Torricelli's Law and Conservation of Momentum

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Abstract: This TFG discusses Conservation of Momentum in the context of Torricelli's law. We consider a large tank filled with water with a tiny hole near its base, and compute the horizontal force exerted on the fluid by the container walls in two ways, finding that the results differ by a factor of 2. We obtain that $F_x = 2\rho g h S_2$, where S_2 is the cross-sectional area of the hole, after applying conservation of momentum, and $F_x = \rho g h S_2$ using Cauchy's hypothesis. We analyze the assumptions made in these calculations, and perform experiments to confirm that some of them are not realized, identifying the sources of the discrepancy. Specifically, the assumption that the pressure is hydrostatic everywhere in the tank except at the hole and that the velocity is strictly horizontal are the key reasons for our discrepancy. Finally, we discuss Borda's Mouthpiece and redefine the selected control volume as a way to reconcile both approaches to obtain the force.

I. INTRODUCTION

The conservation of momentum is a well-known principle in mechanics and it is a fundamental tool at finding solutions in complex problems. When it comes to hydrodynamics, the conservation of momentum is also really useful at providing results in stationary fluids. Choosing the right control volume will lead to obtain results that would be more difficult to obtain via alternate means. In this document I apply the conservation of momentum to a classical problem, the discharge of an ideal fluid from a large tank through a tiny hole in its base.

The system of study, shown in Fig. 1, is a tank with a large section S_1 filled with an ideal fluid up to a height y_0. The tank has a tiny hole with section $S_2 \ll S_1$, at a distance h from the free surface. The exit velocity v at S_2 is given by the well-known Torricelli's Law, which is a classical result more than three hundred years old. Explicitly Torricelli's law states that [1-6]

$$v = \sqrt{2gh}, \qquad (1)$$

where g is the gravitational acceleration. This result expresses energy conservation: potential energy density of material particle at free surface ($U = \rho g h$) transforms into kinetic energy density at the orifice ($K = \frac{1}{2}\rho v^2$). This transformation, from potential to kinetic energy, is only possible in absence of friction effects which is justified since we are considering ideal fluid flow.

To obtain Torricelli's Law more formally, it is common to start from Bernoulli's equation [2]

$$\frac{1}{2}v^2 + \Phi + \frac{P}{\rho} = \text{const}, \qquad (2)$$

where v is the velocity of the fluid in a streamline, Φ is the gravitational potential energy, P is the pressure of the fluid along the streamline, and ρ its density. This expression applied along a streamline between point 1 on the free surface and point 2 at the exit hole where the fluid exits the tank, Bernoulli's equation gives

$$\frac{1}{2}v_1^2 + \Phi_1 + \frac{P_1}{\rho_1} = \frac{1}{2}v_2^2 + \Phi_2 + \frac{P_2}{\rho_2}. \qquad (3)$$

Now, considering that $P_1 = P_2 = P_0$, the atmospheric pressure, that $\vec{v}_1 \approx 0$, because $S_1 \gg S_2$, and that $\Phi_1 - \Phi_2 = hg$, we obtain:

$$hg = \frac{1}{2}v_2^2 \rightarrow v_2 = \sqrt{2gh}$$

which is the Torricelli's Law.

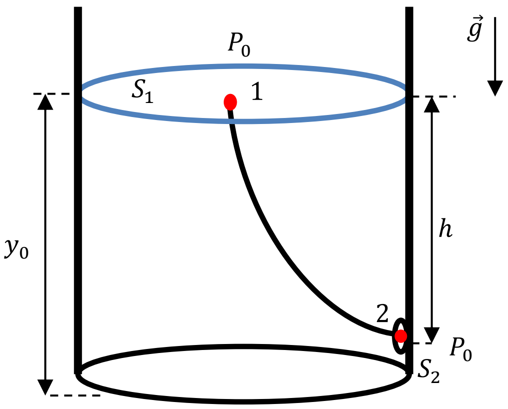

FIG. 1: The system of study: a tank with cross section S_1 and a tiny hole with section S_2 in its lateral surface, where $S_1 \gg S_2$. The tank is filled with an ideal fluid and it is drawn the streamline that it is taken to apply the Bernoulli equation.

II. CONSERVATION OF MOMENTUM

Conservation of momentum applied to a control volume V that is fixed in space can be written as [3]

$$\frac{d}{dt}\int_V \vec{\rho}_p dV = -\oint_S \underline{\underline{u}} \cdot d\vec{S} + \int_V \rho \vec{F}_V dV, \qquad (4)$$

where S is the surface surrounding V, $\vec{\rho}_p = \rho\vec{v}$ is the density of momentum, $\vec{F}_V$ is a volume force, and $\underline{\underline{u}}$ is the momentum flow tensor, defined as [3]:

$$\underline{\underline{u}} = \rho\vec{v}\vec{v} - \underline{\underline{\sigma}}$$

where

$$\underline{\underline{\sigma}} = -P\underline{\underline{I}} + \underline{\underline{\sigma}}'$$

is the stress tensor, with $\underline{\underline{\sigma}}'$ the viscosity stress tensor. For an ideal fluid $\underline{\underline{\sigma}}' = 0$. The $\rho\vec{v}\vec{v}$ term in $\underline{\underline{u}}$, is a velocity tensor that represents the transport of momentum $\rho\vec{v}$ by particles with velocity $\vec{v}$.

Equation (4) tells us that the time variation of momentum in a control volume V occurs because of the momentum loss across the surface S, or due to the momentum sources, which are related to the volume forces.

It is remarkable that in steady state, we can apply this theorem after knowing the velocity field in *S*. This is especially useful in cases where the only information that it is known is in the boundaries of the control volume.

In steady state, the derivative of the momentum drops and there we are only left with the surface and volume integrals. If we consider this case and assume that the only volume force is gravitational the conservation of momentum applied to an ideal fluid becomes:

$$\oint_S \underline{\underline{\sigma}} \cdot d\vec{S} = \oint_S \rho\vec{v}\vec{v} \cdot d\vec{S} - \int_V \rho\vec{g}dV$$

In this equation we identify the left side integral as the contact force exerted on the fluid, so that

$$\vec{F} = \oint_S \rho\vec{v}\vec{v} \cdot d\vec{S} - \int_V \rho\vec{g}dV. \qquad (5)$$

We can therefore use momentum conservation to obtain the contact force exerted on the fluid, without needing to know the velocity field in *V*; we only need to know it in the surface. This is different from the more common way to calculate forces through Cauchy's Hypothesis after finding $\underline{\underline{\sigma}}$ via the Navier-Stokes Equations.

III. THE HORIZONTAL COMPONENT OF THE FORCE EXERTED ON THE FLUID

To prove the usefulness of the Conservation of Momentum Theorem we will use it to compute the horizontal force (projection of $\vec{F}$ onto the horizontal axis) exerted on the fluid by the container walls. We will also do this using the Cauchy's Hypothesis.

A. Using Conservation of Momentum Theorem

To evaluate the force exerted by the container on the fluid, we choose the control volume shown in Fig. 2. This control volume is bounded by the following surfaces: S_L (lateral surface of the tank wet by the fluid), S_1 (free surface of the fluid), and S_B (base of tank) and S_2 (cross sectional area of the hole). Therefore, the total surface that surrounds *V* is

$$S = S_1 \cup S_B \cup S_L \cup S_2$$

The horizontal component of $\vec{F}$, F_x, is given by Eq. (5):

$$F_x = \oint_S \rho v_x \vec{v} \cdot d\vec{S}.$$

Note that the volume force has dropped out since it has no *x*-component (gravity is along the vertical axis, $\vec{F}_V = -g\hat{j}$). We are left with the velocity integral, which can be easily computed. Since $S_1 \gg S_2$, the velocity of the free surface as well as that on S_L and S_B can be neglected. Hence, the velocity is zero at almost all S, except at S_2 where it is given by Torricelli's Law. As a result

$$F_x = \oint_S \rho v_x \vec{v} \cdot d\vec{S} = \int_{S_2} \rho v_x^2 dS = \rho v_x^2 S_2$$
$$\to F_x = 2\rho g h S_2. \qquad (6)$$

Since S_2 is small enough, we have considered that the exit velocity is constant thorough this section.

B. Using Cauchy's Hypothesis

Let's now find this force using Cauchy's Hypothesis [5]:

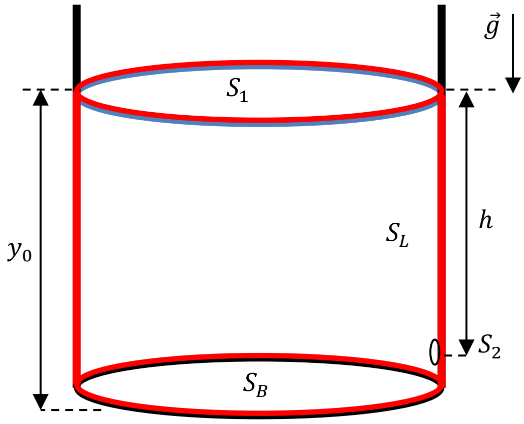

FIG. 2: The chosen control volume of our system in red.

$$\vec{F} = \oint_S \underline{\underline{\sigma}} \cdot d\vec{S}. \qquad (7)$$

The *x*-component is:

$$F_x = \oint_S \left(\underline{\underline{\sigma}} \cdot \hat{n}\right)_x dS.$$

The unitary vector $\hat{n}$ is the normal vector to the surface. As it was mentioned in the introduction, the fluid can be considered ideal so that $\underline{\underline{\sigma}}$ is just related to the pressure:

$$\underline{\underline{\sigma}} = -P\underline{\underline{I}}.$$

Thus:

$$F_x = -\oint_S \left(P\underline{\underline{I}} \cdot \hat{n}\right)_x dS. \qquad (8)$$

Due to the geometry of the problem, the only surface where $\hat{n}$ has a not null projection along the *x*-axis is the lateral surface S_L and S_2. Consequently, the pressure integral is just

$$\oint_S \left(P\underline{\underline{I}} \cdot \hat{n}\right)_x dS = \int_{S_L \cup S_2} P n_x dS. \qquad (9)$$

This integral can be easily computed if we consider that the fluid is at rest at S_L. This means that the pressure on the lateral surface is just the hydrostatic pressure (P_H). This is not the case on S_2, where the velocity is nonzero. The exit hole is open to the atmosphere so the pressure there is just the atmospheric pressure (P_0). Hence:

$$\int_{S_L \cup S_2} P n_x dS = \int_{S_L \cup S_2} P_H n_x dS - \int_{S_2} P_H dS + \int_{S_2} P_0 dS.$$

I first compute the pressure as if all the fluid in $S_L \cup S_2$ is at rest, so that the pressure would be P_H, and then I subtract the hydrostatic pressure and add the atmospheric pressure contribution on S_2. The first integral (P_H in S_L) is

$$\int_{S_L \cup S_2} P_H n_x dS = \int_{S_L \cup S_2} (P_0 + \rho g y) n_x dS = 0.$$

That is easy to see because of the symmetry of the system, the same pressure acts in both sides of the surface. Using this,

$$\int_{S_L \cup S_2} P n_x dS = -\int_{S_2} (P_0 + \rho g y) dS + \int_{S_2} P_0 dS =$$
$$= -\int_{S_2} \rho g y dS = -\rho g h \int_{S_2} dS = -\rho g h S_2.$$

So, by Eq. (8) and (9) the horizontal force is

$$F_x = \rho g h S_2. \qquad (10)$$

I emphasize that in these steps we have used the expression of hydrostatic pressure ($P_H = P_0 + \rho g y$), where we have been assumed that the hole is tiny enough for the vertical coordinate to be considered constant.

Comparing the forces obtained by the two different methods (Eqs. (6) and (10)), we see that there is a factor of two that differs between them. The question now is why.

IV. DISCUSSION

In this section I address assumptions made in the prior section and see if they are realized in experiments. Let us emphasize that despite Torricelli's law is a classical result, the analysis performed reflects it contains interesting subtleties. So is the case that, in fact, there is recent published work addressing the issue [7,8].

A. Pressure Integral

In evaluating the pressure integral when using Cauchy's Hypothesis, we took the velocity equal to zero in all S_L. This allowed assuming $P = P_H$ in that region. To address whether this is true, we set up an experiment in the lab. We used a plastic container filled with water with a tiny hole close in its base. We introduced little drops of ink at the wall in the proximities of the hole to identify streamlines and track the motion of the fluid. In Fig. 3, there is a snapshot of the result. It is notorious how the light transmission decreases near the hole, indicating there is a non-zero velocity in that area. Since there is velocity in the container surface near S_2, I cannot longer assume that the pressure is hydrostatic there; hence,

$$\int_{S_L \cup S_2} P n_x dS \neq \int_{S_L \cup S_2} P_H n_x dS - \int_{S_2} P_H dS + \int_{S_2} P_0 dS$$

B. Fluid Velocity at S_2

Another source of error is to consider that the velocity is horizontal at S_2. We assumed this in Eq. (6). In reality, the streamlines at S_2 are not horizontal; this can be seen by realizing the exit jet stretches, as illustrated in Fig. 4. As a result:

$$\int_{S_2} \rho v_x^2 dS \neq \rho v_x^2 \int_{S_2} dS = \rho v_x^2 S_2$$

We conclude that to capture reality, and obtain a correct x-component of the force, we would need to know the right pressure in the proximities of S_2 and the correct velocity distribution at the exit hole.

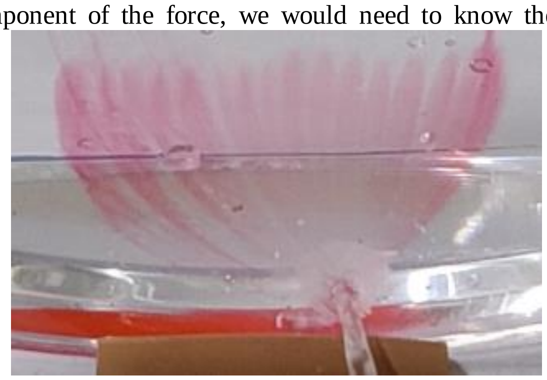

FIG. 3: A snapshots of the experiment where is easy to see streamlines in the surface surrounding S_2, therefore, we conclude that there is a certain velocity in that region.

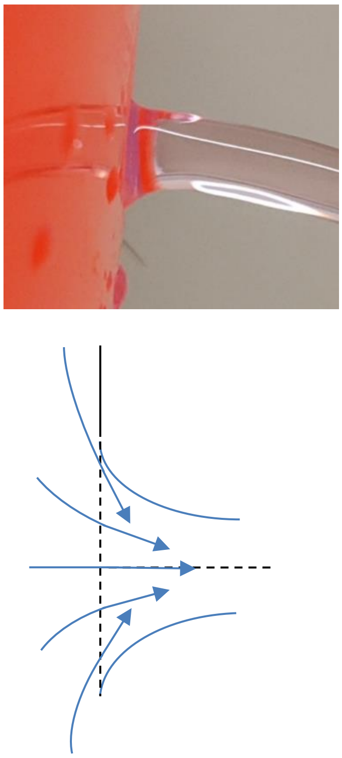

FIG. 4: TOP: a snapshot of the experiment where is possible to observe the contraction of the jet. BOTTOM: a sketch of the exit jet and the streamlines getting out of the container.

V. AN EXACT EXAMPLE: BORDA'S MOUTHPIECE

There is an experimental geometry for a mouthpiece and a better consideration for control volume that makes both assumptions correct. With respect to the pressure integral, we introduce *Borda's Mouthpiece*. This configuration for the opening in S_2 is designed to reenter into the volume, as shown in Fig. 5. By using this structure, we assure that the fluid is expelled only from the inner part of V and, for that reason, that there is no velocity in S_L. Second, the chosen control volume will be slightly different, and will follow the jet until its cross section is constant; see Fig. 5. Since there is a contraction of the fluid jet that exits the tank, the *Vena Contracta*, we extend our control volume up to when the jet straightens. This means that in the velocity integral, Eq. (6), the effective surface where there is a constant velocity v_x is S_f. In this situation,

$$F_x = \oint_{S_T} \rho v_x \vec{v} \cdot d\vec{S} = \int_{S_f} \rho v_x^2 dS = \rho v_x^2 S_f$$

$$\rightarrow F_x = 2\rho g h S_f.$$

For the pressure integral, we still have the same result:

$$F_x = -\oint_{S_T} (PI \cdot \hat{n})_x dS = -\int_{S_L} P n_x dS$$

$$\rightarrow F_x = \rho g h S_0.$$

Equating them, we get

$$2\rho g h S_f = \rho g h S_0$$

$$\rightarrow S_f = \frac{1}{2} S_0$$

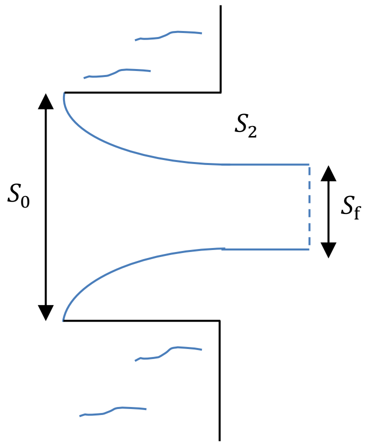

FIG. 5: Borda's Mouthpiece at S_2. Now the velocity in S_L is truly zero.

As we see, using Borda's Mouthpiece we reconcile our two analyses and also found another result: the contraction of the fluid tube is in this case half the original cross section.

One may observe that we could have done this redefinition of the control volume in our original system (without Borda's Mouthpiece). Had we done it, we would have obtained the same result already found. The same contraction factor of 1/2 would arise, but this would not be a correct result: we would still have velocity in the region surrounding S_2, therefore the pressure would not be hydrostatic, and the pressure integral would be wrong. In fact, it is shown experimentally that the contraction factor is approximately 0.6 [3,4], the jet contracts less than with Borda's Mouthpiece. This happens because there is a lower pressure in the neighborhood of S_2 (due to the non-zero velocity), so the force exerted on the fluid by the walls is higher.

VI. CONCLUSIONES

In this TFG it has been shown some intricacies related to Torricelli's experiment. We found them by calculating F_x via two different methods, and obtaining different results. As a consequence, I revised the procedure and discussed that certain assumptions could be wrong.

It has been shown by experiment, that the velocity in the vicinity of S_2 is not zero. This conclusion has motived discussing a geometry and analysis that allowed me to reconciliate both results.

Finally, I emphasize that even though this setup is hundreds of years old, nowadays the theorical study of a tank with a tiny hole near its base still raises interesting problems. It encouraged us to study its behavior and design different mouthpieces. In fact, in current days, the setup is still under investigation [7,8] and there are motivations to keep exploring it.

Acknowledgments

I want to thank Alberto Fernandez-Nieves and Ramon Planet for their help and guidance. For showing me that even the most common things always have hidden gems yet to discover.

[1] E. Torricelli, Opera Geometrica, Florence: Amatoris Masse & Laurent de Landis, 1644.

[2] D. J. Acheson, Elementary Fluid Dynamics, New York: Oxford University Press Inc., 1990.

[3] E. Guyon, J. P. Hulin, L. Petit, C. D. Mitescu, Physical hydrodynamics, New Yor: Oxford University Press Inc., 2001.

[4] Y. Nakayama, Introduction to Fluid Mechanics, Second Edition, Oxford: Butterworth-Heinemann, 2018.

[5] D. Lautrup, Physics of Continuos Matter, Boca Ratón: CRC Press, 2011.

[6] R. Feynman, The Feynman Lectures on Physics Vol 2, New York: Basic Books, 1964.

[7] A. Malcherek, «A new approach to hydraulics based on the momentum balance: sharp edged outflows and sluices» *Proceedings of the 37th IAHR World Congress*, pp. 1515-1521, 2017.

[8] V. Ferro, I. Aydin, «Testing the outflow theory of Malcherek by slit weir data» *Flow Measurement and Instrumentation* vol. 59, pp. 114-117, 2018.