. This fact suggests that $\{X_t^{(n)}; t \geq 0\}$ is approximately a Brownian motion. With the following results we show that, although the random variables ξ_j are not necessary normal, the Central Limit Theorem dictates that the limiting distributions of the increments of $X^{(n)}$ are normal.

Theorem 2.40. *Let $\{X^{(n)}\}$ be defined by the expression in* (2.33) *and $0 \leq t_1 < \cdots < t_d < \infty$, we have*

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{\mathcal{D}} (B_{t_1}, \ldots, B_{t_d}), \text{ as } n \to \infty,$$

where $\{B_t, \mathcal{F}_t^B; t \geq 0\}$ is a standard, one-dimensional Brownian motion.

Proof. In order to make the notation simpler, we take the case $d = 2$. We set, $s = t_1$ and $t = t_2$. Thereby, we want to show that $(X_s^{(n)}, X_t^{(n)}) \xrightarrow{\mathcal{D}} (B_s, B_t)$. From equations (2.32) and (2.33) we have that,

$$\left| X_t^{(n)} - \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} \right| = \frac{1}{\sigma\sqrt{n}} (nt - \lfloor nt \rfloor) \left| \xi_{\lfloor nt \rfloor + 1} \right| \leq \frac{1}{\sigma\sqrt{n}} \left| \xi_{\lfloor nt \rfloor + 1} \right|,$$

and by the Chebyshev inequality,

$$P\left[\left| X_t^{(n)} - \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} \right| > \epsilon \right] \leq \frac{1}{\epsilon^2 n} \to 0, \text{ as } n \to \infty.$$

Therefore,

$$\| (X_s^{(n)}, X_t^{(n)}) - \frac{1}{\sigma\sqrt{n}} (S_{\lfloor ns \rfloor}, S_{\lfloor nt \rfloor}) \| \to 0 \text{ in probability.}$$

So, by the Proposition 2.37, it suffices to show that

$$\frac{1}{\sigma\sqrt{n}} (S_{\lfloor ns \rfloor}, S_{\lfloor nt \rfloor}) \xrightarrow{\mathcal{D}} (B_s, B_t).$$

From Proposition 2.38 we observe that it is equivalent to proving

$$\frac{1}{\sigma\sqrt{n}} \left(\sum_{j=1}^{\lfloor ns \rfloor} \xi_j, \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_j \right) \xrightarrow{\mathcal{D}} (B_s, B_t - B_s).$$

Since $\{\xi_j\}_{j=1}^{\infty}$ are independent random variables, we know that,

$$\begin{aligned}
&\lim_{n\to\infty} E\left[exp\{\frac{iu}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor ns\rfloor}\xi_j + \frac{iv}{\sigma\sqrt{n}}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_j\}\right] \\
&= \lim_{n\to\infty} E\left[exp\{\frac{iu}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor ns\rfloor}\xi_j\}\right]\cdot \lim_{n\to\infty} E\left[exp\{\frac{iv}{\sigma\sqrt{n}}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_j\}\right],
\end{aligned} \tag{2.34}$$

whenever both limits on the right-hand side exist. We start studying the first limit, and the other one is treated the same way. Since

$$\left|\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor ns\rfloor}\xi_j - \frac{\sqrt{s}}{\sigma\sqrt{\lfloor ns\rfloor}}\sum_{j=1}^{\lfloor ns\rfloor}\xi_j\right| \to 0, \quad \text{in probability},$$

and by the Central Limit Theorem $(\sqrt{s}/\sigma\sqrt{\lfloor ns\rfloor})\sum_{j=1}^{\lfloor ns\rfloor}\xi_j$ converges in distribution to a normal random variable with mean zero and variance s, we have then

$$\lim_{n\to\infty} E\left[exp\{\frac{iu}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor ns\rfloor}\xi_j\}\right] = e^{\frac{-u^2 s}{2}}.$$

Likewise,

$$\lim_{n\to\infty} E\left[exp\{\frac{iv}{\sigma\sqrt{n}}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_j\}\right] = e^{\frac{-v^2(t-s)}{2}}.$$

Finally, if we substitute this two equations into (2.34) we complete the proof. □

To show that the sequence $\{X^{(n)}\}$ defined in (2.33) converges to a Brownian motion in distribution we need tightness (recall Theorem 2.36). To see that tightness is assured, we need these two following lemmas, whose proofs can be found in [1] (pages 68-70).

Lemma 2.41. *Set $S_k = \sum_{j=1}^{k}\xi_j$, being $\{\xi_j\}_{j=1}^{\infty}$ a sequence of independent, identically distributed random variables, with mean zero and finite variance $\sigma^2 > 0$. Then, for any $\epsilon > 0$,*

$$\lim_{\delta\to 0^+}\overline{\lim_{n\to\infty}}\frac{1}{\delta}P\left[\max_{1\le j\le\lfloor n\delta\rfloor+1}|S_j| > \epsilon\sigma\sqrt{n}\right] = 0$$

Lemma 2.42. *Under the assumptions on Lemma 2.41, we have that for every $T > 0$,*

$$\lim_{\delta\to 0^+}\overline{\lim_{n\to\infty}}\frac{1}{\delta}P\left[\max_{\substack{1\le j\le\lfloor n\delta\rfloor+1\\ 0\le k\le\lfloor nT\rfloor+1}}|S_{j+k}-S_k| > \epsilon\sigma\sqrt{n}\right] = 0.$$

With these results we can establish the convergence in distribution of the sequence of normalized random walks in (2.33) to Brownian motion. This result is also known as the *invariance principle*.

Theorem 2.43. ***(The Invariance Principle of Donsker).*** *Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is given a sequence $\{\xi_j\}_{j=1}^{\infty}$ of independent, identically distributed random variables with mean zero and variance $\sigma^2 > 0$. Define $X^{(n)} = \{X_t^{(n)}; t \geq 0\}$ as* (2.33), *and let P_n be the measure induced by $X^{(n)}$ on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$. Then, $\{P_n\}_{n=1}^{\infty}$ converges weakly to a measure P_* under which the coordinate mapping process $W_t(\omega) \triangleq \omega(t)$ on $C[0,\infty)$ is a standard, one-dimensional Brownian motion.*

Proof. Taking into account Theorem 2.36 and Theorem 2.40, now it only remains to show that $\{X^{(n)}\}_{n=1}^{\infty}$ has the property of tightness. To do so, we use Theorem 2.33. Observe that $X_0^{(n)} = 0$ a.s. for every n (see the way we defined it in expressions (2.32) and (2.33)). Thereby, we only need to show that, for arbitrary $\epsilon > 0$ and $T > 0$,

$$\lim_{\delta \to 0^+} \sup_{n \geq 1} P\left[\max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |X_s^{(n)} - X_t^{(n)}| > \epsilon\right] = 0. \tag{2.35}$$

Here we may replace $\sup_{n\geq 1}$ in this expression by $\overline{\lim}_{n\to\infty}$, since for a finite number of integers n we can make the probability in (2.35) as small as we choose, by reducing δ. But,

$$P\left[\max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |X_s^{(n)} - X_t^{(n)}| > \epsilon\right] = P\left[\max_{\substack{|s-t| \leq n\delta \\ 0 \leq s,t \leq nT}} |Y_s - Y_t| > \epsilon\sigma\sqrt{n}\right],$$

and

$$\max_{\substack{|s-t| \leq n\delta \\ 0 \leq s,t \leq nT}} |Y_s - Y_t| \leq \max_{\substack{|s-t| \leq \lfloor n\delta \rfloor + 1 \\ 0 \leq s,t \leq \lfloor nT \rfloor + 1}} |Y_s - Y_t| \leq \max_{\substack{1 \leq j \leq \lfloor n\delta \rfloor + 1 \\ 0 \leq k \leq \lfloor nT \rfloor + 1}} |S_{j+k} - S_k|$$

where the last inequality follows from the fact that Y is piecewise linear and changes slope only at integer values of t. Then, (2.35) follows from Lemma 2.42. □

Definition 2.44. The probability measure P_* on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$, under which the coordinate mapping process $W_t(\omega) \triangleq \omega(t)$, $t \geq 0$, is a standard, one dimensional Brownian motion, is called ***Wiener measure***.

We can think of a standard, one-dimensional Brownian motion defined on any probability space, as a random variable with values in $C[0,\infty)$. This way, Brownian motion induces the Wiener measure on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$. Because of that, $(C[0,\infty), \mathcal{B}(C[0,\infty)), P_*)$ is named *the canonical probability space* for Brownian motion.

Chapter 3

Properties of Brownian Motion

In Section 2.2 we observed the continuity, for almost all ω, of the sample function $t \to B_t(\omega)$, $t \geq 0$, of Brownian motion. Questions naturally arise concerning the more refined properties such as differentiability or asymptotic behaviour as $t \to \infty$. So, in this section some typical results in this area are presented.

Throughout this section, $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard linear Brownian motion on $(\Omega, \mathcal{F}, P)$. In particular, $W_0 = 0$ a.s. P. For fixed $\omega \in \Omega$, we denote by $W_\cdot(\omega)$ the sample path $t \mapsto W_t(\omega)$.

Proposition 3.1. *Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard linear Brownian motion. Then the following properties hold:*

a) (Symmetry). The process $-W = \{-W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, is a Brownian motion.

b) (Scaling). For every $c > 0$, the process $X = \{X_t, \mathcal{F}_{ct}; 0 \leq t < \infty\}$ defined by $X_t = (1/\sqrt{c})W_{ct}$ is a Brownian motion.

c) (Time-inversion). The process $X = \{X_t, \mathcal{F}_t^X; 0 \leq t < \infty\}$ defined by $X_0 = 0$, $X_t = tW_{1/t}$, for $t > 0$, is a Brownian motion.

The prove of these properties can be found on [12].

Remark 3.2. Observe that a consequence of *c)* is the Law of Large Numbers for the Brownian motion, namely $P[\lim_{t\to\infty} t^{-1}W_t = 0] = 1$.

Proposition 3.3. *Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard one dimensional Brownian motion. Define, for each $\omega \in \Omega$ the new process*

$$M(t) = \max_{0 \leq s \leq t} W_s(\omega).$$

Then, for all $t > 0$ and $a > 0$, $P[M(t) > a] = 2 \cdot P[W_t(\omega) > a]$.

Proof. We define $\tau_a = \min\{t \geq 0; W_t(\omega) = a\}$. We will use the notation $W_t(\omega) = \omega(t)$. We have the following equality,

$$P[(\omega(t) - \omega(\tau_a)) > 0 | t > \tau_a] = P[(\omega(t) - \omega(\tau_a)) < 0 | t > \tau_a]. \tag{3.1}$$

Then,

$$\begin{aligned}
&P[M(t) > a] = P[t > \tau_a] \\
=&P[\{(\omega(t) - \omega(\tau_a)) > 0\} \cap \{t > \tau_a\}] + P[\{(\omega(t) - \omega(\tau_a)) < 0\} \cap \{t > \tau_a\}] \\
=&2 \cdot P[\{(\omega(t) - \omega(\tau_a)) > 0\} \cap \{t > \tau_a\}] = 2 \cdot P[\{(\omega(t) - a) > 0\} \cap \{t > \tau_a\}] \\
=&2 \cdot P[\omega(t) > a \cap \{t > \tau_a\}] = 2 \cdot P[\omega(t) > a],
\end{aligned}$$

where we have used equation (3.1) and the fact that $\omega(\tau_a) = a$. Finally, the last equality holds because if we know that $\omega(t) > a$, it means that necessarily $t > \tau_a$, by the definition of τ_a. □

3.1 Markov Property and Strong Markov Property

Proposition 3.4. ***(Markov Property).*** *Let W_t be a Brownian motion. For any $s > 0$, the process $X = \{X_t, \mathcal{F}_t^X; 0 \leq t < \infty\}$, defined by $X_t = W_{s+t} - W_s$, with $0 \leq t \leq s$, is a Brownian motion independent of $\sigma(W_u; u \leq s)$.* ([12], [1] Chapter 2.5.)

Proposition 3.5. ***(Strong Markov Property).*** *For every almost surely finite stopping time T, the process $W_{T+t} - W_T$ is a standard Brownian motion independent of $\mathcal{F}_t$.* ([12], [1] Chapter 2.6.)

3.2 The zero set

Definition 3.6. The *zero set of the Brownian path* is defined as follows,

$$\mathcal{Z} \triangleq \{(t, \omega) \in [0, \infty) \times \Omega; W_t(\omega) = 0\}.$$

And for fixed $\omega \in \Omega$, *the zero set of $W_\cdot(\omega)$* is defined as,

$$\mathcal{Z}_\omega \triangleq \{0 \leq t < \infty; W_t(\omega) = 0\}.$$

Proposition 3.7. *Almost surely, W_t has infinitely many zeros in every time interval $(0, \epsilon)$, with $\epsilon > 0$.*

Proof. We induct on the number of zeros in $(0, \epsilon)$. First, we show that there must be a zero in this interval. Let M_t^+ and M_t^- be the maximal and the minimal processes, respectively. We fix $\epsilon > 0$. By Proposition 3.3, we know that $P(M_\epsilon^+ > a) = 2 \cdot P(W_\epsilon > a)$, for $a > 0$. By taking the limit as $a \to 0^+$ we obtain that $P(M_\epsilon^+ > 0) = 2 \cdot P(W_\epsilon^+ > 0) = 2 \cdot (1/2) = 1$, since W_t is symmetric. By symmetry, $P(W_\epsilon^- < 0) = 1$. Due to the fact that W_t is almost surely continuous we employ the intermediate value theorem and conclude that $W_t = 0$ for some $t \in (0, \epsilon)$.

Now take some finite set S of zeros of W_t on the interval $(0, \epsilon)$. We define $T = \min\{t \geq 0; t \in S\}$. Since ϵ was arbitrary when we established that W_t had a zero in $(0, \epsilon)$, by the same argument we can see that W_t almost surely has a zero in $(0, T)$. By the minimality of T, this zero must not be in S, thus there is no finite set containing all the times W_t hits zero, so $\mathcal{Z}$ is almost surely infinite. $\square$

Remark 3.8. Thus, $W_t(\omega)$ crosses the t-axis infinitely often.

Theorem 3.9. *For $P-a.e.\ \omega \in \Omega$, the zero set $\mathcal{Z}_\omega$*

(i) has Lebesgue measure zero,

(ii) is closed and unbounded,

(iii) has no isolated point in $(0, \infty)$.

Proof. (i) Let $|\mathcal{Z}_\omega|$ be the Lebesgue measure of $\mathcal{Z}_\omega$. By Fubini's theorem, we compute the expectation,

$$E(|\mathcal{Z}_\omega|) = E\int_0^\infty \chi_{\{0\}}(W_t)dt = \int_0^\infty (1 \cdot P(W_t = 0) + 0 \cdot P(W_t \neq 0))dt$$
$$= \int_0^\infty P(W_t = 0)dt = 0.$$

We know that $|\mathcal{Z}_\omega|$ is non-negative since it is a measure. We now show that non-negative random variables with expectation zero are almost surely zero. Suppose $X \geq 0$ is a random variable with $E(X) = 0$ and fix $a > 0$. Then,

$$0 = E(X) = \int_\Omega X \cdot dP \geq \int_{\{X \geq a\}} \geq a \cdot P(X \geq a) \geq 0.$$

So, $P(X \geq a) = 0$ for all $a > 0$. Letting $a \to 0^+$, we see that $P(X > 0) = 0$, and therefore $X = 0$ almost surely. We conclude that if $E(|\mathcal{Z}_\omega|) = 0$, then $|\mathcal{Z}_\omega| = 0$.

We should have expected this result because W_t is almost surely nonzero for all nonzero t since the increment $W_t - W_0 = W_t$ is a normal random variable. Nonetheless, W_t hits 0 infinitelly many times in any interval to the right of the origin.

(ii) For P- a.e., $\omega \in \Omega$, the mapping $t \mapsto W_t(\omega)$ is continuous, then $\mathcal{Z}_\omega = W_t^{-1}(\{0\})$ is closed because it is the inverse image of the closed set $\{0\}$. It follows from an application of the reflection principle that $P(\sup_{t\geq 0} W_t/\sqrt{t} = +\infty) = 1$ and $P(\inf_{t\geq 0} W_t/\sqrt{t} = -\infty) = 1$ (see [9], [10]). Therefore, given any $T > 0$ there must be a time instant $t > T$ such that $W_t = 0$. If there were a finite last time such that $W_t = 0$, then for the sample path, the supremum and the infimum cannot simultaneously be infinite. This means that the zero set is unbounded.

(iii) To show that the zero set has no isolated points in $(0, \infty)$ consider the time $\tau_q = \inf\{t \geq q; W_t = 0\}$ where $q \in \mathbb{Q}$. It is clearly a stopping time and it is almost surely finite because of Proposition 3.7. Moreover, the infimum is a minimum because $\mathcal{Z}_\omega$ is almost surely closed. We apply the Strong Markov property at τ_q and we get that $W_{t+\tau_q} - W_{\tau_q}$ is a standard Brownian motion. Because we already know that a Brownian motion crosses 0 in every small interval to the right of the origin, τ_q is not isolated from the right in $\mathcal{Z}_\omega$.

Now suppose that we have some $z \in \mathcal{Z}_\omega$ that is not in $\{\tau_q | q \in \mathbb{Q}\}$. We take some sequence $\{q_n\}_{n=1}^{\infty}$ of rational numbers that converges to z. For each q_n there must exist some $t_n \in \mathcal{Z}_\omega$ such that $q_n \leq t_n < z$ since $z \neq \tau_{q_n}$. Because $q_n \to z$, we have that $t_n \to z$. So, z is not isolated from the left in $\mathcal{Z}_\omega$. $\square$

3.3 Non-differentiability

As we will see in Theorem 3.11, $W_t(\omega)$ is nowhere differentiable. So, we can't use standard Differential Calculus with Brownian sample paths. Instead, Stochastic Calculus (also named Itô's Calculus) is used.

Definition 3.10. Let $f : [0, \infty) \to \infty$ be a continuous function. We denote by

$$D^{\pm} f(t) = \overline{\lim_{h \to 0^{\pm}}} \frac{f(t+h) - f(t)}{h}$$

the *upper (right and left) Dini derivates at t* , and by

$$D_{\pm} f(t) = \underline{\lim_{h \to 0^{\pm}}} \frac{f(t+h) - f(t)}{h}$$

the *lower (right and left) Dini derivates at t* . The function f is said to be *differentiable at t from the right* (respectively, *the left*) if $D^+ f(t)$ and $D_+ f(t)$ (respectively, $D^- f(t)$ and $D_- f(t)$) are finite numbers and are equal. The function f is said to be *differentiable* at $t > 0$ if it is differentiable from both right and left, and the four Dini derivates are equal. At $t = 0$, differentiability is defined as from the right.

Theorem 3.11. ***(Paley, Wiener and Zygmund).*** *For almost every $\omega \in \Omega$, the Brownian sample path $W_{\cdot}(\omega)$ is nowhere differentiable.*

Before giving the proof of the theorem we present the underlying idea. For small $h > 0$ the increment $W_{t+h}(\omega) - W_t(\omega)$ is a Gaussian random variable with mean 0 and variance h, from where it follows that $h^{-1/2}[W_{t+h}(\omega) - W_t(\omega)]$ is a standard (mean 0 and variance 1) Gaussian random variable and so can be thought of as being of ordinary magnitude, however small the value of h. If we then consider the ratio $h^{-1}[W_{t+h}(\omega) - W_t(\omega)]$ and let h tend to 0, we see that the variance of this ratio will become arbitrarily large, and so we would never expect the existence of a limit of the ratio for each ω, which would have to be the case to have a time derivative of $W_t(\omega)$.

Proof. By the Markov property it suffices to prove the theorem for $t \in [0,1]$. Suppose that $W_t(\omega)$ was differentiable at some point $s \in [0,1)$. Then, since $W_t(\omega)$ is differentiable from the right at s, there exists $\epsilon > 0$ and an integer $l \geq 1$ such that for $0 < t - s < \epsilon$,

$$|W_t(\omega) - W_s(\omega)| < l(t-s).$$

Now we take a larger integer n, and set $i = \lfloor ns \rfloor + 1$. Let j run over $i+1$, $i+2$, $i+3$, successively. Then we can obtain the following inequality,

$$W_{\frac{j}{n}}(\omega) - W_{\frac{j+1}{n}}(\omega) < \frac{7 \cdot l}{n}, \text{ j=i+1, i+2, i+3,} \tag{3.2}$$

where it will become clear in the computations which follow why we take three successive j's.

Let $A^{i,j}_{l,n}$ be the set of all ω satisfying inequality (3.2), which is a Borel measurable set. Let us consider the Borel measurable set

$$A = \bigcup_{l\geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} \bigcup_{0<i\leq n} \bigcap_{i<j\leq i+3} A^{i,j}_{l,n}$$

This set is the event that there exists an integer l such that for all n sufficiently large, the inequality (3.2) holds at some point i/n. Thus, A includes every ω for which $W_{\cdot}(\omega)$ is differentiable at some point t. So, if we can prove that $P(A) = 0$, the proof will be completed. We note that,

$$P\left(\bigcap_{n\geq m} \bigcup_{0<i\leq n} \bigcap_{i<j\leq i+3} A^{i,j}_{l,n}\right) \leq \lim_{n\to\infty} \inf nP\left(\left|W_{\frac{1}{n}}(\omega)\right| < \frac{7l}{n}\right)^3 =$$

$$= \lim_{n\to\infty} \inf nP\left(|W_1(\omega)| < \frac{7l}{\sqrt{n}}\right)^3 \leq \lim_{n\to\infty} n\left\{\frac{1}{\sqrt{2\pi}} \frac{14l}{\sqrt{n}}\right\}^3 = 0.$$

Thus the set A is the union of a countable number of sets of probability zero, as we wanted to prove. □

We have followed reference [2] (Chapter 2, Section 2) in order to prove the theorem above.

Remark 3.12. Many problems in the natural, social, and biological sciences could be studied when Newton and Leibniz invented the calculus. The primary components of this invention were the use of differentiation to describe rates of change, the use of integration to pass to the limit in approximating sums, and the fundamental theorem of calculus. All of this gave rise to the concept of ordinary differential equations. Stochastic calculus appeared because of the need to assign meaning to ordinary differential equations involving continuous stochastic processes. The most important stochastic process, Brownian motion, cannot be differentiated, so that stochastic calculus plays a different role to that of classical calculus, which is the stochastic integral. In stochastic calculus, differential has no meaning apart from that assigned to it when it enters an integral.

3.4 Quadratic variation of the Brownian motion paths

Theorem 3.13. *Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard one dimensional Brownian motion. Let $T > 0$ and the time interval $[0, T]$. Suppose a subdivision of $[0, T]$ into n parts, $\Delta_n[0, T] = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$, where $t_k^n = (k/n)T$, with $k = 0, \ldots, n$. Then, the following convergence takes place in probability,*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(W_{\frac{k}{n}T} - W_{\left(\frac{k-1}{n}\right)T} \right)^2 = T. \tag{3.3}$$

Proof. Let's define $t_i = (i/n)T$ with $i = 0, \ldots, n$. Observe that $t_{i+1} - t_i = T/n$. So, $(W_{t_{i+1}} - W_{t_i}) \sim N(0, T/n)$. Therefore,

$$\sum_{i=0}^{n-1} \left(W_{t_{i+1}} - W_{t_i}\right)^2 = \sum_{i=0}^{n-1} X_i^2$$

where $X_i \sim N(0, T/n)$. We denote $Y_i = X_i^2$. Then, $E(Y_i) = E(X_i^2) = T/n$ because $T/n = Var(X_i) = E[(X_i - E(X_i))^2] = E(X_i^2)$. Thus,

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W_{t_{i+1}} - W_{t_i}\right)^2 = \lim_{n\to\infty} \sum_{i=0}^{n-1} X_i^2 = \lim_{n\to\infty} \sum_{i=0}^{n-1} Y_i = \lim_{n\to\infty} n \left(\frac{1}{n} \sum_{i=0}^{n-1} Y_i \right) = n \cdot \frac{T}{n} = T$$

because the random variables Y_i are independent and identically distributed, and due to the Strong Law of Large Numbers we have

$$\frac{1}{n} \sum_{i=0}^{n-1} Y_i \xrightarrow[n\to\infty]{} \frac{T}{n}.$$

□

Remark 3.14. Remember that the Strong Law of Large Numbers states that if X is a real-valued random variable, $\{X_1, X_2, \dots\}$ is an infinite sequence of independent and identically distributed copies of X, and $\overline{X}_n$ is the average of this sequence, then, $P(\lim_{n\to\infty} \overline{X}_n = E(X)) = 1$. The proof can be found in [7].

See Figure 3.1 in Section 3.7 in order to observe the quadratic variation of a Brownian sample path.

Remark 3.15. Suppose now that we have a continuous and differentiable function f and we compute the quadratic variation.

$$\begin{aligned}\sum_{i=0}^{n-1} (f(t_{i+1}) - f(t_i))^2 &\leq \sum_{i=0}^{n-1} \left(f'(s_i)\cdot(t_{i+1}-t_i)\right)^2 \leq \\ \leq \left(\max_{0\leq s\leq T} f'(s)^2\right)\cdot \sum_{i=0}^{n-1}(t_{i+1}-t_i)^2 &= A\cdot\frac{T^2}{n} \xrightarrow[n\to\infty]{} 0,\end{aligned} \tag{3.4}$$

unlike what we found in Theorem 3.13, because in this case the function f is differentiable.

1. In the first inequality in (3.4) we have used the Mean Value Theorem and $s_i \in [t_i, t_{i+1}]$.
2. In the last equality we have used that $A = \max_{\{0\leq s\leq T\}} f'(s)^2$.

As a consequence of equation (3.3) the paths of the process W almost surely have an infinite variation on the interval time $[0, T]$. To show that, we only need to prove that there exist a sequence of subdivisions $\Delta_n[0, T]$ such that almost surely $\lim_{n\to\infty} \sum_{k=1}^{n} |W_{t_k^n} - W_{t_{k-1}^n}| = +\infty$. Reasoning by absurd, let us assume that the supremum on all the subdivisions of the time interval $[0, T]$ of the sums $\lim_{n\to\infty} \sum_{k=1}^{n} |W_{t_k^n} - W_{t_{k-1}^n}|$ may be bounded from above by a positive M. From the above result, since the convergence in probability implies the existence of an almost surely convergent subsequence, we can find a sequence of subdivisions $\Delta_n[0, T]$ whose mesh tends to zero and such that almost surely

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(W_{t_k^n} - W_{t_{k-1}^n}\right)^2 = T.$$

We get then,

$$\sum_{k=1}^{n} \left(W_{t_k^n} - W_{t_{k-1}^n}\right)^2 \leq M \sup_{1\leq k\leq n} |W_{t_k^n} - W_{t_{k-1}^n}| \xrightarrow[n\to+\infty]{} 0,$$

which is a contradiction.

3.5 No monotonicity

Theorem 3.16. *For almost every $\omega \in \Omega$, the sample path $W_{\cdot}(\omega)$ is monotone in no interval.*

Proof. If we denote by F the set of $\omega \in \Omega$ with the property that $W_{\cdot}(\omega)$ is monotone in some interval, then we have,

$$F = \bigcup_{\substack{s,t \in \mathbb{Q} \\ 0 \le s < t < \infty}} \{\omega \in \Omega, W_{\cdot}(\omega) \text{ is monotone in [s,t]}\},$$

since every nonempty interval includes one with rational endpoints. Therefore, it suffices to show that on any such interval, say on $[0,1]$, the path $W_{\cdot}(\omega)$ is monotone for almost no ω. Thanks to the symmetry property on Proposition 3.1 we only need to show that the event

$$A \triangleq \{\omega \in \Omega, W_{\cdot}(\omega) \text{ is nondecreasing on [0,1]}\}$$

is in $\mathcal{F}$ and has probability zero. $A = \bigcap_{n=1}^{\infty} A_n$, where

$$A_n \triangleq \bigcap_{i=0}^{n-1} \{\omega \in \Omega, W_{(i+1)/n}(\omega) - W_{i/n}(\omega) \ge 0\} \in \mathcal{F}$$

has probability $P(A_n) = \prod_{i=0}^{n-1} P[W_{(i+1)/n} - W_{i/n} \ge 0] = 2^{-n}$. Thus, $P(A) \le \lim_{n\to\infty} P(A_n) = 0$. □

3.6 Law of the Iterated Logarithm

The Law of the Iterated Logarithm describes the oscillations of Brownian motion near $t = 0$ and as $t \to \infty$.

Theorem 3.17. ***(Law of the Iterated Logarithm).*** *For almost every $\omega \in \Omega$, we have*

$$(i)\ \overline{\lim_{t\to 0}} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = 1, \qquad (ii)\ \underline{\lim_{t\to 0}} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = -1, \tag{3.5}$$

$$(iii)\ \overline{\lim_{t\to \infty}} \frac{W_t(\omega)}{\sqrt{2t \log\log(t)}} = 1, \qquad (iv)\ \underline{\lim_{t\to \infty}} \frac{W_t(\omega)}{\sqrt{2t \log\log(t)}} = -1, \tag{3.6}$$

By symmetry, property (ii) follows from (i), and by time-inversion properties (iii) and (iv) follow from (i) and (ii), respectively (see Proposition 3.1). Then , it only suffices to prove (i).

Proof. We want to prove (i), so we are going to prove the two following inequalities.

$$P\left(\overline{\lim_{t\to 0}}\frac{W_t(\omega)}{\sqrt{2t\log\log(1/t)}}\geq 1\right)=1,\quad P\left(\overline{\lim_{t\to 0}}\frac{W_t(\omega)}{\sqrt{2t\log\log(1/t)}}\leq 1\right)=1.$$

First, we recall the upper and lower bounds on the tail of the normal distribution. For every $x>0$, we have [11]

$$\frac{x}{1+x^2}e^{-x^2/2}\leq\int_x^\infty e^{-u^2/2}du\leq\frac{1}{x}e^{-x^2/2}. \tag{3.7}$$

We name $h(t)=\sqrt{2t\log\log(1/t)}$. Let $\alpha,\beta>0$, and we apply the Doob's Maximal Inequality [5] to the martingale $\left(e^{\alpha W_t-\frac{\alpha^2}{2}t}\right)$, $t\geq 0$. Then, we have for $t\geq 0$:

$$P\left(\sup_{0\leq s\leq t}(W_s-\frac{\alpha}{2}s)>\beta\right)=P\left(\sup_{0\leq s\leq t}e^{\alpha W_t-\frac{\alpha^2}{2}t}>e^{\alpha\beta}\right)\leq e^{-\alpha\beta}.$$

Let now $\delta,\theta\in(0,1)$. Taking into account the previous inequality, for every $n\in\mathbb{N}$ with $t=\theta^n$, $\alpha=[(1+\delta)h(\theta^n)]/\theta^n$ and $\beta=(1/2)h(\theta^n)$, yields when $n\to\infty$,

$$P\left(\sup_{0\leq s\leq\theta^n}\left(W_s-\frac{(1+\delta)h(\theta^n)}{2\theta^n}s\right)>\frac{1}{2}h(\theta^n)\right)=\mathcal{O}\left(\frac{1}{n^{1+\delta}}\right).$$

Therefore, from the Borel-Cantelli Lemma (Lemma 2.19), for almost every $\omega\in\Omega$, we may find $N(\omega)\in\mathbb{N}$ such that for $n\geq N(\omega)$,

$$\sup_{0\leq s\leq\theta^n}\left(W_s-\frac{(1+\delta)h(\theta^n)}{2\theta^n}s\right)\leq\frac{1}{2}h(\theta^n). \tag{3.8}$$

Equation (3.8) implies that for $\theta^{n+1}\leq t\leq\theta^n$,

$$W_t(\omega)\leq\sup_{0\leq s\leq\theta^n}W_s(\omega)\leq\frac{1}{2}(2+\delta)h(\theta^n)\leq\frac{(2+\delta)h(t)}{2\sqrt{\theta}}. \tag{3.9}$$

We conclude that,

$$P\left(\overline{\lim_{t\to 0}}\frac{W_t}{\sqrt{2t\log\log(1/t)}}\leq\frac{2+\delta}{2\sqrt{\theta}}\right)=1.$$

With $\theta\to 1$ and $\delta\to 0$ we have,

$$P\left(\overline{\lim_{t\to 0}}\frac{W_t}{\sqrt{2t\log\log(1/t)}}\leq 1\right)=1.$$

Let us now prove that

$$P\left(\overline{\lim_{t\to 0}}\frac{W_t}{\sqrt{2t\log\log(1/t)}}\geq 1\right)=1.$$

Let $\theta\in(0,1)$. We denote $A_n=\{\omega, W_{\theta^n}(\omega)-W_{\theta^{n+1}}(\omega)\geq(1-\sqrt{\theta})h(\theta^n)\}$ for $n\in\mathbb{N}$. We want to prove that $\sum_{n=1}^{\infty}P(A_n)=+\infty$. From equation (3.7) we have,

$$P(A_n)=\frac{1}{\sqrt{2\pi}}\int_{a_n}^{+\infty}e^{-\frac{u^2}{2}}du\geq\frac{a_n}{1+a_n^2}e^{-\frac{a_n^2}{2}},$$

with $a_n=\frac{(1-\sqrt{\theta})h(\theta^n)}{\theta^{n/2}\sqrt{1-\theta}}$ taking into account that $(W_{\theta^n}-W_{\theta^{n+1}})\sim N(0,\theta^n(1-\theta))\sim\theta^{n/2}\sqrt{1-\theta}N(0,1)$.

When $n\to\infty$, $\frac{a_n}{1+a_n^2}e^{-\frac{a_n^2}{2}}=\mathcal{O}\left(\frac{1}{n^{\frac{1+\theta-2\sqrt{\theta}}{1-\theta}}}\right)$. Therefore, $\sum_{n=1}^{\infty}P(A_n)=+\infty$. As a consequence of the independence of the Brownian increments and of Borel-Cantelli Lemma, we know that the event $W_{\theta^n}-W_{\theta^{n+1}}\geq(1-\sqrt{\theta})h(\theta^n)$, will occur almost surely for infinite many n's. Moreover, from the first part of the proof we know that, for almost every ω, we may find $N(\omega)$ such that for $n\geq N(\omega)$,

$$W_{\theta^{n+1}}>-2h(\theta^{n+1})\geq-2\sqrt{\theta}h(\theta^n).$$

Thus, almost surely, the event $W_{\theta^n}>h(\theta)(1-3\sqrt{\theta})$ will occur almost surely for infinite many n's. This implies that

$$P\left(\overline{\lim_{t\to 0}}\frac{W_t}{\sqrt{2t\log\log(1/t)}}\geq 1-3\sqrt{\theta}\right)=1.$$

By letting $\theta\to 0$ we finlly get that,

$$P\left(\overline{\lim_{t\to 0}}\frac{W_t}{\sqrt{2t\log\log(1/t)}}\geq 1\right)=1.$$

□

3.7 Sample path simulations using Python

In this section we simulate a one dimensional Brownian sample path in the time interval $[0,T]$, with $0<T<\infty$. To carry out the simulations we divide the time interval $[0,T]$ into N subintervals, each one of length T/N. In other words, let $0=t_0<t_1<\cdots<t_N=T$. Then, assuming that $\{B_t, t\geq 0\}$ is a Brownian motion and knowing that $B_{t_{i+1}}-B_{t_i}\sim B_{t_{i+1}-t_i}\sim N(0,t_{i+1}-t_i)$, we have that

$$B_{t_{i+1}}=B_{t_i}+(B_{t_{i+1}}-B_{t_i})=B_{t_i}+\sqrt{t_{i+1}-t_i}Z_i, \tag{3.10}$$

where $Z_i \sim N(0,1)$ and $t_{i+1} - t_i = T/N$ which will be named the length of the step. With all this information, we implement the following Python code. Firstly, we import all the packages that we will need.

```
import random
import math
import numpy as np
from bokeh.io import show, output_notebook
from bokeh.plotting import figure
random.seed(17)
output_notebook(hide_banner=True)
```

Then, using (3.10) we define a function to simulate the Brownian path, taking into account that the numpy function named *np.random.randn* returns a sample (or samples) from the standard normal distribution, and the function *np.cumsum* returns the cumulative sum of the elements along a given axis.

```
def brownian_path(N):
    dt_sqrt = math.sqrt(T / N)
    Z = np.random.randn(N)
    Z[0] = 0
    B = np.cumsum(dt_sqrt * Z)
    return B
```

Once we have defined the Brownian path function we are able to plot it in a graphic, by setting previously the parameters T and N.

```
T = 15
N = 1000
steps = [x for x in range(N)]
B = brownian_path(N)
p = figure(title='Sample path of a brownian motion', plot_width=600, plot_height=400)
p.xaxis.axis_label = "Step"
p.yaxis.axis_label = "B"
p.line(steps, B)
show(p)
```

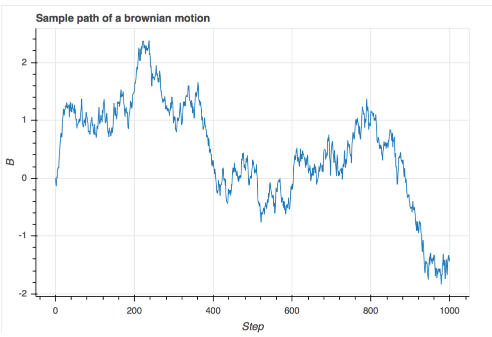
Additionally, we want to observe graphically the quadratic variation convergence (expression (3.3)). With this purpose we define a function which sums the squares of the differences between consecutive values of the Brownian path.

```
def ssq(B):
    dB = np.diff(B)
    dBsq = np.square(dB)
    dBqv = np.sum(dBsq)
    return dBqv
```

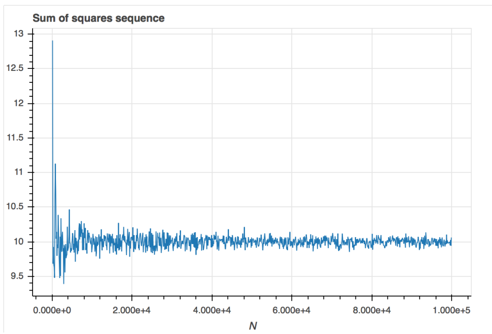
On the next lines of code, we generate a sequence with the values of the sums of the quadratic increments of the Brownian path as the number of subdivisions N increases.

```
N_lim = 100000
N_seq = np.arange(100, N_lim, step=100)
dBqv_seq = np.empty(N_seq.shape)
for k, n in enumerate(N_seq):
    B = brownian_path(n)
    dBqv_seq[k] = ssq(B)

opts = dict(plot_width=450, plot_height=450, min_border=0)
p_qv = figure(**opts, title='Sum of squares sequence')
p_qv.xaxis.axis_label="N"
p_qv.line(N_seq, dBqv_seq)
show(p_qv)
```



(a) Simulation of Brownian sample path, for $T = 10$, and $N = 1000$.



(b) $\sum_{k=1}^{N}\left(B_{\frac{k}{N}T} - B_{\left(\frac{k-1}{N}\right)T}\right)^2$ as a function of N.

Figure 3.1: Observe that in image (b) $\sum_{k=1}^{N}\left(B_{\frac{k}{N}T} - B_{\left(\frac{k-1}{N}\right)T}\right)^2 \to T$ as the value of N increases.

Conclusions

Brownian motion is the most studied real valued continuous-time stochastic process, due to its numerous applications. The results shown in this project are of very importance in pure and applied mathematics, economics and finance.

Firstly, in order to study the Brownian motion, we have introduced the branch of probability known as Theory of stochastic processes. Then, we have provided the construction of a probability measure from a consistent family of finite-dimensional distributions (Daniell-Kolmogorov Theorem) as well as the existence of a continuous modification of the coordinate mapping process (Kolmogorov's Continuity Theorem). Furthermore, we have seen the Wiener process as a limit of random walks and we have defined the Wiener measure.

Some properties of Brownian sample paths are detailed in the last chapter of this work because they are of great interest, especially the nondifferentiability which leads to stochastic calculus.

Elaborating this project I have acquired a lot of new concepts about stochastic processes and Brownian motion that I had never studied before.

Bibliography

[1] Karatzas, Ioannis, et al. *Brownian Motion and Stochastic Calculus.* Second edition. Springer. 1998.

[2] Hida, Takeyuki. *Brownian Motion.* Translated by the author and T.P. Speed. Springer. 1980.

[3] Revuz, Daniel, et al. *Continuous Martingales and Brownian Motion.* Third Edition. Springer. 2005.

[4] Revuz, Daniel. *Markov Chains.* Second Edition. North-Holland. 1984.

[5] Baudoin, Fabrice. *Research and Lecture Notes: Stochastic Calculus Lectures*. Fabrice Baudoin Blog. `https://fabricebaudoin.wordpress.com`. Posted on 23 May, 2012. [Consultation: 22/03/2020].

[6] Winkelbauer, Andreas. *Moments and Absolute Moments of the Normal Distribution*. arXiv:1209.4340v2 [math.ST]. Revised version. 15 July 2014. `https://arxiv.org/abs/1209.4340v2`.

[7] Terence, Tao. *The strong law of large numbers*. Terence Tao's Blog. Posted on 18 June, 2008. `https://terrytao.wordpress.com/2008/06/18/the-strong-law-of-large-numbers/` [Consultation: 18/05/2020].

[8] Anonymous. *Lecture 7: The Reflection Principle. The Distribution of the Maximum. Brownian motion with drift.* Massachusetts Institute of Technology. MIT OpenCourseWare. 25 September, 2013. `https://ocw.mit.edu/courses/sloan-school-of-management/15-070j-advanced-stochastic-processes-fall-2013/lecture-notes/MIT15_070JF13_Lec7.pdf`

[9] Zitkovic, Gordan. *Lecture 15: First Properties of the Brownian Motion.* Course: Theory of Probability ll. Department of Mathematics, The University of Texas at Austin. 16 February, 2015. `https://web.ma.utexas.edu/users/gordanz/notes/first_properties.pdf`

[10] Zitkovic, Gordan. *Lecture 17: Brownian Motion as a Markov Process.* Course: Theory of Probability ll. Department of Mathematics, The University of Texas at Austin. 7 March, 2015. https://web.ma.utexas.edu/users/gordanz/notes/brown_markov.pdf

[11] Peres, Yuval, et al. *Brownian motion*. First edition. Cambridge University Press. 2010.

[12] Dahl, Andy. *A Rigorous Introduction to Brownian Motion.* Department of Mathematics, The University of Chicago. 19 August, 2010. http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Dahl.pdf