Evolution of a non-minimally coupled scalar field in a LTB metric

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Abstract: We analyzed a theory of gravity with a non-minimally coupled scalar field, writing down its equation of motion and the Einstein equations in a spherically symmetric Lemaître-Tolman-Bondi (LTB) metric. This is motivated by the possibility that a non-minimally coupled scalar could alleviate the "Hubble tension" that exists between measurements of the Hubble parameter H_0, using Supernovae or CMB. In particular, the aim of this work is to provide the necessary equations, in order to investigate the relation between the cosmological value of the scalar field and its local value within a cosmic structure, described here by the LTB metric.

I. INTRODUCTION

In this work we study a scalar-tensor theory of gravity, where the metric is coupled non-minimally to a scalar field. Our motivation lays in the fact that in the standard cosmological model (ΛCDM) within Einstein's theory, there is a disagreement between different methods of measuring the Hubble constant [1], in particular when comparing late-time local measurements (using supernovae) with early time measurements (using CMB). This is the so-called "H_0 tension", and it could be alleviated by a non-minimal scalar field that evolves changing in time [2]. This represents a variation in time of the Planck mass, or equivalently of the Newton's constant. These theories, however, are subject to a set of constraints for the scalar field inside the solar system that constrain significantly the possibility of alleviating the Hubble tension. However, in [2] the cosmological value of the field has been used to include such a constraint, and it is not clear whether the local value is close to the cosmological value. We will thus start to investigate this issue: what is the relation between the inner and outer scalar field values in a cosmic structure (having in mind a cluster, a galaxy or even smaller structures). Moreover, it could be of general interest to understand how to match the local and the cosmological value of the scalar field in modified gravity models that include an extra scalar. To do so, we use an exact solution of the Einstein equations that describes a non-homogeneous blob inside a homogeneous universe. They will be represented by the spherically symmetric Lemaître-Tolman-Bondi (LTB) and the Friedmann-Lemaître-Robinson-Walker (FLRW) metrics respectively, which will match smoothly at the border. We introduce the Einstein's equations and Klein-Gordon's equation for such scalar field in these metrics.

As we will see the scalar field value will be directly related to Newton's gravitational constant, G, which implies that G is no longer a constant, but depends on space-time position. For simplicity, we will consider that the scalar field has only dependence on temporal and radial coordinates, $\phi = \phi(r,t)$. Throughout the report we will use natural units: $c = 1$, $\hbar = 1$, and we used the software *Mathematica* to compute some of our results.

II. THE MODEL

Our model is described by a scalar field non-minimally coupled to the Ricci scalar $\mathcal{R}$. We assume that the scalar field does not have a potential contribution. The action reads:

$$S = \int d^4x\sqrt{-g}[\tfrac{1}{2}(\mathcal{M}^2+\beta\phi^2)\mathcal{R}-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\mathcal{L}_{mat}], \tag{1}$$

where $\mathcal{L}_{mat}$ represents any matter contribution. One should notice that in this model M does not coincide with the Planck Mass that we observe today, since the effective Planck mass squared is given instead by $M_{Pl}^2 = \mathcal{M}^2 + \beta\phi^2 \equiv 1/(8\pi G)$ and thus the cosmological effective Newton's constant will depend on the value of ϕ. By varying the action with respect to the metric and the scalar field we have verified that the Einstein equations and the equation of motion for ϕ, or Klein-Gordon equation, that follow from this action are respectively [3]:

$$\begin{aligned} G_{\mu\nu} &= \tfrac{1}{\mathcal{M}^2+\beta\phi^2}[T_{\mu\nu}^{mat} + T_{\mu\nu}^{\phi} \\ &\quad + \beta(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi^2)], \end{aligned} \tag{2}$$

$$\beta\phi\mathcal{R} + \Box\phi = 0, \tag{3}$$

where $G_{\mu\nu}$ is Einstein's tensor, ∇_μ stands for the covariant derivative and $\Box \equiv \nabla^\mu\nabla_\mu$.This is a general result for any given metric. We will study their behaviour for FLRW and LTB metrics.

III. FLRW METRIC

As we said, we will consider the scalar field outside the blob to be homogeneous, $\phi = \phi(t)$. The FLRW (spatially flat) metric is $g_{\mu\nu} = diag(-1, a^2(t), a^2(t), a^2(t))$. If we introduce the Hubble ratio, $H \equiv \dot{a}(t)/a(t)$, the Ricci scalar reads:

$$\mathcal{R} \;=\; 12H^2 + 6\dot{H}. \tag{4}$$

If we develop further eqs. (2) and (3) we get:

$$H^2 \;=\; \frac{1}{3(\mathcal{M}^2+\beta\phi^2)}[\rho + \tfrac{1}{2}\dot{\phi} - 6\beta H\phi\dot{\phi}], \tag{5}$$

$$\begin{aligned}\dot{H} \;=&\; \frac{-1}{2(\mathcal{M}^2+\beta\phi^2)}[\rho - p + \dot{\phi} - 8\beta H\phi\dot{\phi} \\ &+ 2\beta^2\phi^2\mathcal{R} + 2\beta\dot{\phi}^2],\end{aligned} \tag{6}$$

$$\ddot{\phi} + 3H\dot{\phi} - \beta\phi\mathcal{R} \;=\; 0. \tag{7}$$

where ρ is the energy density and p the pressure of matter. We shall solve eq. (7) for late times, when the universe is dominated by non-relativistic matter (dust), with $p = 0$. We ignore for simplicity the contribution of a cosmological constant. This gives $a(t) \propto t^{2/3}$ and thus:

$$\ddot{\phi} + \frac{2}{t}\dot{\phi} - \frac{4}{3t^2}\phi \;=\; 0. \tag{8}$$

This equation has power-law solutions $\phi(t) = Ct^{\alpha}$, where C is an arbitrary constant related to the initial conditions and

$$\alpha \;=\; -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{4}{3}\beta}. \tag{9}$$

One solution decays faster than the other and can thus be neglected. Note also that for $\beta < -3/16$ has an imaginary part, which means that the solution has oscillations. Such solutions have been used in [2] to alleviate the Hubble tension.

IV. LTB METRIC

We will use this metric to describe a blob with a generic density profile. This metric is spherically symmetric but inhomogeneous, and given by:

$$ds^2 \;=\; -dt^2 + S^2(r,t)dr^2 + R^2(r,t)(d\theta^2 + \sin^2\theta d\psi^2). \tag{10}$$

We consider here only dust, in addition to the scalar field. We have chosen coordinates (r, θ, ψ) comoving with dust and proper time t. $S(r,t)$ and $R(r,t)$ are arbitrary functions. It is interesting to mention that the FLRW metric is a particular case of the LTB metric, the homogeneous case, and can be recovered if: $S(r,t) = \frac{a(t)}{\sqrt{1-kr^2}}$ and $R(r,t) = a(t)r$, where k is the spatial curvature constant (0, 1 or -1).

From now on we will use: $\partial_r A \equiv A'$ and $\partial_t A \equiv \dot{A}$. The Einstein tensor is given by:

$$\begin{aligned}G^0_0 \;=&\; -\left(\tfrac{R'}{SR}\right)^2 - 2\tfrac{R''}{SR^2} + \tfrac{\dot{S}\dot{R}}{SR} \\ &+ \tfrac{1}{R^2} + \tfrac{\dot{R}^2}{R^2} + 2\tfrac{S'R'}{S^3R},\end{aligned} \tag{11}$$

$$G^1_1 \;=\; -\left(\tfrac{R'}{SR}\right)^2 + \tfrac{1}{R^2} + \tfrac{\dot{R}^2}{R^2} + 2\tfrac{\ddot{R}}{R}, \tag{12}$$

$$G^2_2 \;=\; G^3_3 \;=\; \tfrac{\dot{S}\dot{R}}{SR} + \tfrac{\ddot{S}}{S} + \tfrac{\ddot{R}}{R} + \tfrac{S'R'}{S^3R} - \tfrac{R''}{S^2R}, \tag{13}$$

$$G_{01} \;=\; \tfrac{2\dot{S}R' - 2S\dot{R}'}{SR}. \tag{14}$$

We have expressed the non-diagonal component with both indices lowered for later convenience. Computing the right-hand side of eq. (2), we get the dust and field contributions:

$$\begin{aligned}G^0_0 \;=&\; \tfrac{-1}{\mathcal{M}^2+\beta\phi^2}\big[\rho + \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\tfrac{1}{S^2}\phi'^2 \\ &+ 2\beta\big(\tfrac{1}{S^2}\phi\phi'' - \left(\tfrac{\dot{S}}{S} + 2\tfrac{\dot{R}}{R}\right)\phi\dot{\phi} \\ &+ \tfrac{1}{S^2}\left(\tfrac{S'}{S} + 2\tfrac{R'}{R}\right)\phi\phi' + \tfrac{1}{S^2}\phi'^2\big)\big],\end{aligned} \tag{15}$$

$$\begin{aligned}G^1_1 \;=&\; \tfrac{1}{\mathcal{M}^2+\beta\phi^2}\big[-p + \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2S^2}\phi'^2 \\ &+ 2\beta\left(-\phi\ddot{\phi} - \dot{\phi}^2 - 2\tfrac{\dot{R}}{R}\phi\dot{\phi} + \tfrac{1}{S^2}\tfrac{R'}{R}\phi\phi'\right)\big],\end{aligned} \tag{16}$$

$$G^2_2 \;=\; G^3_3 \;=\; \tfrac{1}{\mathcal{M}^2+\beta\phi^2}\big[-p + \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2S^2}\phi'^2\big], \tag{17}$$

$$G_{01} \;=\; \tfrac{1}{\mathcal{M}^2+\beta\phi^2}\phi'\dot{\phi}. \tag{18}$$

The non-diagonal term usually vanishes in LTB with dust, but it does not vanish now in presence of ϕ.

For further developments we will consider that the scalar field is a 'test' field, which feels the LTB metric (sourced by dust), but such that the metric is not affected by the scalar field. In other words, we will assume that the scalar field is negligible compared to matter. We will also be considering late times, where the universe is dominated by dust. Thus, the only non-vanishing term of the Einstein's tensor is G^0_0 and the Einstein equations reduce to $G^\mu_\nu = 8\pi G(T_{\rm mat})^\mu_\nu$. Now we are in the case of the standard LTB solution with dust [7]; the non-diagonal component G_{01} is zero, which implies:

$$\dot{S}/S \;=\; \dot{R}'/R' \;\Longrightarrow S^2(r,t) \;=\; \frac{R'^2(r,t)}{1+2E(r)}. \tag{19}$$

Where we have conveniently chosen the integration constant and $E(r)$ is an arbitrary function. Using eq. (19), one can prove that just three of the above equations are independent: eqs. (12) and eq. (13) are equivalent under this approximation. Then, if we use eq. (19) in eq. (11) and we integrate eq. (12) we get, respectively:

$$\frac{\dot{R}^2 - 2E(r)}{R^2} + 2\frac{\dot{R}'\dot{R} - E'(r)}{RR'} \;=\; 8\pi G\rho, \tag{20}$$

$$\dot{R}^2 \;=\; \frac{2GM(r)}{R} + 2E(r). \tag{21}$$

Then if we use eq. (21) and its derivatives in eq. (20) we get:

$$S^2(r,t) \;=\; \frac{R'^2(r,t)}{1+2E(r)}, \tag{22}$$

$$\frac{1}{2}\dot{R}^2 - \frac{GM(r)}{R} \;=\; E(r), \tag{23}$$

$$4\pi\rho(r,t) \;=\; \frac{M'(r)}{R^2(r,t)R'(r,t)}, \tag{24}$$

where $M(r)$ is also an arbitrary function, that corresponds to the mass inside the comoving sphere of radial coordinate r. $E(r)$ is intuitively an r-dependent spatial curvature. These equations have different solutions depending on the sign of $E(r)$. We will study the case where $E(r) < 0$, which corresponds to a closed universe. The solution is given implicitly in terms of a parameter u as in [4]:

$$R = \frac{GM(r)}{-2E(r)}(1-\cos u), \tag{25}$$

$$t - t_b(r) = \frac{GM(r)}{[-2E(r)]^{3/2}}(u - \sin u). \tag{26}$$

where $t_b(r)$ is an arbitrary function that can be interpreted as a Big Bang singular surface at which $R = 0$. One of the three arbitrary functions, $E(r), M(r)$ and $t_b(r)$ is a gauge degree of freedom. For instance, by reparameterizing the radial coordinate we can give to $M(r)$ a simple form as in [4]:

$$M(r) = \frac{4\pi}{3}M_0^4 r^3, \tag{27}$$

with M_0 an arbitrary mass scale factor. One can see that $t_b(r)$ will be negligible at late times and for simplicity we will ignore it. If we define $\tilde{M} \equiv M_0^2/M_{Pl}$, the equations read:

$$R = \frac{4\pi(\tilde{M}r)^2}{-6E(r)}r(1-\cos u), \tag{28}$$

$$t = \frac{4\pi(\tilde{M}r)^2}{3[-2E(r)]^{3/2}}r(u - \sin u). \tag{29}$$

This is a parametric solution; as we will discuss it is possible to get a simpler explicit form for $R(r,t)$, using an approximation, but first let us discuss the conditions that the blob has to follow.

A. Junction conditions and restrictions on curvature

We want the blob to match the homogeneous universe at the border. This requirement sets additional restrictions over the values of M_0 and $E(r)$. Let us first introduce the "curvature function"

$$k(r) \equiv \frac{E(r)}{\tilde{M}^2 r^2}. \tag{30}$$

The metric of the blob has to match the homogeneous universe at the border, $r = L$, and so the curvature function must satisfy the following restrictions. Its derivative has to vanish so the junction is achieved smoothly. In addition, if we want it to match the *flat* FLRW universe, it must also vanish at the border. Finally, the curvature must not have a cusp at the origin. Thus the curvature function has to verify:

$$k'(L) = k(L) = k'(0) = 0. \tag{31}$$

Then, any curvature profile that fulfills these conditions may be studied with this metric.

On the other hand, we may choose M_0 such that M_0^4 coincides with the average density ρ_0 at present time t_0:

$$M_0^4 = \rho_0 = \frac{M_{pl}^2}{6\pi t_0^2} \iff t_0\tilde{M} = \frac{1}{\sqrt{6\pi}}. \tag{32}$$

We find a relation between the Hubble radius and $\tilde{M}$: $R_H^{-1} = H_0 = \frac{2}{3t_0} = \sqrt{\frac{8\pi}{3}}\tilde{M}$.

B. Small u approximation

It is possible to introduce an approximation assuming that $u \ll 1$ in eqs. (28) and (29), as in [5]; this approximation can describe quite precisely the dynamics even for $\delta\rho/\rho \gg 1$ and it simplifies our expressions. Expanding eq. (29) we get:

$$t \approx \frac{4\pi(\tilde{M}r)^2}{3[-2E(r)]^{3/2}}r\big(\frac{u^3}{6} - \frac{u^5}{5!}\big) \tag{33}$$

$$\Rightarrow \big(\frac{9[-2E(r)]^{3/2}}{4\pi\tilde{M}^2 r^3}t\big)^{1/3} = u\big(1 - \frac{u^2}{20}\big)^{1/3}. \tag{34}$$

The approximation is valid if $u \ll 1$, and it is equivalent to:

$$v \equiv \big(\frac{9[-2E(r)]^{3/2}}{4\pi\tilde{M}^2 r^3}t\big)^{1/3} = \sqrt{-k(r)}\big(\frac{9\sqrt{2}\tilde{M}t}{\pi}\big)^{1/3} \ll 1. \tag{35}$$

One can see that is valid either at small times or if the curvature $|k(r)|$ is small, or both. Under these conditions, one gets:

$$v\big(1 + \frac{u^2}{60}\big) \simeq u \implies u \approx v + \frac{v^3}{60}, \tag{36}$$

which has been solved iteratively and we have neglected terms of order $\sim \mathcal{O}(v^5)$. In order to have a more compact notation, we define also:

$$\gamma \equiv \big(\frac{9\sqrt{2}\tilde{M}t}{\pi}\big)^{\frac{1}{3}};\ R_2 \equiv \frac{1}{20};\ \tau \equiv (\tilde{M}t)\frac{1}{3}, \tag{37}$$

$$\implies v \equiv \gamma\tau\sqrt{-k(r)}. \tag{38}$$

The approximation will be good when:

$$u \approx u_0 \equiv v = \gamma\tau\sqrt{-k(r)} \ll 1. \tag{39}$$

If we stick to first order in v we recover the homogeneous case, that is why we need higher orders. If we develop further the equations we get:

$$u^2\big(\frac{1}{2} - \frac{u^2}{4!}\big) + \mathcal{O}(v^6) \approx \frac{1}{2}\big(v^2 - \frac{1}{20}v^4\big). \tag{40}$$

Finally, if we substitute eq. (40) in eq. (28) we get:

$$R(r,t) \approx \frac{\pi}{3}\gamma^2\tau^2 r[1 + R_2\gamma^2\tau^2 k(r)]. \tag{41}$$

It is interesting to note that if we had chosen the case where $E(r) > 0$ as in [5] the approximation leads to the exact same expression for $R(r,t)$. With this result we can analyse both situations simply changing the sign of the function $k(r)$ that we choose.

C. Choice of k(r) and density profile

We are looking for density profiles that resemble the structures that we find in the late time universe, i.e. profiles that simulate compact structures, with a growing density contrast due to gravity. Thus, in the center one expects to find a higher density than near the borders. The profile density must match the average density outside the blob at the border, in order to have an exact solution everywhere. To compute the density we need first the value of $R'(r,t)$:

$$R'(r,t) \;=\; \tfrac{\pi}{3}\gamma^2\tau^2[1+\gamma^2\tau^2 A(r)], \qquad (42)$$

where we have defined $A(r) \equiv (rk(r))'$. The density is obtained using eq. (24) and eq. (27):

$$\rho \;=\; \frac{M_0^4}{6\pi(\tilde{M}t)^2[1+R_2\gamma^2\tau^2k(r)]^2[1+R_2\gamma^2\tau^2A(r)]}. \qquad (43)$$

If we use the approximation $|k(r)| \ll 1$ we can neglect the term proportional to $k(r)$ but we cannot neglect its derivative, which is not necessarily small. Then it follows:

$$\rho \;=\; \tilde{\rho}\,\frac{1}{[1+R_2\gamma^2\tau^2A(r)]}, \qquad (44)$$

where we have defined $\tilde{\rho} \equiv \frac{M_0^4}{6\pi(\tilde{M}t)^2}$. Let us now choose a function $k(r)$; we propose:

$$k(r) \;=\; -k_0\Big[1-\big(\frac{r}{L}\big)^2\Big], \qquad (45)$$

where k_0 is an arbitrary constant and L is the radius of the blob. It is interesting to study the density contrast, defined by $\delta \equiv \frac{\rho-\langle\rho\rangle}{\langle\rho\rangle}$. First we need to compute the value of the average density inside a sphere of radius $\bar{r}$:

$$\begin{aligned}\langle\rho\rangle \;&=\; \frac{M_{tot}}{V_{tot}} \;=\; \frac{\int d\bar{r}\sqrt{-g}\rho(r,t)}{\int d\bar{r}\sqrt{-g}} \approx \frac{\int drR^2R'\rho(r,t)}{\int drR^2R'}\\ &=\; M_0^4\frac{\int drr^2}{\int dRR^2} \;=\; M_0^4\frac{\bar{r}^3}{R^3}.\end{aligned} \qquad (46)$$

Under the small u approximation, the density contrast is related to:

$$\delta \;=\; -\frac{R_2\gamma^2\tau^2A(r)}{1+R_2\gamma^2\tau^2A(r)}. \qquad (47)$$

If $\delta > 0$ we will have an overdensity, otherwise we will have an underdensity. When δ is close to -1 we will have a void-like situation. We need this density contrast to be exactly zero at the border, provided that is has to match the homogeneous universe and $\langle\rho\rangle$ is computed with the average universe density. An example of a density profile is given in Fig. (1) for two different times. We can see

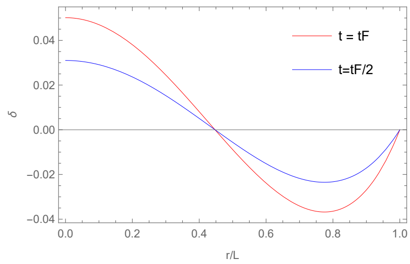

FIG. 1: Density contrast, δ, as a function of the radial coordinate. Here we used $k_0 = 0.1$ and we used units of Mpc = 1, so that $H_0 = 1/3000$, , $t_F \equiv \frac{2}{3H_0^{-1}}$ and a void size of $L = 1$ that fulfills $LH_0 \ll 1$.

in Figure (1) that at the center of the blob matter will tend to accumulate due the gravitational attraction. On the other hand, after a critical radius the density contrast turns negative, which corresponds to an underdensity. One should notice as well that as time evolves, the density contrast increases.

D. Klein-Gordon's equation

We now write down the Klein-Gordon's equation, eq. (3) in the LTB metric under the 'test' field approximation. We need first to compute the value of $\Box\phi$, given by:

$$\begin{aligned}\Box\phi \;=\; &-\ddot{\phi} + \tfrac{1+2E(r)}{R'^2}\phi'' - \big(2\tfrac{\dot{R}}{R}+\tfrac{\dot{R}}{R'}\big)\dot{\phi}\\ &+ \tfrac{1+2E(r)}{R'^2}\big[2\tfrac{R'}{R}+\tfrac{R''}{R'}-\tfrac{E'(r)}{1+2E(r)}\big]\phi'.\end{aligned} \qquad (48)$$

We also need the value of the Ricci scalar $\mathcal{R}$:

$$\begin{aligned}\mathcal{R} \;=\; 2\big[&2\tfrac{\ddot{R}}{R}+\tfrac{\ddot{R}'}{R'}+2\tfrac{\dot{R}}{R}\tfrac{\dot{R}'}{R'}+\tfrac{\dot{R}^2}{R^2}+\tfrac{1}{R^2}\\ &+\tfrac{1+2E(r)}{R'^2}\big(\tfrac{R'}{R}\tfrac{2E'(r)}{1+2E(r)}-\tfrac{R'^2}{R^2}\big)\big].\end{aligned} \qquad (49)$$

We obtain thus:

$$\begin{aligned}&\big[2\tfrac{\ddot{R}}{R}+\tfrac{\ddot{R}'}{R'}+2\tfrac{\dot{R}}{R}\tfrac{\dot{R}'}{R'}+\tfrac{\dot{R}^2}{R^2}+\tfrac{1}{R^2}\\ &+\tfrac{1+2E(r)}{R'^2}\big(\tfrac{R'}{R}\tfrac{2E'(r)}{1+2E(r)}-\tfrac{R'^2}{R^2}\big)\big]2\beta\phi\\ &-\ddot{\phi}+\tfrac{1+2E(r)}{R'^2}\phi''-\big(2\tfrac{\dot{R}}{R}+\tfrac{\dot{R}}{R'}\big)\dot{\phi}\\ &+\tfrac{1+2E(r)}{R'^2}\big[2\tfrac{R'}{R}+\tfrac{R''}{R'}-\tfrac{E'(r)}{1+2E(r)}\big]\phi' \;=\; 0.\end{aligned} \qquad (50)$$

This equation is one of our main results, in a suitable form for a numerical study. In order to have some analytical intuition, we can simplify further these expressions in the "small u approximation". Furthermore, we may also use a Taylor expansion for small k_0. We will cut the series expansion at $\mathcal{O}(k_0^2)$. In addition, we will consider the scalar field inside the blob to be a perturbation of the homogeneous scalar field, as:

$$\phi(r,t) \ = \ \phi_h(t) + \delta\phi(r,t). \tag{51}$$

One must notice that using this expression will let us simplify order $\mathcal{O}(k_0^0)$ terms, as it is the homogeneous case and $\phi_h(t)$ fulfills such equations, but terms of order $\mathcal{O}(k_0^1)$ will not vanish. Under these approximations, we get an equation of the form:

$$\begin{aligned}
&\delta\ddot{\phi} - (A_0 + k_0 A_1)\delta\phi'' - \frac{2}{r}(B_0 + k_0 B_1)\delta\phi' \\
&\quad + \left(\frac{C_0}{t} + \frac{1}{t}k_0 C_1\right)\delta\dot{\phi} - \beta\left(\frac{R_0}{t^2} + \frac{1}{t^2}k_0 R_1\right)\delta\phi \\
&\quad = \ \beta\frac{1}{t^2}k_0 R_1\phi_h - \frac{1}{t}k_0 C_1\dot{\phi}_h.
\end{aligned} \tag{52}$$

where all coefficients are functions of r. Terms with subscript 0 are the same as in the homogeneous equation, and the terms with subscript 1 correspond to the coefficients of first order in k_0, where we have made explicit the dependency on k_0. A full numerical solution of this partial differential equation is needed to find out the behaviour of the solution, with the appropriate boundary conditions. However, if we take into account that the order of magnitude of the various terms is controlled by $1/r \sim 1/L; 1/t \sim H; \partial/(\partial r) \sim 1/L; \partial/(\partial t) \sim H$, we can attempt to understand some properties of the behaviour of the solutions. At small times, gradient terms are negligible, recovering an equation similar to the homogeneous case. Then, as time grows, time derivative terms lose weight compared to gradient terms. The right side of the equation acts a a source: if it is positive, as there is a second derivative in time, we may expect that solutions would try to grow (they would collapse), otherwise solutions would be decreasing. One has to consider also that coefficients, such as C_1 and R_1, have a sign switch at a given radius. Finally, gradient terms would act as an effective pressure, preventing solutions to collapse or to decrease, giving instead oscillating solutions.

V. CONCLUSIONS

We have successfully written Einstein's equation for a scalar field non-minimally coupled to the Ricci scalar, in an arbitrary LTB metric. We have also found an expression for Klein-Gordon's equation, and we wrote down its coefficients explicitly under the 'test' field approximation, assuming that the metric is not affected by the scalar field. In addition, we have set out Klein-Gordon's equation under *small u* approximation and discussed briefly its behaviour and the possible forms of its solutions. Our work can be used straightforwardly to study numerical solutions of such a system and describe thus the evolution of a non-minimally coupled field inside a collapsing cosmic structure. This is an important preliminary study in order to understand how to impose local constraints on scalar-tensor theories of gravity.

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