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GRAU DE MATEMÀTIQUES

Treball final de grau

Knots and Seifert surfaces

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Agraïments

Voldria agraïr primer de tot al Dr. Javier Gutiérrez Marín per la seva ajuda, dedicació i respostes immediates durant el transcurs de tot aquest treball. Al Dr. Ricardo García per tot i no haver pogut tutoritzar aquest treball, haver-me recomanat al Dr. Javier com a tutor d'aquest. A la meva família i als meus amics i amigues que m'han fet costat en tot moment (i sobretot en aquests últims mesos força complicats). I finalment a aquelles persones que "m'han descobert" aquest curs i que han fet d'aquest un de molt especial.

Abstract

Since the beginning of the degree that I think that everyone should have the oportunity to know mathematics as they are and not as they are presented (or were presented) at a high school level. In my opinion, the answer to the question "why do we do this?" that a student asks, shouldn't be "because is useful", it should be "because it's interesting" or "because we are curious". To study mathematics (in every level) should be like solving an enormous puzzle. It should be a playful experience and satisfactory (which doesn't mean effortless nor without dedication).

It is this idea that brought me to choose knot theory as the main focus of my project. I wanted a theme that generated me curiosity and that it could be attractive to other people with less mathematical background, in order to spread what mathematics are to me. It is because of this that i have dedicated quite some time to explain the intuitive idea behind every proof and definition, and it is because of this that the great majority of proofs and definitions are paired up with an image (created by me).

In regards to the technical part of the project I have had as main objectives: to introduce myself to knot theory, to comprehend the idea of genus of a knot and know the propeties we could derive to study knots.

Resum

Des que vaig començar la carrera que penso que tothom hauria de tenir la oportunitat de conèixer les matemàtiques tal i com són i no com són (o almenys eren) presentades a nivell de secundària. En la meva opinió, la resposta a la pregunta "per què fem això?" que formula un alumne no hauria de ser "perquè és útil", si no més aviat "perquè ens interessa" o "perquè tenim curiositat". Estudiar matemàtiques (en tots els nivells) hauria de ser com resoldre un trencaclosques gegant. Hauria de ser una experiència juganera i satifactoria (que no vol dir que no requereixi esforç ni dedicació).

És aquesta idea la que m'ha portat a escollir la teoria de nusos com a branca principal del meu treball. Volia una temàtica que em despertés curiositat i que a la vegada pogués ser atractiva per a gent sense gaire coneixement matemàtic, per així poder difondre al màxim el què són per a mi les matemàtiques. És per això que he dedicat força temps a explicar la idea intuïtiva darrera de cada definició i demostració, i és per això que la gran majoria d'aquestes va acompanyada d'una imatge (fetes per mi).

Pel que fa a la part tècnica del treball, com a principals objectius he tingut: introduïr-me en la teoria de nusos, comprendre la idea del gènere d'un nus i veure quines propietats ens proporcionava per a estudiar nusos.

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Contents

Chapter 1

Basic concepts

In this chapter we will give the necessary definitions to work with knots, we will define the equivalence between knots, we will give a brief explanation why is it natural to consider such definition of equivalence and not others and we will define the connected sum of two knots.

1.1 Knots and diagrams

Loosely speaking, a mathematical knot is a piece of string entangled with itself, with no ends, inside a three dimensional space. The easiest way to imagine this is to take a piece of string, tie a knot, and glue the ends together.

Of course, this is not a rigorous definition and we cannot work whit it. The mathematical definition is as follows.

Definition 1.1. A *knot* K is a subspace of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$.

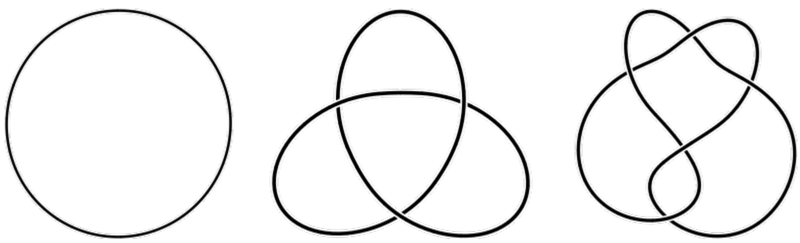


Figure 1.1: Examples of knots projected on a plane

We can also think a knot K, as the image of a continuous map $\alpha\colon [0,1] \to \mathbb{R}^3$ such that $\alpha|_{[0,1)}$ is injective and $\alpha(0) = \alpha(1)$. Such a map will be called a *parametrization* of K.

The examples in Figure 1.1 are not knots, in rigorus terms, they are what we will call *knot diagrams*.

Definition 1.2. Let K be a knot, α a parametrization of K and $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ a projection. Supose there exists $t_0, t_1 \in [0,1)$ such that $\pi \circ \alpha(t_0) = \pi \circ \alpha(t_1)$ (that is, the plane curve $\pi \circ \alpha$ has a self intersection). We say that such self intersection is *transversal* if $\pi \circ \alpha$ is differentiable at t_0 and t_1, and the tangent vectors are linearly independent.

Definition 1.3. Given a knot K, a projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a *regular projection of* K if:

i) For every point $p \in \pi(K)$, we have $|\pi^{-1}(p)| \leq 2$.

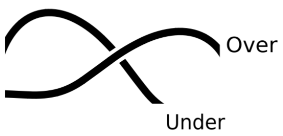
ii) Every self intersection of $\pi(K)$ is transversal.

The points $p \in \pi(K)$ where $|\pi^{-1}(p)| = 2$ are called *crossing points*.

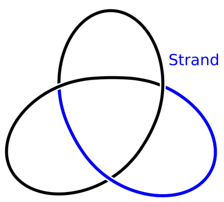
$|A|$ denotes the cardinality of the set A.

For us, a projection of a knot will always be regular unless is otherwise indicated. Then, a *knot diagram* (or a *knot projection*) for a knot K is a picture of $\pi(K)$, where π is a regular projection and where we indicate that a line is above another at a crossing point as follows:

At each crossing point p we have two lines, say A and B, that intersect at p. To indicate that A is over B, we will draw A as a connected line, and B as two disconnected lines. Each connected line in $\pi(K)$ is called a *strand*.



(a) A crossing point



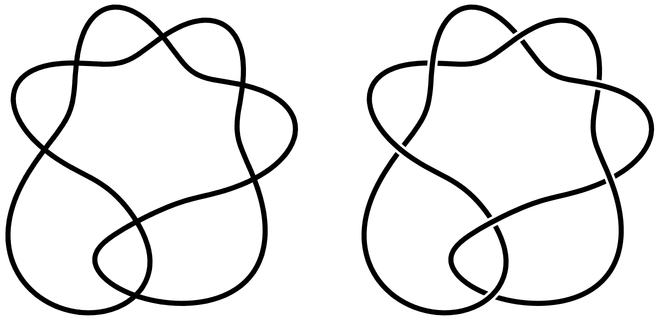
(b) A blue strand

Figure 1.2: Example of a crossing and a strand

Figure 1.3b illustrates the explanation of a knot diagram.

Definition 1.4. We say that a projection of a knot K is *alternating* if each strand contains three crossing points.

A knot K is said to be *alternating* if there exists an alternating projeccion of K.



(a) Not a knot diagram (b) A proper knot diagram

Figure 1.3: Difference between a knot diagram and a shadow of a knot

Examples of alternating projections can be found in Figure 1.2b and in Figure 1.3b.
A first measure of the complexity of a knot K is the *crossing number* of K, usually denoted by $c(K)$, which is the minimum number of crossings needed to represent a diagram of K. In fact, most tables of knots are ordered by crossing number and using the Alexander-Biggs notation, which works as follows. Every knot K is designated by c_i where $c = c(K)$ and i is just a natural number to enumerate different knots with the same crossing number (although the choice of i is arbitrary, mathematicians have come to a consent as shown in Appendix A). This notation was introduced by James W. Alexander and Garland B. Briggs in 1926 and it is known as the Alexander-Briggs notation (the paper were the notation was first introducced can be found in [2]). Some knots even have names, for example 3_1 is called the *trefoil knot* and 4_1 *figure eight knot.*

1.2 Equivalence of knots

So far we haven't introduced the idea of equivalence between knots. We want the definition of equivalence between two knots to corespond to the idea of untangling (or tangling) one knot to get the other. We are going to see that the concepts of homotopy equivalence and homeomorphic equivalence are not enough to descrive what we want.

Let's just say that two knots K_1 and K_2 are equivalent if they are homeomorphic. Since cutting K_1 by an arbitrary point, untangling it and gluing back together the points previuosly cutted, is an homeomorphism, we would have that every knot could be untangled, in other words, all knots would be equivallent to each other! Then homeomorphic equivalence is not what we need. This, should have been clear from the definition of knot.

Now, since all knots are homeomorphic to each other, they are also homotopy equivalent. Therefore homotopy equivalence is not good enough either.

What we want is a definition that takes into account what happens to the space around the knot while we deform it. This fact of taking into account how the space around is deformed is encapsulated with the concept of *ambient isotopy.*

Definition 1.5. Let X, Y be topological spaces, $I = [0, 1]$, and f, g embeddings from X to Y. Then, f and g are said to be *ambient isotopic* if there exists a continuous map $H\colon Y \times I \to Y$ such that:

i) For all $t \in I$, $H(\cdot, t)$ is an homeomorphism from Y to itself.

ii) $H(\cdot, 0) = id_Y$.

iii) For all $x \in X$, $H(f(x), 1) = g(x)$.

We say that H is an *ambient isotopy* between f and g.
Here id_Y denotes the identity map in Y.

With this in mind the definition of eqivalence between knots is the following.

Definition 1.6. Given two knots K_1 and K_2, we say that K_1 and K_2 are *equivalent* and we will write $K_1 \sim K_2$, if there exists an ambient isotopy $H\colon \mathbb{R}^3 \times I \to \mathbb{R}^3$ between $id_{\mathbb{R}^3}\colon \mathbb{R}^3 \to \mathbb{R}^3$ and an homeomorphism $H_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that, $H(K_1, 1) = H_1(K_1) = K_2$
It is common to write $H_t(x)$ instead of $H(x, t)$.

We are now ready to define the unknot, also known as the trivial knot.

Definition 1.7. We say that a knot K is the *unknot* (or the *trivial knot*) if it is equivalent to the subspace $\{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = 1,\ \ z = 0\}$. We will denote the unknot by $\mathcal{O}$

A diagram of the unknot can be found in the first picture of Figure 1.1.

We are not interested in all possible knots, we want to avoid knots with rather complicated structures like those with infinite crossings. To avoid these type of knots we will work only with *tame* knots.

Definition 1.8. A *polygonal knot* K is the union of a finite collection of line segments that are disjoint or that intersect at their end points in such a way that K is homeomorphic to $\mathbb{S}^1$

Definition 1.9. A knot is *tame* if it is equivalent to a polygonal knot. A knot is *wild* if it's not tame.

From now on, all knots considered will be tame unless is otherwise indicated.

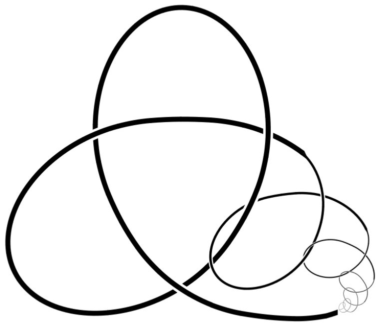


Figure 1.4: A picture of a wild knot

1.3 Connected sum

The idea behind the connected sum of two knots is that of cutting two knots and gluing the ends so that from two knots we get a new one. But, to rigorously define this operation we first need to define what an oriented knot is.

Definition 1.10. Given a knot K and two homeomorphisms $f, g\colon \mathbb{S}^1 \to K$, we say that an *oriented knot* is a class of equivalence of the pair $\big(K, f\big)$, where

$$\Big[\big(K, f\big)\Big] = \Big[\big(K, g\big)\Big] \text{ if and only if } \quad deg(f^{-1} \circ g) = 1,$$

where $[a]$ denotes the equivalence class of a and $deg(h)$ denotes the degree of the continuous map $h\colon \mathbb{S}^1 \to \mathbb{S}^1$

An oriented knot is in some sense, a knot equiped with a direction to traverse the knot. In a diagram we will idicate the orientation of a knot with arrows.

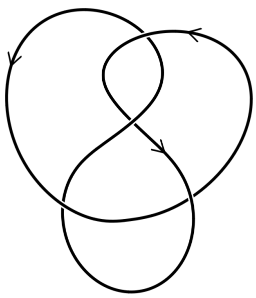


Figure 1.5: Diagram of an oriented knot

Every knot can have two different orientations. If $\Big[\big(K, f\big)\Big]$ is an oriented knot then the same knot with the reversed orientation is $\Big[\big(K, f \circ i\big)\Big]$ (where $i\colon \mathbb{S}^1 \to \mathbb{S}^1$ is given by $z \mapsto z^{-1}$).

Let's see that they are in fact two different orientations. What we need to see is that $deg(f^{-1} \circ f \circ i) \neq 1$, but that is clear since

$$deg(f^{-1} \circ f \circ i) = deg(i) = -1$$

If K denotes a knot, K^+ will denote the same knot with a fixed orientation and we will denote the same knot with the other orientation by K^-. Ocasionally we will not use the superscript to indicate the orientation, whether we are talking about a knot or an oriented knot will be clear from context.

Equivalence of knots translates to equivalence of oriented knots as follows:

Definition 1.11. Let K_1^+ and K_2^+ be two oriented knots given by the classes $\Big[\big(K_1, f_1\big)\Big]$, $\Big[\big(K_2, f_2\big)\Big]$. Then we say that the *oriented knots are equivalent*, and we will write $K_1^+ \sim K_2^+$ if there is an ambient isotopy between the two that respects the orientation. That is $K_1 \sim K_2$ via the ambient isotopy H, and $\Big[\big(K_2, f_2\big)\Big] = \Big[\big(K_2, H_1 \circ f_1\big)\Big]$, i.e, $deg(f_2^{-1} \circ H_1 \circ f_1) = 1$. We say that a knot K is *invertible* if $K^+ \sim K^-$.

As an example, the unknot is invertible. An ambient isotopy H_t between $\mathcal{O}^+$ and $\mathcal{O}^-$ is the one that for every $t \in I$, H_t is a rotation around the x axis of angle πt radians, that is

$$H_t(x, y, z) = (x, cos(\pi t)y + sin(\pi t)z, -sin(\pi t)y + cos(\pi t)z) \qquad (x, y, z) \in \mathbb{R}^3$$

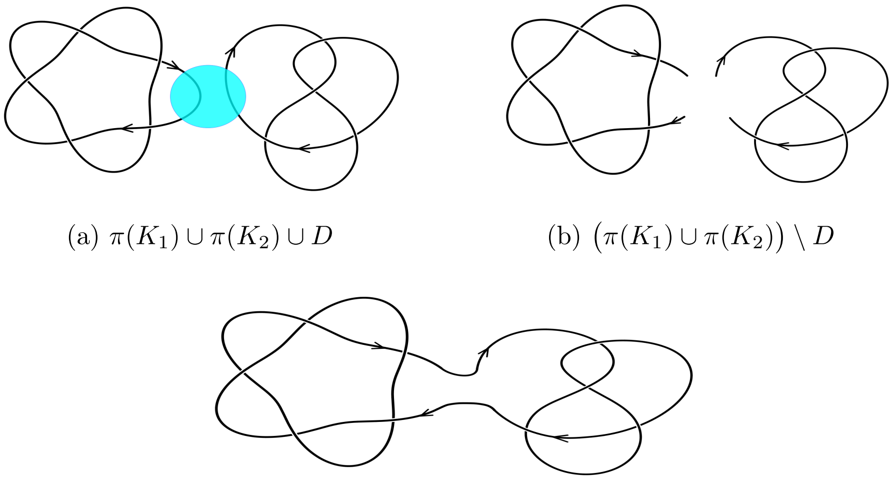
We are now ready to define the connected sum of two knots.

Definition 1.12. Let K_1, K_2 be two oriented knots, and consider their diagrams (given by regular projections π_1, π_2, respectively) in such a way that they do not intersect. Consider a closed disk D such that:

i) $D \cap \pi_1(K_1)$ and $D \cap \pi_2(K_2)$ have no crossing points and are connected.

ii) $|\partial D \cap \pi_1(K_1)| = |\partial D \cap \pi_2(K_2)| = 2$

If needed bring $\pi_1(K_1)$ and $\pi_2(K_2)$ closer or further apart via an ambient isotopy so that the conditions are satisfied (keeping in mind they cannot intersect!). Then remove $Int(D)$ from both K_1 and K_2. Finally, identify each point of $\partial D \cap \pi_1(K_1)$ with a point of $\partial D \cap \pi_2(K_2)$ in such a way that orientations match up.
This process gives us a new diagram. The *connected sum* of K_1 and K_2, denoted $K_1 \# K_2$, is an oriented knot with that diagram.



(a) $\pi(K_1) \cup \pi(K_2) \cup D$

(b) $\big(\pi(K_1) \cup \pi(K_2)\big) \setminus D$

(c) $K_1 \# K_2$

Figure 1.6: Connected sum of 5_1 and 4_1

The pictures in Figure 1.6 help to understand the process.

It is important to notice that if we had not required the knots to be oriented, then there would be two possible diagrams of $K_1 \# K_2$ that could end up beeing not equivalent!

Although the connected sum is a well defined operation between oriented knots, the result does depend on the orientation of both K_1 and K_2. But if one of the knots is invertible, say K_1 then $K_1^+ \# K_2 \sim K_1^- \# K_2$.

The name *connected sum* suggests there must be some connection between this new operation and the sum of numbers as we know it. The next result shows us the connection between the two.

Proposition 1.13. *The following staments are true:*

i) The connected sum is commutative.

ii) The connected sum is associative.

iii) For every oriented knot K, $K \# \mathcal{O} \sim K$. That is, $\mathcal{O}$ is the neutral element of the connected sum.

Proof.

i) Given the sum $K_1 \# K_2$, we can get to $K_2 \# K_1$ via an ambient isotopy by sliding one knot through the other (see an example in Figure 1.7).

ii) It is clear that the sum $(K_1\#K_2)\#K_3$ gives the same diagram as $K_1\#(K_2\#K_3)$, and hence they are equivalent.

iii) The procedure of the sum of $\mathcal{O}$ with any knot K doesn't change the knot K since we are replacing a piece of an untangled strand of K with another piece of an untangled strand of K (since $\mathcal{O}$ has no crossings). $\square$

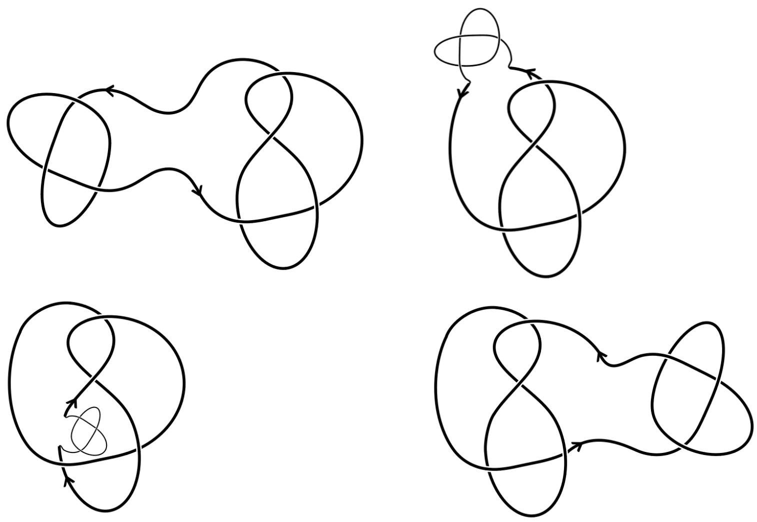


Figure 1.7: Ambient isotopy from $3_1^+\#4_1^+$ to $4_1^+\#3_1^+$

To end this chapter we introduce two interesting concepts.

Definition 1.14. A knot K is *irreducible* if $K \sim K_1\#K_2$ implies that $K_1 \sim \mathcal{O}$ or $K_2 \sim \mathcal{O}$.
A knot K is *prime* if it is irreducible and it is not equivalent to $\mathcal{O}$.

With this, we can ask ourselves different questions:
How can we tell if a knot is prime or not? Is there a fundamental theorem of arithmetic-type theorem for the sum of knots?. That is, *can every knot (except the unknot) be uniquely represented by a finite sum of prime knots?*.
The answer of the second question is a theorem proved by Schubert in 1949 and can be found in [4] page 96.
In chapter four, we will give a sufficient condition for a knot to be prime, and we will be able to see that every knot is a finite sum of prime knots (we won't prove the uniqueness).

Chapter 2

Seifert surfaces and genus

In this chapter we first recall some notions regarding surfaces and we will introduce the concept of Seifert surfaces of a knot in order to define a knot invariant, the genus of a knot.

2.1 Topological surfaces

We first recall some definitions about topological surfaces

Definition 2.1. A *topological surface* is a topological space S such that:

i) S is a Hausdorff space.

ii) S is a second countable space i.e, S has a countable base.

iii) For every $p \in S$, there exists an open neighborhood U_p of p, and an homeomorphism $\varphi_p \colon U_p \longrightarrow \varphi(U_p) \subseteq \mathbb{R}^2$ where $\varphi(U_p)$ is open in $\mathbb{R}^2$.

The pair $\big(U_p, \varphi_p\big)$ is called a *chart* of S. And a family of charts $\mathcal{A} = \big\{\big(U_i, \varphi_i\big)\big\}_{i\in I}$ is called an *atlas* of S if $S = \bigcup_{i\in I} U_i$.

In Figure 2.1 there are two examples of surfaces.

Definition 2.2. A *topological surface with boundary* is a topological space S such that:

i) S is a Hausdorff space.

ii) S is a second countable space i.e, S has a countable base.

iii) For every $p \in S$, there exists an open neighborhood U_p of p, and an homeomorphism $\varphi_p \colon U_p \longrightarrow \varphi(U_p) \subseteq \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$ where $\varphi(U_p)$ is open in $\mathbb{R}^2_+$.

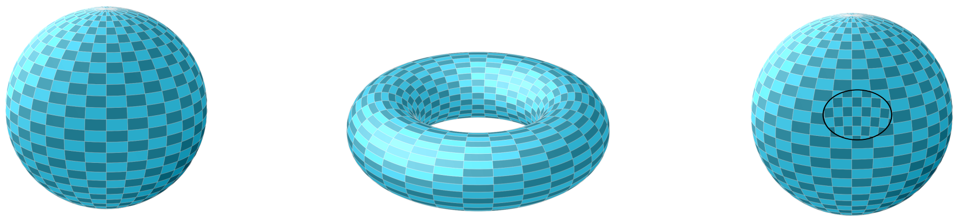


Figure 2.1: From left to right: a sphere, a torus and a sphere with one boundary component

The *interior* of S, denoted $Int(S)$, is the set of points in S which have neighborhoods homeomorphic to an open subset of $\mathbb{R}^2$. The *boundary* of S, denoted ∂S is the compliment of $Int(S)$ in S.

A *boundary component* B of ∂S is a connected component of ∂S.

It is important to observe that in the case of a topological surface with boundary S, ∂S is locally homeomorphic to $\mathbb{R}$.

We will only work with compact connected topological surfaces, so from now on every topological surface will be compact and connected (unless is otherwise indicated).

Definition 2.3. Given a topological surface S (with or without boundary), a *triangle* T in S is a closed subset of S such that there exists an homeomorphism $f\colon \Delta \longrightarrow T$ where Δ is the following subspace of $\mathbb{R}^2$, $\Delta = \big\{(x, y) \in \mathbb{R}^2 |\ x, y \geq 0 \text{ and } x + y \leq 1\big\}$ (which is a triangle in $\mathbb{R}^2$).

The *vertices* of T are the images (by f) of the vertices of Δ and the *edges* of T are the images (also by f) of the edges of Δ.

A *triangulation* of S is a family of triangles in S, $\mathcal{T} = \{T_i\}_{i\in I}$ such that:

i) $S = \bigcup_{i\in I} T_i$

ii) If $T_i \cap T_j \neq \emptyset$ $(i \neq j)$ then $T_i \cap T_j$ is either just one vertex of both T_i and T_j or just one edge of both T_i and T_j.

It is a well known theorem that every topological surface S can be triangulated. Moreover if the surface is compact, the triangulation consists of finetly many triangles. A proof of this theorem can be found in [9]

Definition 2.4. Let S be a topological surface (with or without boundary) and $\mathcal{T} = \big\{T_i\big\}_{i\in I}$ a triangulation of S. If v denotes de number of vertices in $\bigcup_{i\in I} T_i$, e

the number of edges in $\bigcup_{i\in I} T_i$ and f the number of triangles of $\mathcal{T}$ (also known as *faces*), then the *Euler-Poincaré characteristic* of $\mathcal{T}$ is

$$\chi(\mathcal{T}) = v - e + f.$$

It can be proved (using homology groups) that the Euler-Poincaré characteristic is a topological invariant and therefore does not depend on the triangulation, so it is common to talk about the Euler-Poincaré characteristic of a topological surface instead of the characteristic of a triangulation of that surface. We will write $\chi(S)$ to denote the Euler-Poincaré characteristic of the surface S.

In the same way that there is a notion of connected sum for knots, there is also a notion of connected sum of topological surfaces.

Definition 2.5. Given two surfaces S_1 and S_2. Let D_1 be a closed disk in a chart of S_1 and D_2 a closed disk in a chart of S_2. Let $h\colon \partial D_1 \longrightarrow \partial D_2$ be a homeomorphism.

Then the *connected sum* of S_1 and S_2 is another topological surface defined as

$$S_1 \# S_2 = (S_1 \setminus \overset{\circ}{D}_1) \sqcup (S_2 \setminus \overset{\circ}{D}_2)/\sim$$

where for all $x \in \partial D_1$ and all $y \in \partial D_2$, $x \sim y$ if and only if $y = h(x)$

Proposition 2.6. *Let S_1 and S_2 be two topological surfaces (with or without boundary) then the following equation holds:*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations of S_1 and S_2 (respectively) and v_i, e_i and f_i the number vertices, edges and faces of $\mathcal{T}_i$ respectively (with $i = 1, 2$).

We can think of $S_1 \# S_2$ as the surface resulting of removing a face of a triangle T_1 of S_1 and a face of a triangle T_2 of S_2 and then identifying each edge of T_1 with an edge of T_2 (we can do this because every closed triangle is homeomorphic to a closed disk). From this, it is clear that there is a triangulation $\mathcal{T}$ of $S_1 \# S_2$ such that the number vertices of $\mathcal{T}$ are $v_1 + v_2 - 3$, the number of edges of $\mathcal{T}$ are $e_1 + e_2 - 3$ and the number of faces of $\mathcal{T}$ are $f_1 + f_2 - 2$. Therefore the Euler-Poincaré characteristic of $S_1 \# S_2$ is:

$$\chi(S_1 \# S_2) = v_1 + v_2 - 3 - e_1 - e_2 + 3 + f_1 + f_2 - 2 = \chi(S_1) + \chi(S_2) - 2. \qquad \square$$

We now recall the calssification theorem of compact connected surfaces.

Theorem 2.7 (Classification of compact connected surfaces)**.** *Every compact connected topological surface is homeomorphic to one and only one of the following surfaces:*

i) $\mathbb{S}^2$

ii) $g\mathbb{T}^2 = \mathbb{T}^2 \# \overset{g}{\cdots} \# \mathbb{T}^2$

iii) $\# g\mathbb{P}^2_{\mathbb{R}} = \#\mathbb{P}^2_{\mathbb{R}} \# \overset{g}{\cdots} \# \mathbb{P}^2_{\mathbb{R}}$

Where $g \in \mathbb{N} \setminus \{0\}$*.* $\mathbb{S}^2$ *denotes the two dimensional sphere,* $\mathbb{T}^2$ *the two dimensional torus and* $\mathbb{P}^2_{\mathbb{R}}$ *the real projective plane.*

A proof of this theorem can be found in [6]

Definition 2.8. A topological surface S is *orientable* if it is either homeomorphic to $\mathbb{S}^2$ or to $g\mathbb{T}^2$ with $g \in \mathbb{N} \setminus \{0\}$.

The *genus* of an orientable surface S, denoted $g(S)$, is 0 if S is homeomorphic to $\mathbb{S}^2$ or n if S is homeomorphic to $n\mathbb{T}^2$ (with $n \in \mathbb{N}$).

In some sense, a surface is orientable if it has two different sides.

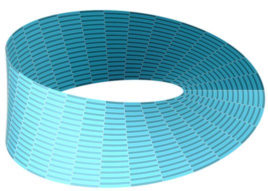


Figure 2.2: The Möbius strip, a non oriantable surface with one boundary component

Taking into account the formula in Proposition 2.6 and the fact that $\chi(\mathbb{S}^2) = 2$ and $\chi(\mathbb{T}^2) = 0$, we can conclude that $\chi(g\mathbb{T}^2) = 2 - 2g$ (understanding $0\mathbb{T}^2$ as $\mathbb{S}^2$).

We end this section with the theorem of classification of compact connected surfaces with boundary.

Theorem 2.9 (Classification of compact connected surfaces with boundary)**.** *Let* S *be a compact connected topological surface with boundary. Then* S *is homeomorphic to one and only one of the following surfaces:*

i) $\mathbb{S}^2 \setminus (D_1 \cup D_2 \cup \ldots \cup D_b)$ *where* b *is the number of boundary components of* S *and each* D_i *is homeomorphic to an open disc disjoint to every other disc.*

ii) $g\mathbb{T}^2 \setminus (D_1 \cup D_2 \cup \ldots \cup D_b)$ *where b is the number of boundary components of S and each D_i is homeomorphic to an open disc disjoint to every other disc.*

iii) $n\mathbb{P}^2_{\mathbb{R}} \setminus (D_1 \cup D_2 \cup \ldots \cup D_b)$ *where b is the number of boundary components of S and each D_i is homeomorphic to an open disc disjoint to every other disc.*

Proof. Let S be a compact connected topological surface with boundary. Since ∂S is closed in S and S is both Hausdorff and compact, it follows that ∂S is compact. From this, we know that ∂S has a finite number of boundary components, say b. Each boundary component is closed and therefore also compact. And since every boundary component is compact, connected and locally homeomorphic to $\mathbb{R}$ it follows that every boundary component is homeomorphic to $\mathbb{S}^1$.

Now for every boundary component C_i consider a closed disk $\overline{D_i}$ and consider a homeomorphism $h_i\colon C_i \longrightarrow \partial D_i$ $(i = 1, \ldots, b)$. Finally consider the topological space $\hat{S} = S \sqcup \overline{D_1} \sqcup \ldots \sqcup \overline{D_b}/\sim$ where for every pair of points $x \in C_i$ and $y \in \partial D_i$, $x \sim y$ if and only if $y = h_i(x)$. It is clear from the construction that $\hat{S}$ is Hausdorff, has a countable base, is compact and connected. What it may not be obvious at first glance is that it is locally homeomorphic to an open set of $\mathbb{R}^2$. Let us see that it is indeed the case.

Take a point $p \in \hat{S}$. If $p \in Int(S) \cup Int(D_1) \cup \ldots \cup Int(D_b)$ then by definition p has an open neighborhood homeomorphic to an open set of $\mathbb{R}^2$. Supose then that $p \in C_i$ for a particular $i \leq b$. Since C_i is identyfied with ∂D_i we can think of p as beeing both in C_i and ∂D_i. Because $p \in C_i$ and $p \in \partial D_i$ there exist two open neighborhoods U, V of p with $U \subseteq S$ and $V \subseteq \overline{D_i}$ and two homeomorphism $f\colon U \longrightarrow f(U) \subseteq \mathbb{R}^2_+$ and $g\colon V \longrightarrow g(V) \subseteq \mathbb{R}^2_+$. Without loss of generality we can supose that $f(p) = g(p)$ (if nedded bring $g(p)$ to $f(p)$ with a translation). Choose a neighborhood I of p in C_i such that $I \subseteq U$ and $I \subseteq V$. Consider now $U' = f(Int(U)) \cup f(I)$ and $V' = g(Int(V)) \cup g(I)$, then $f^{-1}(U')$ and $g^{-1}(V')$ are open in S and $\overline{D_i}$ (respectively) and they overlap only in I. Therefore $W = f^{-1}(U') \cup g^{-1}(V')$ is an open neighborhood of p homeomorphic to an open subset of $\mathbb{R}^2$.

With all of this, $\hat{S}$ is a compact connected topological surface and by Theorem 2.7 we have that $\hat{S}$ is homeomorphic to either $\mathbb{S}^2$, $g\mathbb{T}^2$ or $n\mathbb{P}^2_{\mathbb{R}}$ via H. So S is homeomorphic to either $\mathbb{S}^2 \setminus (H(D_1) \cup H(D_2) \cup \ldots \cup H(D_b))$, $g\mathbb{T}^2 \setminus (H(D_1) \cup H(D_2) \cup \ldots \cup H(D_b))$ or $n\mathbb{P}^2_{\mathbb{R}} \setminus (H(D_1) \cup H(D_2) \cup \ldots \cup H(D_b))$. □

With the previous theorem in mind it is easy to see that for every surface S with boundary, $\chi(\hat{S}) = \chi(S) + b$ where b is the number of boundary components of S.

Definition 2.10. Given a surface with boundary S, we say that S is *orientable* if $\hat{S}$ is orientable.

The *genus* of S (denoted $g(S)$) is the genus of $\hat{S}$.

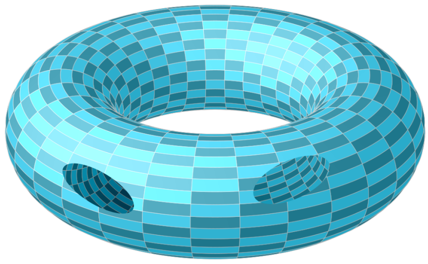


Figure 2.3: A torus with two boundary components

In the end, for every orientable surface with boundary S we have the following formula (which will be of use later on)

$$\chi(S) = 2 - 2g(S) - b \tag{2.1}$$

2.2 Seifert surfaces

Now that we are familiarized with orientable surfaces with boundary we are ready to define the concept of Seifert surface.

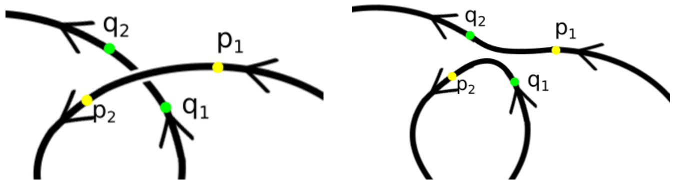
Definition 2.11. Given a knot K, a *Seifert surface* for K is an orientable compact connected surface $S \subset \mathbb{R}^3$ such that $\partial S = K$. We will also say that S is a *Seifert surface spanning* K.

It is not clear from the definition that such surfaces must exist, even less clear is how to construct them (if they even exist of course). The next theorem proves the existence of Seifert surfaces and gives a way to construct them.

Theorem 2.12. *Given a knot K, there exists a Seifert surface S spanning K.*

Proof. This proof relies on the so-called *Seifert algorithm.* Begin by fixing an orientation in K, say K^+, and consider a diagram of K^+. At each crossing point choose four points, the first two p_1, p_2 from the over strand in such a way that p_1 comes before the crossing point and p_2 after (as indicated by the orientation previously fixed) and q_1, q_2 from the under strand in such a way that q_1 comes before the crossing point and q_2 after (see Figure 2.4a).

Remove the line going from p_1 to p_2 (the one that intersects the crossing point) and the one going from q_1 and q_2 (which also intersects the crossing point) and connect p_1 with q_2 with a continuous line and p_2 with q_1 with another continuous



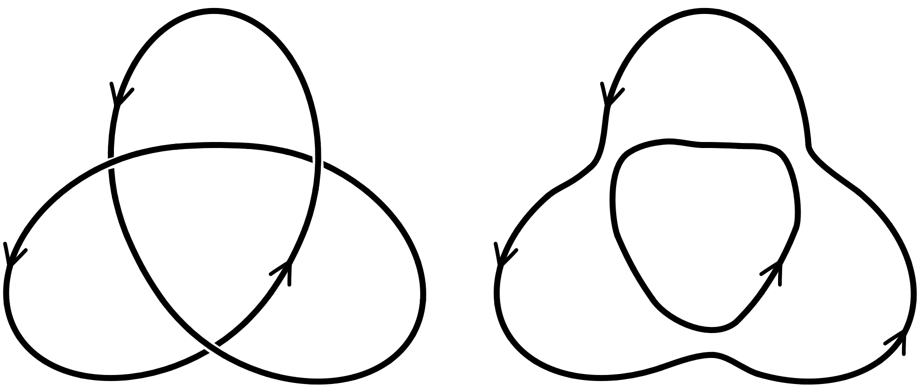
(a) (b)

Figure 2.4: Choosing points in a crossing to create Seifert circles

line that does not intersect the previus one (see Figure 2.4b). The result is a diagram of disjoint loops with an induced orientation, these loops are called *Seifert circles*.

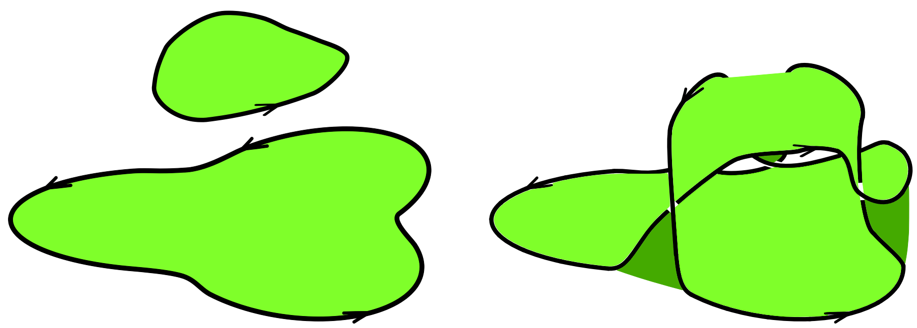
Now consider each Seifert circle c contained in a plane $z = z_c$ in $\mathbb{R}^3$, where z_c is different for every Seifert circle (thus every Seifert circle is at a different height). Continue by considering, for each Seifert circle c, a disc (or subset homeomorphic to a disc) in $z = z_c$ whose boundary is c (the process is similar to that of the proof of Theorem 2.9). Finally connect each disc with a twisted band begining in the line previously attached to p_1 and q_2 and ending in the line previously attached to p_2 and q_1. Connect it in such a way that the orientations on the boundries of the discs match up, that is p_1 must be connected with a line (an "edge" of the band) to p_2 that does not contain q_1 and q_2 (and the same must happen with q_1 and q_2).

The result is a compact connected orientable surface (because it is a union of discs joined by strips) with one boundary component which coincides with K. The fact that the surface is orientable comes from considering the bands in such a way that orientations match up (see figures 2.5 and 2.6 for an example of the Seifert algorithm applied to the knot 3_1). □



(a) 3_1 with a fixed orientation (b) Seifert circles

Figure 2.5: Creating Seifert circles from an oriented diagram of a knot



(a) Discs at different heights (b) Seifert surface for 3_1

Figure 2.6: Joining Seifert circles with bands

A question one might ask is if Seifert surfaces are unique to every knot, the answer is no, one can construct many different Seifert surfaces for a same knot.

What will be of interest is the genus of a Seifer surface.

2.3 Genus

Now we are ready to fulfill the objective of this chapter, to define the genus of a knot.

Definition 2.13. Let K be a knot. Then the *genus of* K, denoted $g(K)$, is the minimal genus of any Seifert surface spanning K. We will say that a Seifert surface S spanning K is *minimal* if $g(K) = g(S)$.

The next result shows us that the genus of a knot is a knot invariant, in other words, if two knots are equivalent then their genera is equal.

Proposition 2.14. *Let K_1 and K_2 be two knots. If $K_1 \sim K_2$ then $g(K_1) = g(K_2)$*

Proof. Consider a minimal Seifert surface S_1 spanning K_1. Since $K_1 \sim K_2$ then there must exist an homeomorphism $h\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $h(K_1) = K_2$. Then $h(S_1)$ is a Seifert surface spanning K_2 and $g(h(S_1)) = g(S_1) = g(K_1)$. Even more, $h(S_1)$ is a minimal Seifert surface spanning K_2 and then $g(K_1) = g(K_2)$. If $h(S_1)$ were not a minimal Seifert surface for K_2, then there would be a minimal Seifert surface S_2 spanning K_2 such that $g(S_2) < g(h(S_1))$, and then $h^{-1}(S_2)$ would be a Seifert surface spanning K_1 whose genus is less than $g(S_1)$, but that is impossible since S_1 is a minimal Seifert surface spanning K_1. $\square$

We now give a characterization of the unknot in terms of the genus.

Theorem 2.15. *Let K be a knot. Then K is the unknot if and only if $g(K) = 0$.*

Proof. If K is the unknot, then a disc with boundary K is clearly a minimal Seifert surface spanning K. And since a disc is a sphere with one boundary component it follows that $g(K) = g(\mathbb{S}^2) = 0$.

Conversely, if K is a knot with $g(K) = 0$, then there is a minimal Seifert surface S of K with $g(S) = 0$, and so S is a sphere with one boundary component, i.e, a disc. We can view K in S as a poligonal knot and the vertices of K as vertices of a triangulation $\mathcal{T}$ of S. Now for each vertex in the boundary of S if the vertex is attatched to only two edges, we remove those two edges and end up with the remaining edge of the triangle (this can be viewed as pushing the vertex inside the triangle along with its edges, which is an ambient isotopy). This process is done until we end up with just on triangle (the fact that we can do that is because $\mathcal{T}$ is in fact a triangulation of a disc). Since the boundary of a triangle and $\mathcal{O}$ are ambient isotopic we have that $K \sim \mathcal{O}$ (see an example in Figure 2.7). $\square$

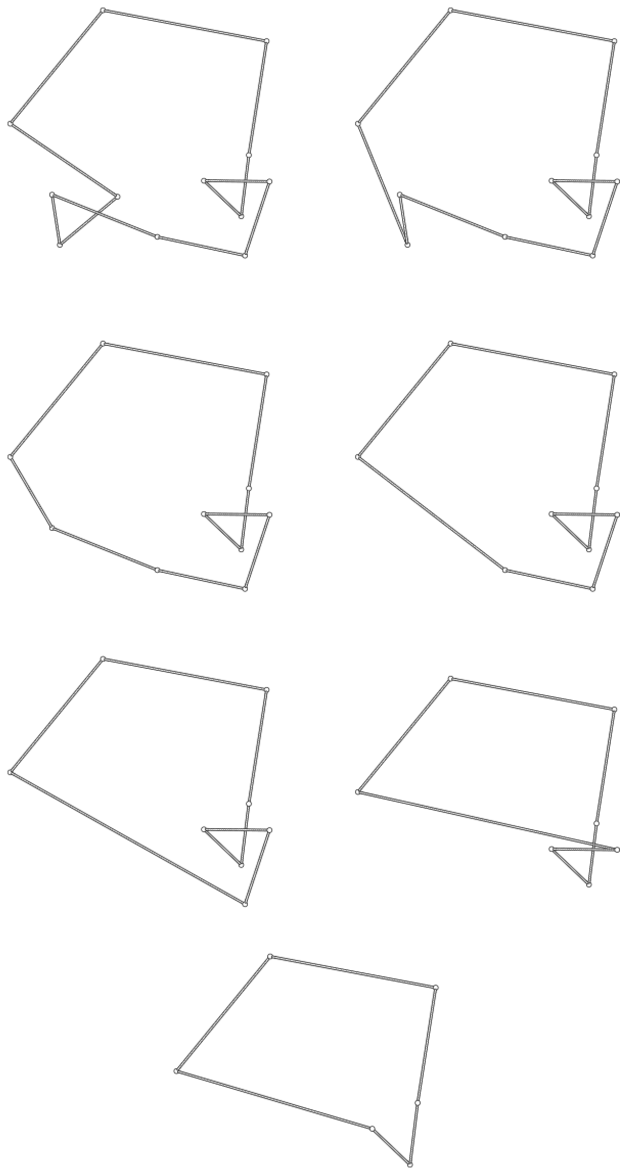


Figure 2.7: A sequence of pictures (from left to right and up to down) illustrating the proof in Theorem 2.15

Chapter 3

Minimal genus theorem for alternating knots

In Chapter 2 we have seen that the genus of a knot is a knot invariant. But if we try to calculate the genus of a knot K from a Seifert surface obtained using Seifert's algorithm, we end up with an upper bound for $g(K)$ and not necessarily $g(K)$. Then how do we actually calculate the genus of a knot?

In this chapter we will prove that applying Seifert's algorithm to an alternating projection of a knot K does in fact yield a minimal Seifert surface for K.

3.1 The theorem

The proof we are presenting is an adaptation of a proof presented by David Gabai (a mathematician at *Princeton University*) in an article of the *Duke Mathematical Journal*. In his article, David Gabai proves the result for *alternating links* (links essentially are finite collection of knots tangled between each other). Since we are only interested in knots, we have adapted the proof to our case. The complete proof can be found in [5].

To prove the theorem we first need the following result:

Lemma 3.1. *Let K be a knot. If S is a Seifert surface for K, which is not minimal, then there exists a Seifert surface T for K such that $Int(S) \cap Int(T) = \emptyset$ and $g(T) < g(S)$.*

It is important to remark that in his article David Gabai writes $\chi(S') > \chi(S)$ instead of $g(S') < g(S)$. The two facts are equivalent if one considers formula (2.1) from Section 1 Chapter 2.

We are not going to prove this lemma here since it relies on the notions of tubular neighborhoods, intersection numbers and how these concepts are related to homology groups, and this falls out of the scope of our project.

Theorem 3.2. *Let K be an alternating knot. If S is a Seifert surface for K obtained by applying Seifert's algorithm to an alternating projection of K then S is a minimal Seifert surface for K.*

Proof. Let K be an alternating knot. Consider an alternating projection of K (given by a regular projection π) with n crossings and the Seifert surface S obtained through Seifert's algorithm. We are going to prove the theorem by induction on n.

Base case: $n = 1$

On one hand, since the projection has only one crossing, by applying an ambient isotopy to K we can remove the crossing (by twisting the knot) then $K \sim \mathcal{O}$, and so $g(K) = 0$.

On the other hand, S is two discs joined by a twisted band. By performing an homeomorphism (the one induced by the ambient isotopy above is enough) we have that S is homeomorphic to a disc, and thus $g(S) = 0$.

So in the case $n = 1$ Seifert's algorithm applied to an alternating knot yields a minimal Seifert surface.

Induction step

Begin by considering an alternating projection of K with $n+1$ crossings, and let S be the Seifert surface obtained by applying Seifert's algorithm to that projection. Suppose S is not minimal.

By an ambient isotopy we can deform K so that K lies in the sphere $\mathbb{S}^2$ except for a small neighborhood of each crossing. Now via an ambient isotopy deform K slightly in a way that K intersects $\mathbb{S}^2$ in exactly $2n+2$ points (we are separating the crossing from the knot with $\mathbb{S}^2$). For an example see Figure 3.1

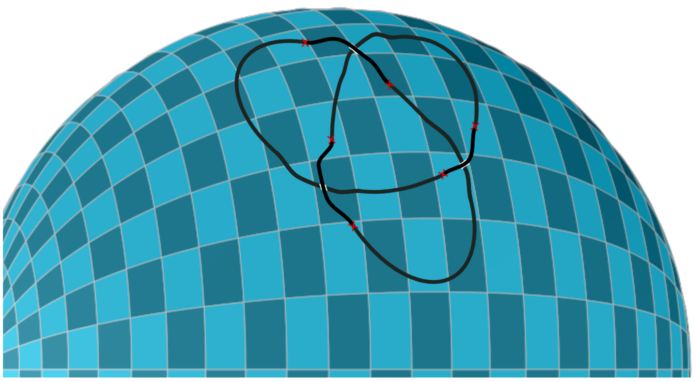


Figure 3.1: A knot that intersects $\mathbb{S}^2$ at $2n+2$ points shown in red

Let T be the Seifert surface of K obtained by applying Lemma 3.1 to S, and let D be an open disc in $\mathbb{S}^2 \setminus S$. Perform an ambient isotopy so that $T \cap \overline{D}$ is exactly k arcs (here $\overline{D}$ denotes the closure of D in $\mathbb{R}^3$), where k is such that $K \cap \overline{D}$ is exactly $2k$ points. Any innermost arc β in $T \cap \overline{D}$ (that is, any arc in $T \cap \overline{D}$ closest to the center of D) is parallel to an arc α in S, here by parallel we mean that their intersections are only the endpoints (for an example see Figure 3.2).

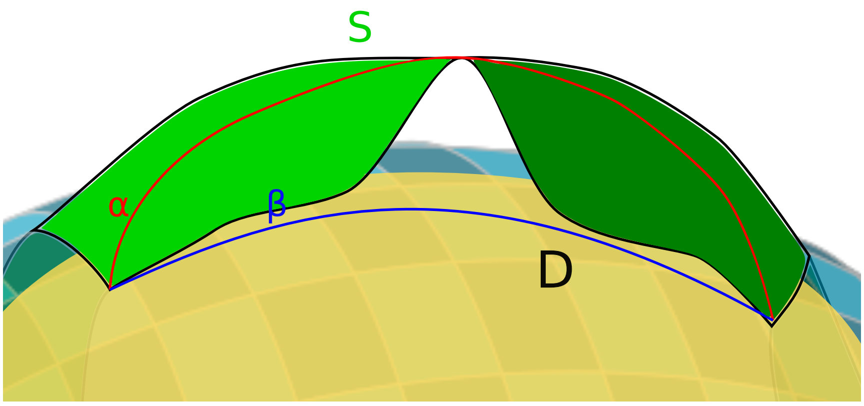


Figure 3.2: A Seifert surface S, a disc D and the arcs α and β

Now consider $S' = S \setminus Int(\alpha)$ (that is, cut S along α) and $T' = T \setminus Int(\beta)$. We can now remove a crossing from S' as we have done in the previous case. By doing this, we end up with a new surface S^* which is the Seifert surface obtained by applying Seifert's algorithm to a projection of n crossings of a knot K', and by the induction hypothesis S^* is minimal. T' is also a Seifert surface spanning K', but removing α and β from S and T, respectively, results in either adding two vertices and one edge to each triangulation or adding four vertices, two edges and two faces to each triangulation. In either case $\chi(T) - \chi(S) = \chi(T') - \chi(S^*)$ and since $g(T) < g(S)$ it follows that $g(T') < g(S^*)$ which contradicts the fact that S^* is minimal. Therefore we conclude that S must be minimal. $\square$

With the proof of the theorem ended we give some examples of calculations of genera of knots.

3.2 Examples

We start by calculating the genus of the trefoil knot. Let S be the Seifert surface obtained by applying Seifert's algorithm to 3_1. It will be of some use if we choose to view the seifert circles in S as a collection of polygons with some edges identified (see Figure 3.3 for an example). Let us denote by s the number of Seifert circles.

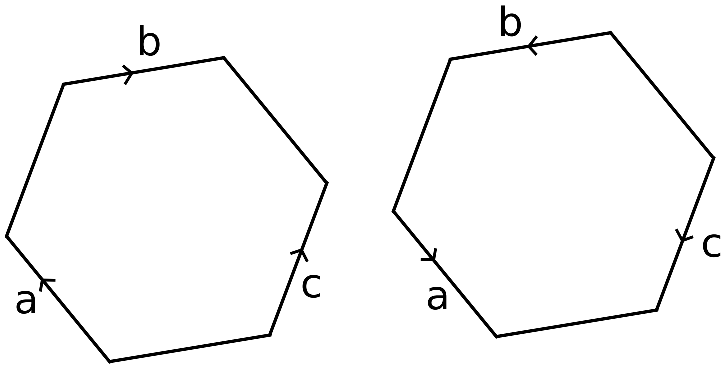


Figure 3.3: A Seifert surface for 3_1 made of polygons

Now consider the triangulation of S whose vertices are the vertices of each polygon plus one point in the interior of each polygon and whose edges are the edges of each polygon plus one line for each polygon connecting a vertex of the polygon with the vertex in the interior of the polygon (we are essentialy chopping each polygon with smaller triangles that have two vertices at the boundary and one in the interior as shown in Figure 3.4).

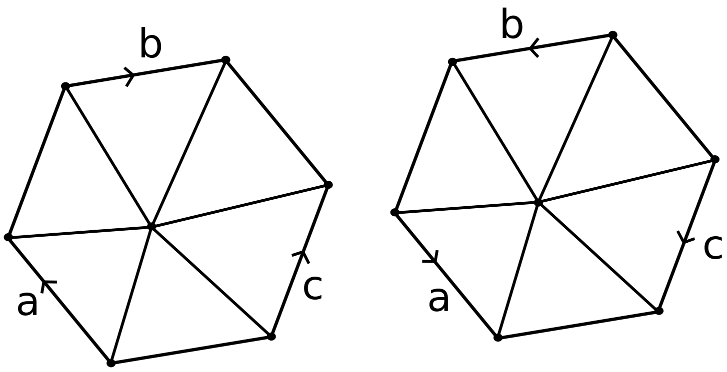


Figure 3.4: A triangulation of the Seifert surface of 3_1

This triangulation can be achieved with $2c(K) + s$ vertices, $7c(K)$ edges and $4c(K)$ faces, and so

$$\chi(S) = 2c(K) + s - 7c(K) + 4c(K) = s - c(K) = 2 - 3 = -1.$$

And using equation (2.1) we have that

$$g(K) = g(S) = \frac{1 - (-1)}{2} = 1.$$

In fact this procedure can be applied to every alternating knot K to find its genus, giving the following formula:

$$g(K) = g(S) = \frac{1 - \chi(S)}{2} = \frac{1 + c(K) - s}{2}, \tag{3.1}$$

where s is the number of Seifert circles. Moreover for any knot K we have the following inequality:

$$g(K) \leq \frac{1 + c(K) - s}{2}$$

Lets calculate now the genus of the figure eight knot. Using the notation previously described, as it can be seen in Figure 3.5 we have that $c(4_1) = 4$ and $s = 3$ therefore

$$g(4_1) = \frac{1 + 4 - 3}{2} = 1$$

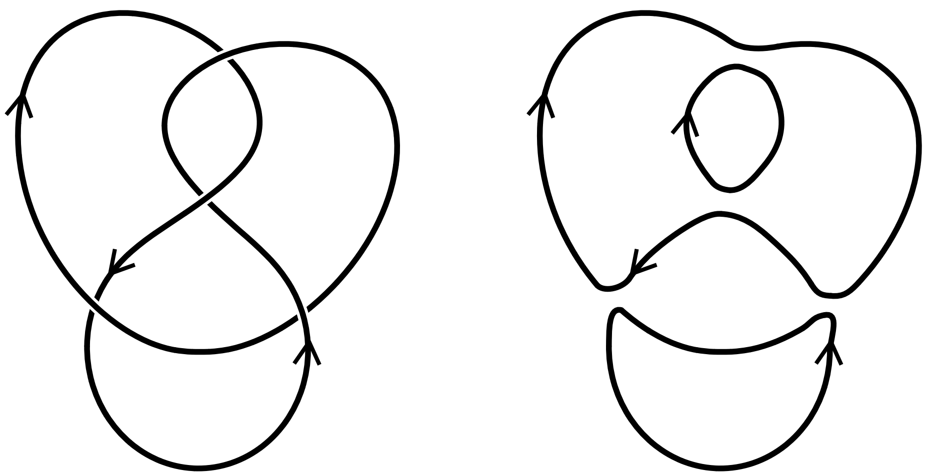


Figure 3.5: Seifert circles for 4_1

Chapter 4

Genus additivity and aplications

In this chapter we are going to see that the genus of a knot is additive with respect to the connected sum of knots. This property will allow us to answer the questions asked in Chapter 1 and thus conclude with one of the objectives of the project.

4.1 Additivity

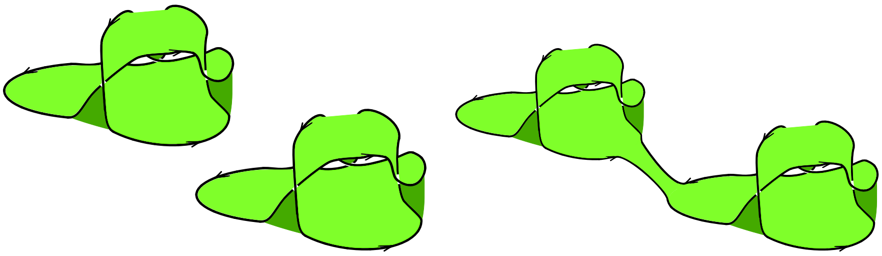
Theorem 4.1. *For any two knots K_1 and K_2, $g(K_1 \# K_2) = g(K_1) + g(K_2)$*

Proof. We will prove the theorem in two parts, the first one will prove $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ and the second one $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

Part 1: $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$

Consider K_1 and K_2 in such a way that there is a plane that separetes them and consider S_1 and S_2 two minimal Seifert surfaces spanning K_1 and K_2 respectively (S_1 and S_2 also separeted by the plane). Give orientations to both K_1 and K_2 and consider $K_1 \# K_2$ making sure that $K_1 \# K_2$ does not intersect $Int(S_1)$ nor $Int(S_2)$.

Now consider a band B connecting K_1 and K_2 whose boundary is the two lines introduced when creating $K_1 \# K_2$ and such that B does not intersect $Int(S_1)$ nor $Int(S_2)$ (see Figure 4.1 for an example). Then B connects S_1 and S_2, and therefore $S = S_1 \cup S_2 \cup B$ is an orientable compact connected surface whose boundary is $K_1 \# K_2$, that is S is a Seifert surface spanning $K_1 \# K_2$. Finally since $g(S) = g(S_1) + g(S_2) = g(K_1) + g(K_2)$, we have that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$.



(a) Disjoint seifert surfaces of 3_1 (b) Seifert surface for $3_1\#3_1$

Figure 4.1: Connecting two Seifert surfaces with a band

Part 2: $g(K_1) + g(K_2) \leq g(K_1\#K_2)$

Consider a minimal Seifert surface S spanning $K_1\#K_2$, and a sphere Σ that intersects $K_1\#K_2$ at two points (the points identified at the definition of $K_1\#K_2$) transverally, that is, the tangent space of $K_1\#K_2$ and the tangent space of Σ at the point of intersection span $\mathbb{R}^3$ (here we are assuming $K_1\#K_2$ to be smooth, which is no restriction). Then Σ separates $K_1\#K_2$ in two arcs α_1, α_2, one in the bounded component of $\mathbb{R}^3 \setminus \Sigma$, and the other in the unbounded comonent (we know that those components exist because of the Jordan-Brower theorem). Then if β is an arc in Σ connecting the two previous points, it follows that $\alpha_1 \cup \beta$ is a copy of K_1 and $\alpha_2 \cup \beta$ is a copy of K_2 (see Figure 4.2 for an example).

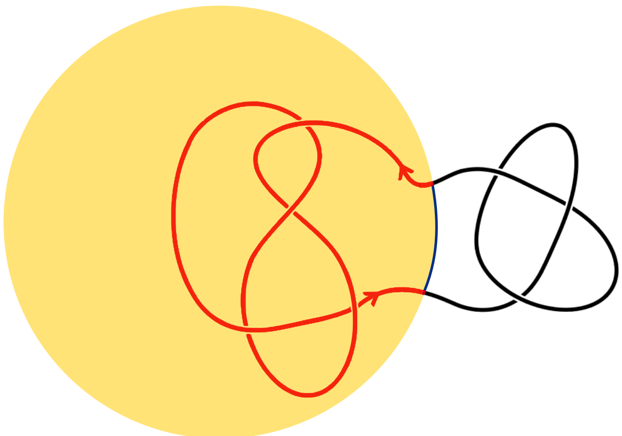


Figure 4.2: Σ separating K_1 and K_2

Without loss of generality, we can assume that S and Σ intersect transversaly (they intertect transversaly at each point in $S \cap \Sigma$) if needed we deform Σ in a neighborhood of $\Sigma \cap S$ so that the assumption holds. It follows that $S \cap \Sigma$ must be a one dimensional manifold i.e, a finite collection of loops and β (for an example see Figure 4.4).

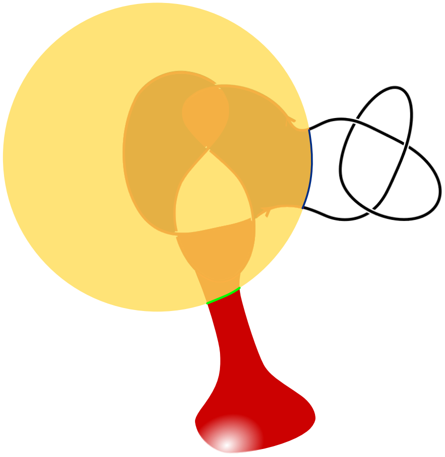


Figure 4.3: $\Sigma \cap S$ a collection of loops (in green) and β (in blue)

The idea now is to do a sequence of deformations of S for each loop in a way that the genus does not change but at each step $S \cap \Sigma$ has fewer loops to finally conclude that $S \cap \Sigma = \beta$.

Let C be a loop in $S \cap \Sigma$ and D a disc in Σ that bounds C such that $D \cap S = \emptyset$.

Now, let $\hat{S}$ be the surface resulting of the following process. Remove from S a small annular neighborhood U of C (a neighborhood homeomorphic to an annulus) and consider two discs D_1 and D_2, D_1 in the bounded component of $\mathbb{R}^3 \setminus \Sigma$ and D_2 in the unbounded one (we know that such components exist because of the Jordan-Brower theorem). Attach D_1 to the boundary component of U included in the bounded component of $\mathbb{R}^3 \setminus \Sigma$ (the "inside" of Σ) and D_2 to the boundary component of U included in the unbounded component of $\mathbb{R}^3 \setminus \Sigma$ (the "outside" of Σ).

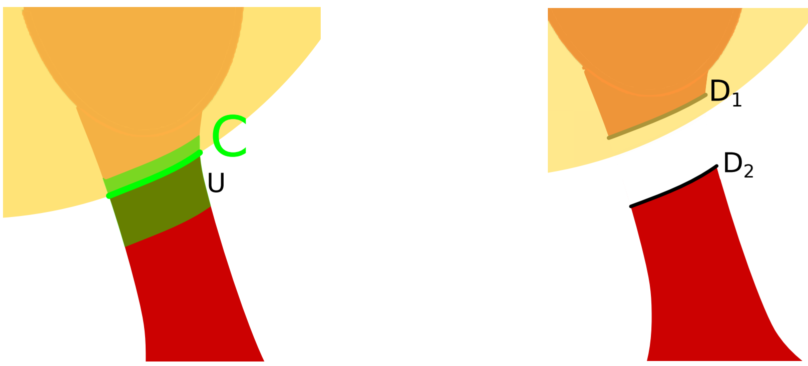


Figure 4.4: Removing the annular neighborhood and attaching discs

If $S \setminus C$ has only one connected component then $\hat{S}$ is also a Serifert surface spanning $K_1\#K_2$ whose genus is less than the genus of S, but that is impossible since S is minimal. Then $S \setminus C$ has more than one connected component. Consider the connected component fo $\hat{S}$ that contains $K_1\#K_2$, this is a Seifert surface spanning $K_1\#K_2$ whose genus is the same as S and that intersects Σ in fewer loops (at least we have eliminated C).

By repeating this process a finite number of times (because the number of loops is finite), we end up with a Seifert surface S^* for $K_1\#K_2$ that intersects at Σ only at β. Then if we denote B_1 the bounded component of $\mathbb{R}^3 \setminus \Sigma$ and B_2 the unbounded one, we have that $S_1^* = (B_1 \cap S^*) \cup \beta$ is a Seifert surface for K_1 and $S_2^* = (B_2 \cap S^*) \cup \beta$ is a Seifert surface for K_2 and so we have

$$g(K_1) + g(K_2) \leq g(S_1^*) + g(S_2^*) = g(S^*) = g(K_1\#K_2)$$

and thus we complete the proof. □

4.2 Aplications

Theorem 4.1 has a good amount of impications which we now state and prove.

Corollary 4.2. *Given any two knots K_1 and K_2, then if $K_1\#K_2 \sim \mathcal{O}$ then $K_1 \sim K_2 \sim \mathcal{O}$.*

Proof. By Theorem 4.1 we have that $0 = g(K_1\#K_2) = g(K_1) + g(K_2)$ and since $g(K) \geq 0$ for any knot K, it must be that $g(K_1) = g(K_2) = 0$ and by Theorem 2.15 we have that $K_1 \sim K_2 \sim \mathcal{O}$. □

This tells us that the unknot is not a connected sum of two non trivial knots (intuitively a rope with two knots tied in it where the ends are joined, will only untangle if the rope is cutted).

Corollary 4.3. *For every knot K if $g(K) = 1$ then K is a prime knot*

Proof. If $K = K_1\#K_2$ then by Theorem 4.1 either $g(K_1) = 0$ or $g(K_2) = 0$ and then either $K_1 \sim \mathcal{O}$ or $K_2 \sim \mathcal{O}$. □

Corollary 4.4. *Every knot K is a sum of at least $g(K)$ prime knots, and thus a finite sum of prime knots.*

Proof. If K is prime then the result is obvious. Suppose then that K is not prime. By definition there exist K_1, K_2 both different from the unknot such that $K = K_1\#K_2$ and from Theorem 4.1 it follows that $g(K_1), g(K_2) < g(K)$. Now we do the same process for K_1 and K_2 in order to end up with new knots (assuming K_1 and K_2

are no both prime knots) whose connected sum is K. By repeating this process we conclude that K is a sum of prime knots. The fact that the number of prime knots is at most $g(K)$ is immediate from the additivity of the genus and the fact that all prime knots have genus greater or equal than 1. □

Corollary 4.5. *There are non trivial knots with arbitrarily large crossing number.*

Proof. For any knot K, by equation (3.1), since the number of seifert circles will always be at least 1 we have that

$$g(K) \leq \frac{c(K)}{2}.$$

Let K be a non trivial knot, and K_n the sum of n copies of K. Then

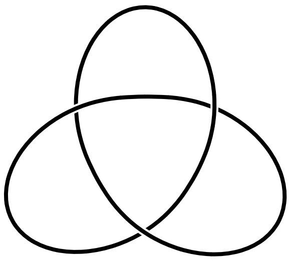
$$n \leq g(K_n) \leq \frac{c(K_n)}{2}$$

and as n approaches infinity so does $c(K_n)$. □

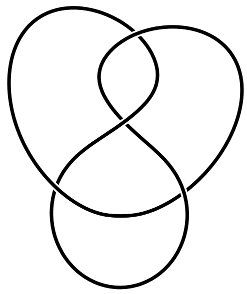
Appendices

Appendix A

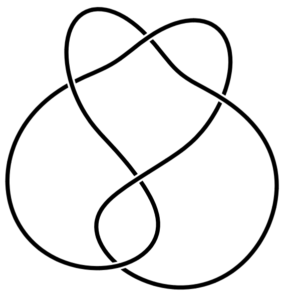
This appendix contains a table with projections of prime knots with up to 6 crossings, and does not contain its mirror images (that is, the same projection changing every crossing so that the strand that was over the crossing will now be under, and the one that was under will now be over). We have also included the genus of every knot.



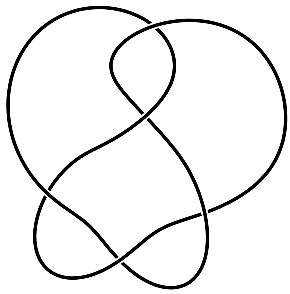
3_1 $g(3_1) = 1$



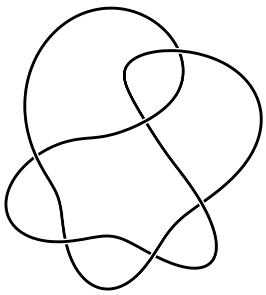
4_1 $g(4_1) = 1$



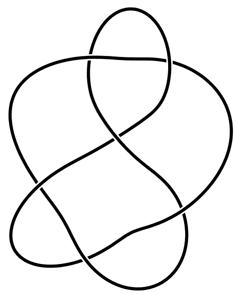
5_1 $g(5_1) = 2$



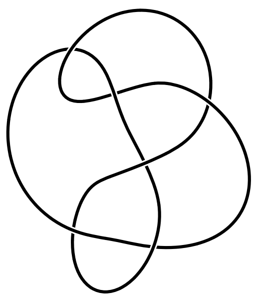
5_2 $g(5_2) = 1$



6_1 $g(6_1) = 1$



6_2 $g(6_2) = 2$



6_3 $g(6_3) = 2$

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