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A Geometric Approach to Quantum Entanglement Classification

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Abstract

Quantum entanglement represents one of the fundamental differences between classical and quantum physics, with crucial roles in quantum information theory, superdense coding and quantum teleportation among others. A particularly simple description of entanglement of quantum states arises in the setting of complex algebraic geometry, via the Segre embedding. This is a map of algebraic varieties that serves as a tensor product and allows to detect separable (non-entangled states). In this thesis, we review the main fetaures of the geometric approach to entanglement. We focus on SLOCC equivalence, which is defined as the set of possible states that a quantum state may transform into. We construct generalisations of previous results for concrete instances, giving a classification formula for all states. Some applications concerning quantum information are also given.

Resum

L'entrallaçament quàntic representa una de les diferències fonamentals entre els models clàssic i quàntic de la física, amb rol crucial als camps de la informació quàntica, el *superdense coding* i la teletransportació quàntica, entre d'altres. Una descripció particularment simple de l'entrallaçament en estats quàntics sorgeix en el camp de la geometria algebraica complexa, mitjançant l'embedding de Segre, que és una aplicació entre varietats algebraiques que funciona de producte tensorial i permet detectar estats separables (no entrellaçats). En aquesta tesi presentem les principals característiques d'aquest tractament de l'entrellaçament. Ens centrem en l'equivalència SLOCC entre estats, definida com el conjunt de possibles estats en que un estat pot transformar-se. S'han construit generalitzacions de resultats anteriors sobre casos concrets, obtenint un resultat de classificació per tots els estats. Finalment, s'han donat algunes possibles aplicacions del resultat en el camp de la teoria de la informació quàntica.

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Contents

1 Introduction

From the time when the first drafts of a quantum physics theory were drawn in the first half of the twentieth century, our mathematical understanding of these models has been broadly enriched. While, when presented with this theory in the 1930s, Einstein asserted that this bizarre phenomena led into mathematical conclusions deemed impossible by the basic laws of physics, this same impossible conclusions were in fact what was later defined to be the defining trait of quantum physics, especially after Bell further formalized the results in the form of inequalities, and became more established as they were increasingly supported by experimental evidence.

At the core of these results that incur in a physical impossibility by the classical paradigm lies the concept of quantum entanglement. For a set of particles to be entangled entails that a change done locally to one of the particles has the potential to incur in a change for all particles in the set, regardless of their position in space. And, as a matter of fact, a system of particles incurs in a classical impossibility (i.e. it causes a violation of Bell's inequalities) if and only if it has some kind of entanglement present ([HHHH09]).

Several direct applications of entanglement are known. These include physical phenomena, such as new interpretations of the concept of entropy (see, for example, [Sch95]). Other applications delve into the informational realm, where entanglement can be used to teleport information (this concept is looked upon in Section 2.4, and was first observed by [BBC$^+$93]). This is highly related by the distribution of safe keys in cryptography, firstly studied by [Eke91], which lies at the core of the field of quantum cryptography. There are suggested applications of quantum entanglement even in the field of biology, where it has been suggested that the efficient transfer of energy occurring during plant photosynthesis cannot be explained without it ([WSI11]).

The aim of this monograph is to study entanglement by classifying in what ways the particles may be entangled. We will focus on quantum binary terms, i.e. particles that may only conceive two possible states, which will be conveniently labeled 0 and 1. In order to study the way this particles are entangled, we will consider how they may change under local transformations. For this purpose, local transformations on each of the particles of the system will be performed on classical channels, and we will consider the set of possible results that the transformation of a given state may yield (i.e. all the states which the current system may become with non-vanishing probability). This is known as the Stochastic Local Operations and Classical Comunications (SLOCC) equivalence class. It was introduced by [LP01] and [Vid99], and developped in the famous [DVC00] paper, which gives the classification for 3 particles. Further developments include the case for 4 particles (see [VDMV02]).

For the study of equivalence classes of quantum states the tools provided by classical algebraic geometry will be put into use. The study of entanglement using geometric tools is not new, and some instances can be found, among others, in [BH01], [BBC$^+$19] and [Sb17].

In recent years, further developments have been made to further classify SLOCC equivalence classes. Beyond the particular cases for three and four binary particles, the ranks of coefficient matrices, as well as polynomials invariant under the special linear group, have been used to classify some instances of SLOCC classes ([GW13]). In [LL12] local ranks of matrices serve to classify some families of states, and in [ZZH16] the result is generalised to include more states.

In this monograph we will use these tools, which mostly come from recent developments, in order to generalise the result given in [DVC00] to the general case of n binary particles.

The basic state of a quantum binary particle is normalised vector $(a, b) \in \mathbb{C}^2$. When measured by an external agent, it will collapse to either 0 or 1 by a probability given by $P(0) = a^2$, $P(1) = b^2$. This process, uncertain at first, is what is popularly denoted by *superposition of states*, as a single particle may a priori yield both possibilities. The fact that when the probabilities are taken into account the sign does not affect the state of measurements causes the global phase not to be taken into account, and this allows a bijection between the possible states for one quantum bit and points in the complex projective line $\mathbb{P}^1$.

On systems consisting of more than one binary particle, the joint state of n particles is considered to lie in the categorical product $\mathbb{C}^{2^n} \cong \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ of the spaces of states for every single particle. We consider the projectivisation $\mathbb{P}^{2^n-1} \cong \mathbb{P}^1 \otimes \ldots \otimes \mathbb{P}^1$ of this space, as it allows for the natural inclusion of the states into the tensor product via the *Segre embedding*

$$\begin{aligned} \varphi : \mathbb{P}^k \times \mathbb{P}^m &\to \mathbb{P}^k \otimes \mathbb{P}^m \cong \mathbb{P}^{(k+1)(m+1)-1} \\ [x_0 : \cdots : x_k], [y_0 : \cdots : y_m] &\mapsto [x_0 y_0 : x_0 y_1 : \cdots : x_i y_j : \cdots : x_n y_m]. \end{aligned}$$

The image Σ of the Segre embedding is called the *Segre variety*. It is a projective algebraic variety, whose elements are the simple tensors $x \otimes y$.

In this context, a transformation applied to a quantum binary term corresponds to a linear map on $\mathbb{C}^2$, transforming the state into another state. In order to preserve dimensions and the normalisation condition, this map is required to have maximal rank and to be unitary. When given not a single one but a set of particles, a local change in each of the particles, characterised by a unitary matrix M_i, may be considered. By taking the tensor product $M_1 \otimes \ldots \otimes M_n$ of all these operators, a change on the global system is achieved. Because of the phenomenon of entanglement, if it is present in the system, a particle shall not only be affected by the change applied on it, but also by the changes applied to all other particles in the system.

We approach the study of this phenomenon by considering an invariant known as the *tensor rank*. The tensor rank of a vector v in a tensor product $\mathbb{C}^n \otimes \mathbb{C}^m$ is the minimum number of non-zero summands in the expression

$$v = \sum_{i=0}^{k} v_i \otimes w_i.$$

In this manner, elements of tensor rank 1 correspond precisely to those points in the Segre variety Σ after projectivisation.

Elements of tensor rank 1 have the property that, when an operator is applied to one of the sides, the other side remains unaffected. Translating this idea to quantum physics, when considering a system of particles, if they are in the Segre variety, it follows that they are not entangled. These states are called *separable*.

The main contribution of this monograph is Theorem 4.9, stating quatum states of n particles are SLOCC equivelent if and only if the subsets of particles that are entangled are the same and have the same tensor rank.

This result is used to summarise the conclusions given in [DVC00]. Furthermore, some bounds for the number of SLOCC classes of n particles deduced.

Similarly to the results stated in [LL12] and [ZZH16], we classify SLOCC states in regards to constructions on their coefficients as vectors on a complex vector spaces, which in our case are represented by the tensor rank. The contribution of our theorem is to bring projective geometry, such as the Segre embedding, as a tool to study entanglement and distinguish entangled and non-entangled states.

Lastly, we present present some applications on other areas of quantum physics in Section 5. We focus, on the one hand, on the persistency of entanglement, which is, for a quantum state, the number of qubits that need to be removed in order for the system to become separated, and its relation to the tensor rank. On the other hand, on some possible applications of our work in the field of quantum information by generalising the algorithms of quantum teleportation.

We briefly review the contents of this work. Section 2 serves as a mathematical introduction to the models of quantum mechanics. In the section, which motivates many of the notation that will come to use later, and requires as little as background knowledge as necessary in order to make it adequate to both mathematicians and physicists , we present two settings of special interest that are very particular to quantum physics: the EPR paradox, and quantum teleportation. Section 3 presents the mathematical tools to be used in the classification efforts. We review the geometric preliminaries in order to understand entanglement via Segre embeddings, the Schmidt decomposition and the tensor rank.

These developments come into use in Section 4, which mathematically characterizes SLOCC equivalence in order to arrive finally at the classification Theorem 4.9, which as seen defines all equivalence classes up to the tensor rank, upper bounds to which are given. Finally, applications, further work ideas, such as a revisitation of the teleportation phenomenon, and general conclusions are given in Sections 5 and 6.

2 Quantum bits and entanglement

In order to study entanglement and its role in quantum mechanics, we will properly present how it is understood in a mathematical context. Beginning with the standard representation of quantum bits as normalized vectors in the complex plane, in this section we motivate and present the most important concepts related to quantum physics that will form the backbone of the work in the sections afterwards.

2.1 Quantum bits and the complex projective line

In quantum mechanics, the general quantum state of a quantum bit (qubit for short) is represented by a linear combination of its two orthonormal basis vectors, usually denoted using the "ket" notation:

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These two orthonormal basis vectors generate the two-dimensional complex vector space $\mathcal{H}$ of the qubit space state, which is in fact a Hilbert space[1].

A general qubit state is written as

$$|\psi\rangle = z_0 |0\rangle + z_1 |1\rangle ,$$

where z_0 and z_1 are complex numbers satisfying $|z_0|^2 + |z_1|^2 = 1$. When measured, the qubit has a probability $|z_0|^2$ of having the value 0 and a probability $|z_1|^2$ of having the value 1. In particular, such a state is, in a physical sense, uniquely determined by the pair of complex numbers (z_0, z_1) and the normalization condition $|z_0|^2 + |z_1|^2 = 1$ ensures $(z_0, z_1) \neq (0, 0)$. This allows one to consider the corresponding equivalence class $[z_0 : z_1]$ in the complex projective line $\mathbb{P}^1$. We recall that this space is defined as the set of lines of $\mathbb{C}^2$ that go through the origin and may be formally described via the quotient

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{z \sim \lambda z}, \ \lambda \in \mathbb{C}^*.$$

[1] We recall that a Hilbert space is a vector space together with an inner product such that the space is complete in regards to the metric it induces

Hence the class $[z_0 : z_1]$ denotes the the set of all points $(z'_0, z'_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that there is a non-zero complex number λ with $(z_0, z_1) = \lambda(z'_0, z'_1)$. Therefore we have

$$[z_0 : z_1] = [\lambda z_0 : \lambda z_1] \text{ for all } \lambda \in \mathbb{C}^*.$$

The above discussion shows that every quantum bit state defines a unique point in $\mathbb{P}^1$. Conversely, given a point $[z_0 : z_1] \in \mathbb{P}^1$ we may choose a representative (z_0, z_1) such that $|z_0|^2 + |z_1|^2 = 1$, unique up to a global phase which does not affect any state measurements, and so it determines a unique quantum bit state.

This one-to-one correspondence between qubit states and points in the projective space generalizes analogously to several particle states, as we shall later see.

The notion of a qubit is connected with those physical particles which may only present two distinct states (i.e. those with spin $\frac{1}{2}$), but the notion can be expanded to the generalised concept of qudit. Thus for example a qutrit has a basis with three three orthonormal vectors, often denoted $|0\rangle$, $|1\rangle$ and $|2\rangle$. In this way, qutrit states are in one-to-one correspondence with points in the complex projective plane $\mathbb{P}^2$. More generally, a qudit is the generalization to base d of a qubit, for which there is a basis $|0\rangle, \cdots, |d\rangle$ of orthonormal vectors.

2.2 Two-particle entanglement

Let us now study entanglement for two qubits. The initial set-up consists in two particles which can be shared between two different observers that can perform quantum measures to each of the particles. A quantum state for two qubits can be written as as

$$|\psi\rangle = z_0 |00\rangle + z_1 |01\rangle + z_2 |10\rangle + z_3 |11\rangle ,$$

where z_i are complex numbers that satisfy the normalization condition $\sum |z_i|^2 = 1$. Here, the vectors

$$|ij\rangle := |i\rangle \otimes |j\rangle , \text{ for } i, j \in \{0, 1\}$$

determine a basis of the 2-qubit vector space (where each of the components in the tensor product belongs to each of the two observers). Hence, they correspond to the vectors

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Reasoning analogously as in the one-particle case, we obtain a one-to-one correspondence bewteen two-qubit states and points in the complex projective space $\mathbb{P}^3$ (the set of lines in

$\mathbb{C}^4$ going through the origin), where the above state $|\psi\rangle$ is sent to the point

$$[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3.$$

A two-qubit state is said to be *separable* if it can be written as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

where $|\psi_i\rangle$ are states for a single qubit. Otherwise it is called *entangled.* For instance, the well-known qubit states

$$|\mathrm{Sep}\rangle := |00\rangle \text{ and } |\mathrm{EPS}\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right),$$

corresponding to the points

$$[\mathrm{Sep}] = [1 : 0 : 0 : 0] \text{ and } [\mathrm{EPS}] = [1 : 0 : 0 : 1] \text{ in } \mathbb{P}^3$$

are examples of separated and entangled states respectively.

The characterization of entanglement has a simple geometric interpretation via the *Segre embedding.* This is the map

$$\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$

defined by the products of coordinates

$$[x_0 : x_1], [y_0 : y_1] \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$

The Segre map is the categorical product of projective spaces, describing how to take products on projective Hilbert spaces. The word *embedding* accounts for the fact that this map is injective: it embedds the product $\mathbb{P}^1 \times \mathbb{P}^1$, which has complex dimension 2, inside $\mathbb{P}^3$, which has complex dimension 3 (i.e. is an embedding in the geometric sense, as will be proven in 3.7). The image of this map

$$\Sigma := \mathrm{Im}(\varphi)$$

is called the *Segre variety.* This is a complex algebraic variety of dimension 2 and is given by the set of points $[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3$ satisfying the single quadratic polynomial equation

$$z_0z_3 - z_1z_2 = 0.$$

The state $|\psi\rangle$ is separable if and only if its corresponding class $[\psi]$ in $\mathbb{P}^3$ is in the Segre variety Σ. It is entangled if and only if $[\psi] \notin \Sigma$.

For example, it is straightforward to verify that $[\mathrm{Sep}] = [1 : 0 : 0 : 0] \in \Sigma$, since $z_0z_3 = z_1z_2 = 0$, while $[\mathrm{EPS}] = [1 : 0 : 0 : 1] \notin \Sigma$, since in this case $z_0z_3 - z_1z_2 = 1 \neq 0$.

We will see in Section 3 that this geometric interpretation of entanglement is also valid in the case of more than two qubits. For that, we will introduce generalized Segre embeddings.

2.3 The EPR paradox

In this section we will see a proof that the model of quantum physics conveys the information in a fundamentally distinct way that what classical models of physics do. We shall proceed by means of the famous EPR paradox, which states a thought experiment (or game, in the mathematical sense) that has different solution on each model. We present Bell's version of the game, as presented on [AB09].

This will lean on the concept of two-qubit entanglement as we saw in the previous section, in order to show bits and qubits yield fundamentally discinct results. The game presented is known as the EPR paradox in regards to Einstein, Podolsky and Rosen, who in 1935 published a paper ([EPR35]) where they showed that using the tools provided by the new quantum paradigm one came to conclusions impossible in the physics hitherto known (hence, a paradox), claiming to disprove the concept of quantum physics by means of this. It was later J.Bell in the 1960s (see [Bel64]) who *solved* the paradox by showing that this impossible conclusions where indeed correct, and the fact that they could not be obtained in a classical channel marked a fundamental difference between both models.

The game involves two players, Alice and Bob, which cannot comunicate during the game, and an external arbiter, Charlie. Charlie chooses two random bits $x, y \in \{0,1\}$, and shares x with Alice and y to Bob. Alice then chooses a bit a, and Bob a bit b ($a, b \in \{0,1\}$). Alice and Bob win the game if and only if

$$a \oplus b = x \wedge y.$$

- Here the $\oplus$ symbol stands for the XOR operand: the function on bits that takes value 1 if a and b take different value and 0 otherwise.
- The $\wedge$ symbol stands for the AND operator: the function on bits that takes value 1 when $x = y = 1$ and 0 otherwise.

Example 2.1. *Take, as an example, the case $a = 1; b = 0; x = 1; y = 1$. Then $a \oplus b = 1 \oplus 0 = 1$, $x \wedge y = 0 \wedge 1 = 0$, and hence Alice and Bob would lose. On the other hand, if Alice chose 0, we have $a \oplus b = 0 \oplus 0 = 0$, and they would win.*

As $x \wedge y$ takes value 0 three times out of four, by choosing $a = b = 0$ Alice and Bob win with probability $\frac{3}{4}$, and it can be proven ([Bel64]) that this is an optimal strategy. But let us consider now this game in a quantum context.

- Assume now that Alice and Bob share a state of two qubits which, as we have seen, corresponds to a point in $\mathbb{P}^3$. Assume as well that, before starting the game, Alice and Bob have shared the entangled state

$$|\mathrm{EPS}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

In physics, it is customary to label the basis

$$|ij\rangle = |i\rangle_A \otimes |j\rangle_B$$

to denote that the first component of the tensor product belongs to Alice, while the second component belongs to Bob. In this case, each has a qubit to which apply transformations when necessary.

- When Alice receives x from Charlie, she applies to her shared qubit a rotation of $\frac{\pi}{8}$ if it is 1, and does nothing otherwise.
- Similarly, Bob applies a rotation of $-\frac{\pi}{8}$ to his shared qubit if $y = 1$ and does nothing otherwise.
- Then, Alice and Bob measure their shared qubits, and send them to Charlie.

We note that, even though Alice and Bob share the qubits i and j and apply transformations to them within a quantum framework, the original bits x and y are entirely classical, and the quantum transformation is done in regards to their value, with them partaking no direct involvement.

Lemma 2.1. With the strategy described above, both players Alice and Bob win with probability $\frac{4}{5}$.

Proof. Note that by homogeneity, any element of $\mathbb{P}^1$ can be expressed with coordinates of the form $(\cos(t) : \sin(t))$, therefore allowing to define rotations and angles in terms of coordinates.

We observe firstly that, if $x = y = 0$, then $a = b$ with probability 1, as no alterations have been done to the initial state of a and b.
Secondly, if $x \neq y$, then $a = b$ with probability greater than 0.85. For this, by symmetry, it suffices to analyse the case $x = 0, y = 1$. In this scenario, only Bob applies a rotation. Therefore, the angle between the two qubits is $\frac{\pi}{8}$. Thus when Alice measures her qubit, she will obtain the same result as Bob with probability $cos\frac{\pi}{8} > 0.85$.
Lastly, if $x = y = 1$, $a = b$ with probabiliy $\frac{1}{2}$, which can be shown by direct computation.
Therefore, the total winning probability is greater than $\frac{1}{4}1 + 2(\frac{1}{4}0.85) + \frac{1}{4}\frac{1}{8} = 0.8$ $\square$

Therefore, Alice and Bob have conceived a strategy that allows them to win with a probability impossible in a classical model.

2.4 The Teleportation Phenomenon

In this section an example of what is known as *quantum teleportation* will be shown. Specifically, it will be worked how Alice can send a qubit to Bob by using quantum entanglement and two classical bits of communication.

Let $|\psi\rangle = z_0 |0\rangle + z_1 |1\rangle$ be a qubit state. Alice wants to send Bob the state $|\psi\rangle$.

For this purpose, Alice applies a transformation, applying to $|\psi\rangle$ a state $|EPS\rangle$ which Alice and Bob have prepared and is shared between them.

$$|v\rangle = |\psi\rangle \otimes |EPS\rangle = |\psi\rangle |0\rangle |0\rangle + |\psi\rangle |1\rangle |1\rangle = (z_0 |0\rangle + z_1 |1\rangle) |0\rangle |0\rangle + (z_0 |0\rangle + z_1 |1\rangle) |1\rangle |1\rangle =$$
$$= z_0 |000\rangle + z_1 |100\rangle + z_0 |011\rangle + z_1 |111\rangle$$

Or, in projective notation:

$$[\psi] = [z_0 : 0 : 0 : z_1 : z_0 : 0 : 0 : z_1]$$

We introduce the notation

$$u_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}; u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}; u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

We note that the set $\{u_i\}$ that has just been introduced is an orthonormal basis of $\mathbb{C}^4$. In ket notation, they correspond to

$$|u_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle); |u_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle); |u_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle); |u_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

The above four states are prototypical examples of *Bell states*, specific states of two qubits that represent the simplest and maximal examples of entanglement. They form a maximally entangled basis, known as the *Bell basis*, of the four-dimensional Hilbert space for two qubits. Moreover, it can be obtained with the cannonical basis as

$$\begin{aligned}
|00\rangle = \quad & (1\ 0\ 0\ 0) = & \tfrac{1}{2}((1\ 0\ 0\ 1) + (1\ 0\ 0\ -1)) = & \quad \sqrt{2}(u_0 + u_1) \\
|01\rangle = \quad & (0\ 1\ 0\ 0) = & \tfrac{1}{2}((0\ 1\ 1\ 0) + (0\ 1\ -1\ 0)) = & \quad \sqrt{2}(u_2 + u_3) \\
|10\rangle = \quad & (0\ 0\ 1\ 0) = & \tfrac{1}{2}((0\ 1\ 1\ 0) - (0\ 1\ -1\ 0)) = & \quad \sqrt{2}(u_2 - u_3) \\
|11\rangle = \quad & (0\ 0\ 0\ 1) = & \tfrac{1}{2}((1\ 0\ 0\ 1) - (1\ 0\ 0\ -1)) = & \quad \sqrt{2}(u_0 - u_1)
\end{aligned}$$

So we can rewrite our last expression as

$$z_0 |000\rangle + z_1 |100\rangle + z_0 |011\rangle + z_1 |111\rangle =$$
$$= z_0(u_0 + u_1) |0\rangle + z_1(u_2 + u_3) |1\rangle + z_0(u_2 - u_3) |0\rangle + z_1(u_0 - u_1) |1\rangle =$$
$$= u_0(z_0 |0\rangle + z_1 |1\rangle) + u_1(z_0 |0\rangle - z_1 |1\rangle) + u_2(z_1 |0\rangle + z_0 |1\rangle) + u_3(-z_1 |0\rangle + z_0 |1\rangle)$$

At this point, Bob can apply a partial measurement on the system, which will fix on of the states u_i, making the system fall back to one of the states

$$z_0 |0\rangle + z_1 |1\rangle \quad z_0 |0\rangle - z_1 |1\rangle \quad z_1 |0\rangle + z_0 |1\rangle \quad -z_1 |0\rangle + z_0 |1\rangle$$

Knowing which of the states the system is in, Bob may apply a transformation in order to recover the original state $|\psi\rangle$. We can use, for instance, the matrices

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We can see that taking the state $x_0 |0\rangle + x_1 |1\rangle$ as the vector $\binom{x_0}{x_1}$, and applying one of the matrices, the image results in the original state $z_0 |0\rangle + z_1 |1\rangle$. Therefore, using the properties of entanglement, if Alice prepares the states conveniently and takes a measurement (which can be sent to Bob via classical channels), Bob can recover the original state by doing a simple transformation, therefore efectively teleporting a quantum state.

We note that this process uses classical bits in order to send the information. A classical communicacion channel is also needed for Alice and Bob to communicate between them, and therefore this phenomena does not incur in a violation of the causality principle. Quantum teleportation has been experimentally verified, with qubits being sent on distances over 100km (see [XSb12] for further details).

3 Geometric interpretation of entanglement

In the last section we saw that quantum states can be presented both in *ket* and projective notations. By using the geometric tools developed in both affine and projective geometry, this section aims to, on the one hand, generalize the definitions of qubit states and entanglement to the general case of n qubits and, on the other, introduce geometric tools and invariants which are able to convey information about those in a meaningful way.

We will start by defining the projective $\mathbb{P}^m$ space in Section 3.1. In Section 3.2 we will present a generalized version of the Segre embedding, a particular case of which was seen in Section 2.2. Section 3.3 will present a general definition of quantum entanglement, while the section closes with a presentation of the tensor rank and Schmidt decomposition factorisations, which play a key role in the classification of quantum states, in Sections 3.4 and 3.5.

3.1 Complex projective space and projective varieties

We briefly recall the classic definition of the projective space $\mathbb{P}^n$ over $\mathbb{C}$ (or, in general, an arbitraty field $\mathbb{K}$, see for example [Har77], [GH78]) as the quotient of the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ that identifies all points on the same line through the origin. We can write said relation as

$$z \sim z' \iff z = \lambda z'; \quad \lambda \neq 0 \quad z, z' \in \mathbb{K}^{n+1} - \{0\}$$

Given $z \in \mathbb{C}^{n+1} \setminus \{0\}$, we denote by $[z]$ the corresponding point in $\mathbb{P}^n$. If $z = (z_0, \cdots, z_n)$ we denote by $[z] = [z_0 : \cdots : z_n]$ its *homogeneous coordinates*, noting that $[z_0 : \cdots : z_n] = [\lambda z_0 : \cdots : \lambda z_n]$ for all $\lambda \in \mathbb{C}^*$.

In order to define projective algebraic varieties , we briefly introduce the notion of homogeneous algebraic sets.

We recall that a polynomial in several variables is called *homogeneous* if and only if all of its nonzero terms have the same degree.

Definition 3.1. Let I be an ideal of $\mathbb{C}[x_1 \dots x_n]$ generated by homogeneous polynomials. The zero-locus $V(I)$ of I is the set of points in $\mathbb{P}^n$ on which elements in I vanish:

$$V(I) := \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \,;\, P(z_0, \cdots, z_n) = 0 \text{ for all } P \in I\}.$$

Then $V(I)$ is called a *homogeneous algebraic set.* Such an algebraic set is said to be *irreducible* if it cannot be expressed as the union of two algebraic sets.

We also recall the definition of the Zariski topology on $\mathbb{P}^n$ defined by taking closed sets to be the homogeneous algebraic sets. Checking that the Zariski topology is indeed a topology is an immediate aplication of Hilbert's basis theorem (see [CLO15]). We can now define:

Definition 3.2. Homogeneous irreducible algebraic sets are called *projective algebraic varieties.* An open subset of a projective algebraic variety is called *quasi-projective variety.*

Generalising the above construction, given a vector space V we may consider its *projectivisation* $\mathbb{P}(V)$ as the quotient of the set $V \setminus \{0\}$ of non-zero vectors by the action of the multiplicative group of the base field by scalar transformations. If $\dim V = n$ then we have an isomorphism $\mathbb{P}(V) \cong \mathbb{P}^{n-1}$.

Projectivization is functorial with respect to injective linear maps: if $f : V \to V'$ is a linear map with $\mathrm{Ker}(f) = \{0\}$ then it induces an algebraic map of the corresponding projective spaces $\mathbb{P}(f) : \mathbb{P}(V) \to \mathbb{P}(V')$

We consider a special type of varieties.

Definition 3.3. A *projective linear variety* is a projective variety such that its defining polynomials have degree one.

A *projectivity* or *homography* is a bijective map $f : \mathbb{P}(V) \to \mathbb{P}(V')$ such that $dim\ \mathbb{P}(V) = dim\ \mathbb{P}(V') = n$, and

$$\forall F \subset \mathbb{P}^n |\ \text{F is a linear variety } \to f(F) \subset \mathbb{P}'^n \text{ is a linear variety.}$$

We note that projective linear varieties can be expressed as the solution of a system of homogeneous linear equations. Linear varieties and projectivities have certain desirable properties due to the fact that they can be related to linear subspaces of the underlying vector space. In order to further explore this concept, we take a closer look into projective coordinates.

Definition 3.4. Let $\mathbb{P}^n$ be a projective space of dimension n, and $\mathbb{C}^{n-1} - \{0\}$ be the domain for its original definition. Let $\mathcal{B} = \{v_0 \dots v_n\}$ be a basis for $\mathbb{C}^{n+1}$ as a vector space.

Let $\{z_0 \dots z_n\} \subset \mathbb{P}^n$ such that, for each i, $z_i = [v_i]$. Let $A \in \mathbb{P}^n$ such that, if we consider the simplex $\{z_0 \dots z_n\}$, A does not lie in any of the faces of such object. We will say then that

$$\Delta = \{z_0 \dots z_n; A\}$$

is a reference of $\mathbb{P}^n$, and $\mathcal{B}$ is a *basis adapted* to it. A point A that satisfies this conditions will be called the *unit point* of the reference.

For a more detailed construction of projective references, see Chapter 2 of [CA14].

We note that, given a vector space basis, one can construct a projective reference by letting an arbitrary point in $\mathbb{P}^n$ be the unit point. Via the use of references, we may further characterise projectivities:

Proposition 3.5 (Fundamental Theorem of Projective Geometry)**.** Let $\mathbb{P}^n$ be a projective spaces, and Δ, Δ' references of $\mathbb{P}^n$. Then, there exists a unique projectivity $f : \mathbb{P}^n \to \mathbb{P}^n$ such that $f(\Delta) = \Delta'$ and $\forall z \in \mathbb{P}^n$, the coordinates of p under the reference Δ are the same as the coordinates of $f(z)$ under Δ'.

See pages 52-53 of [CA14] for the proof. The fundamental theorem of projective geometry states that there is a bijection between projectivities and changes of references. This is analogous to the known result in vector spaces stating that isomorphisms of vector spaces can be viewed as changes of basis. It is a known fact that this isomorphisms have matrix representations. We will try to construct matrices for projectivities that work the same way.

On may start, for instance, by considering a $(n+1) \times (n+1)$ matrix M whose columns are $f(z_i)$. This raises the problem that the coordinates $f(z_i)$ are not uniquely defined (as $z_i = \lambda z_i$). To solve this issue, we impose

$$\sum_{i=0}^{n} f(z_i) = f(A)$$

where A is the unit point. It is easy to check that this system has a unique solution, hence determining a unique matrix M. We can prove that, just as matrices for linear maps in vector spaces do, this matrix helps find the image of an arbitrary point under the map.

Proposition 3.6. In the notation above, let $z \in \mathbb{P}^n$. Then,

$$f(z) = Mz$$

(where any of the two sides may be multiplied by a constant because of homogeneity)

What this results allow us is for the consideration of the points both in a vector context and in the projective context. Hence, a duality is achieved, in which both the tools for vector spaces such as basis representations and linear maps and matrices, and projective tools, such as the Segre embedding which will be introduced in the following section, will be given a place in our arsenal of techniques for the development of a geometric model for quantum bit states.

3.2 Segre embeddings

Given positive integers k and m, the *Segre embedding* $\varphi_{k,m}$ is the map

$$\varphi_{k,m} : \mathbb{P}^k \times \mathbb{P}^m \longrightarrow \mathbb{P}^{K(k,m)}$$

where $K(k,m) = (k+1)(m+1)-1$ defined by sending a pair of points $x = [x_0 : \dots : x_k] \in \mathbb{P}^k$ and $y = [y_0 : \dots : y_m] \in \mathbb{P}^m$ to the point of $\mathbb{P}^{K(k,m)}$ whose homogeneous coordinates are the pairwise products of the homogeneous coordinates of x and y:

$$\varphi_{n,m}(x,y) = [z_{00} : \dots : z_{ij} : \dots : z_{km}] \text{ with } z_{ij} := x_i y_j,$$

where we take the lexicographical order.

The image of the Segre Embedding is called the *Segre Variety.*

Proposition 3.7. The Segre embedding is indeed an embedding (and hence the Segre variety is a variety).

Proof. To show that the map defined above is an embedding, we must show that it is well-defined, it is injective, and that its image defines an irreducible projective variety.

For the first statement, it is immediate that, for arbitrary $(x,y) \in \mathbb{P}^k \times \mathbb{P}^m$, $\varphi_{k,m}(x,y) \neq 0$, as there must exist $x_i \neq 0$, $y_j \neq 0$, and therefore the coordinate $x_i y_j$ of $\varphi(x,y)_{k,m}$ is non-zero.
Moreover,

$$\varphi_{k,m}(\lambda[x_0 : \dots : x_n]\mu[y_0 \dots : y_m]) = [\lambda\mu x_0 y_0 : \dots : \lambda\mu x_i y_j \dots : \lambda\mu x_n y_m] = \lambda\mu[\dots : x_i y_j \dots]$$

thus $\varphi_{k,m}$ does not depend on the class representative.
To test injectivity, suppose $[\dots : x_i y_j \dots] = [\dots : x'_i y'_j \dots]$. Then, $x_0 y_j = \lambda x'_0 y'_j \forall j$, and, suposing without loss of generality $x_0, x'_0 \neq 0$

$$[y_0 : \dots : y_m] = x_0[y_0 : \dots : y_m] = \lambda x'_0[y'_0 : \dots : y'_m] = [y'_0 : \dots : y'_m]$$

With the same argument we get equality of x's and x' 's.
Lastly we need to prove that the image of $\varphi_{k,m}$ is an irreducible variety. For that purpose, let us define the family of degree 2 homogeneous polynomials

$$\Sigma = \{x_{i(k+1)+j}x_{s(k+1)+l} - x_{s(k+1)+j}x_{i(k+1)+l}\}_{i,s\leq k,j,l\leq m}$$

It is clear that any point on $\Sigma_{k,m}$:=Im(φ) is in the null set of Σ, and viceversa, therefore proving that $\Sigma_{k,m}$ is a projective variety. □

In other words, we note that we just stated that the Segre variety can be explicitly written as the null set of all polynomials resulting from the 2×2 minors of the matrix

$$\begin{pmatrix} z_{00} & \cdots & z_{0n} \\ \vdots & \ddots & \vdots \\ z_{m0} & \cdots & z_{mn} \end{pmatrix} \tag{1}$$

A combinatorial argument shows that there is a total of

$$\xi_{k,\ell} := \binom{k+1}{2} \cdot \binom{\ell+1}{2} = \frac{k \cdot (k+1) \cdot \ell \cdot (\ell+1)}{4}$$

minors of size 2×2 in such matrix.

We will present now some examples of Segre Embeddings and varieties. The most basic example is that in which $k = m = 1$. In that case, the map would correspond to

$$\begin{aligned} \varphi_{1,1} : \qquad \mathbb{P}^1 \times \mathbb{P}^1 \qquad & \to \mathbb{P}^3 \cong \mathbb{P}^1 \otimes \mathbb{P}^1 \\ [x_0 : x_1], [y_0 : y_1] \quad & \mapsto [z_0 : z_1 : z_2 : z_3] := [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1] \end{aligned}$$

And the Segre variety (as seen in the proof of 3.7) is defined by the null set of the polynomial

$$z_0z_3 - z_1z_2$$

It hence corresponds to a quadric in $\mathbb{P}^3$, which is seen to be an hyperbolic paraboloid. We already saw this variety in Section 2.2. Another example would be taking dimensions $k = 1, m = 3$. In that case, the Segre embedding $\varphi_{1,3}$ translates to

$$\begin{aligned} \varphi_{1,3} : \mathbb{P}^1 \times \mathbb{P}^3 \quad & \to \mathbb{P}^7 \\ [x_0 : x_1][y_0 : y_1 : y_2 : y_3] \quad & \mapsto [x_0y_0 : x_0y_1 : x_0y_2 : x_0y_3 : x_1y_0 : x_1y_1 : x_1y_2 : x_1y_3] \end{aligned}$$

The corresponding variety is defined, as we have seen, by the minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 & z_7 \end{pmatrix}$$

which yield the system of equations

$$\begin{cases} z_0z_1 - z_4z_5 & = 0 \\ z_1z_2 - z_5z_6 & = 0 \\ z_2z_3 - z_6z_7 & = 0 \\ z_0z_2 - z_4z_5 & = 0 \\ z_1z_3 - z_5z_7 & = 0 \\ z_0z_3 - z_4z_7 & = 0 \end{cases}$$

We can also consider compositions of Segre embeddings, as the following proposition shows.

Proposition 3.8. Let $\mathbb{P}^k, \mathbb{P}^m, \mathbb{P}^r$ be projective spaces. Then, the following diagram commutes.

$$\begin{tikzcd} \mathbb{P}^k \times \mathbb{P}^m \times \mathbb{P}^r \arrow[r, "\varphi_{k,m} \times Id_r"] \arrow[d, "Id_r \times \varphi_{m,r}"] & (\mathbb{P}^k \otimes \mathbb{P}^m) \times \mathbb{P}^r \arrow[d, "\varphi_{K_\varphi(k,m),r}"] \\ \mathbb{P}^k \times (\mathbb{P}^m \otimes \mathbb{P}^r) \arrow[r, "\varphi_{k,K(m,r)}"] & \mathbb{P}^k \otimes \mathbb{P}^m \otimes \mathbb{P}^r \end{tikzcd}$$

The proof consists on explicitly writing the formulas. By induction the result can be extended to any finite number of spaces, and we can then define Segre embeddings (and alongside them Segre varieties) with more than two subindices. Therefore $\varphi_{n_1 \ldots n_k}$ wil be given by the iteration of Segre embeddings in any particular order.

Given positive integers $k_1, \cdots, k_n$, let

$$K(k_1, \cdots, k_n) := (k_1 + 1) \cdots (k_n + 1) - 1.$$

For $1 \leq j \leq n$, let $[x_0^j : \cdots : a_{k_j}^j]$ denote coordinates of $\mathbb{P}^{k_j}$.

Definition 3.9. The *generalized Segre embedding*

$$\varphi_{k_1,\cdots,k_n} : \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n} \longrightarrow \mathbb{P}^{N(k_1,\cdots,k_n)}$$

is defined by letting

$$\varphi_{k_1,\cdots,k_n}([\cdots : x_{i_1}^1 : \cdots], \cdots, [\cdots : x_{i_n}^n : \cdots]) := [\cdots : z_{i_1 \cdots i_n} : \cdots] \text{ where } z_{i_1 \cdots i_n} = x_{i_1}^1 \cdots x_{i_n}^n$$

and the lexicographical is assumed. Denote the *generalized Segre variety* by

$$\Sigma_{k_1,\cdots,k_n} := \operatorname{Im}(\varphi_{k_1,\cdots,k_n}).$$

It follows from the definition that every generalized Segre embedding may be written as compositions of maps of the form

$$\mathbb{I}_m \times \varphi_{k,l} \times \mathbb{I}_{m'}$$

for certain values of m, k, l and m', where $\mathbb{I}_m$ denotes the identity map of $\mathbb{P}^m$. These compositions may be arranged in a directed $(n-1)$-dimensional cube, where the initial vertex is $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n}$ and the final vertex is $\mathbb{P}^{N(k_1,\cdots,k_n)}$. Note that the $(n-1)$ final edges of the cube (those edges whose target is the final vertex $\mathbb{P}^{N(k_1,\cdots,k_n)}$) are given by Segre embeddings of the form

$$\varphi_{N(k_1,\cdots,k_j),N(k_{j+1},\cdots,k_n)} : \mathbb{P}^{N(k_1,\cdots,k_j)} \times \mathbb{P}^{N(k_{j+1},\cdots,k_n)} \longrightarrow \mathbb{P}^{N(k_1,\cdots,k_n)}.$$

where $1 \leq j \leq n-1$.

As a matter of example, we present the cube resulting from imposing $n = 4; k_1 = k_2 = k_3 = k_4 = 1$

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, "\mathbb{I}\times\mathbb{I}\times\varphi_{1,1}"] \arrow[dr, "\mathbb{I}\times\varphi_{1,1}\times\mathbb{I}"] \arrow[dd, "\varphi_{1,1}\times\mathbb{I}\times\mathbb{I}"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \arrow[dr, "\mathbb{I}\times\varphi_{1,3}"] \arrow[dd, "\varphi_{1,1}\times\mathbb{I}"] & . \\
& \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^1 \arrow[rr, "\mathbb{I}\times\varphi_{3,1}"] \arrow[dd, "\varphi_{1,3}\times\mathbb{I}"'] & & \mathbb{P}^1 \times \mathbb{P}^7 \arrow[dd, "\varphi_{1,7}"] \\
\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, "\mathbb{I}\times\varphi_{1,1}"] \arrow[dr, "\varphi_{3,1}\times\mathbb{I}"] & & \mathbb{P}^3 \times \mathbb{P}^3 \arrow[dr, "\varphi_{3,3}"] & \\
& \mathbb{P}^7 \times \mathbb{P}^1 \arrow[rr, "\varphi_{7,1}"] & & \mathbb{P}^{15}
\end{tikzcd}$$

We end this section by relating the Segre embedding with the tensor product of vector spaces and projective spaces. This is motivated by the fact that, as seen in the computations for two qubits throughout Section 2, the joint space for the state of 2 qubits is defined by taking the tensor product of the spaces of the individual qubits. Moreover, as we will see in Section 3.3, this generalises for the general case of n qubits.

We recall that, for vector spaces V, W with bases $\{v_0 \dots v_k\}$ and $\{w_0 \dots w_m\}$, the *tensor product* $V \otimes W$ is defined as the space generated by the vectors $\{v_i \otimes w_j\}$, where

$$v_i \otimes w_j := (v_0 w_0, v_0 w_1, \dots, v_n w_n)$$

In the case where

$$M = \begin{pmatrix} m_{11} & \dots & \dots m_{1n} \\ \vdots & & \vdots \\ m_{1n} & \dots & \dots m_{nn} \end{pmatrix}; \quad N = \begin{pmatrix} n_{11} & \dots & \dots n_{1n} \\ \vdots & & \vdots \\ n_{1n} & \dots & \dots n_{nn} \end{pmatrix}$$

Are the matrices for linear maps in V, W respectively, we define their *Kronecker product* as

$$M \otimes N = \begin{pmatrix} m_{11}N & \dots & \dots m_{1n}N \\ \vdots & & \vdots \\ m_{1n}N & \dots & \dots m_{nn}N \end{pmatrix}$$

Example 3.1. *Let, for instance,*

$$M = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix};\ N = \begin{pmatrix} 1 & -3 \\ -1 & 0 \end{pmatrix}$$

In that case,

$$M \otimes N = \begin{pmatrix} 4*1 & -1*-3 & -1*1 & -1*3 \\ 4*-1 & 4*0 & -1*-1 & -1*0 \\ 0*1 & 0*-1 & 2*1 & 2*-3 \\ 0*-1 & 0*0 & 2*-1 & 2*0 \end{pmatrix} = \begin{pmatrix} 4 & 3 & -1 & -3 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

We note that, in both the cases of vectors and matrices, the operator $\otimes$ is associative. Therefore, if given an indeterminate number n of vector spaces $V_1 \dots V_n$, we can define

$$V_1 \otimes \dots \otimes V_n := ((V_1 \otimes V_2) \otimes \dots) \otimes V_n$$

As these developments are motivated by the necessity to define state spaces for qubits, it is also necessary to bring the projective space into the equation. Moreover, it will allow for the introduction of the Segre embedding. Given projective spaces $\mathbb{P}^k, \mathbb{P}^m$, which are in turn the projectivisation of vector spaces V, W, we define the space $\mathbb{P}^k \otimes \mathbb{P}^m$ as the projectivisation of the space $V \otimes W$. Similarly, the tensor space for n projective spaces is the projectivisation of the tensor product of their respective underlying vector spaces.

Now, focusing on the 2 space case, we defined it by being spanned by the set $\{v_i \otimes w_j\}$. Define the sets $\{x_0 \dots x_k\} \subset \mathbb{P}^k, \{y_0 \dots y_m\} \subset \mathbb{P}^m$ such that, for all i, j, $x_i = [v_i]$ and $y_j = [w_j]$. The following proposition, which is a simple restatement of the definition of the Segre embedding, connects the generators for the tensor in the vector space with the theory seen in this section.

Proposition 3.10. In the notation above,

$$[v_i \otimes w_j] = \varphi(x_i \otimes y_j)$$

where φ denotes the Segre embedding, and $[v_i \otimes w_j]$ denotes the projective class of $v_i \otimes w_j$.

3.3 The phenomena of quantum entanglement

Let us consider n distinct qubits. For each $i \in \{1 \dots n\}$, we define $\mathcal{H}_i$ as the space of possible states $|\psi_i\rangle = z_0 |0\rangle + z_1 |1\rangle$, or in projective notation $[\psi_i] = [z_0 : z_1] \in \mathbb{P}(\mathcal{H}_i) \cong \mathbb{P}^1$ for each qubit.

Following what we have seen in Section 2.1, the vectors $|0\rangle, |1\rangle$ formed a basis for the space of states for each qubit. We will consider the projectivisation of this space $\mathbb{P}(\mathcal{H}_{q_i}) \cong \mathbb{P}^1$. We introduce then the notation

$$|k_1 \cdots k_n\rangle_{1 \dots n} = |k_1\rangle_1 \otimes \cdots \otimes |k_n\rangle_n$$

where $k_i \in \{0, 1\}$. This set forms then a basis for the possible states of the qubits. As such, we define the projectivisation of the cojoint state space of n-qubits as

$$\mathbb{P}(\mathcal{H}) := (\mathbb{P}(\mathcal{H}_1) \otimes \cdots \otimes \mathbb{P}(\mathcal{H}_n)) \cong \mathbb{P}^{2^n-1}$$

in the above, we have ommited labels for the basis of each qubit, but one should be aware that a chosen order of the product basis has been fixed unless stated otherwise.

By virtue of what we have seen about the Segre embeddings and varieties we may add some remarks about this definition. If we consider the set of possible states for the qubits considered individually, we yield the space $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. We can apply then a generalized Segre embedding

$$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{2^n-1}$$

From this exercise it can be inferred that, firstly, the set of states in $\mathcal{H}$ is bigger than the set of states considered individually, and secondly, that the states in $\mathbb{P}_{q_1} \times \cdots \times \mathbb{P}_{q_n}$ map into the Segre variety.

In Section 2.2 we talked about quantum entanglement for two qubits. Now we are prepared to offer a mathematically rigorous definition of this concept.

Definition 3.11. Let $1 \leq q \leq n$ be an integer. An n-particle state $|\psi\rangle$ is said to be *q-partite* if it can be written as

$$|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_q\rangle ,$$

where $|\psi_i\rangle$ are n_i-particle states, with $n_i > 0$ and $n_1 + \cdots + n_q = n$.

The two extreme cases are well-known: 1-partite states are called *entangled*, while n-partite states are called *separable.*

A basic observation is that a state is separable if and only if it lies in the generalized Segre variety of Definition 3.9 (see for instance [BBC+19]). Likewise, a state is q-partite if and only if its corresponding projective point on $\mathbb{P}^{N_n}$ lies in a Segre variety of the form

$$\Sigma_{N_{m_1},\cdots,N_{m_q}}$$

with $m_1, \cdots, m_q$ positive integers such that $m_1 + \cdots + m_q = n$.

Proposition 3.12. Let $[\psi] \in \mathbb{P}^{2^n-1}$ be q-partite, and $[\psi] = [\psi_1] \otimes \ldots \otimes [\psi_q]$. For each i, let $[\psi_i]$ be a state of k_i qubits, and let $S_i := 2^{k_i} - 1$ be the projective dimension of the space of states $[\psi_i]$ is in. Then $[\psi] \in \Sigma_{S_1 \ldots S_n}$.

Example 3.2. *In the case of four qubits, consider a state that can be factored as*

$$|\psi\rangle = |\psi_{123}\rangle \otimes |\psi_4\rangle ,$$

where $|\psi_{123}\rangle$ is a three-particle state corresponding to a point in $\mathbb{P}(\mathcal{H}_1)\otimes\mathbb{P}(\mathcal{H}_2)\otimes\mathbb{P}(\mathcal{H}_3) \cong \mathbb{P}^7$ and and $|\psi_4\rangle$ is mono-particle corresponding to a point in $\mathbb{P}(\mathcal{H}_4) \cong \mathbb{P}^1$. In particular, $|\psi\rangle$ lives in the Segre variety $\Sigma_{7,1}$.

We focus now on what it entails to be entangled. Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be the state space of n-qubits and consider linear operators $M_i \in \mathcal{H}_i$, so that $M := \bigotimes_i M_i$ is a linear operator on $\mathcal{H}$. If $|\psi\rangle \in \mathcal{H}$ is a state such that $[\psi] \in \Sigma_{1,\cdots,1}$ lies in the generalized Segre variety, then $M[\psi] = M_1[\psi_1] \otimes \cdots \otimes M_n[\psi_n] = \varphi(M_1[\psi_1], \cdots, M_n[\psi_n])$. Therefore the transformation of ψ by M can be factored as an individual transformation for each of the qubits. In the case $[\psi] \notin \Sigma_{1,\cdots,1}$ such a decomposition is not possible and this means that the transformation M_i will not only affectits corresponding qubit, but also some other of the qubits. In the case where $|\psi\rangle$ is q-partite we will be able to find a partial factorization due to 3.12, and in the case there $|\psi\rangle$ is entangled no such factorization is a priori possible (it may happen of course depending on the M_i).

Therefore entanglement means that a transformation on a qubit will also affect those qubits entangled to it.

3.4 Schmidt Decomposition

The aim of this section is to provide a brief overview of Schmidt's Decomposition Theorem, as well as some important consequences. Schmidt Decomposition states that, when considering a tensor space $V \otimes W$, for every element v there is a basis such that it is formed by elements of the form $v_i \otimes v_w$ such that, if V, W have dimension k, m, the number of nonzero coordinates of v in such base is no greater than $\min(k, m)$. Put formally, we present the result as stated in [Pat13]:

Proposition 3.13 (Schmidt Decomposition)**.** Let V, W be complex vector spaces of dimension $k+1, m+1$ respectively, and assume $k \geq m$. Then, there exist linearly independent sets $\{v_0 \dots v_k\} \subset V$, $\{w_0 \dots w_m\} \subset W$ such that, $\forall v \in V \otimes W$

$$v = \sum_{i=0}^{m} a_i v_i \otimes w_i \tag{2}$$

where $a_i \in \mathbb{C}$

Schmidt Decomposition is essentialy a restatement of Singular Value Decomposition (SVD) for matrices (see [TB97]), which states any matrix can be decomposed via a product of unitary matrices and a diagonal positive definite matrix.

As a direct consequence, we have that

Corollary 3.14. The joint state of two qubits can always be written as

$$|\psi\rangle = \sin(\theta)\,|00\rangle + \cos(\theta)\,|11\rangle\,.$$

or, equivalently, there exists a reference of $\mathbb{P}^3$ such that $[\psi] = [a : 0 : 0 : b]$.

Proof. We know that the joint space of both qubits can be identified with $\mathbb{C}^2 \otimes \mathbb{C}^2$ (using physical notation), or $\mathbb{P}^3$ (using projectivised notation). In either case, applying the Schmidt decomposition to the state yields a basis with at most two non-zero coordinates (which is adapted to a certain reference in $\mathbb{P}^3$), corresponding to the first and last coordinate by 2 □

Proposition 3.15. The joint state of three qubits can be always expressed, given an adequate basis formed by simple tensor elements, with at most 3 nonzero coordinates.

Proof. This proof is an adaptation of the proof of the similar result given in [Ab00]. Let $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ be complex vector spaces of dimension 2. Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. We will denote, in a given basis,

$$|\psi\rangle = (z_{000}, z_{001}, z_{010}, z_{011}, z_{100}, z_{101}, z_{110}, z_{111})$$

and consider the 2×2 matrices

$$T_0 = (z_{0ij}), T_1 = (z_{1ij}).$$

Consider now the linear isomorphism

$$\begin{aligned} T_0' = &\quad x_{00}T_0 + x_{01}T_1 \\ T_1' = &\quad x_{10}T_0 + x_{11}T_1 \end{aligned}$$

where $x_{ij} \in \mathbb{C}$, and impose

$$0 = detT_0' = x_{00}^2 k_0 + x_{00} b_{01} k_1 + x_{01}^2 k_2$$

where the k_i depend on the coefficients of $|\psi\rangle$, and are therefore fixed. By taking the change of basis that diagonalizes T_0', the matrix fill only have one non-zero coefficient in the new basis (as its determinant vanishes). Therefore, the new coordinates of v in such basis are

$$v = (y_0, 0, 0, 0, y_4, y_5, y_6, y_7)$$

and therefore

$$|\psi\rangle = (y_0, 0) \otimes (1, 0) \otimes (1, 0) + (0, 1) \otimes (y_4, y_5) \otimes (1, 0) + (0, 1) \otimes (y_6, y_7) \otimes (0, 1)$$

proving the result. □

As a corollary, we have

Corollary 3.16. The joint state $|\psi\rangle$ of three qubits can always be expressed via an equation of the form

$$|\psi\rangle = k\,|000\rangle + a\,|100\rangle + b\,|110\rangle + c\,|101\rangle + d\,|111\rangle$$

3.5 Tensor Rank

Using the tools provided by the Segre embedding, we can consider operations on tensor vector and projective spaces respectively. This section is devoted to exploring the idea of the partial trace of an operator on a tensor space, and its uses providing invariants on various kinds of relationships between vectors, such as the tensor rank.

The partial trace and the tensor rank have been explored, among others, in [Maz17], [BFZ20] and in the fifth chapter of [BBC$^+$19] and serves as a kind of dimension for vectors in tensor spaces, showing how many simple tensor terms are necessary to define such vector. We start by the definition:

Let V_A, V_B be complex vector spaces of dimension $k+1, m+1$ respectively. Let us consider the space $V = V_A \otimes V_B$, let $\mathrm{End}(V)$ denote the space of endomorphisms of V. Let $f \in \mathrm{End}(V)$

Given the basis for V $\{v_i \otimes w_j\}$, where $0 \leq i \leq k; 0 \leq j \leq m$, we denote by A_f as the matrix of f in this basis. We can write $A_f = (a_{i_A i_B, j_A j_B})$, where $\leq i_A, j_A \leq k$, $0 \leq i_B, j_B \leq m$. The row of the matrix will be $i_A * (k+1) + i_B$, while the column will be denoted by $j_A * (k+1) + j_B$

Definition 3.17. Given $f \in \mathrm{End}(\mathcal{H})$ and $A_f = (a_{i_A i_B, j_A j_B})$, define

$$b_{ij} = \sum_{k=1}^{n} a_{ik,jk}$$

$$\rho(A) := (b_{ij}) 1 \leq i \leq n, 1 \leq j \leq m$$

We will say that $\rho(A)$ is the *extended partial trace* of A with respect to V^B.

Example 3.3. *Put $k = m = 2$, $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. Let*

$$A = \begin{pmatrix} a_{11,11} & a_{11,12} & a_{11,21} & a_{11,22} \\ a_{12,11} & a_{12,12} & a_{12,21} & a_{12,22} \\ a_{21,11} & a_{21,12} & a_{21,21} & a_{21,22} \\ a_{22,11} & a_{22,12} & a_{22,21} & a_{22,22} \end{pmatrix} = \begin{pmatrix} 2 & 7 & 1 & -2 \\ 3 & -1 & 0 & 1 \\ 4 & -1 & 2 & 0 \\ 0 & 2 & 5 & 1 \end{pmatrix}$$

We can compute

$$\begin{cases} b_{11} = & a_{11,11} + a_{12,12} = & 2 - 1 = 1 \\ b_{12} = & a_{11,21} + a_{12,22} = & 1 + 1 = 2 \\ b_{21} = & a_{21,11} + a_{22,12} = & 4 + 2 = 6 \\ b_{22} = & a_{21,21} + a_{22,22} = & 2 + 1 = 3 \end{cases}$$

and therefore

$$\rho(A) = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}$$

Definition 3.18. Let V_A, V_B be vector spaces of dimension n, m respectively, and $V = V_A \otimes V_B$. Let $v \in V$, and consider $v \otimes v \in \mathrm{End}(V)$. Then, the *tensor rank* of v is defined as $\mathrm{rank}(\rho(v \otimes v))$.

We note that this ranks stay stable during the projectivisation process, and hence the tensor rank is invariant wether we are in a vector space or projective context.

Example 3.4. *Take a generic vector* $v = (a\ b\ c\ d) \in \mathbb{C}^4$. *We apply the transformation*

$$v \otimes v = \begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}$$

Then,

$$\begin{cases} b_{11} = & a_{11,11} + a_{12,12} = & aa + bb \\ b_{12} = & a_{11,21} + a_{12,22} = & ac + bd \\ b_{21} = & a_{21,11} + a_{22,12} = & ca + db \\ b_{22} = & a_{21,21} + a_{22,22} = & cc + dd \end{cases}$$

As we are given a matrix of dimension 2×2, *we compute the determinant to find the rank:*

$$det\rho(v) = (aa + bb)(cc + dd) - (ac + bd)(ca + db) =$$
$$= aacc + bbcc + aadd + bbdd - aacc - abcd - abcd - bbdd = bbcc + aadd - 2abcd =$$
$$= -bc(ad - bc) + ad(ad - bc) = (ad - bc)^2$$

We conclude that the tensor rank of v *is 1 if and only if* $ad - bc = 0$, *and 2 otherwise. One can observe that, if we impose* $v = v_1 \otimes v_2$, *and put*

$$\begin{cases} v_1 = & (a_1\ a_2) \\ v_2 = & (b_1\ b_2) \end{cases}$$

Then $v = (a_1b_1\ a_1b_2\ a_2b_1\ a_2b_2)$, *and the determinant we computed is always*

$$a_1b_1a_2b_2 - a_1b_2a_2b_1 = 0$$

showing that vectors of the form $v_1 \otimes v_2 \in \mathbb{C}^4$ *always have tensor rank 1.*

Our aim now is to prove that the tensor rank of a vector does not depend on the choice of basis. The tool used for that result is the following proposition, which allows us to characterise the extended partial trace by means of a universal property.

Proposition 3.19. The extended partial trace is the only operator

$$\rho_B : \mathrm{End}(V_A \otimes V_B) \to \mathrm{End}(V_B)$$

such that, $\forall M \in \text{End}(V_A) \quad N \in \text{End}(V_B)$,

$$\rho_B(M \otimes N) = MTr(N)$$

Where Tr() denotes the trace operator.

Proof. Let $v_1 \dots v_n$ be a basis for V_A, and $E_{ij} \in \text{End}(V_A)$ be the operator that sends v_i to v_j and the rest of basis elements to 0. It is a known fact that the set $\{E_{ij}\}$ forms a basis for $\text{End}(V_A)$ as a vector space. We define in a similar fashion the basis $\{F_{rs}\}$ of $\text{End}(V_B)$ with respect to a basis $w_1 \dots w_m$ of V_B.
We consider the set $\{E_{ij} \otimes F_{rs}\}$, which again is well known to be a basis of $\text{End}(V_A \otimes V_B)$. Those operators send the basis element $e_i \otimes v_r$ to the basis element $v_j \otimes w_s$, and the rest of them to zero. Hence the matrix representation of such operator is

$$(E_{ij} \otimes F_{rs})_{ab,cd} = \begin{cases} 1 & \text{if } \{a, b, c, d\} = \{i, j, r, s\} \\ 0 & \text{otherwise} \end{cases}$$

and it follows by computation

that the identity

$$\rho_B(E_{ij} \otimes F_{rs}) = E_{ij}Tr(F_{rs})$$

holds for this kind of matrix. We wish now to reproduce this result with an arbitrary operator $M \otimes N = \sum_{i,j} a_{ij}E_{ij} \otimes \sum_{r,s} b_{rs}F_{rs} = \sum_{i,j}\sum_{r,s} a_{ij}b_{rs}(E_{ij} \otimes F_{rs})$. Using the definitions for the extended partial trace,

$$\rho_B(\textstyle\sum_{i,j}\sum_{r,s} a_{ij}b_{rs}(E_{ij} \otimes F_{rs})) = \rho_B((a_{ij}b_{rs})) = (\sum_{k=1}^{n} a_{ij}b_{kk}) =$$
$$(a_{ij}(\textstyle\sum_{k=1}^{n} b_{kk})) = (a_{ij})(\sum_{k=1}^{n} b_{kk}) = MTr(N)$$

By the fact that the trace is invariant by basis changes, we can generalise the result to any basis for the spaces, hereby obtaining what was desired. □

This proposition shows that the partial trace works as an extension of the trace operator, allowing to, using a term usually used by physicists, *trace out* one of the components of the product $M \otimes N$. This motivation for this operation relies then in the ability to, in some sense, recover one of the components of the product.

Now we aim, focusing on vectors and the tensor ranks of vectors, to relate the notion of the rank of this matrix onto something which actually gives valuable information about the vectors in the space.

From now on, instead of complex vector spaces, we will consider them projectivised. This is a natural step, as ranks of matrices are also a projective invariant. This will allow us, on the one hand, to consider normalized vectors and not having to worry about constants and, on the other hand, to deploy all the theory about Segre embeddings developped on the previous sections.

Proposition 3.20. A vector $v \in \mathbb{C}^{k+1} \otimes \mathbb{C}^{m+1}$ has tensor rank 1 if and only it is of the form

$$v = v_A \otimes v_B$$

Proof. Let $v \in \mathbb{C}^{k+1} \otimes \mathbb{C}^{m+1}$, $v = (v_{i_A,i_B})$ such that v has tensor rank 1. Then, if we consider $v \otimes v = (v_{i_A,i_B}, v_{j_A,j_B},)$, the matrix

$$b_{ij} = \sum_{k=1}^{n} v_{i,j} v_{k,k} = v_{i,j} \sum_{k=1}^{n} v_{k,k} = v_{i,j} ||v^2||$$

has rank one .

We can see that this resulting matrix is just our initial vector, but put in matrix form with rows and columns corresponding to which coordinate on the original spaces is being referenced. Hence, it being of rank one means that all 2×2 minors of the matrix have determinant zero.

We will now consider the projectivisation $\mathbb{P}^k \otimes \mathbb{P}^m$ of our vector space, point $z = [v]$. For z, the matrix of the partial trace looks in its projectivised form,

$$b'_{i,j} = [v_{i,j} ||v^2||] = z_{i,j}$$

Recalling the definition of the Segre variety, the condition for being a part of $\Sigma_{A,B}$ is that the vector is in the null set of all 2×2 minors of the matrix

$$\begin{pmatrix} z_{00} & \cdots & z_{0k} \\ \vdots & \ddots & \vdots \\ z_{m0} & \cdots & z_{mk} \end{pmatrix}$$

(see (1)). It is clear that both matrices are the same for z. Hence, z is a part of said Segre variety and, by 3.10 we have that z is of the form $z = z_A \otimes z_B$. We can now go back to our original vector space in order to obtain the desired result. □

Now we aim to generalise a notion very similar to what we saw in this last proposition for vectors of tensor rank k.

Proposition 3.21. Let $v \in \mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$ be a vector of tensor rank k. Then there exists a decomposition of v of the form

$$v = \sum_{i=1}^{k} c_i (v_i \otimes w_i)$$

where $c_i \neq 0$, $v_i, w_i \in \mathbb{C}^{m_1+1}, \mathbb{C}^{m_2+1}$, and such k is minimal.

Proof. From the proof of 3.20 we saw that the tensor rank of a vector is the rank of the matrix

$$\begin{pmatrix} v_0 w_0 & \dots & v_0 w_{m_1} \\ \vdots & \ddots & \vdots \\ v_{m_2} w_0 & \dots & v_{m_2} w_{m_1} \end{pmatrix}$$

Given that the matrix has rank k, we can consider a basis change (the one obtained via Gaussian elimination, for instance) in which only the first k rows are non-zero.

If we denote the rows of this new matrix by v_i', we can consider the decomposition

$$v = \sum_{i=1}^{k} v_i \otimes e_i$$

where e_i denotes the canonical basis of $\mathbb{C}^{m_2+1}$, proving the statement. □

Applying the result to projective spaces:

Corollary 3.22. Let $z \in \mathbb{P}^{m_1} \otimes \mathbb{P}^{m_2}$ be a vector of tensor rank k. Then there exists a basis $\{v_i \otimes w_i\}$ adapted to a reference of $\mathbb{P}^{m_1} \otimes \mathbb{P}^{m_2}$ such that, in that reference, z has k non-zero coordinates, and that number is minimal.

Proof. With the notation used on the proof of the proposition, if we take the vectors $v_i \otimes e_i$ and expand them to a basis (which we can do as they are clearly linearly independent), it is clear that z will have k non-zero coordinates in that basis, and minimality follows by the same reason. □

Up to this point we only considered ranks for vectors on spaces of the form $V_A \otimes V_B$. We would like now to expand the notions we worked on to a general multitensor space $V = V_1 \otimes \ldots \otimes V_n$.

We will rely on the invariant introduced in 3.21, which can be expanded to the multitensor context by considering the minimal number k such that there exists a decomposition

$$v = \sum_{i=1}^{k} v_{i,1} \otimes \ldots \otimes v_{i,n} \tag{3}$$

We can define then

Definition 3.23. Let $V = V_1 \otimes \ldots \otimes V_n$, $v \in V$. We will say that the *tensor rank* of v is k if and only if there exists a minimal decomposition of v of the form (3).

4 Classification of quantum bit states

Using the mathematical tools presented in Section 3, we will consider the problem of classification of the entanglement of quantum states. In Section 4.1 we will formally present the definition of SLOCC equivalence. In Section 4.2 we state the SLOCC equivalence classes for two qubits. Section 3.3 presents a general classification of all SLOCC classes for the general n qubit case, and in Section 4.3.1 this is used to classify 3-qubit states by the SLOCC equivalence relation. Finally, Section 4.4 presents upper bounds for the tensor rank of states of qubits.

4.1 SLOCC equivalence

When dealing with entanglement in quantum information, one studies a certain particle subsystem isolated from the rest, in order to safely operate without the need to consider a bigger picture that would render practical calculations too complex. Therefore, one operates locally in a group of n particles, and shall thus be restricted to studying their conjoint state space $\mathcal{H} \cong \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ (or, equivalently, the projectivised counterpart $\mathbb{P}^{2^n-1} \cong \mathbb{P}^1 \otimes \ldots \otimes \mathbb{P}^1$).

As, clearly, there is an infinity of distinct states, a clustering or classification problem of this states arises naturally as a means to obtain relational information on those.

Out of many possible approaches to this problem, one which is prominent in the literature is to group together those states which can be transformed between them (in the sense that the probability of getting from the former state to the latter is not null). As we have discussed, a physical transformation between states is modelled as a linear operator, onto which we will force maximum rank as else the resulting system would not be of the required dimension.

We can hence define SLOCC (or Stochastic Local Operations and Classical Comunication) equivalence between states this way.

Definition 4.1. Let $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$ be two conjoint states of n particles. $|\psi_1\rangle$ and $|\psi_2\rangle$ are said to be *SLOCC equivalent* if there exists a rank 2^n linear operator $M = M_1 \otimes \ldots \otimes M_n \in GL(\mathcal{H})$ such that $|\psi_1\rangle = M |\psi_2\rangle$.

We note that, from the fact that those linear operators have a representation (given a basis) as an invertible matrix, and that composition of operators respects maximality of rank and linearity, that SLOCC equivalence is indeed an equivalence relation. The definition of SLOCC-equivalence and its study on particular cases has been done in [DVC00] and [VDMV02], among others.

The entanglement for two particles was studied in section 2.2. Now we will revisit this topic and will use the matematical tools provided in section 3 to provide the classification of SLOCC classes for two qubits. This will serve as a foundation and motivation to work towards a general classification in the following section.

4.2 Classification for two particles

In the two particle case, the projective version of our state space is identified with $\mathbb{P}^3 \cong \mathbb{P}^1 \otimes \mathbb{P}^1$. Our problem is therefore the classification of points $[x_0 : x_1 : y_0 : y_1]$ by orbits of projectivities in $\mathbb{P}^3$. We need to decide, the, when, for states $|\psi_1\rangle, |\psi_2\rangle$ there exists an invertible matrix $M = M_1 \otimes M_2$ such that $|\psi_1\rangle = M |\psi_2\rangle$.

By means of the Schmidt decomposition of Proposition 3.14, one can always, via transformation by an unitary matrix, transform a state $|\psi\rangle$ onto $a|00\rangle + b|11\rangle$ or, in projective notation, a point of the form $[a : 0 : 0 : b]$

Recalling what was seen in section 2.2, the states

$$\begin{cases} |Sep\rangle = & |00\rangle \\ |EPS\rangle = & |00\rangle + |11\rangle \end{cases}$$

were introduced. Using projective notation,

$$\begin{cases} [Sep] = & [1 : 0 : 0 : 0] \\ [EPS] = & [1 : 0 : 0 : 1] \end{cases}$$

Another distinction was made: those who belonged to the Segre variety Σ, defined by the equation $x_0y_1 - x_1y_0 = 0$, and those who did not. Having now the definition of SLOCC equivalence, we can prove

Proposition 4.2. *Two qubit SLOCC Classification*

1. All 2-qubit states in the Segre variety are SLOCC-equivalent to [Sep]

2. All 2-qubit states not in the Segre variety are SLOCC-equivalent to [EPS]

Proof. To start with, we may assume $[\psi] = (a : 0 : 0 : b)$ for all our states, for $a, b \in \mathbb{C}$. In this notation, $v \in \Sigma \iff ab - 0 = 0 \iff a = 0$ or $b = 0$. As $(a : 0 : 0 : 0) = (a' : 0 : 0 : 0)$, we only need to prove that $[\psi_1] = (a : 0 : 0 : 0)$ is SLOCC equivalent to $[\psi_2] = (0 : 0 : 0 : b)$. We see then that

$$\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{b}{a} \\ 0 & 0 & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & 0 & 0 \\ \frac{b}{a} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix} = (\begin{pmatrix} 0 & \frac{b}{a} \\ \frac{b}{a} & 0 \end{pmatrix} \otimes Id) \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix}$$

Proving the first part.

For the second part, we may assume that we have states

$$[\psi_1] = \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} \; ; \; [\psi_2] = \begin{pmatrix} a' \\ 0 \\ 0 \\ b' \end{pmatrix}$$

and, clearly,

$$\begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{a'} & 0 & 0 & 0 \\ 0 & \frac{a}{a'} & 0 & 0 \\ 0 & 0 & \frac{b}{b'} & 0 \\ 0 & 0 & 0 & \frac{b}{b'} \end{pmatrix} \begin{pmatrix} a' \\ 0 \\ 0 \\ b' \end{pmatrix}$$

Where the matrix is by non-nullity of the coefficients of maximum rank, and moreover

$$\begin{pmatrix} \frac{a}{a'} & 0 & 0 & 0 \\ 0 & \frac{a}{a'} & 0 & 0 \\ 0 & 0 & \frac{b}{b'} & 0 \\ 0 & 0 & 0 & \frac{b}{b'} \end{pmatrix} = \begin{pmatrix} \frac{a}{a'} & 0 \\ 0 & \frac{b}{b'} \end{pmatrix} \otimes Id$$

hence proving the proposition. □

As a conclusion, we have seen that there are only 2 SLOCC classes for two qubits: the class of separated states, and the class of entangled states.

4.3 General SLOCC classification

We move now onto the terrain of higher dimensions. We would like to use the same decomposition argument, but now the thing gets tricky as there is not only one Segre variety now, but a set of them, each coming from one way of constructing the Segre embedding. For instance, if we have 3 qubits, the order in which the tensor product is taken produces different varieties, as the following diagram shows:

$$\begin{tikzcd}
 & (\mathbb{P}_A \otimes \mathbb{P}_B) \times \mathbb{P}_C \arrow[dr] & \\
\mathbb{P}_A \times \mathbb{P}_B \times \mathbb{P}_C \arrow[ur] \arrow[r] \arrow[dr] & (\mathbb{P}_A \otimes \mathbb{P}_C) \times \mathbb{P}_B \arrow[r] & \mathbb{P}_A \otimes \mathbb{P}_B \otimes \mathbb{P}_C \\
 & (\mathbb{P}_B \otimes \mathbb{P}_C) \times \mathbb{P}_A \arrow[ur] &
\end{tikzcd}$$

As we know, this diagram commutes, but each branch will yield a different decomposition. As now states may not only be separated or entangled, but also q-partite, it shall prove useful to consider all branches. We recall the definition 3.11, and we will try to connect it with the branches of the diagram.

The arrows in said diagram represent the Segre embeddings. By definition, a n-qubit will be entangled if it is not on any of the Segre varieties arising from the arrows on the rightmost step of the diagram. Any state in any of this will be therefore be separated or q-partite. Those, in turn may be in other Segre varieties or, in the case when it is separated, in the general Segre variety $\Sigma_{1,\cdots,1}$.

One may come up the idea that different branches (i.e. the belonging of the state to different Segre varieties) may yield different SLOCC classes. We prove via the following proposition that this is indeed true.

Proposition 4.3. A necessary condition for q-partite n-qubit states $[\psi_1], [\psi_2]$ to be SLOCC-equivalent is that they are SLOCC-equivalent as $(n-k)$-qubits in all branches where they are defined.

Proof. Lets assume that $[\psi_1], [\psi_2] \in (\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_k) \times \mathbb{P}_{k+1} \times \cdots \times \mathbb{P}_n$ are SLOCC-equivalent, and let $[\psi_1'], [\psi_2']$ denote their preimages in $\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_k$. Lets assume $[\psi_1'], [\psi_2']$ not SLOCC-equivalent.

Let M be a SLOCC linear operator such that $[\psi_1] = M[\psi_2]$. By the definition of SLOCC-equivalence, $M = M_1 \otimes \ldots \otimes M_{n-1} \otimes M_n$, for $M_i \in \mathbb{P}_i$. Considering the branch

$$\mathbb{P}_1 \times \cdots \times \mathbb{P}_{n-1} \times \mathbb{P}_n \longrightarrow (\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_k) \times \mathbb{P}_{k+1} \times \cdots \times \mathbb{P}_n \longrightarrow \mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_{n-1} \otimes \mathbb{P}_n$$

Where φ denotes the Segre embedding. We consider how our operator is embedded through the diagram and it is clear that

$$\varphi^{-1}(M[\psi_2]) = ((M_1 \otimes \ldots \otimes M_k) \times M_{k+1} \times \cdots \times M_n)[\psi_2] = (M_1 \otimes \ldots \otimes M_k)[\psi_2'] \times (M_{k+1} \times \cdots \times M_n)[\psi_2'']$$

And therefore in $\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_k$;

$$[\psi_1'] = (M_1 \otimes \ldots \otimes M_k)[\psi_2']$$

contradicting our initial statement, and therefore proving our proposition. □

As a corollary, we obtain

Corollary 4.4. All n-qubit separated states are SLOCC-equivalent

Moreover, not only the different branches produce different classes, in the case of q-partite estates. but the number of such classes is determined by the number of classes of each of the entangled components.

Proposition 4.5. Let $\mathbb{P}^{2^{n+m}-1}$ be the set of states for $(n+m)$-qudits, and $[\psi]$ be a (n+m)-qudit such that its first n qubits, denoted by S_n, are entangled, the last m qubits S_m are entangled, and both subsets are separated. Then, if S_n has χ_n possible SLOCC states, and S_m has χ_m, the set of possible SLOCC classes for $[\psi]$ has order $\chi_n \chi_m$.

Proof. Put $[\psi] = [\psi_N] \otimes [\psi_M]$ Let $M^j = \bigotimes M_i^j$ denote the SLOCC linear operators for ψ_N, and $N^k = \bigotimes N_i^k$ the ones for ψ_M. It is clear then that $\{L^{jk} = M^j \otimes N^k\}$ is a set of linear operators for SLOCC equivalence in our space that has cardinality $\chi_n \chi_m$.

Moreover, if we had an operator not in this set, it has to be of the form

$$M = (M_1 \otimes \ldots \otimes M_n) \otimes (M_{n+1} \otimes \ldots \otimes M_{n+m})$$

And, if we consider the branch

$$\mathbb{P}_1 \times \cdots \times \mathbb{P}_{n+m} \longrightarrow (\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n) \times (\mathbb{P}_{n+1} \otimes \cdots \otimes \mathbb{P}_{n+m}) \longrightarrow$$

$$\longrightarrow \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{n+m}$$

we will eventually have a preimage of M by the Segre embedding corresponding to the entanglement state of the first n particles, and hence $(M_1 \otimes \ldots \otimes M_n)$ corresponds to one of the M^j. Using a symmetric argument on the second part, we obtain that M must equal one of the L^{jk}, hereby proving the result. □

Corollary 4.6. Let $[\psi] = [\psi_1] \otimes \ldots \otimes [\psi_s]$ be a q-partite state, with each $[\psi_i]$ being entangled and consisting of k_i qubits. Let $\chi(k)$ denote the number of SLOCC classes for entangled k-qubit states. Then, the number of classes $[\psi]$ may belong to is given by

$$\chi = \prod_{i=1}^{s} \chi(k_i)$$

Therefore, the results proven up to this point characterize the SLOCC classes by they q-partite composition. The only work remaining is then to classify and quantily how many classes are in each of these clusters. For this purpose, the tensor rank invariant defined in 3.18 will prove useful. We see, on the one hand,

Proposition 4.7. The tensor rank is SLOCC-invariant

Proof. By 3.19 we saw that we can characterise the extended partial trace by the equation

$$\rho_B(M \otimes N) = MTr(N)$$

in the case where the operator can be decomposed as $R \otimes S$. It is clear by definition that all SLOCC-equivalence operators are of this form, and therefore the tensor rank is a SLOCC-invariant. □

And on the other direction,

Proposition 4.8. If two entangled states have the same tensor rank, they are SLOCC-equivalent.

Proof. Let $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H} := \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$, and

$$\begin{aligned} |\psi_1\rangle = &\quad v_{11} \otimes \ldots \otimes v_{1n} + \cdots + v_{n1} \otimes \ldots \otimes v_{nn} \\ |\psi_2\rangle = &\quad w_{11} \otimes \ldots \otimes w_{1n} + \cdots + w_{n1} \otimes \ldots \otimes w_{nn} \end{aligned}$$

be minimal decompositions for $|\psi_1\rangle$,$|\psi_2\rangle$ in adequate basis. Such basis fix underlying bases $\{v_ij\}$ and $\{w_ij\}$ in each of the spaces $\mathcal{H}_{A_j}$. We define the matrices C_j as the change of basis matrix from $\{v_ij\}$ to $\{w_ij\}$ in each of such spaces. Therefore the matrices C_j are defined by being the ones satifying the equation

$$C_j v_{ij} = w_{ij}$$

We define now

$$C = C_1 \otimes \ldots \otimes C_n$$

We want to prove that

$$C\,|\psi_1\rangle = |\psi_2\rangle \iff C_1 \otimes \ldots \otimes C_n\,|\psi_1\rangle = |\psi_2\rangle$$

Rewriting the left side the above equation yields

$$(C_1 \otimes \ldots \otimes C_n)(v_{11} \otimes \ldots \otimes v_{1n} + \cdots + v_{n1 \otimes \ldots \otimes v_{nn}}) =$$

$$= C_1 \otimes \ldots \otimes C_n v_{11} \otimes \ldots \otimes v_{1n} + \cdots + C_1 \otimes \ldots \otimes C_n v_{n1} \otimes \ldots \otimes v_{nn} =$$

$$= \textstyle\sum_{i=1}^n C_1 \otimes \ldots \otimes C_n v_{i1} \otimes \ldots \otimes v_{in} = \sum_{i=1}^n C_1 v_{i1} \otimes \ldots \otimes C_n v_{in}.$$

By the definition of the matrices C_j

$$\sum_{i=1}^n C_1 v_{i1} \otimes \ldots \otimes C_n v_{in} = \sum_{i=1}^n w_{i1} \otimes \ldots \otimes w_{in} = |\psi_2\rangle$$

Hence, in conclusion

$$C\,|\psi_1\rangle = |\psi_2\rangle$$

Proving that $|\psi_1\rangle$ and $|\psi_2\rangle$ are SLOCC-equivalent, and hence proving our proposition. □

Given all this results, we can finally state and prove the general SLOCC classification theorem:

Theorem 4.9. [SLOCC Classification] Two n-qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$ are SLOCC-equivalent if and only if the following two conditions are satisfied:

1. If $|\psi_1\rangle = |\psi_{11}\rangle \otimes \ldots \otimes |\psi_{1s}\rangle$ is s-partite and $|\psi_{1i}\rangle$ is a state of k_i qubits, then $|\psi_2\rangle = |\psi_{21}\rangle \otimes \ldots \otimes |\psi_{2s}\rangle$ is also s-partite and $|\psi_{2i}\rangle$ is also a state of k_i qubits.

2. For each i, $|\psi_{1i}\rangle$ and $|\psi_{2i}\rangle$ have the same tensor rank as states of k_i qubits.

Proof. Part 1 follows from Proposition 4.3, while part 2 is a consequence of Propositions 4.7 and 4.8. $\square$

4.3.1 Classification for three particles

As a practical example of the theory developped in this section, we will classify the possible SLOCC states for the 3-qubit case. This classification was done in the famous [DVC00] paper, and we will use the SLOCC classification theorem 4.9 to recover the same results.

First we will draw our attention to the possible branches for the 1 and 2-partite cases. We denote the spaces for our 3 qubits by $\mathbb{P}^1_A, \mathbb{P}^1_B, \mathbb{P}^1_C$, and recall the diagram

$$\begin{tikzcd}
 & (\mathbb{P}^1_A \otimes \mathbb{P}^1_B) \times \mathbb{P}^1_C \arrow[dr] & \\
\mathbb{P}^1_A \times \mathbb{P}^1_B \times \mathbb{P}^1_C \arrow[ur] \arrow[r] \arrow[dr] & (\mathbb{P}^1_A \otimes \mathbb{P}^1_C) \times \mathbb{P}^1_B \arrow[r] & \mathbb{P}^1_A \otimes \mathbb{P}^1_B \otimes \mathbb{P}^1_C \\
 & (\mathbb{P}^1_B \otimes \mathbb{P}^1_C) \times \mathbb{P}^1_A \arrow[ur] &
\end{tikzcd}$$

Using Proposition 4.3 we can distinguish the following entanglement situations, each of which yield distinct SLOCC classes.

1. A, B and C are separated

2. A and B are entangled , C is not

3. A and C are entangled , B is not

4. B and C are entangled , A is not

5. A, B and C are entangled

Our aim now is to compute how many classes are in each of this cases.

For the first one, we already saw (4.4) that all separated states are equivalent. For cases 2-4, in which one of the particles is separated and the other two entangled, we already saw in section 4.2 that those two particles can be entangled in a single (SLOCC) way. Moreover, Proposition 4.5 assures that this uniqueness persists when going into the 3-particle setting.

We have then a single SLOCC class for each of the cases 1-4. In order to study case 5, we define

Definition 4.10. The 3-qubit states $|W\rangle$ and $|GHZ\rangle$ are defined as

$$\begin{aligned} |GHZ\rangle :=& \ \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ |W\rangle :=& \ \tfrac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \end{aligned}$$

In projective notation, they read

$$\begin{aligned} [GHZ] =& \ [1:0:0:0:0:0:0:1] \\ [W] =& \ [0:1:1:0:1:0:0:0] \end{aligned}$$

This states are presented in the ket notation as reported in [DVC00]. The name GHZ follows historical conventions, corresponging to *Greenberger–Horne–Zeilinger*, in honor of the first physicists to study it (see [GHZ89]). We aim to see, via the following propositions, that these states belong to distinct SLOCC classes while being both entangled:

Proposition 4.11. Both the states $|W\rangle$ and $|GHZ\rangle$ are entangled.

Proof. We will consider our factorization in the branch

$$\mathbb{P}_A \times \mathbb{P}_B \times \mathbb{P}_C \longrightarrow (\mathbb{P}_A \otimes \mathbb{P}_B) \times \mathbb{P}_C \longrightarrow \mathbb{P}_A \otimes \mathbb{P}_B \otimes \mathbb{P}_C$$

as the other cases are symmetric (as is clear by the definition of our states). We shall simply prove that the qudits are not part of the image of the Segre embedding resulting from the last step of the branch (and are hence entangled in that state) In order to show that, our states must not be in the null set of the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \end{pmatrix}$$

which correspond to

$$\begin{pmatrix} x_0 & x_1 \\ x_4 & x_5 \end{pmatrix}; \begin{pmatrix} x_1 & x_2 \\ x_5 & x_6 \end{pmatrix}; \begin{pmatrix} x_2 & x_3 \\ x_6 & x_7 \end{pmatrix}; \begin{pmatrix} x_0 & x_2 \\ x_4 & x_5 \end{pmatrix}; \begin{pmatrix} x_1 & x_3 \\ x_5 & x_7 \end{pmatrix}; \begin{pmatrix} x_0 & x_3 \\ x_4 & x_7 \end{pmatrix}$$

Hence the result follows by a simple polynomial evaluation. □

Proposition 4.12. $|W\rangle$ has tensor rank 3, while the tensor rank of $|GHZ\rangle$ is 2.

Proof. Being $|W\rangle$ and $|GHZ\rangle$ states of 3-qubits, their tensor rank will be at most 3 by Proposition 3.15. Moreover, it cannot by one by Proposition 3.20.

The rank of $|GHZ\rangle$ is at most 2, as it is defined by a sum of two simple tensors, and as it cannot be one we conclude that it is indeed 2.

The rank of $|W\rangle$ is at most 3, and to prove that it is indeed 3 it sufficces to see that $|W\rangle$ cannot be expressed as the sum of two simple tensors. Suppose that, in a certain basis,

$$|W\rangle = a(v_1 \otimes v_2 \otimes v_3) + b(w_1 \otimes w_2 \otimes w_3) \tag{4}$$

Computing the partial trace of W respect to the first qubit yields

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Which is clearly a rank 2 matrix. This implies that all decompositions

$$W = a(v_A \otimes v_{BC}) + b(w_A \otimes w_{BC})$$

where $v_A, w_A \in \mathbb{P}_A$, $v_{BC}, w_{BC} \in \mathbb{P}_{BC}$ have rank 2 (i.e. $a, b \neq 0$).

The only way we could have a decomposition of the form 4 is in the case where both v_{BC} and w_{BC} are simple tensors. This would imply, however, that both states belong to the Segre variety Σ_{BC}, and therefore W would also belong to this Segre variety, making $|W\rangle$ a 2-partite state, which leads to a contradiction. □

By means of 4.9, as we have obtained two entangled states with distinct tensor rank, they will result in two distinct maximally entangled SLOCC classes. We prove now

Corollary 4.13. All entangled 3-qubit states are equivalent to either $|W\rangle$ or $|GHZ\rangle$.

Proof. As all entangled states will have rank 2 or 3 (rank-1 states are separated, and we saw in 3.15 that the rank of a 3-qubit is at most 3), the statement follows by 4.9 □

Finally, we state a representative of all SLOCC 3-qubit classes:

Class	*Abreviature*	*Representatives: Ket*	*Projective*
Separated State	(A,B,C)	$\|000\rangle$	[1:0:0:0:0:0:0:0]
A separated, B-C entangled	(A,BC)	$\|000\rangle + \|011\rangle$	[1:0:0:1:0:0:0:0]
B separated, A-C entangled	(AC,B)	$\|010\rangle + \|101\rangle$	[0:0:1:0:0:1:0:0]
C separated, A-B entangled	(AB,C)	$\|000\rangle + \|110\rangle$	[1:0:0:0:0:1:0:0]
GHZ class	(GHZ)	$\|000\rangle + \|111\rangle$	[1:0:0:0:0:0:0:1]
W class	(W)	$\|001\rangle + \|010\rangle + \|100\rangle$	[0:1:1:0:1:0:0:0]

4.4 Bounds on the number of SLOCC classes

Our aim now, to conclude the section, will be directed at giving bounds on the number of distinct SLOCC classes of n-qubits for each n. Proposition 4.5 allows us to calculate this number in function of the number for a smaller n in the case of q-partite states ($q \geq 2$). Therefore, we can generalize it as

Proposition 4.14. Let $n \in \mathbb{N}$. Let $r(n)$ be the maximum tensor rank for n-qubits. Then, an upper bound for the number of SLOCC classes for n-qubits is

$$\chi(n) = (r(n) - 1) + \sum_{i_1 + \cdots + i_k = n; i_1 \leq \cdots \leq i_k} \prod_{j=1}^{k} (r(i_j) - 1)$$

The proof is direct from 4.5, and the fact that separated states have rank 1.

Finally, we present two upper bounds to the tensor rank of a n-qubit system.

Proposition 4.15 (First Upper Bound for r(n))**.** Let $\mathbb{P}^{2^n-1}$ be the space of states n qubits. Then, an upper bound for r(n) is $2^{n-1}-1$

Proof. For this proof, we will generalise the proof given in Proposition 3.15, which gives an upper bound for the tensor rank of 3 qubits.

To start, we may consider $|\psi\rangle \in \mathbb{C}^{2^n}$. Let $|\psi\rangle$ have coordinates $(z_0, \dots z_{2^n-1})$. Using binary representation, we will present the coordinate z_i in the binary expression of i. Let then consider the 2×2 matrices

$$T_{d_1 \dots d_{n-2}} = (z_{d_1 \dots d_{n-2} ij})$$

where the coordinates z_i are distributed in such way that all coordinates except for the last two determine which matrix they belong to, and the last two determine the row and column.

Therefore, we group the coordinates of $|\psi\rangle$ into 2×2 matrices, by grouping the indexes that, in their binary expression, differ only on the last two indices.

We then follow the same procedure as in 3.15, applying a linear transformation which nullifies the rank of the first matrix, and using that we can separate the coordinates two-foldly (i.e. using that $(a, b, c, d) = (1, 0) \otimes (a, b) + (0, 1) \otimes (c, d)$), and that the first matrix $T_{0\dots0}$ has rank one, we obtain the result. □

We note that naively separating the coordinates two-foldly gives an upper bound of 2^{n-1}, and therefore this proposition does not yield a massive improvement. We can, however, refine the result in the odd case by using the same idea:

Proposition 4.16 (Second Upper Bound for r(n))**.** Let $\mathbb{P}^{2^n-1}$ be the space of states for n qubits, where n is an odd number. Then, an upper bound for r(n) is $2^{n-2} + \frac{n^2-1}{4} - 1$

Proof. The proof of this result lies in a similar idea of the one for the first upper bound. However, the matrices now defined are only two, being

$$T_0 = (z_{0i_1 \dots i_n}), T_1 = (z_{1i_1 \dots i_n})$$

which have dimension $2^{n-1} \times 2^{n-1}$. We use a linear transformation as before, we impose that the rank of the first matrix shall not be maximal, and diagonalise the system. Using the two-fold coordinate argument we obtain 2^{n-2} terms for the second matrix, while, for the first one, a combinatorial argument shows the number to be

$$2(1 + \dots + \frac{n-1}{2}) - 1 = \frac{n^2-1}{4} - 1$$

obtaining the desired bound. □

For the cases of 3 to 10 qubits, the bounds read as

Number	First Bound	Second Bound
3	3	3
4	7	-
5	15	13
6	31	-
7	63	43
8	127	-
9	255	147
10	511	-

To end the section, we compare the bounds obtained to the study made in [VDMV02] of the SLOCC classes of four qubits. In that paper, the states are transformed into an unique expression of at most 8 nonzero coefficients, which is analogous to the technique presented in this section. As the tecniques used in classification are not related to the tensor rank, no further minimization of the number of coordinates is attempted there. However, the nine classes present in the total, which include also nonentangled states, are consistent with an upper bound of seven or lower.

5 Applications

In this section we review some possible applications of the SLOCC Classification Theorem 4.9 and the mathematical tools of Section 3. In Section 4.4 we look into the notion of the persistency of entanglement, which states the number of qubits that must be removed from a state to break the entanglement. Section 5.2 takes a further look into the quantum teleportation algorithm presented in Section 2.4, and explores how the knowledge of the SLOCC classes may contribute to a generalisation of the algorithm to the n qubit case, as well as some possible implications this may have regarding quantum information.

5.1 Persistency of Entanglement

A notion closely knit to how particles are entangled is the persistency of entanglement. Roughly speaking, it states how many particles must be removed from a system in order for it to become completely disentangled (see pag. 226 of [RP14]).

Before being able to define mathematically what this means, we must briefly stop onto defining what it entails to remove a particle from a system.

Example 5.1. *Take a system of 2 qubits in the state*

$$|EPS\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Our objective is to remove the first qubit, making the system fall to a single-qubit system. The ket notation ilustrates that $|EPS\rangle$ is in a state of superposition between $|00\rangle$ and $|11\rangle$. As the second index relates to the second qubit, which is the one that will remain in the system, we would expect to obtain a state which is either in a state $|0\rangle$ or a state $|1\rangle$. Therefore, as the first qubit will either take the value 0 *or* 1*, in the case where it collapses to 0, the second qubit takes the value $|0\rangle$, and in the case where it collapses to 1, the second qubit will be at state $|1\rangle$.*

We note that, in this example, the system becomes disentangled (which was certain to happen, as all 1-qubit systems are disentangled). This motivates the definition:

Definition 5.1. Let $|\psi\rangle$ be an entangled state of n qubits. We will say that $|\psi\rangle$ has *persistency* p if and only if p is the minimal amount of qubits that need to be removed from $|\psi\rangle$ in order for it to become separated.

As an example, we will compute the persistency for the entangled 3-qubit states.

Example 5.2. *Let us consider the states*

$$\begin{aligned} |GHZ\rangle &= \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ |W\rangle &= \tfrac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \end{aligned}$$

We observe that, as they are symmetrical on all qubits, it makes no difference which qubit is first removed. We will study then what happens when the first qubit is removed in each case.

In the case of $|GHZ\rangle$, the system will fall onto $|00\rangle$ or $|11\rangle$ respectively, depending on whether the qubit takes value 0 *or* 1*. We see that both states are separated, and thus the persistency of the state is 1.*

In the case of $|W\rangle$, if the qubit takes value 0*, the system will fall into the state $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$, which is an entangled state. Therefore, we will have to remove 2 qubits to make the system separated, and the persistency of the state is 2.*

We would like now to connect this notion to the tensor rank. By using that the tensor rank is the number of terms of the minimal representation of the state in the form

$$|\psi\rangle = \sum_{i=1}^{k} z_i \, |t_{i1} \dots t_{in}\rangle \tag{5}$$

where $t_{ij} \in \{0, 1\}$, we can prove:

Proposition 5.2. Let $|\psi\rangle$ be a state of n qubits of tensor rank k. The persistency of $|\psi\rangle$ is at most $k-1$.

Proof. In order for $|\psi\rangle$ to be entangled, the expression (5) of $|\psi\rangle$ must have, on the first qubit, both terms with $|0\rangle$ and $|1\rangle$. Therefore, removing the qubit will yield an expression of at most $k-1$ terms. Repeating $k-1$ times this process, we arrive at a tensor rank 1 state, which is separated by Proposition 3.20. □

Therefore, the tensor rank can be used to give bounds on how resistent the entanglement is to the removal of part of the system. Certain classes of highly-persistent states have been studied in [BR01].

5.2 Revisiting quantum teleportation

Many quantum algorithms for two qubits make use of the fact that the entangled $|EPS\rangle$ state can be transformed into any other two-qubit entangled state by a local operation, as was seen in Section 4.2. As a consequence of the SLOCC classification theorem, we can know into which states a particular quantum state may transform, which may come useful in generalising some of this algorithms to a more general case.

We fill focus on the teleportation algorithm described in Section 2.4. We showed that, given a qubit in state $|\psi\rangle = z_0 |0\rangle + z_1 |1\rangle$, Alice could send this state to Bob by using a pre-shared entangled $|EPS\rangle$ state. The algorithm relies on being able to rewrite the 2-partite state $|\psi\rangle \otimes |EPS\rangle$ as a combination of the product between $|\psi\rangle$ and the states of the Bell basis, and making it collapse to a state $\pm z_0 |0\rangle \pm z_1 |1\rangle$, which can be transformed back into the original one with a local operation on it. This final trick relies on being able to transform this state into the original one.

We could study, then, what would happen if, instead of a single qubit, Alice and Bob wanted to teleport a state of n qubits. Let us take then a state

$$|\psi\rangle = \sum_{i=1}^{k} z_i |t_{i1} \dots t_{in}\rangle$$

where $t_{ij} \in \{0,1\}$, and suppose that the tensor rank of $|psi\rangle$ is k. If we mirror the algorithm in Section 2.4, we can take

$$|\psi\rangle \otimes |EPS\rangle = (\sum_{i=1}^{k} z_i |t_{i1} \dots t_{in}\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\sum_{i=1}^{k} z_i |t_{i1} \dots t_{in}00\rangle + \sum_{i=1}^{k} z_i |t_{i1} \dots v_{in}11\rangle)$$

If, following the notation in Section 2.4, we use that

$$\begin{aligned}
|00\rangle = &\ (1\ 0\ 0\ 0) = & \tfrac{1}{2}((1\ 0\ 0\ 1) + (1\ 0\ 0\ \ -1)) = &\ \sqrt{2}(u_0 + u_1) \\
|01\rangle = &\ (0\ 1\ 0\ 0) = & \tfrac{1}{2}((0\ 1\ 1\ 0) + (0\ 1\ \ -1\ 0)) = &\ \sqrt{2}(u_2 + u_3) \\
|10\rangle = &\ (0\ 0\ 1\ 0) = & \tfrac{1}{2}((0\ 1\ 1\ 0) - (0\ 1\ \ -1\ 0)) = &\ \sqrt{2}(u_2 - u_3) \\
|11\rangle = &\ (0\ 0\ 0\ 1) = & \tfrac{1}{2}((1\ 0\ 0\ 1) - (1\ 0\ 0\ \ -1)) = &\ \sqrt{2}(u_0 - u_1)
\end{aligned} \tag{6}$$

and apply it to factor the last qubit in $|\psi\rangle$ and the first in $|EPS\rangle$, we will obtain an expression of the form

$$u_0 |\psi_0\rangle + u_1 |\psi_1\rangle + u_2 |\psi_2\rangle + u_2 |\psi_3\rangle$$

where $|\psi_i\rangle$ are states of n qubits. If a partial measurment is applied on the system, it will fall back to one of the states $|\psi_i\rangle$. Now, if we can prove that, like the single-qubit case, those states can be transformed between them, the algorithm will be valid.

We will prove, in order to obtain a similar result, that the four states have the same tensor rank. To see this, we will study what form do the states take. In state $|\psi_0\rangle$, we have the terms that had a $|0\rangle$ in the last qubit, with this qubit replaced by a $|0\rangle$, and the ones that had a $|1\rangle$, replaced this time with a $|1\rangle$. Hence, $|\psi_0\rangle = |\psi\rangle$. On the other cases, one can see that $|0\rangle$ and $|1\rangle$ will be replaced by $\pm|0\rangle$ or $\pm|1\rangle$. This comes from taking a closer look on how the basis transforms using equation (6)

We see thus that the number of terms in each of the $|\psi_i\rangle$ is k. As coordinates of $|\psi_i\rangle$ are permutations of those in $|\psi\rangle$, an expression with less than k terms for $|\psi_i\rangle$ would imply one for $|\psi\rangle$. We can use this to prove that, for all i, the tensor rank of $|\psi_i\rangle$ is k.

We would like now to use Theorem 4.9 to prove that all states are SLOCC-equivalent. This would show that the states *could*, at least, be transformed into each other. There is, however, one remark to be made: because of the coordinate transpositions and qubit manipulations, it may well happen that the qubits that are entangled are not the same. For example, recalling the 3-qubit SLOCC classes of Section 4.3.1, it could happen that one of the states was in class A-BC and other in class AB-C. We cannot, therefore, prove that the states are SLOCC equivalent, though it may be interesting to study whether the condition of being q-partite is maintained through the states $|\psi_i\rangle$, as it would provide a solution practically equal to the one we are seeking (as, in a practical context, it is not critically relevant if the separated qubit is the first or the last, as long as you know which qubit it is).

Instances of concrete algorithms for multiple qubit teleporting have been given, in fact, for two qubits in [PRS+15], and for the $|GHZ\rangle$ and $|W\rangle$ states seen in Section 4.3.1 in [SHCC18].

As an application of this, we will look upon quantum key distribution protocols. The general idea between this protocols is that entanglement can be used for the exchange of secure keys in cryptography. The first proponent of a method was [Eke91], who used Bell's inequalities in order to test the safety of the distribution. This algorithms aim at generating one bit of a secret key from every entangled pair that Alice and Bob share. This takes into account that, on the one hand, all teleportation protocols need a classical communication channel, which is a priori unsafe and may be intercepted by an external auditor, and that, as well as classical algorithms do, quantum computing incurs in systemic errors and losses of information, as described by [Ren13].

With this in mind, we may explore how this possible generalisations of the teleportation algorithms may yield interesting results in the field of key distribution. We may argue that the algorithms described for the teleportation of more than one qubit make the process far more efficient, but we cannot escape the limitations described.

On the other hand, the knowledge of a concise classification of SLOCC classes could help develop a safe key-exchanging system. While very far from the current technological capabilities, as the class of SLOCC equivalent states is the set of those states that may transform into each other, if there was a way to safely measure the SLOCC class of a state, we can assure that throughout the process the states will remain, as long as the number of qubits stays stable, in the same SLOCC class. Thus measuring the SLOCC class of a state could provide, in ideal circumstances, a channel for safe communication. Moreover, as measuring the state would efectively destroy it, it could help with possible external intrusions in the channel.

6 Conclusions

In this monograph we have introduced a classification theorem for the entanglement quantum states. Focusing on the SLOCC classification, we used the projective notion of the Segre variety, along with the vector invariant of the tensor rank, to build a characterisation of equivalence classes of said relation. Yielding a result similar to contemporary research such as [LL12] and [ZZH16], we introduced the capabilities of algebraic geometry in order to move towards a comprehensive classification that leads to a better understanding of how entanglement of quantum states functions.

This two foundational legs of our work, the algebraic projective geometry of the Segre variety and the tensorial algebra of the tensor rank, the Schmidt decomposition and, the ket notation, come together in the main result, Theorem 4.9, and along the applications in Section 5. We have thus focused on the purely mathematical faces of the quantum physics paradigm, hoping to, as a mathematician, be able to bring some new perspectives into the field, as a part of a wave of further mathematical formalisation of quantum physics that has been developing within the last years.

Being a field with many applications, both currently available and subject to the capabilities of technology, especially in the construction of quantum computers, we hope to be able to contribute in laying a solid mathematical foundation for all the real world implementations of the quantum paradigm. In Section 5 we hinted some places of application that may be of interest, and that can directly benefit from being able to understand SLOCC equivalence. Moreover, we hope that further work is able to improve the upper bounds given for the tensor rank in Section 4.4, in order to obtain a concrete number of the number of possible SLOCC classes.

In conclusion, in this work we provided tools to work towards a complete characterisation of the SLOCC equivalence relation, which can lead to a better understanding of the quantum entanglement phenomenon, which is one of the foundational pillars of quantum phyisics. Moreover, we hinted some applications which we think may be of interest, and directly use the concepts introduced in the monograph. We hope that we are able to motivate further work in the area, both to refine the results obtained here as to get an overall better knowledge on the field.

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