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THE BARGAINING PROBLEM: NASH AND OTHER SOLUTIONS

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Abstract

The main objective of this project is to analyze the bargaining problem from a mathematical perspective, as Nash (1950) did. A bargaining problem is a situation where a (finite) set of agents may cooperate to their mutual benefit. They must reach an unanimous agreement. If they do not reach such an agreement, they get the *status quo*, the disagreement outcome. This problem was first analyzed in an axiomatic form by Nash (1950). Our object of study in bargaining theory is to find a bargaining solution and characterize it. We consider a set of axioms, motivated by a particular application, and we identify the bargaining solution that satisfy them.

To this end, we provide some mathematical tools and concepts, mainly separation theorems and some concepts of preferences and utility functions. This is needed for the following chapters. Next, we present the bargaining model of Nash (1950). We prove that the unique solution he proposes of the bargaining problem, the Nash solution, is the one that is characterized by four axioms, which have a nice interpretation. Finally, other bargaining solutions are presented such as the Kalai-Smorodinsky (1975) solution, or other solutions proposed in the literature, which use different sets of axioms.

Resum

L'objectiu principal d'aquest projecte és analitzar el problema de negociació des d'una perspectiva matemàtica, tal com va fer Nash (1950). Un problema de negociació és una situació en què un conjunt (finit) d'agents tenen l'oportunitat de cooperar en el seu benefici mutu. Han d'arribar a un acord unànime. Si no arriben a aquest acord, obtindran la situació de *status quo*, punt de desacord. Aquest problema va ser analitzat per primera vegada en forma axiomàtica per Nash (1950). El nostre objectiu d'estudi en la teoria de negociació és trobar una solució de negociació i caracteritzar-la. Considerem un conjunt d'axiomes, motivats per una aplicació determinada, i identifiquem la solució de negociació que els satisfà.

Amb aquesta finalitat, proporcionem algunes eines i conceptes matemàtics, principalment teoremes de separació i alguns conceptes de preferències i funcions d'utilitat. Això és necessari per als capítols següents. A continuació, presentem el model de negociació de Nash (1950). Demostrem que la solució única que proposa del problema de negociació, la solució de Nash, és la que es caracteritza per quatre axiomes, que tenen una bona interpretació. Finalment, es presenten altres solucions de negociació com la solució Kalai-Smorodinsky (1975) o altres solucions proposades en la literatura, que utilitzen diferents tipus d'axiomes.

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"Reserve your right to think, for even to think wrongly is better than not to think at all."

Hypatia of Alexandria

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Chapter 1

Introduction

One of the most influential paradigms in cooperative game theory is the bargaining problem, in which a group of two or more agents is faced with a set of possible outcomes, any one of which will be the result if it is specified by the unanimous agreement of all the agents. When a unanimous agreement cannot be reached, there is a disagreement outcome or a *status quo*. In order to give room to the agents to reach an agreement, we assume there is a feasible outcome which all agents prefer to the disagreement outcome. Nonetheless, so long as at least two of the agents differ over which outcome is most preferable, there is a need for bargaining over which outcome should be agreed upon. Each agent has veto power to any outcome different than the disagreement point, since unanimity is required for any other result.

The axiomatic approach to the bargaining problem was originated in a fundamental paper by John F. Nash (1950) [24]. He presents a framework which provides an unique possible outcome as the solution of a given bargaining problem. He formulates a list of axioms that a solution should satisfy, and he establishes the existence and uniqueness of a solution satisfying all of these axioms. This solution is called the *Nash solution.* He focus his attention on a two-person bargaining problem, but this solution can easily be extend to the case of a n-person bargaining problem.

Nash approach was in contrast to earlier approaches within the game-theoretic tradition: the von Neumann-Morgenstern (1944) [39] solution to the bargaining problem coincides with Edgeworth's (1881) [9] "contract curve", and is equal to the entire set of individually rational, Pareto Optimal outcomes.[1]

The Nash bargaining model is based on the theory of rational individual choice behavior as initiated by von Neumann and Morgenstern. For situations in which an individual must choose from a set of feasible alternatives, von Neumann and Morgenstern define rational behavior as a behavior that can be modelled with the assumption that the individual has a consistent preference relation over alternatives, and always chooses the alternative with the most preferred feasible. They consider situations in which the feasible alternatives may involve chance events, and show that a simple set of consistency conditions on the preference makes the process rational choice equivalent to picking the alternatives which maximizes the expected value of some real-valued *utility function.*

[1]These terms will be defined in the following chapters.

John Forbes Nash was born on June 13, 1928, in Bluefield, West Virginia. He got the Nobel Memorial Prize in Economic Sciences in 1994, joint with John Harsanyi and Reinhard Selten, “for their pioneering analysis of equilibria in the theory of non-cooperative games”. He was a mathematician and he got the Abel Prize in 2015, joint with Louis Nirenberg, “for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis”. On May 23, 2015, Nash and his wife died in a traffic accident in New Jersey on their way home from the airport, after receiving the Abel Prize. His life was explained in the book *A beautiful mind* of Sylvia Nasar (1998) [23]. It inspired the 2001 film by the same name which won numerous awards, including the Academy Award for Best Picture and Best Adapted Screenplay for 2001 at the 74th Academy Awards.

However, some of the axioms that the Nash Solution must obey have been the object of some criticism since they are not suitable for certain problems. Due to criticism of the axioms, several economists have suggest alternative sets of axioms which leads to different unique solutions of the bargaining problem. So, other solutions were introduced which led to different characterizations and the theory expanded in several directions. Kalai and Smorodinsky (1975) [16] and Kalai (1977) [14] present an alternative axiom which leads to another unique solution to Nash's bargaining problem. These three solutions which are the Nash solution, the Kalai-Smorodinsky solution and the Egalitarian solution, are the classic solutions with characterizations that occupy the center stage in the theory as it stands today. However, there is a large variety of different available bargaining solutions.

An alternative justification for Nash's solution of the bargaining problem (1950) [24] is the Zeuthen-Harsanyi theory of bargaining. The Zeuthen-Harsanyi theory is a non-cooperative negotiation approach that was originally developed by Frederik Zeuthen (1930) [41] and later by John Harsanyi (1956) [12]. Harsanyi showed how the Zeuthen theory expresses the bargaining process as a sequence of moves that eventually converge to the Nash bargaining solution. This Zeuthen-Harsanyi mechanism is based upon the concept of a critical risk ratio. The critical risk ratio measures the probability of defecting or choosing the conflict outcome. Moreover, the agents should satisfy two rules: within a bargaining process, the player with the lower critical risk ratio makes a concession to the player with the higher critical risk ratio such that the inequality is reversed and when the critical risk ratios of both players are equal, both make a concession. Harsanyi shows that if both agents satisfy these two rules, the bargaining process will converge to the Nash solution.

Many economists, as well as other empirical social scientists, will probably find Zeuthen's argument more convincing, since it is based on a rather plausible psychological model of the bargaining process, and, at the same time, may look with some hesitation at Nash's theoretical method, which is based on abstract mathematical postulates whose empirical relevance may be less obvious. But, in reality, the two approaches are complementary. Zeuthen's approach supplements Nash's in at least two important respects. It furnishes a more detailed analysis of the actual bargaining process. Moreover, it explains how the result of the bargaining process depends on their attitudes towards risk-taking a dependence which in Nash's theory must be assumed without the possibility of being explained. On the other hand, due to its higher level of abstraction and mathematical rigor, Nash's theory has the important advantage of complete generality, which assures us that no particular case within the scope of the theory is likely to be overlooked. While Zeuthen's argument in its original form is restricted to symmetric bargaining situations, Nash's theory covers the asymmetric case.

Nowadays, bargaining situations are everywhere, both in the Western and the Eastern hemisphere. It is an important aspect of economic, political and social life. There are bargaining situations with just two agents like when a buyer and a seller have to agree on a price of an object. Also, this situations can be extended with more agents such as the wage negotiation between a trade union and a group of employers, trade agreements between countries or unions (e.g. Spain and China, European Union and United States) and environmental negotiations between developed and less developed countries. These are just a few examples of bargaining that have been important over the years. Thus, it is no surprise that researchers from a wide range of disciplines have studied bargaining theory.

To sum up, the bargaining problem is the choice of a feasible alternatives by a group of agents or nations or associations which often have conflicting preferences in a framework of cooperation. As Kalai (1985) [15, p. 77] said, this "may be viewed as a theory of consensus, because when it is applied it is often assumed that a final choice can be made if and only if every member of the group supports this choice". He continues saying "because this theory deals with the aggregation of peoples' preferences over a set of feasible alternatives, it bears close similarities to theories of social choice and the design of social welfare functions". The final result that the agents (or groups of agents) involved fight for can be achieved by the parties themselves. However, the final result will sometimes be achieved through the mediation of an outside person, an arbitrator.

There is a characteristic that distinguishes the bargaining problem from almost all other approaches to social choice. It is the existence of a disagreement outcome which it comes when the agents involved in bargaining do not reach an agreement. In consequence, the gains that the agents (may) achieve are measured with reference to the disagreement point or *status quo*.

About this work

The project is divided into three different chapters. The Chapter 2, Preliminary notions, presents concepts and mathematical tools needed for the development of this project, mainly the separation theorems and some notions of preferences and utility functions.

The Chapter 3 introduces the bargaining model of Nash (1950) [24]. We prove that the unique solution that he proposes of the bargaining problem, the Nash Solution, is the one that satisfies four axioms. These axioms are Independence of Equivalent Utility Representation, Symmetry, Independence of Irrelevant Alternatives and Pareto Optimality. We also discuss each of these axioms separately.

Finally, the Chapter 4 presents other bargaining solutions to show how rich and varied the class of available solutions is. The Kalai-Smorodinsky solution (1975) [16] is explained in a depth manner. Moreover, we explain in a brief way the Egalitarian (1977) [14], Dictatorial, Discrete Raiffa (1953) [30], Perles-Maschler (1981) [28], Equal Area, Utilitarian and Yu (1973) [40] solution.

Chapter 2

Preliminary notions

In this chapter, we provide the concepts and mathematical tools needed for the development of this work. Some of these preliminaries are well-known for any mathematician, but they allow to fix notation and the results we need. In all cases, we put the proofs of the results or the references where they can be consulted. We will use $\mathbb{R}^n$ as our reference set.

2.1 Convex sets and separation theorems

Convexity is a basic notion in geometry but also is widely used in other areas of mathematics such as calculus of variations, complex analysis, partial differential equations, optimization and so on. For example, a subfield of mathematical optimization is the convex optimization that studies the problem of minimizing convex functions over convex sets. Also, convexity have importance outside mathematics, such as physics, chemistry, economics and other sciences. For a thorough history on the concept of convexity along the years, see Dwilewicz (2009) [8].

The Greeks introduced the concept of convexity with the development of geometry, culminating in Euclid's *Elements*.[1] However, the first rigorous definition of convexity was given by Archimedes[2] in his treatise *On the sphere and cylinder*. From Archimedes to the end of the 19th century there were no major contributions on convexity. Due to the growing interest in convex geometry and thanks to mathematicians like Brunn[3] or Minkowski,[4] great advances were made in this field. In 1934 the first systematic book on convexity *Theorie der konvexen Körper* [2] was published by Bonnesen[5] and Fenchel.[6] Thereafter, numerous important discoveries of convex sets were made that

[1]Euclid of Alexandria (c. 325 BC–c. 265 BC) was a Greek mathematician known as "the father of geometry".

[2]Archimedes of Syracuse (c. 287 BC–c. 212 BC) was a Greek mathematician, physicist, engineer, inventor and astronomer.

[3]Karl Hermann Brunn (1862–1939) was a German mathematician known for his work in convex geometry and in knot theory.

[4]Hermann Minkowski (1864–1909) was a German mathematician who created and developed the geometry of numbers which uses geometry for the study of algebraic numbers. He focused his studies on number theory and mathematical physics and his works were used to found the theory of relativity.

[5]Tommy Bonnesen (1873–1935) was a Danish mathematician who did research on convex geometry.

[6]Moritz Werner Fenchel (1905–1988) was a German mathematician known for his contributions to geometry and to optimization theory.

have consolidated everything we know so far on this concept.

Intuitively, if we think on $\mathbb{R}^2$ or $\mathbb{R}^3$, or any Euclidean space, a convex set is a set of elements from an Euclidean space such that all the points on the segment between any two points of the set are also contained in the set (see Figure 2.1). We state next the definition of a convex set in $\mathbb{R}^n$.

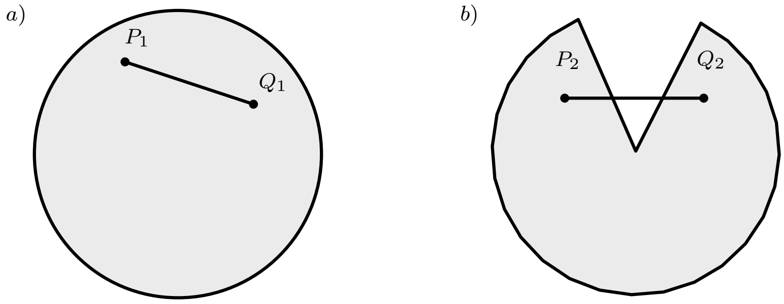

Figure 2.1: a) A convex set b) A non-convex set

Definition 2.1. A set $S \subseteq \mathbb{R}^n$ is called *convex* if $\forall\ x, y \in S$ and $\forall\ \lambda \in [0,1]$, we have $\lambda x + (1-\lambda)\, y \in S$.

An example of convex set is the hyperplane. Let $h \in \mathbb{R}^n$ be a vector with $h \neq 0$ and $\beta \in \mathbb{R}$, the *hyperplane* $H_{h,\beta}$ generated by h and β is the set

$$H_{h,\beta} = \{x \in \mathbb{R}^n \mid h \cdot x = \beta\}\,. \tag{2.1}$$

The *half-space above* $H_{h,\beta}$ is the set $\{x \in \mathbb{R}^n \mid h \cdot x \geq \beta\}$ and *half-space below* $H_{h,\beta}$ is the set $\{x \in \mathbb{R}^n \mid h \cdot x \leq \beta\}$. Also, the half-spaces are convex sets. Figure 2.2 illustrates the hyperplane and half-spaces.

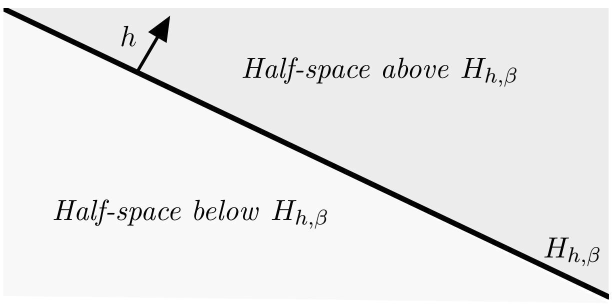

Figure 2.2: Hyperplane and half-spaces

Convex sets have many useful properties. Notice that the empty set and the whole set are convex. Also, a set consisting of one single point is convex too. Observe that the intersection of any number of convex sets is convex, but the union of convex sets does not have to be convex.

A function f from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}$ is called convex (concave) if it is defined on a convex set and the line segment joining any two points on the graph is never below (above) the graph.

Definition 2.2. A function $f : S \to \mathbb{R}$ defined on a convex set $S \subseteq \mathbb{R}^n$ is *convex* on S if

$$f(\lambda x + (1-\lambda)\, y) \leq \lambda f(x) + (1-\lambda)\, f(y), \quad \forall\, x, y \in S \text{ and } \forall\, \lambda \in [0,1]. \tag{2.2}$$

Function f is *concave* if (2.2) holds with $\leq$ replaced by $\geq$.

The function f is convex on S if $-f$ is concave. Notice that for a concave function $f : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$, the set $\{(z,t) \in \mathbb{R}^{n+1} \mid z \in S,\ t \leq f(z)\}$ is a convex set. Also, if f is a convex function then the set $\{(z,t) \in \mathbb{R}^{n+1} \mid z \in S,\ t \geq f(z)\}$ is a convex set. The first set is named as the *hypograph* and the second set as *epigraph*.

For more information on the notion of convexity, check the books of Eggleston (1958) [10], Lay (1992) [18] and Boyd and Vandenberghe (2004) [3].

Separating hyperplane theorems

One of the most important results about convex sets is the separating hyperplane theorem that has different versions. In fact there are several different results, and here we state those we need. The main idea of this theorem is that given a convex closed set S and a point $x \notin S$, one should be able to trace a straight line that separates on one side the point x and on the other side the set S. Figure 2.3 illustrates this result.

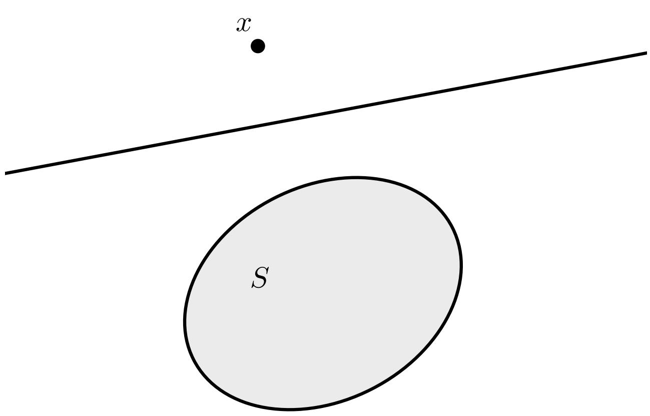

Figure 2.3: Separating hyperplane theorem

In order to prove the separating hyperplane theorem, we need a previous lemma about the distance between a set and an outside point.

Lemma 2.1. *Let $S \subseteq \mathbb{R}^n$ be a compact set not containing the origin. Then, there is a point $x_{min} \in S$ such that $d(x_{min}, 0) = \inf_{x \in S} d(x, 0) > 0$.*

Proof. This Lemma follows by the Weierstrass theorem[7] and the continuity of the distance.[8] □

The compactness of the set is a very important hypothesis for the previous Lemma 2.1. Notice that x_{min} of the Lemma is not unique. For example, if S is a circle of radius 1 then all points in the circle can be taken as x_{min} since all of them are at the same minimum distance, equal to 1.

The first theorem we show is the strict separating hyperplane theorem and our hypotheses are that the set is closed and convex, and it can be separated from a point that is not in that set.

Theorem 2.1 (Strict separating hyperplane theorem). *Let $S \subseteq \mathbb{R}^n$ be a closed convex set and $y \notin S$. Then, there is a hyperplane $H_{h,\beta}$ such that*

$$h \cdot y < \beta < h \cdot x, \quad \forall\, x \in S.$$

Proof. If $S = \emptyset$ then the statement is trivial. Now we assume that $S \neq \emptyset$. We may assume that $y = 0$ by a translation of the coordinates. In order to apply the previous Lemma 2.1 we need to select a compact set. So, let us pick any $x' \in S$ and denote $S' = S \,\cap\, \{x \in S \mid d(x,0) \leq d(x',0)\}$. Notice that S' is a compact set, since it is the intersection of two closed and bounded sets and moreover $0 \notin S'$.

By Lemma 2.1 there is a point $x_{min} \in S'$ such that $d(x_{min}, 0) = \inf_{x\in S'} d(x,0) > 0$. Moreover, because of the definition of S' we have $d(x,0) \geq d(x_{min},0) > 0$ for all $x \in S$. This point is unique, since if there were two different points at the same distance, the midpoint of the segment joining them will be at a lesser distance. Indeed, we have an isosceles triangle.

Define $m \in \mathbb{R}^n$ as the midpoint of the line joining 0 to x_{min}, i.e., $m = x_{min}/2$. Choose $H_{h,\beta}$ to be the hyperplane that goes through m and is perpendicular to the line joining 0 to x_{min}. Therefore, $h \in \mathbb{R}^n$ is the vector x_{min} scaled by $d(x_{min},0)$, i.e., $h = x_{min}/d(x_{min},0)$ and $\beta = h \cdot m$. An illustration is given in Figure 2.4.

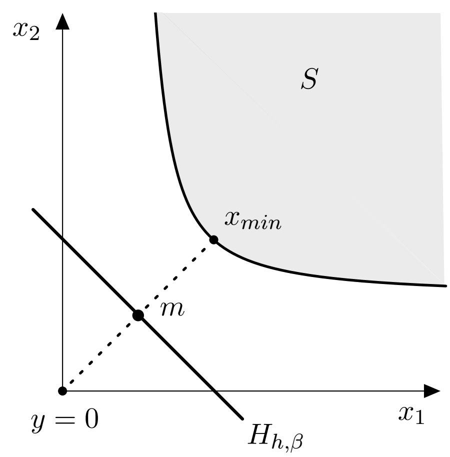

Figure 2.4: Illustration of the proof of Theorem 2.1 when $n = 2$

[7]**Weierstrass theorem or Extreme Value Theorem**: Let S be a compact set and $f : S \to \mathbb{R}$ is continuous function, then f is bounded and there exist $x, y \in S$ such that $f(x) = \sup_{z\in S} f(z)$ and $f(y) = \inf_{z\in S} f(z)$.

[8]The Euclidean distance to the origin is defined as $d(x,0) = \sqrt{x \cdot x}$, $\forall x \in \mathbb{R}^n$. Obviously, it is a continuous function.

Note that

$$h \cdot m = \frac{x_{min}}{d(x_{min}, 0)} \cdot \frac{x_{min}}{2} = \frac{d(x_{min}, 0)}{2} > 0,$$

since $x_{min} \cdot x_{min} = d(x_{min}, 0)^2$.

Now, we show that $y = 0$ is on one side of the hyperplane $H_{h,\beta}$ and x_{min} is on the other side. Indeed, $h \cdot y < h \cdot m < h \cdot x_{min}$ since

$$h \cdot y = 0 < \frac{d(x_{min}, 0)}{2} = h \cdot m < d(x_{min}, 0) = x_{min} \cdot \frac{x_{min}}{d(x_{min}, 0)} = h \cdot x_{min}.$$

Now, we prove that $\beta < h \cdot x$ for all $x \in S$, that is, all $x \in S$ are on the same side of $H_{h,\beta}$ as x_{min}.

Let's pick any $x \in S$. Thanks to the convexity of S we have that for all $\lambda \in (0, 1]$, $\lambda x + (1 - \lambda)\, x_{min} \in S$. Notice that $\lambda x + (1 - \lambda)\, x_{min} = x_{min} + \lambda(x - x_{min})$.

Notice also that from the choice of x_{min} we have $d(x_{min}, 0) \leq d(x_{min} + \lambda(x - x_{min}), 0)$, and then we also have $d(x_{min}, 0)^2 \leq d(x_{min} + \lambda(x - x_{min}), 0)^2$. Therefore,

$$\begin{aligned} d(x_{min}, 0)^2 \leq\ & (x_{min} + \lambda(x - x_{min})) \cdot (x_{min} + \lambda(x - x_{min})) \\ &= x_{min} \cdot x_{min} + 2\lambda x_{min} \cdot (x - x_{min}) + \lambda^2 (x - x_{min}) \cdot (x - x_{min}) \\ &\leq d(x_{min}, 0)^2 + 2\lambda x_{min} \cdot (x - x_{min}) + \lambda^2 d(x - x_{min}, 0)^2. \end{aligned} \tag{2.3}$$

The inequality (2.3) is reduced to

$$0 \leq 2\lambda x_{min} \cdot (x - x_{min}) + \lambda^2 d(x - x_{min}, 0).$$

Since $\lambda \neq 0$ and can be made arbitrarily small we obtain that

$$0 \leq x_{min} \cdot (x - x_{min}), \quad \forall\, x \in S.$$

We can rewrite that last inequality using the fact that $h = x_{min}/d(x_{min}, 0)$ and $x_{min} = 2 \cdot m$ as:

$$0 \leq h \cdot x - 2m \cdot h,$$

which reduces to

$$m \cdot h < 2m \cdot h \leq h \cdot x.$$

Finally, we have seen that $h \cdot y < \beta < h \cdot x, \quad \forall\, x \in S.$ □

The previous Theorem 2.1 says that if we have a closed convex set S and a point y that is not in that set there is a hyperplane that strictly separates the set from the point. Now, we state a separation theorem for a set which is not closed. Consequently, the point y could be a boundary point of the set S. In this last case, the hyperplane is called a *supporting hyperplane* to S at y.

Theorem 2.2 (Weak separating hyperplane theorem). *Let $S \subseteq \mathbb{R}^n$ be a convex set and $y \notin S$. Then, there is a hyperplane $H_{h,\beta}$ such that*

$$h \cdot y \leq \beta \leq h \cdot x,\ \forall\, x \in S.$$

Proof. If $y \notin \overline{S}$, then we can consider S as $\overline{S}$ which is a closed convex set because the closure of a convex set is convex. Now, we can apply the previous Theorem 2.1.

If $y \in \overline{S}$, then y is on the boundary of S, i.e., $y \in \partial S$. We know that there exists a sequence of vectors $(y_k)_{k\in\mathbb{N}}$ outside $\overline{S}$ that converge to y. We can apply our previous Theorem 2.1 to each of these points, and we obtain a sequence of vectors $(h_k)_{k\in\mathbb{N}}$, with $h_k \neq 0$, such that for all $k \in \mathbb{N}$,

$$h_k \cdot y_k < h_k \cdot x, \quad \forall\, x \in \overline{S}.$$

Without loss of generality these vectors can be normalized to one, i.e., $||h_k|| = 1$. These vectors form a sequence on the unit ball, which is a compact set. Therefore, there exists (by the Bolzano-Weierstrass theorem[9]) a convergent subsequence $(h_{k_i})_{i\in\mathbb{N}}$, and let h be its limit, i.e., $\lim_{i\to\infty} h_{k_i} = h$. This vector satisfies

$$h \cdot y = \lim_{i\to\infty} h_{k_i} \cdot y \leq \lim_{i\to\infty} h_{k_i} \cdot x = h \cdot x, \quad \forall\, x \in \overline{S}.$$

Moreover since $||h_k|| = 1$ for all $k \in \mathbb{N}$, we have $||h|| = 1$, and then $h \neq 0$. □

We can extend Theorem 2.2 to two disjoint convex sets. Let S and T be disjoint convex sets. That is, there is a hyperplane that separates S and T, leaving S and T on different sides of it.

Theorem 2.3 (Separating hyperplane theorem). *Let $S, T \subseteq \mathbb{R}^n$ be non-empty, disjoint, convex sets. Then, there is a hyperplane $H_{h,\beta}$ such that*

$$h \cdot y \leq \beta \leq h \cdot x, \ \forall\, x \in S \text{ and } \forall\, y \in T.$$

Proof. Let $R \subseteq \mathbb{R}^n$ be the set of the vectors that are defined as a difference between a vector in S and one in T, i.e.,

$$R = \{z \in \mathbb{R}^n \mid z = x - y,\ x \in S,\ y \in T\}.$$

By the definition of R we have that is convex and since S and T are disjoint we know that $0 \notin R$. Thus, by Theorem 2.2 there is a hyperplane $H_{h,\beta'}$ with $h \in \mathbb{R}^n, h \neq 0$ and $\beta' \in \mathbb{R}$ such that $h \cdot 0 \leq \beta' \leq h \cdot z$, $\forall\, z \in R$.

For any $x \in S$ and $y \in T$ we have $x - y \in R$. Therefore, we can write the following inequality:

$$0 \leq h \cdot x - h \cdot y, \ \forall\, x \in S \text{ and } \forall\, y \in T.$$

In particular, we can rewrite the previous inequality as:

$$h \cdot y \leq \sup_{y' \in T} h \cdot y' \leq \inf_{x' \in S} h \cdot x' \leq h \cdot x, \ \forall\, x \in S \text{ and } \forall\, y \in T.$$

Finally, if any $\beta \in [\inf_{x'\in S} h \cdot x', \sup_{y'\in T} h \cdot y']$ the theorem is proved. □

In this part, we have seen different separating hyperplane theorems following Vohra (2005) [38] and Sydsæter et al. (2006) [33].

[9]**Bolzano-Weierstrass theorem**: Every sequence in a closed and bounded (compact) set S in $\mathbb{R}^n$ has a convergent subsequence.

2.2 Preferences and utility functions

In this section, we will revise the theory of individual decision–making in a preference–based approach. We assume that the agents have a preference relation over some alternatives. They are rational and we will explain which is the meaning of rational in this context. We will mainly follow Mas-Colell et al. (1995) [20] and Jehle and Reny (2001) [13].

Later, we move to express the preference relation as a numerical function on the alternatives, the utility function. We revise the conditions for the existence of this function and introduce lotteries as a way to expand our set of alternatives. In this way, we introduce the idea of expected utility. The expectation for a lottery was initiated by Bernouilli[10] in the 18th century. John von Neumann[11] and Oskar Morgenstern[12] (1944) [39] define the concept of utility in the third edition of their book and use the expected utility to compare lotteries.

Preference relations

In the approach to modeling the individual choice behavior based in preferences, the decision maker's goals are summarized in a preference relation, which we denote by $\succsim$. Let X be the set of possible alternatives which the individual must choose. We will take X as a (convex) subset of the Euclidean space $\mathbb{R}^n$.

A *preference relation* $\succsim$ is a binary relation on the set of alternatives X, allowing the comparison of pairs of elements $x, y \in X$. We read $x \succsim y$ as "x is at least as good as y". The *strict preference* relation $\succ$ is defined by $x \succ y$ if $x \succsim y$ but not $y \succsim x$ and, it is read as "x is preferred to y". The *indifference* relation $\sim$ is defined by $x \sim y$ if $x \succsim y$ and $y \succsim x$ and, it is read as "x is indifferent to y".

In economic theory, it is common to assume that individual preferences are rational. There are two basic assumptions about preference relations which are completeness and transitivity to satisfy the rationality hypothesis.[13]

The preference relation $\succsim$ is *rational* if it satisfies the following properties:

- *Completeness*: $\forall\, x, y \in X$, we have $x \succsim y$ or $y \succsim x$.
- *Transitivity*: $\forall\, x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

There are some implications for the strict and indifference relations by the assumptions of completeness and transitivity in the preference relation. Let it be $X \subseteq \mathbb{R}^n$ and $x, y, z \in X$. If the preference relation $\succsim$ is rational then:

[10] Daniel Bernoulli (1700–1782) was a Swiss mathematician and physicist known for his applications of mathematics to mechanics, particularly fluid mechanics, and for his pioneering work in probability and statistics.

[11] John von Neumann (1903–1957) was a Hungarian-American mathematician who integrated both pure and applied sciences. He made major contributions to a number of fields, including mathematics, physics, economics, computing and statistics.

[12] Oskar Morgenstern (1902–1977) was a German-American economist and with the collaboration of von Neumann founded the mathematical field of game theory and its application to economics.

[13] The assumption that $\succsim$ is *reflexive* (defined as $x \succsim x,\ \forall\, x \in X$) is added to the completeness and transitivity assumptions. In fact, this property is redundant because it is implied by the completeness.

- The strict preference relation $\succ$ is irreflexive and transitive.[14]
- The indifference relation $\sim$ is reflexive, transitive and symmetric, i.e., an equivalence relation.[15]
- If $x \succ y \succsim z$, then $x \succ z$.

Utility function

We can describe the preference relation of an agent by means of a utility function. The utility function ranks the elements of X in accordance with the individual's preference, allocating a numerical value to each element in X.

Definition 2.3. Let $X \subseteq \mathbb{R}^n$ be equipped with a preference relation $\succsim$. It is said $x \succsim y$ is *representable* if there exists a real-valued function $u : X \to \mathbb{R}$ such that $\forall\, x, y \in X$,

$$x \succsim y \iff u(x) \geqslant u(y).$$

The mapping u is called the *utility function* representing the preference relation $\succsim$.

This utility representation is not unique, because if we only require of a preference relation to rank alternatives in a meaningful way, and the only requirement of the utility function representing that preference relation is that it reflects the ordering of alternatives. Then, any other function that assigns numbers to alternatives in the same order as the utility function does will also represent that preference relation. It is clear, then, that we cannot have uniqueness of utility functions.

The assumption of rationality is important to represent preferences by a utility function. It is clear that if a utility function represents a preference relation, this one has to be rational. However, not every rational preference relation can be described by some utility function. If the set X is a finite or countable set, then every rational preference can be described by a utility function. But when X is non-countable, things get more complicated. We need continuity to obtain some results. Debreu[16] (1954) [6] proved that any binary relation that is complete, transitive, and continuous[17] can be represented by a continuous real-valued utility function.

Proposition 2.1. *Let $X \subseteq \mathbb{R}^n$ be equipped with a preference relation $\succsim$. If the binary relation $\succsim$ is rational and continuous there exists a continuous real-valued function, $u : X \to \mathbb{R}$, which represents $\succsim$.*

Proof. See Debreu (1954) [6]. □

Therefore, the Proposition 2.1 allow us to represent preferences relation $\succsim$ by continuous utility function.

[14] $\succ$ is *irreflexive* if $x \succ x$ never holds and $\succ$ is *transitive* if $x \succ y$ and $y \succ z$, then $x \succ z$.

[15] $\sim$ is *reflexive* if $x \sim x$, $\sim$ is *transitive* if $x \sim y$ and $y \sim z$, then $x \sim z$ and $\sim$ is *symmetric* if $x \sim y$, then $y \sim x$.

[16] Gérard Debreu (1921–2004) was a French-American mathematician and economist who won the 1983 Nobel Prize in Economic Sciences.

[17] The preference relation $\succsim$ is *continuous* if $\forall\, x \in X$, the set $\succsim(x) = \{y \mid y \in X, x \succsim y\}$ and the set $\precsim(x) = \{y \mid y \in X, y \succsim x\}$ are closed in X.

Lotteries and expected utility theorem

Until now, we studied choices that result in perfectly certain outcomes. However, many economic decisions involve an element of risk. We are considering a setting in which alternatives with uncertain outcomes are described by means of objectively known probabilities defined on an abstract set of possible outcomes. These representations of risky alternatives are named as lotteries. Formally, we denote the set of all possible outcomes by Z. To avoid some technicalities, we assume in this section that the set Z is finite, and $Z = \{1, \ldots, n\}$. After some definitions, we will present an important theorem, the expected utility theorem. This theorem says that under certain conditions, each possible outcome can be quantified and represented through some utility function.

The first important concept of this section is lottery, a formal device that is used to represent risk alternatives.

Definition 2.4. Let $Z = \{1, \ldots, n\}$ be the set of all possible outcomes. A *simple lottery* $L(Z)$ is a list $L(Z) = (p_1, \ldots, p_n)$ with $p_i \geq 0,\ \forall\, i = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$, where p_i is interpreted as the probability of outcome i occurring.

If no confusion will arise, we will write L instead of $L(Z)$. Notice that the elements of Z can be interpreted as a simple lottery.

We denote by $\mathcal{L}(Z)$ *the set of all simple lotteries over the set of outcomes.* We assume that the decision maker has a rational preference relation $\succsim$ on $\mathcal{L}(Z)$. This preference is a complete and transitive relation that allows comparison of any pair of simple lotteries.

Now, we are going to introduce two additional assumptions about the decision maker's preferences over lotteries which are going to be used in the expected utility theorem.

Definition 2.5. The preference relation $\succsim$ on the space of simple lotteries $\mathcal{L}(Z)$ satisfies the *continuous axiom* if for any L, L', $L'' \in \mathcal{L}(Z)$, the following two sets

$$\left\{\lambda \in [0,1] \mid \lambda \cdot L + (1-\lambda) \cdot L' \succsim L''\right\} \subseteq [0,1]$$
$$\left\{\lambda \in [0,1] \mid L'' \succsim \lambda \cdot L + (1-\lambda) \cdot L'\right\} \subseteq [0,1]$$

are closed.

The continuity axiom says that between two lotteries small changes in probabilities do not change the nature of the ordering.

Definition 2.6. The preference relation $\succsim$ on the space of simple lotteries $\mathcal{L}(Z)$ satisfies the *independence axiom* if $\forall\, L$, L', $L'' \in \mathcal{L}(Z)$ and $\forall\, \lambda \in (0,1)$, we have

$$L \succsim L' \iff \lambda \cdot L + (1-\lambda) \cdot L'' \succsim \lambda \cdot L' + (1-\lambda) \cdot L''.$$

What the axiom of independence tells us is that if we have three lotteries and we mix each of two lotteries with a third one, the ordering of the resulting mixture does not depend on the third lottery used. Thus, the preferences order of the resulting mix is independent of the lottery that we used third.

If preferences satisfy the two previous axioms over lotteries, we can represent such preferences with a continuous, real-valued utility function. The continuity axiom implies the existence of a utility function representing $\succsim$, a function $U : \mathcal{L}(Z) \to \mathbb{R}$ such that $L \succsim L'$ if and only if $U(L) \geq U(L')$, for $L, L' \in \mathcal{L}(Z)$. Moreover, the independence axiom is linked to the representability of preferences over lotteries by a utility function that has a specific form, the expected utility form.

Definition 2.7. Let $Z = \{1, \dots, n\}$ be a set of all possible outcomes. The utility function $U : \mathcal{L}(Z) \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \dots, u_n)$ to n outcomes such that for a simple lottery $L = (p_1, \dots, p_n) \in \mathcal{L}(Z)$, we have

$$U(L) = u_1 \cdot p_1 + \ldots + u_n \cdot p_n.$$

Thus, if U has the expected utility property, this means that it assigns to each lottery the expected value of the probability distribution over the utility of each elementary outcome. The utility function $U : \mathcal{L}(Z) \to \mathbb{R}$ with the expected utility form it is usually called as a *von Neumann-Morgenstern expected utility function.* Notice that the expression $U(L) = \sum_{i=1}^{n} u_i \cdot p_i$ is a general form for a linear function over probabilities $(p_1, \dots, p_n)$.

Now, we can explain the main theorem of this section. The expected utility theorem tells that if the decisions maker's rational preference over lotteries satisfies the axioms of continuity and independence, then his/her preferences are representable by a utility function with the expected utility form.

Theorem 2.4 (**Expected Utility Theorem**). *Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathcal{L}(Z)$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility. That is, we can assign a number u_i to each outcome $i = 1, \dots, n$ in a way that for any two lotteries $L = (p_1, \dots, p_n) \in \mathcal{L}(Z)$ and $L' = (p'_1, \dots, p'_n) \in \mathcal{L}(Z)$, we have*

$$L \succsim L' \iff \sum_{i=1}^{n} u_i \cdot p_i \geq \sum_{i=1}^{n} u_i \cdot p'_i$$

Proof. See Mas-Colell et al. (1995) [20, p. 176]. □

In this previous section, we have assumed the possible outcomes set is finite. It is possible to extend our theorem to an infinite number of possible outcomes. In this case, obviously summations will be changed for integrals.

Chapter 3

The Bargaining Problem and Nash Solution

This chapter analyzes the problem faced by two or more agents who must reach an agreement to choose a point in a feasibility space. Sometimes the problem is to divide a homogeneous good, but usually the utilities of the agents are not comparable. If they do not reach an agreement there is an *status quo* point which is selected.

This problem is called the bargaining problem. It was analyzed in an axiomatic form by John F. Nash[1] (1950) [24] and therefore we analyze, in the first place, the Nash's model of bargaining. Nash [24] proved that the unique solution he proposes of the negotiation problem, the Nash solution, will be the one that is characterized by four properties. These four axioms are Independence of Equivalent Utility Representation, Symmetry, Independence of Irrelevant Alternatives and Pareto Optimality.

This chapter mainly follows Part III of the paper published by Jordi Massó (2017) [21]. The main goal of this paper was to analyse the two most important contributions John F. Nash gave to economics, which are Nash Equilibrium in a non-cooperative game (Part II) and Nash Solution to the Bargaining Problem (Part III). Also, we will follow Roth (1979a) [31].

3.1 An Introduction to Bargaining

A *Bargaining problem* is a situation where a set of *agents* (players or bargainers) may cooperate to their mutual benefit. The agents try to reach an agreement on how they will cooperate, by bargaining. However, it's usually known that the agents' preferences may differ on the set of possible agreements.

There are a lot of examples of bargaining problems, such as: a buyer and a seller have to agree on a price of an object, the wage setting and the working conditions between a trade union and a firm, the resolutions of the conditions of divorce in a couple and the peace conversations to reach an agreement to solve a conflict by supranational institutions

[1] John Forbes Nash (1928–2015) got the Nobel Memorial Prize in Economic Sciences in 1994, joint with John Harsanyi and Reinhard Selten. He was an American mathematician and he got the Abel Prize in 2015, joint with Louis Nirenberg.

and different countries.

We will suppose that all the agreements needs to be reached by *unanimously* because it is an important feature of the (pure) bargaining problem. So, there is no role for partial agreements among a subset of agents nor for the intermediate coalitions. All the agents need to reach a *unanimous* agreement to achieve the situation of mutual benefit. If all agents are not able to reach an agreement, the *status quo* is maintained, i.e., there is a disagreement outcome. Specifically, every agent has veto power over all possible agreements.

In the 50's Nash starts all this literature with two papers: *The Bargaining Problem* [24] and *Two-Person Cooperative Games* [25]. In [24] presents the bargaining problem and its solution (The Nash Bargaining Solution). Nash shows that general assumptions have to be made concerning the individual and group behaviour of two agents in a specific economic context. So, he wants to present a new treatment of this classic economic problem. It is from these assumptions that the author proves and solves the bargaining problem. But in [25], he researches how to solve the Bargaining Problem what has been called the *Nash Program*. He argues about the desirability of justifying the axiomatic solution by obtaining it as the utilities of a Nash equilibrium of a non-cooperative game, where the strategies of the players correspond to their decisions taken in a process of bargaining. Hence, the Nash Program consists of studying cooperative solutions such that they are equilibriums of some non-cooperative game.

The bargaining problem was analyzed by some economists before Nash, in a less formal manner. Francis Y. Edgeworth (1881) [9] came up with a way of representing, using the same axis, indifference curves and the corresponding contract curve that analyzes the possible solution in a problem of choosing bundles for two agents. However, it was Vilfredo Pareto (1906) [26] who developed Edgeworth's ideas into a simpler diagram which is called the Edgeworth box. The Edgeworth box was as a precursor to the bargaining problem that allows a unique solution. The bargaining abilities and thoughness of the agents would dictate the particular chosen solution.

3.2 The Bargaining Problem

The bargaining problem has two fundamental concepts:

- The set $N = \{1, \ldots, n\}$ of agents (players or bargainers) where $n \geq 2$.
- The set Z of feasible outcomes or agreements (Z can be a finite or infinite set).

These two elements are related through the preferences. Each agent $i \in N$ has preferences $\succsim_i$ over the set Z. Let $z, z' \in Z$, $z \succsim_i z'$ indicates that "the z agreement for agent i is at least as preferred as the z' agreement". However, $z \succ_i z'$ indicates that "the z agreement is strictly preferred to the z' agreement for agent i".

For $i \in N$, a *utility function* $u_i : Z \to \mathbb{R}$ represents $\succsim_i$ preferences if for all $z, z' \in Z$,

$$z \succsim_i z' \Leftrightarrow u_i(z) \geq u_i(z').^2$$

We let the agents agree to choose a probability distribution over Z as a solution to the bargaining problem. For this reason, we suppose that each $i \in N$ has preferences $\widehat{\succsim}_i$ on the set of probabilities over Z, represented by the function $h_i : \mathcal{L}(Z) \to \mathbb{R}$, where $\mathcal{L}(Z)$ is the set of probability distributions on Z and h_i satisfies the expected utility theorem.

Let $S \subseteq \mathbb{R}^n$ be the *set of feasible outcomes* in terms of expected utilities:

$$x \in S \Longleftrightarrow \exists\, p \in \mathcal{L}(Z) \text{ such that } \forall\, i \in N,\ h_i(p) = x_i.$$

There are one implicit and four explicit assumptions on the Nash Model (these assumptions are directly formulated over the set S):

- Implicit hypothesis:

  Only utilities (not the outcomes themselves) are relevant to represent and determine the solution of the Bargaining problem, i.e., in the $h_i : \mathcal{L}(Z) \to \mathbb{R}$ function we will pay attention at the image in $\mathbb{R}$ and not at the output set $\mathcal{L}(Z)$. For this reason, what is only matter is the fact that there is an agreement, not the way that it is reached.

- Explicit hypotheses:

  1. S is convex (by allowing for randomizations on the set of feasible outcomes Z).
  2. S is compact (for instance, this is clear if Z is finite).
  3. There is a disagreement point (or *status quo*): $d \in S$.
  4. There is an agreement such that all agents strictly prefer that the disagreement, i.e., there exists $x \in S$ such that $x_i > d_i$ for all $i \in N$.

Let $\mathcal{B}$ be the set of all pairs (S, d) with the above properties; namely, $\mathcal{B}$ is *the set of all Bargaining problems*:

$$\mathcal{B} = \Big\{(S, d) \mid S \subset \mathbb{R}^n \text{ is convex and compact, } d \in S \text{ and } \exists\, x \in S \text{ such that } x_i > d_i,\ \forall\, i \in N\Big\}.$$

We will use this notation $\mathcal{B}$ when the set of N agents is fixed. However, if we want to specify the number of agents n of the problem we will use $\mathcal{B}_n$. For example, for a two-person bargaining problems, we will denote the set of all Bargaining problems as $\mathcal{B}_2$.

[2] The preferences can be represented by a von Neumann-Morgenstern expected utility function, if Z is a finite set. Besides, if Z is a infinite set the preferences can be represented by utility integral function, which is unique except for positive monotonic transformations under very general conditions (See Debreu (1959) [7]).

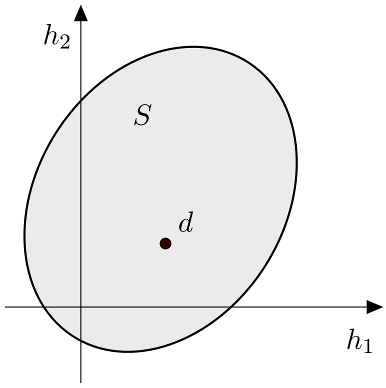

Figure 3.1: Bargaining Problem when $n = 2$

Definition 3.1. A *solution* of the Bargaining problem is a function $f : \mathcal{B} \to \mathbb{R}^n$ such that for all $(S, d) \in \mathcal{B},\ f(S, d) \in S$.

A solution is a rule that assigns to each bargaining problem a feasible vector of utilities. It is a single point. Thus, a solution can be interpreted as an arbitrator responding to a particular set of axioms on how to solve the Bargaining problem.

3.3 Nash Solution

In this section, we will explain the four axioms that characterize the Nash Solution. The Nash Solution is obtained by maximizing the product of utility gains from the disagreement point. Nash (1950) [24] proposes a solution to the bargaining problem, that is a function $f : \mathcal{B} \to \mathbb{R}^n$, with four axioms. He shows that the axioms of Independence of Equivalent Utility Representation, Symmetry, Independence of Irrelevant Alternatives and Pareto Optimality characterize this solution. Moreover, it will be proved that the unique solution of the bargaining problem will be the one that satisfy these four axioms.

AXIOM 1: **Independence of Equivalent Utility Representation (IEUR)**

This property says that the solution of the bargaining problem does not depend of any linear or affine transformation of the utility.

For every $(S, d) \in \mathcal{B}$ and every $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $a_i > 0\ \forall\ i \in N$, define a new bargaining problem $(S', d') \in \mathcal{B}$ as follows

$$d'_i = b_i + a_i \cdot d_i,\ \forall\ i \in N$$
$$S' = \{y = (y_i)_{i \in N} \in \mathbb{R}^n \mid \exists\ x \in S \text{ such that } \forall\ i \in N,\ y_i = b_i + a_i \cdot x_i\}.$$

We say that (S', d') is a *positive affine transformation* (b, a) of (S, d). Now, we are in the position to state our first axiom.

Definition 3.2. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Independence of Equivalent Utility Representation* if for all $(S, d) \in \mathcal{B}$, and all $(S'd') \in \mathcal{B}$ a positive affine transformation (b, a) of (S, d), we have

$$f_i(S', d') = b_i + a_i \cdot f_i(S, d),\ \forall\ i \in N.$$

The axiom of Independence of Equivalent Utility Representation requires that the solution does not take into account the numerical representation of the agents' preferences on the probability distributions on the possible outcomes of the bargaining (*e.g.* the numerical representation on lotteries is unique up to positive affine transformations). The problems (S, d) and (S', d') are equivalent in terms of the agents' preferences over the probability distributions over the possible agreements and hence, the underlying outcome should be the same.

AXIOM 2: **Symmetry (SYM)**

Like the previous property, this one requires that the solution depends only on information contained in the model. If the agents cannot be differentiated on the basis of the information contained in S, then the solution should treat them alike.

A bargaining problem $(S, d) \in \mathcal{B}$ is *symmetric* if:

(i) $d_1 = \ldots = d_n$, and

(ii) for any permutation of the agents $\pi : N \to N$ if $x = (x_1, \ldots, x_n) \in S$ then $y = (y_1, \ldots, y_n) \in S$, where $y_i = x_{\pi(i)}$ for all $i \in N$.

Thus, a bargaining problem is symmetrical if the players' roles in describing problem (S, d) are interchangeable.

Definition 3.3. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Symmetry* if for every symmetric bargaining problem $(S, d) \in \mathcal{B}$,

$$f_1(S, d) = \ldots = f_n(S, d).$$

If (S, d) is symmetric there is no information that distinguishes one player from the another one. Hence, the solution should not distinguish between players.

AXIOM 3: **Independence of Irrelevant Alternatives (IIA)**

The IIA says that the outcome of bargaining depends only on the relationship of the outcome to the disagreement point, and does not depend on other alternatives in the feasible set.

Definition 3.4. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Independence of Irrelevant Alternatives* if for every two problems $(S, d), (T, d) \in \mathcal{B}$ such that $S \subset T$ and $f(T, d) \in S$, then $f(S, d) = f(T, d)$.

If the solution to the problem (T, d) is $f(T, d)$, and $f(T, d)$ is also a possible agreement on the reduced problem (S, d), the Independence of Irrelevant Alternatives requires that the solution to the reduced problem $f(S, d)$ must coincide with $f(T, d)$. Indeed, if alternatives in $T \setminus S$ were not selected when T was available, when they are not available anymore (they are not in S), the solution should select the same outcome. Equivalently, when the set S is enlarged to the set T then, either the solution selects the same outcome $(f(T, d) = f(S, d))$ or else selects a new outcome $(f(T, d) \notin S)$.

The following figure illustrates this axiom graphically when $n = 2$.

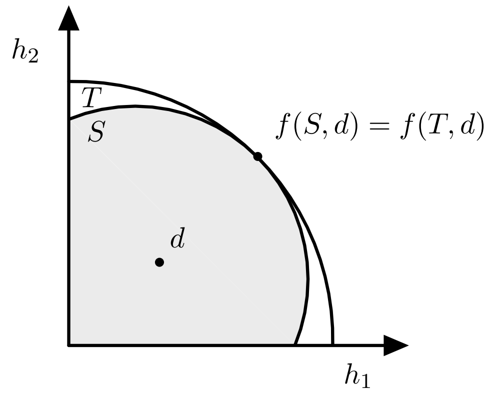

Figure 3.2: Independence of Irrelevant Alternatives when $n = 2$

AXIOM 4: **Pareto Optimality (PO)**

This axiom shows that the solution allocates the profits obtained from cooperation, in such a way that it is not possible to give strictly more to any agent without harming another agent.

First, we introduce the following notation. For $(S, d) \in \mathcal{B}$ we define the set of *Pareto optimal points of* S as

$$PO(S) \equiv \{x \in S \mid \forall\, y \in \mathbb{R}^n,\ y \geq x \text{ and } y \neq x \ \Rightarrow\ y \notin S\}.$$

We use the following vector inequalities: $x \geq y$ if for each $i \in N$, $x_i \geq y_i$; and $x > y$ if for each $i \in N$, $x_i > y_i$.

Definition 3.5. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Pareto Optimality* if for every $(S, d) \in \mathcal{B}$

$$f(S, d) \in PO(S).$$

This property can be thought of as requiring that the agents collectively should behave in a rational way, since it specifies that the solution will select an outcome such that no other feasible outcome is preferred by all of the agents. The outcome must be a maximal element of the "social preference" defined by the intersection of all the individual preferences. The subset of these "collectively rational" outcome contained in a given set S is called the *Pareto Optimal*[3] subset, $PO(S)$. The axiom of Pareto Optimality requires that, for any bargaining game (S, d), the solution f should always select an outcome contained in $PO(S)$.

For example, a company have to allocate a budget of 100.000 euros to make a project with two agents. A possible distribution would be 45.000 euros for each one, but this solution would not be efficient, since all the possible gains of the negotiation would not be exhausted when 10.000 euros remain without being distributed. Therefore, this property avoids the points that are not on the frontier.

[3]The concept is named after Vilfredo Pareto (1848–1923), an Italian engineer, economist and sociologist, who used the concept in his studies of economic efficiency and income distribution.

Nash (1950) [24] shows that the four properties stated above define a unique solution to the bargaining problem that is named as the Nash solution.

Definition 3.6 (Nash Solution). The *Nash Bargaining solution* $\mathcal{N} : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S, d) \in \mathcal{B}$, $\mathcal{N}(S, d)$ is the solution of the maximization problem

$$\arg\max \quad \prod_{i=1}^{n}(x_i - d_i) \tag{3.1}$$

over the set $S \cap \{x \in \mathbb{R}^n \mid x_i \geq d_i,\ i = 1, \ldots, n\}$.

The Nash solution is a function that selects the outcome that maximizes the geometric average[4] of the gains. These gains are earned by players when they reach an agreement rather than being set for the outcome of the disagreement.

Observations.

1. The solution of the maximization problem (3.1) is unique. This is because if two different points x and x' in S such that $x, x' \geq d$ yield the same geometric average, then their mean yields a strictly higher geometric average. This is because their mean is also contained in S since S is convex. So, the maximum is not achieved at either x or x'.

2. Hence, the *Nash Bargaining solution* $\mathcal{N} : \mathcal{B} \to \mathbb{R}^n$ is defined as $\forall\ (S, d) \in \mathcal{B}$, $\mathcal{N}(S, d) = x$ where $x \in S$ is such that $x \geq d$ and

$$\prod_{i=1}^{n}(x_i - d_i) > \prod_{i=1}^{n}(y_i - d_i), \quad \forall\, y \in S \setminus \{x\} \ \text{ and } y \geq d.$$

3. The expression $\prod_{i=1}^{n}(x_i - d_i)$ is called the *Nash Product*.

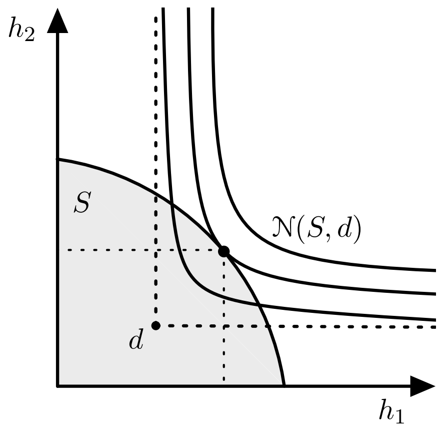

Figure 3.3: Nash's Solution for two agents

Figure 3.3 illustrates geometrically the Nash's solution to the bargaining problem when $n = 2$, where there are drawn three equilateral hyperbolas with d as the origin. The border

[4]The geometric average is $\prod_{i=1}^{n}(x_i - d_i)^{1/n}$. In our case, we are considering the n-th power of the geometric average. But this difference has no consequence, because the geometric average of its n-th power is maximized at the same point.

of the set S in the first quadrant is the set of efficient solutions of the bargaining problem. We must known that the Nash solution lies on this curve, now see which of its points is the solution.

Proposition 3.1. *The Nash solution $\mathcal{N} : \mathcal{B} \to \mathbb{R}^n$ of the bargaining problem (S, d) satisfies the Independence of Equivalent Utility Representation, Symmetry, Independence of Irrelevant Alternatives and Pareto Optimality.*

Proof. Let $\mathcal{N}$ be the function $\mathcal{N} : \mathcal{B} \to \mathbb{R}^n$, the Nash solution of the bargaining problem $(S, d) \in \mathcal{B}$. We will prove that $\mathcal{N}$ satisfies each axiom separately.

1. To see that $\mathcal{N}$ satisfies the Independence of Equivalent Utility Representation, $\forall\ (S, d),\ (S', d') \in \mathcal{B}$ resulting from a positive affine transformation (b, a) of (S, d), the solution $\mathcal{N}(S', d') = b + a \cdot \mathcal{N}(S, d)$ is the Nash solution for the bargaining problem (S', d').

   So, it should be proved that if $x = \mathcal{N}(S, d)$, then $a \cdot x + b = N(S', d')$. It is easy to verify if we look at the following inequality:

   $$\prod_{i=1}^{n} \left[a_i \cdot x_i + b_i - (a_i \cdot d_i + b_i)\right] = \prod_{i=1}^{n} a_i \prod_{i=1}^{n} (x_i - d_i) >$$
   $$> \prod_{i=1}^{n} a_i \prod_{i=1}^{n} (y_i - d_i) = \prod_{i=1}^{n} \left[a_i \cdot y_i + b_i - (a_i \cdot d_i + b_i)\right]$$

   $\forall\ y \in S \cap \{x \in \mathbb{R}^n \mid x_i \geq d_i,\ i = 1, \ldots, n\}$.

2. The Nash solution $\mathcal{N}(S, d)$ clearly satisfies the symmetry axiom. Let $(S, d) \in \mathcal{B}$ be a symmetric bargaining problem and let x be the solution. We have $d_1 = \ldots = d_n = \widetilde{d} \in \mathbb{R}$ and for any permutation of the agents $\pi : N \to N$, if $x = (x_1, \ldots, x_n) \in S$, $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in S$.

   Since the negotiation problem is symmetric, $d_i = \widetilde{d}\ \forall\ i \in N$ and for any permutation of the agents the product will be the same:

   $$\prod_{i=1}^{n} (x_i - d_i) = \prod_{i=1}^{n} (x_i - \widetilde{d}) = \prod_{i=1}^{n} (x_{\pi(i)} - \widetilde{d}).$$

   Therefore, the Nash solution satisfies $\mathcal{N}_1(S, d) = \ldots = \mathcal{N}_n(S, d)$, and $\mathcal{N}(S, d)$ satisfies Symmetry.

3. To see that $\mathcal{N}$ obeys the Independence of Irrelevant Alternatives is the same that to see that given two bargaining problems $(S, d), (T, d) \in \mathcal{B}$ such that $S \subset T$ and $\mathcal{N}(T, d) \in S$, then $\mathcal{N}(S, d) = \mathcal{N}(T, d)$.

   If the solution of Nash $\mathcal{N}(T, d) = x \in S$, then $\prod_{i=1}^{n} (x_i - d_i)$ is by definition the maximization product at T. The restricted Nash product at S need to be equal or less than the Nash product at T because $S \subset T$. So, the solutions of the bargaining problem (S, d) and (T, d) are the same, therefore $\mathcal{N}$ satisfies axiom three.

4. Finally, we will prove that $\mathcal{N}$ satisfies Pareto Optimality. Let $(S,d) \in \mathcal{B}$ be a bargaining problem and every $x, y \in S$ such that $x_i \geq y_i \ \forall \ i \in N$, and $x_j > y_j$ for some $j \in N$, then $\mathcal{N}(S,d) \neq y$.

   Due to $x_j - d_j > y_j - d_j \geq 0$, for some $j \in N$ we have:

$$\prod_{i=1}^{n}(x_i - d_i) > \prod_{i=1}^{n}(y_i - d_i) \geq 0.$$

We have proved that the Nash solution of a bargaining problem satisfies the axioms of Independence of Equivalent Utility Representation, Symmetry, Independence of Irrelevant Alternatives and Pareto Optimality. □

In particular, Nash proved the following remarkable theorem: the unique solution of the negotiation problem will be the one that satisfy the previous axioms.

Theorem 3.1 (Nash, 1950). *A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies the Independence of Equivalent Utility Representation, Symmetry, Independence of Irrelevant Alternatives and Pareto Optimality if and only if f is the Nash Bargaining solution of the bargaining problem, i.e., $f = \mathcal{N}$.*

Proof. The foregoing Proposition 3.1 shows that if $\mathcal{N}$ is the Nash solution then $\mathcal{N}$ satisfies the four axioms. Therefore, only the opposite implication must be proved.

We show now that any bargaining solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfying the four properties is indeed the Nash Bargaining solution $\mathcal{N}$. The proof proceeds in three steps:

- *Step 1*: Independence of Equivalent Utility Representation allows to treat every bargaining problem as with symmetric disagreement point d and a symmetrical solution.
- *Step 2*: By Symmetry and Pareto Optimality f and $\mathcal{N}$ should coincide on any symmetric bargaining game since there is a unique Pareto optimal outcome whose components are all equal.
- *Step 3*: By Independence of Irrelevant Alternatives f and $\mathcal{N}$ coincide on the original problem.

Let f be a solution that satisfies the four axioms and let $(S,d) \in \mathcal{B}$ be an arbitrary bargaining problem. Let $\mathcal{N}(S,d) = x$ denote the expected utility vector selected for the Nash $\mathcal{N}$ solution in the bargaining problem (S,d). Observe that by assumption and the Pareto Optimality of $\mathcal{N}$, $x_i > d_i \ \forall \ i \in N$.

Step 1: Define a new bargaining problem $(S', d') \in \mathcal{B}$ as the following positive affine transformation (b, a) of (S,d): for all $y \in S$,

$$\lambda_i(y_i) = \frac{-d_i}{x_i - d_i} + \frac{1}{x_i - d_i} \cdot y_i \quad \forall \ i = 1, \ldots, n,$$

i.e.,

$$b = \left(\frac{-d_1}{x_1 - d_1}, \ldots, \frac{-d_n}{x_n - d_n}\right) \quad \text{and} \quad a = \left(\frac{1}{x_1 - d_1}, \ldots, \frac{1}{x_n - d_n}\right).$$

Notice that for all $i \in N$:

$$\lambda_i(x_i) = \frac{-d_i}{x_i - d_i} + \frac{1}{x_i - d_i} \cdot x_i = \frac{-d_i + x_i}{x_i - d_i} = 1,$$
$$\lambda_i(d_i) = \frac{-d_i}{x_i - d_i} + \frac{1}{x_i - d_i} \cdot d_i = \frac{-d_i + d_i}{x_i - d_i} = 0.$$

Since $\mathcal{N}$ satisfies the Independence of Equivalent Utility Representation, we have $\mathcal{N}(S', d') = (1, \ldots, 1)$. Figure 3.4 illustrates the transformation from problem (S, d) to problem (S', d') when $n = 2$: the previous affine transformation composes by a translation to put the point d to 0 and by an homothety that places the solution on the coordinate 1 of each axis.

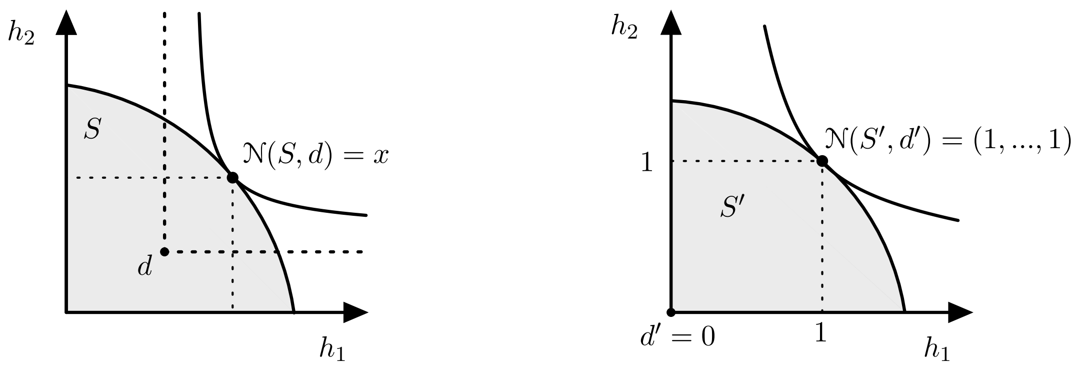

Figure 3.4: Transformation from problem (S, d) to problem (S', d') when $n = 2$

Step 2: The vector $(1, \ldots, 1)$ is the maximizer of the Nash product on S'. Thus, $x' = (1, \ldots, 1)$ is the unique point in the intersection of S' and the convex set

$$H = \left\{ y \in \mathbb{R}^n \;\middle|\; \prod_{i=1}^{n} y_i \geq 1 \right\}.$$

Notice that the boundary (∂) of the set H is differentiable

$$\partial H = \left\{ y \in \mathbb{R}^n \;\middle|\; \prod_{i=1}^{n} y_i = 1 \right\}.$$

Since ∂H is differentiable, the hyperplane

$$T = \left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} y_i = n \right\}$$

is the unique hyperplane that is tangent to H and goes through $x' = (1, \ldots, 1)$.

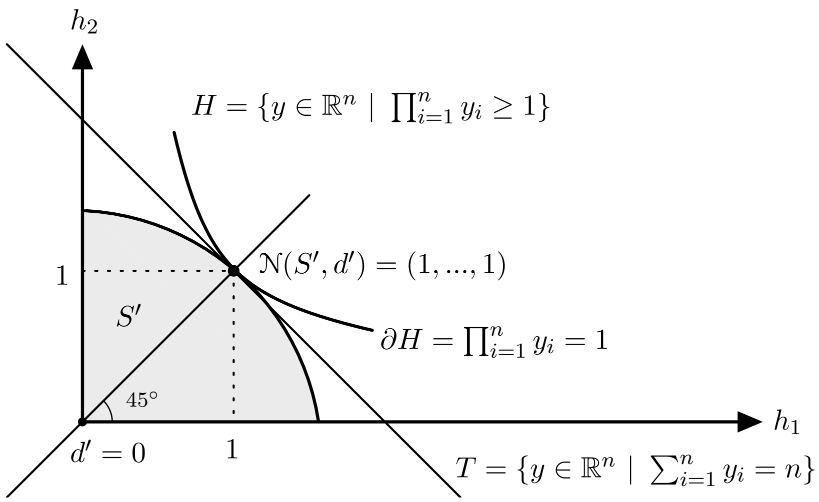

Figure 3.5: Illustration of the hyperplane T that separates the sets H and S' when $n = 2$

Since H and S' are two disjoint nonempty convex subsets of $\mathbb{R}^n$ (their intersection is the point $(1, \ldots, 1)$ that belongs to the borders of both sets), by Theorem 2.3 (Separating Hyperplane Theorem).

$$S' \subseteq \left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} y_i \leq n \right\}.$$

Therefore, since S' is compact, there exists a symmetric set R such that $S' \subseteq R$ and $PO(R) \subseteq T$, where $PO(R)$ is the set of Pareto Optimal subset of R.

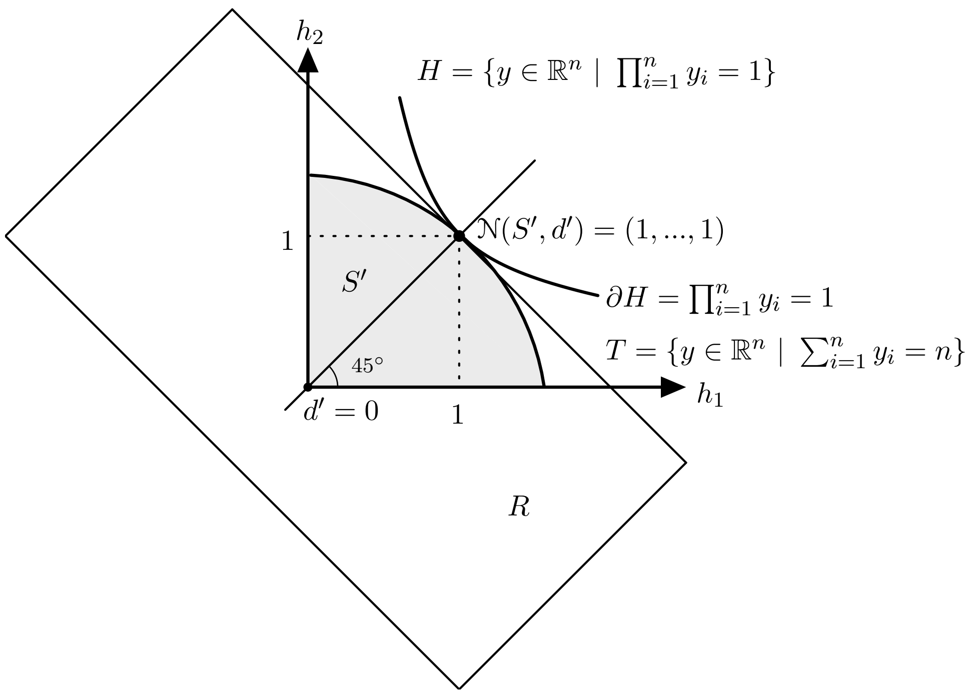

Figure 3.6: Illustration of the set R in our problem when $n = 2$

Figure 3.6 illustrates geometrically at dimension two the problem $(R, 0)$. Thus, by Symmetry and Pareto Optimality, $f(R, d') = (1, \ldots, 1)$.

Step 3: Notice that $S' \subset R$ and $(1, \ldots, 1) \in S'$. Therefore by Independence of Irrelevant Alternatives,

$$f(S', d') = (1, \ldots, 1),$$

and finally, by Independence of Equivalent Utility Representation,

$$f(S, d) = x = \mathcal{N}(S, d).$$

$\square$

To sum up, the best-known solution for the bargaining problem is the one introduced by Nash (1950) [24]. In the Nash solution, compromise is obtained by maximizing the product of utility gains from the disagreement point. Nash shows that the unique solution of the negotiation problem will be the one that satisfies the axioms of the Pareto Optimality, Symmetry, Independence of Equivalent Utility Representation and Independence of Irrelevant Alternatives. But some of these axioms, or the assumptions they express, have been the object of some criticism since they are not suitable for certain problems.

The main point of the bargaining problem is to predict how real-world conflicts are resolved. The axioms of Pareto Optimality is not always desirable, since such conflicts sometimes result in dominated compromises. Likewise, the agents enter symmetrically in the problem but we might want to take into account the differences between the agents of the environment that are not explicitly modeled. So, the axiom of Symmetry is disobeyed. The Independence of Equivalent Utility Representation avoids relying on commitments on interpersonal comparisons of utility, but in many situations these comparisons are made. Finally, there is the issue of how much and what information should be discarded in order to find the solution. For instance, the elimination of only alternatives in which the payment of a particular agent is greater than in the initial solution result and the other agents' payment lower than in the initial solution result, the Independence of Irrelevant Alternatives prevents the solutions from responding to such eliminations. The IIA is the most controversial axiom and many examples have been constructed to illustrate this problem (see [19, p. 128]).

Due to criticism of the axioms, several economists have suggest alternative sets of axioms which leads to different unique solutions of the bargaining problem.

Chapter 4

Other solutions of the Bargaining Problem

In this chapter, we see other possible solutions to the bargaining problem, and look at their characterizations. After John Nash, several economists presented other solutions to Nash's bargaining problem like Ehud Kalai and Meir Smorodinsky in 1975 among others. Since Nash solution is characterized by the four axioms we explained in the previous chapter, it is obvious that these other solutions are based on other axioms that we will present.

For this chapter, we mainly follow Thomson (1994) [36] and Thomson (2010) [37] to present some other solutions, as Kalai–Smorodinsky, Egalitarian, Dictatorial, Discrete Raiffa, Perles-Maschler, Equal Area, Utilitarian and Yu Solution.

From now on, we use a variant to our definition of bargaining problem. We add the d-comprehensive condition which will be defined later.

4.1 Kalai–Smorodinsky Solution

After the consideration by Nash (1950) [24] of the bargaining problem presenting a unique solution under certain axioms, Ehud Kalai[1] and Meir Smorodinsky (1975) [16] present an alternative axiom which leads to another unique solution. There were many objections to Nash's axiom of Independence of Irrelevant Alternatives so under the consideration of a two-person bargaining problem, they present *Other Solutions to Nash's Bargaining Problem* [16]. The characterization of the Kalai-Smorodinsky solution is obtained as a result of replacing IIA by Individual Monotonicity, and maintaining the rest of axioms that characterize the Nash solution. The following example shows one of the objections of the Nash solution with the IIA axiom when $n = 2$.

Example 4.1. Consider the following bargaining problems $(S, d), (S', d) \in \mathcal{B}_2$, where $S =$ convex hull $\{(0, 0), (2, 0), (0, 2)\}$ and $S' =$ convex hull $\{(0, 0), (0, 1), (1, 1), (0, 2)\}$. The disagreement point is equal to $(0, 0)$, i.e., $d = (0, 0)$. Obviously, $S \subset S'$.

Since (S, d) is symmetric, the Nash Solution is $\mathcal{N}(S, d) = (1, 1)$. However, the bargaining problem (S', d) is asymmetric but, by the axiom of Independence of Irrelevant

[1]Ehud Kalai (1942–) is an Israeli-American mathematical economist known for his contributions to the field of game theory.

Alternatives, the Nash Solution is also $(1,1)$, i.e., $\mathcal{N}(S',d) = (1,1)$. An illustration is given in Figure 4.1.

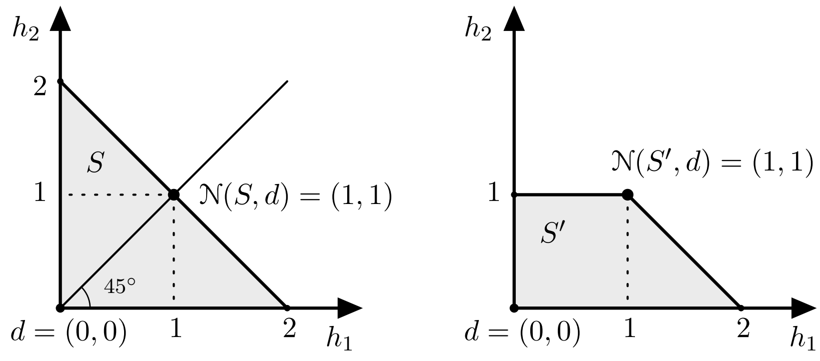

Figure 4.1: Illustration of the Example 4.1

Therefore, the Nash solutions are the same thanks to the axiom of IIA. However, agent one could demand to get more in the bargaining problem (S',d) than he does in (S,d). These solutions do not satisfy player demand.

Kalai and Smorodinsky (1975) [16] present their solution just for a two person bargaining problem. Nearly 50 years later, Karos, Muto and Rachmilevitch (2018) [17] present a characterization of the Kalai-Smorodinsky solution for n-person bargaining problem. Because of the several difficulties concerning the possible generalization of the Kalai-Smorodinsky solution, in this section will assume that $n = 2$, as in the original paper of Kalai and Smorodinsky.

We add to our definition of bargaining problem that the set (S,d) must be d - comprehensive. That means, (S,d) is *d-comprehensive*, if a point x is in S and $d \leq y \leq x$, then y is in S.

To describe the Kalai-Smorodinsky solution it is necessary to first present the ideal point of an agent.

Definition 4.1. Given $(S,d) \in \mathcal{B}_2$ and $i \in \{1,2\}$ the *i's ideal point of* (S,d) is defined as

$$a_i(S,d) = max\{x_i \in \mathbb{R} \mid x \in S,\ x_j \geq d_j\ \forall\, j \neq i\}.$$

The *ideal point of* (S,d) is

$$a(S,d) = (a_1(S,d), a_2(S,d)).$$

That is the best point that each agent can reach without making the situation of the other agent worse than the disagreement point. Notice that by the assumptions of the bargaining problem we have $a_i(S,d) > d_i$ for $i = 1,2$. Also by the compactness of S, $a_1(S,d)$ and $a_2(S,d)$ are well defined and attained at points in S. The following example shows how to find the ideal point of (S,d) when $d = 0$ or $d \neq 0$.

Example 4.2. Consider the following bargaining problems $(S,d),(S,d') \in \mathcal{B}_2$, where $S =$ convex hull $\{(0,0),(3,0),(0,3)\}$. Let it be the disagreement point $d = (0,0)$ and $d' = (1,1)$.

By its definition, the ideal point of (S, d) is $a(S, d) = (3, 3)$ and the ideal point of (S, d') is $a(S, d') = (2, 2)$. An illustration is given in Figure 4.2.

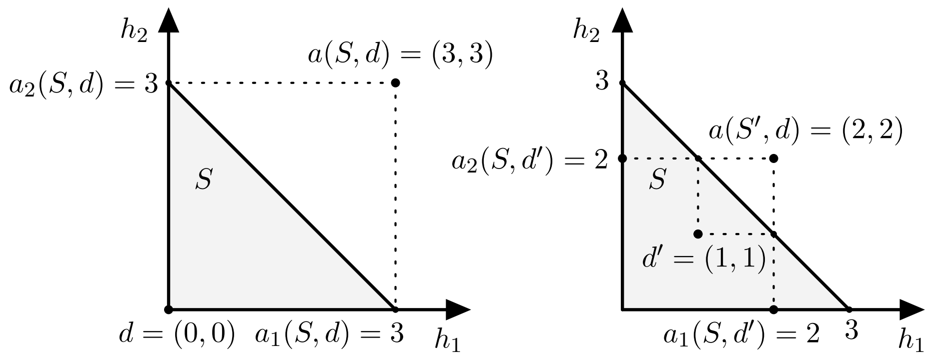

Figure 4.2: How the ideal point is affected by the choice of the disagreement point

Kalai and Smorodinsky believe that a solution to the bargaining problem is a function $f : \mathcal{B}_2 \to \mathbb{R}^2$ that satisfy four axioms. Most of these axioms are explained in the previous chapter, but the axiom of Individual Monotonicity remains to be explained.

AXIOM 5: **Individual Monotonicity (IM)**

This property says that an agent with better options should get a weakly-better agreement.

Definition 4.2. A solution $f : \mathcal{B}_2 \to \mathbb{R}^2$ satisfies *Individual Monotonicity* if for all $(S, d), (S', d) \in \mathcal{B}_2$ and $i \in \{1, 2\}$, with $S \subseteq S'$ and $a_j(S, d) = a_j(S', d)$ for $j \neq i$, then $f_i(S, d) \leq f_i(S', d)$.

This axiom states that if, for every utility level that agent two may demand, the maximum feasible utility level that agent one can simultaneously reach is increased, then it should be also increased the utility level assigned to agent one according to the solution. The following Figure 4.3 represents the IM graphically when $n = 2$.

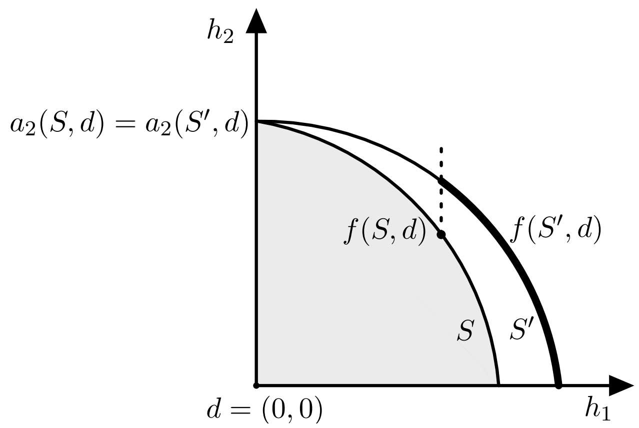

Figure 4.3: Individual Monotonicity when $n = 2$ and $d = (0, 0)$

Notice that Nash's solution does not satisfy the axiom of Individual Monotonicity as the following example shows.

Example 4.3. Consider the following bargaining problems $(S,d),(S',d) \in \mathcal{B}_2$ where $S =$ convex hull $\{(0,0),(40,0),(25,25),(0,40)\}$ and $S' =$ convex hull $\{(0,0),(40,0),(40,20),(0,40)\}$. Let be the disagreement point equal to $(0,0)$, i.e., $d = (0,0)$.

By symmetry, the Nash solution of the problem (S,d) is $(25,25)$, i.e., $\mathcal{N}(S,d) = (25,25)$. Whereas the (S',d) problem is not symmetric, the slope of the Nash product at $(40,20)$ is:

$$\left.\frac{\frac{\partial(x_1 \cdot x_2)}{\partial x_1}}{\frac{\partial(x_1 \cdot x_2)}{\partial x_2}}\right|_{(40,20)} = -\left.\frac{x_2}{x_1}\right|_{(40,20)} = -\frac{1}{2}.$$

The Nash product over (S',d) is maximized at $(40,20)$, therefore $\mathcal{N}(S',d) = (40,20)$. An illustration is given in Figure 4.4.

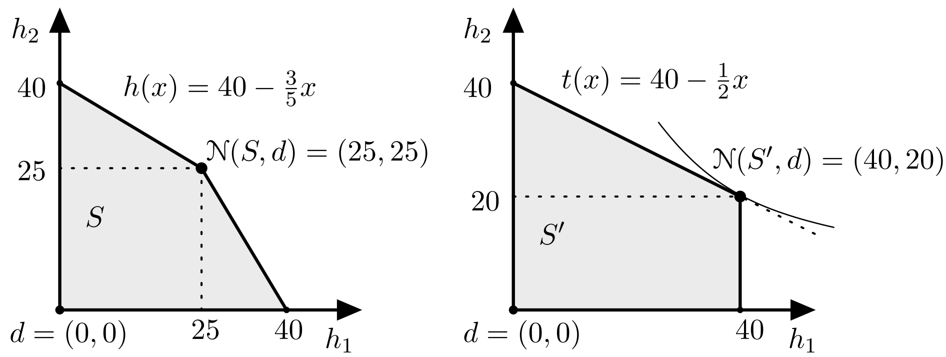

Figure 4.4: Illustration of the Example 4.3

Notice that $S \subset S'$ and $a(S,d) = a(S',d) = (40,40)$ but $\mathcal{N}_2(S,d) = 25 > 20 = \mathcal{N}_2(S',d)$. Thus, the Nash Bargaining Solution does not satisfy the Individual Monotonicity.

After presenting the notion of the ideal point of (S,d) and the axiom of Individual Monotonocity, we can define the Kalai-Smorodinsky solution. Kalai and Smorodinsky (1975) [16] show that the four axioms stated above define an unique solution to the bargaining problem that is named the Kalai-Smorodinsky solution.

Definition 4.3 (Kalai-Smorodinsky Solution). The *Kalai-Smorodinsky Bargaining solution* $\mathcal{K} : \mathcal{B}_2 \to \mathbb{R}^2$ is defined by letting $\mathcal{K}(S,d)$ to be equal to the maximal point of S on the segment connecting d and $a(S,d)$.

We denote $L(a,d)$ the segment connecting d and $a(S,d)$ which equation is given by

$$\frac{x_1 - d_1}{a_1(S,d) - d_1} = \frac{x_2 - d_2}{a_2(S,d) - d_2}$$

where $d_1 \leq x_1 \leq a_1(S,d)$ and $d_2 \leq x_2 \leq a_2(S,d)$.

The Kalai-Smorodinsky solution selects the maximum point of S that is proportional to the profile of maximum payments that agents can achieve separately among the points of S that dominate d. This list of maximum payments is the ideal point of (S, d).

Observation. Definition 4.3 could be rewritten as follows. The *Kalai-Smorodinsky Bargaining solution* $\mathcal{K} : \mathcal{B}_2 \to \mathbb{R}^2$ is defined, for every $(S, d) \in \mathcal{B}_2$, as

$$\mathcal{K}_i(S, d) = \underset{x \,\in\, S \,\cap\, L(a,d)}{\arg\max} \; x_i, \qquad \text{for} \quad i = 1, 2.$$

Geometrically, $\mathcal{K}(S, d)$ is the intersection of the line segment $[d, a(S, d)]$ and the northeast boundary of S.

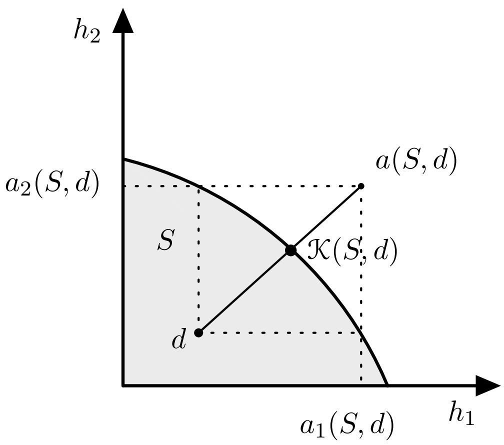

Figure 4.5: Kalai-Smorodinsky solution for two agents

Figure 4.5 illustrates geometrically the Kalai-Smorodinsky solution to the bargaining problem when $n = 2$. As we said before, one of the most controversial axioms of Nash solution was the Independence of Irrelevant Alternatives. The Kalai-Smorodinsky Bargaining solution does not satisfy Independence of Irrelevant Alternatives. The following Example 4.4 shows that the Kalai-Smorodinsky solution does not satisfy IIA axiom.

Example 4.4. Consider the following bargaining problems $(S, d), (S', d) \in \mathcal{B}_2$ where S = convex hull $\{(0, 0), (2, 0), (0, 2)\}$ and S' = convex hull $\{(0, 0), (0, 1), (1, 1), (0, 2)\}$. The disagreement point is $d = (0, 0)$. Obviously, $S' \subset S$.

The Kalai-Smorodinsky solution of (S, d) is $(1, 1)$ that is $\mathcal{K}(S, d) = (1, 1)$. For the bargaining problem (S', d) is $(\frac{4}{3}, \frac{2}{3})$ that is $\mathcal{K}(S', d) = (\frac{4}{3}, \frac{2}{3})$. An illustration is given in Figure 4.6.

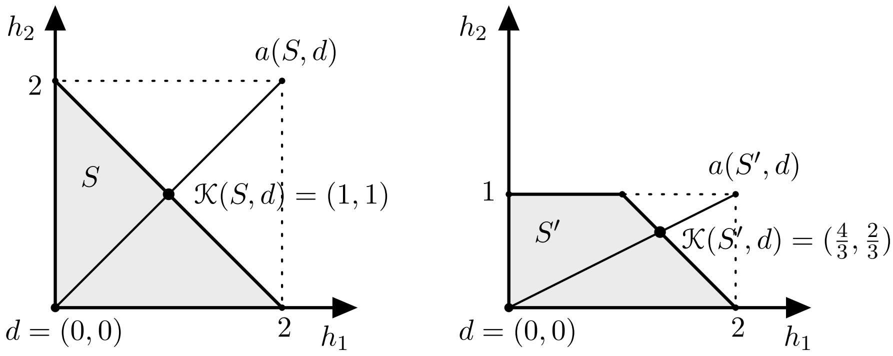

Figure 4.6: Illustration of the Example 4.4

Notice that $\mathcal{K}(S,d) \in S'$. Hence, the axiom of IIA would require that the solution also chooses $(1,1)$ in the problem (S',d), but $\mathcal{K}(S',d) = (\frac{4}{3}, \frac{2}{3})$.

Agent two has less bargaining power in (S',d) than in (S,d), and the Kalai-Smorodinsky solution reflects that. Since the problem (S,d) is symmetric the agreement $(1,1)$ seems reasonable, but it does not for the problem (S',d). Notice that the Kalai-Smorodinsky solution violates the Independence of Irrelevant Alternatives, but it does it in the intuitively good direction.

As a result of replacing Independence of Irrelevant Alternatives by Individual Monotonicity, in the earlier axioms that are shown to characterize the Nash solution we obtain a characterization of the Kalai-Smorodinsky solution.

Theorem 4.1 (**Kalai and Smorodinsky, 1975**). *A solution $f : \mathcal{B}_2 \to \mathbb{R}^2$ satisfies the Individual Monotonicity, Independence of Equivalent Utility Representation, Symmetry and Pareto Optimality if and only if f is the Kalai-Smorodinsky solution of the bargaining problem, i.e., $f = \mathcal{K}$.*

Proof. We first show that the Kalai-Smorodinsky solution $\mathcal{K}$ satisfies the four axioms. Let $\mathcal{K}$ be the function $\mathcal{K} : \mathcal{B}_2 \to \mathbb{R}^2$, the Kalai-Smorodinsky solution of the bargaining problem $(S,d) \in \mathcal{B}_2$.

To see that $\mathcal{K}$ satisfies the Independence of Equivalent Utility Representation, we assume that $(S',d') \in \mathcal{B}_2$ is an affine transformation of (S,d). It is known that an affine transformation maps straight lines into straight lines and $a(S,d)$ into $a(S',d')$. It preserves the partial ordering of $\mathbb{R}^2$. Then, the definition of $\mathcal{K}$ and the previous facts imply that $\mathcal{K}$ satisfies the axiom of Independence of Equivalent Utility Representation.

It is easy to see that $\mathcal{K}$ satisfies the Symmetry axiom. Clearly, for every symmetric bargaining problem the ideal point is symmetric, and also the disagreement point. Hence, for every symmetric bargaining problem $(S,d) \in \mathcal{B}_2$, we have that $\mathcal{K}_1(S,d) = \mathcal{K}_2(S,d)$.

To see that $\mathcal{K}$ satisfies the axiom of Pareto Optimality is the same as $\mathcal{K}(S,d) \in PO(S)$. Since S is compact and convex we have that $\mathcal{K}(S,d) \in PO(S)$.

Finally, we prove that $\mathcal{K}$ satisfies the Individual Monotonicity assuming that $d = (0,0)$. Let $(S,d), (S',d) \in \mathcal{B}_2$ such that $S \subseteq S'$, $i \in \{1,2\}$ and $a_j(S,d) = a_j(S',d)$ for $j \neq i$, then

$\mathcal{K}_i(S', d) \geq \mathcal{K}_i(S, d)$. The monotonicity follows from the following geometric observations. We will see only when the property for $i = 1$, that is $a_2(S, d) = a_2(S', d)$, then $\mathcal{K}_1(S', d) \geq \mathcal{K}_1(S, d)$, because the other proof is similar.

Let L_α be the line with slope α such that $0 \leq \alpha \leq \pi/2$ passing through d and $a(S, d)$ and let L_β be the line with slope β such that $0 \leq \beta < \alpha$ passing through d and $a(S', d)$. Let $(x_\alpha, y_\alpha), (x_\beta, y_\beta) \in \mathbb{R}^2$ be the intersection point of L_α and L_β, respectively, with the boundary of $\{x \in \mathbb{R}^2 \mid x \geq 0 \text{ and } x \leq x' \text{ for some } x' \in S\}$. Then, we have that $x_\beta \geq x_\alpha$. Let (x', y') is the corresponding intersection point of L_β with the boundary of $\{x \in \mathbb{R}^2 \mid x \geq 0 \text{ and } x \leq x' \text{ for some } x' \in S'\}$. Then, $x' \geq x_\beta \geq x_\alpha$. The following Figure 4.7 illustrates that $\mathcal{K}$ satisfies the Individual Monotonicity.

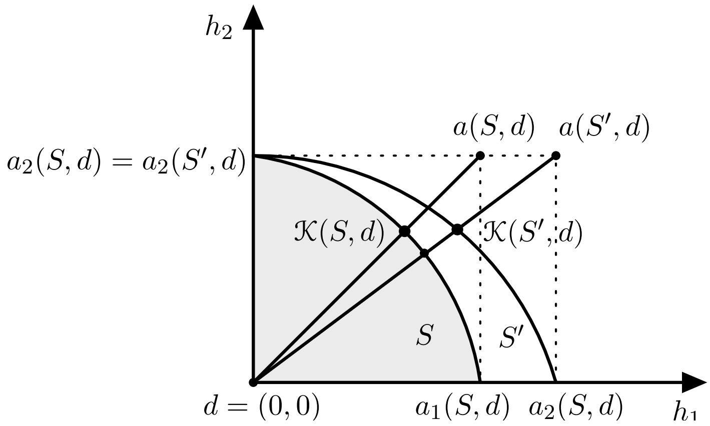

Figure 4.7: $\mathcal{K}$ satisfies the Individual Monotonicity

Now, we show that any bargaining solution $f : \mathcal{B}_2 \to \mathbb{R}^2$ satisfying the four properties is indeed the Kalai-Smorodinsky Bargaining solution $\mathcal{K}$.

Let f be a solution that satisfies the four axioms and let $(S, d) \in \mathcal{B}_2$ be an arbitrary bargaining problem. Since f satisfies Independence of Equivalent Utility Representation, we can assume that $d = (0, 0)$ and $a(S, d)$ has equal coordinates, i.e., $a(S, d) = (1, 1)$. Let $\mathcal{K}(S, d) = x = (x_1, x_2)$ denote the expected utility vector selected for the Kalai-Smorodinsky solution in the bargaining problem (S, d). Clearly, by its definition $x_1 = x_2 \geq 1/2$. Notice that $(1, 0),\ (0, 1) \in S$. The following Figure 4.8 illustrates our problem (S, d).

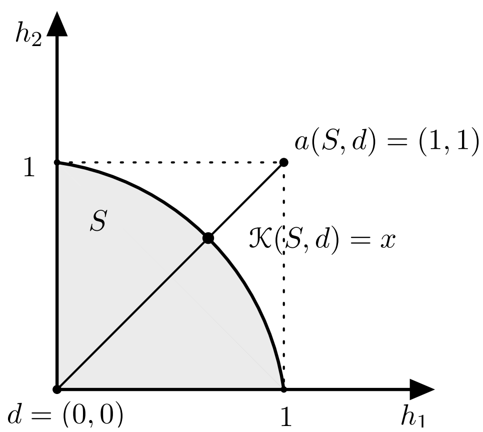

Figure 4.8: Problem (S, d) when $d = (0, 0)$ and $a(S, d) = (1, 1)$

Let S' = convex hull $\{(0,0),(1,0),x,(0,1)\}$. Since $(0,0),(1,0),(0,1)$, $x \in S$ and S is convex then $S' \subseteq S$. The problem (S',d) is symmetric and $x \in PO(S')$ so that by Pareto Optimality and Symmetry, $f(S',d) = x$.

Due to $S' \subseteq S$ and $a(S,d) = a(S',d) = (1,1)$, by the Individual Monotonicity of f applied twice, we conclude that $f(S,d) \geq x$. Since $x \in PO(S)$ then $f(S,d) = x$. Therefore, $f(S,d) = \mathcal{K}(S,d)$. An illustration is given in Figure 4.9.

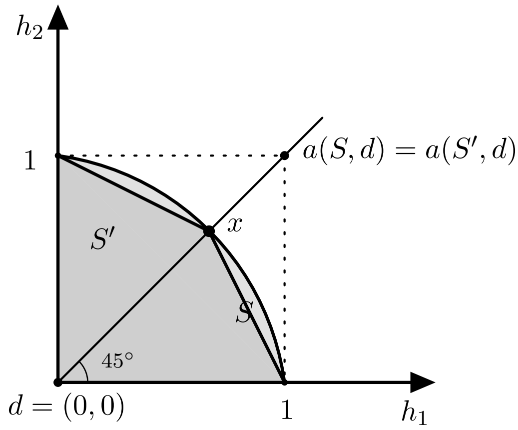

Figure 4.9: Illustration of the Theorem 4.1

□

The Definition 4.3 of the Kalai-Smorodinsky solution can be extended to n-person problems for $n > 2$. In order to make this generalization we need to define the ideal point of (S,d) for $n > 2$: Given $(S,d) \in \mathcal{B}$ and $i \in N$, define *i's ideal point of* (S,d) as

$$a_i(S,d) = max\{x_i \in \mathbb{R} \mid x \in S,\ x_j \geq d_j \ \forall\ j \neq i\}.$$

Therefore, the *Kalai-Smorodinsky Bargaining solution* $\mathcal{K} : \mathcal{B}_n \to \mathbb{R}^n$ is defined by letting $\mathcal{K}(S,d)$ to be equal to the maximal point of S on the segment connecting d and $a(S,d)$.

As we said before, there are several difficulties concerning the possible generalization of the Kalai-Smorodinsky solution to a n-person problems for $n > 2$. Roth (1979b) [32] proved that the solution is not Pareto Optimal on the class of all n-person bargaining games. On such domains the solutions could fail to yield Pareto Optimal points. However, by requiring comprehensiveness of the admissible problems, the Kalai-Smorodinsky solution satisfies the Weakly Pareto Optimal axiom.

AXIOM 6: **Weak Pareto Optimality (WPO)**

The Weakly Pareto Optimal axiom is the weaker version of the Pareto Optimal axiom explained before. This axiom requires that there be no feasible alternative at which all agents are better off than at the solution outcome.

First, we introduce the notation we need. For $(S,d) \in \mathcal{B}$ we define the set of *weakly Pareto optimal points of S* as

$$WPO(S) \equiv \{x \in S \mid \forall\, y \in \mathbb{R}^n,\ y > x \ \Rightarrow\ y \notin S\}.$$

Definition 4.4. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Weak Pareto Optimality* if for all $(S, d) \in \mathcal{B}$

$$f(S, d) \in WPO(S).$$

Therefore, the Kalai-Smorodinsky solution is Pareto Optimal for two agents, but only Weakly Pareto Optimal for more agents. The following example shows that for three agents the Kalai-Smorodinsky solution is Weakly Pareto Optimal but it is not Pareto Optimal.

Example 4.5. Consider the following bargaining problem $(S, d) \in \mathcal{B}_3$, where $S =$ convex hull$\{(0,0,0),(1,1,0),(0,1,1)\} \subset \mathbb{R}^3$ and $d = (0,0,0)$. Notice that S is a triangle into a three-dimensional space.

The Kalai-Smorodinsky solution of the problem (S, d) is $\mathcal{K}(S, d) = (0,0,0)$ which is in fact dominated by all points of S. An illustration of this example is given in Figure 4.10.

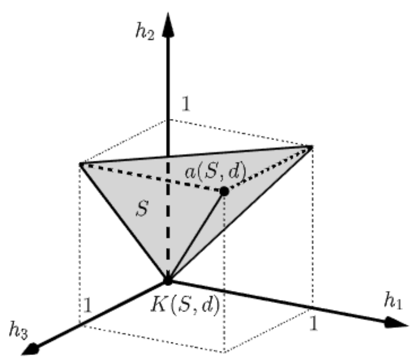

Figure 4.10: Illustration of the Example 4.5

By requiring comprehensiveness of the admissible problems, the solution satisfies the weakening of Pareto Optimality.

The other property that is difficult in extending the Theorem 4.1 to more than two agents is the Individual Monotonicity. There are some possible ways to generalize this axiom but not all of them allow the result to go through. One possibility is simply to write "for all $j \neq i$" in Definition 4.2. Another way presented by Roth (1979b) [32] and Thomson (1980) [34] is to consider expansions that leave the ideal point unchanged.

Another point of view presented in Roth (1979b) [32] is the property of Restricted Monotonicity which is weaker than the property of Individual Monotonicity considered by Kalai and Smorodinsky. This property makes it clear that only this weaker condition is necessary to characterize the solution $\mathcal{K}$ for two-person games.

AXIOM 7: **Restricted Monotonicity (RM)**

The axiom of RM means that if there is an expansion of the feasible set leaving unaffected the ideal point benefits all agents.

Definition 4.5. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Restricted Monotonicity* if for all $(S, d) \in \mathcal{B}$, $S \subseteq S'$ and $a(S, d) = a(S', d)$ imply $f(S, d) \leq f(S', d)$.

An alternative to monotonicity conditions is the Restricted Monotonicity. This conditions says that if the maximal utilities remain the same but the opportunities expand, then all agents should weakly gain. The following figure represents the RM graphically when $n = 2$.

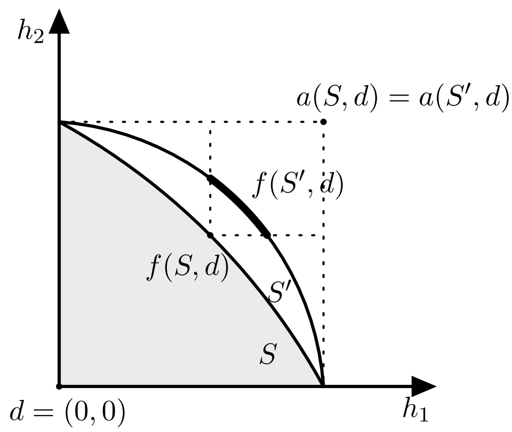

Figure 4.11: Restricted Monotonicity when $n = 2$ and $d = (0, 0)$

It is easy to see that if a solution satisfies Individual Monotonicity then it satisfies Restricted Monotonicity. So, the Kalai-Smorodinsky solution satisfies IM and therefore it satisfies RM as well. Notice that the Nash Bargaining solution does not satisfy Restricted Monotonicity (and consequently it does not satisfy Individual Monotonicity) as the previous Example 4.3 shows. However, the properties of Restricted Monotonicity and Individual Monotonicity are not easily generalize to bargaining problems with $n > 2$.

To sum up, Kalai and Smorodinsky present an alternative unique solution of the bargaining problem (for two person problems). The Kalai and Smorodinsky solution is a result of replacing the axiom of Independence of Irrelevant Alternatives by the Individual Monotonicity, in the characterization of the Nash Solution. However, there are several difficulties to generalize this solution to $n > 2$. For instance, the Kalai-Smorodinsky solution is only Pareto Optimal for two agents, but if there are more agents it is only Weakly Pareto Optimal. Moreover, the Individual Monotonicity could be replaced by a weaker axiom named Restricted Monotonicity.

4.2 Other solutions

In this section, we give other solutions to the bargaining problem so as to show how rich and varied the class of available solutions is. They provide different explanations and rationales. We add a representation to highlight the difference between solutions for two agents and the origin as a disagreement point.

Egalitarian solution

After presenting the characterizations of the Nash (1950) [24] and Kalai and Smorodinsky (1975) [16] of the bargaining problems, Kalai (1977) [14] present an alternative axiom which leads to another unique solution. These three solutions are the classic characterizations that occupy center stage in the theory as it stands today.

Kalai believes that a solution to the bargaining problem is a function $f : \mathcal{B} \to \mathbb{R}^n$ that satisfy three axioms, these axioms are Symmetry, Weak Pareto Optimality and Strong Monotonicity. Two of these axioms are explained in the previous sections, but the axiom of Strong Monotonicity remains to be explained.

AXIOM 8: **Strong Monotonicity (SM)**

The Strong Monotonicity axiom requires that, if the feasible set expands, then all the agents should weakly gain.

Definition 4.6. A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies *Strong Monotonicity* if for all (S, d), $(S', d) \in \mathcal{B}$, $S \subseteq S'$ then $f(S, d) \leq f(S', d)$.

The following figure represents this axiom graphically when $n = 2$

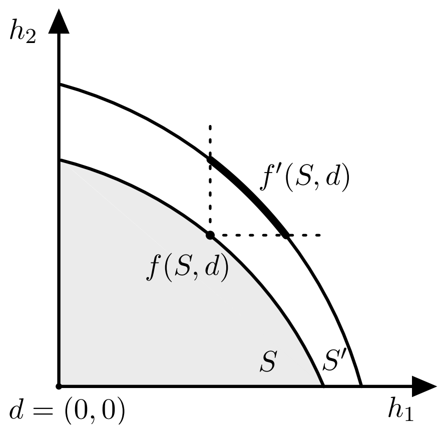

Figure 4.12: Strong Monotonicity when $n = 2$ and $d = (0, 0)$

The Egalitarian solution satisfies the axiom of Strong Monotonicity, but neither the Nash Solution or the Kalai and Smorodinsky Solution does.

After presenting the axiom of Strong Monotonocity, we can define the Egalitarian solution. Kalai (1977) [14] shows that the three axioms stated above define a unique solution to the bargaining problem that is named as the Egalitarian solution.

Definition 4.7 (Egalitarian Solution). The *Egalitarian Bargaining solution* $\mathcal{E} : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S, d) \in \mathcal{B}$, $\mathcal{E}(S, d)$ is a maximal point of S of equal coordinates, i.e.,

$$\mathcal{E}_i(S, d) - d_i = \mathcal{E}_j(S, d) - d_j \ \ \forall\, i, j \in N.$$

The egalitarian solution selects the maximal point of S at which utility gains from d are equal.

Observation. If we introduce the concept of individual rationality, the definition of the egalitarian solution can be rewritten. This concept requires that each agent prefer an agreement to disagreement. Formally, the *individually rational set* for problem (S, d) is

$$I(S, d) = \{x \in S \mid x \geq d\}.$$

So, the Definition 4.7 could be rewritten as follows. The *Egalitarian Bargaining solution* $\mathcal{E} : \mathcal{B} \to \mathbb{R}^n$ is defined, for every $(S, d) \in \mathcal{B}$, as

$$\mathcal{E}_i(S, d) = d_i + \arg \max_{x \in I(S,d)} \big(\min_{k=1,\dots,n} \{x_k - d_k\} \big), \qquad \text{for} \ \ i = 1, \dots, n.$$

Geometrically, $\mathcal{E}(S, d)$ is the intersection of the boundary of S and the half line that starts at d and passes through $d + (1, \dots, 1)$.

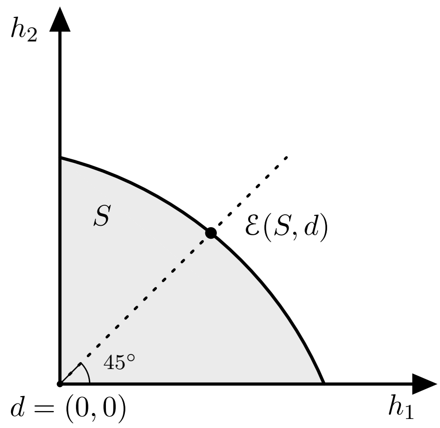

Figure 4.13: Egalitarian solution for two agents and $d = (0, 0)$

Figure 4.13 illustrates geometrically the Egalitarian solution to the bargaining problem when $n = 2$ and $d = (0, 0)$.

In particular, Kalai obtained a characterization of the egalitarian solution of the bargaining problem which satisfies the axioms Strong Monotonicity, Symmetry and Weak Pareto Optimality.

Theorem 4.2 (Kalai, 1977). *A solution $f : \mathcal{B} \to \mathbb{R}^n$ satisfies the Strong Monotonicity, Symmetry and Weak Pareto Optimality if and only if f is the Egalitarian solution of the bargaining problem, i.e., $f = \mathcal{E}$.*

Proof. See Kalai (1977) [14]. □

The comprehensiveness of S is needed to satisfy Weak Pareto Optimality of the egalitarian solution $\mathcal{E}(S,d)$ even if $n = 2$. Moreover, Luce and Raiffa (1957) [19] proves that without comprehensiveness of S the axioms of Weak Pareto Optimality and Strong Monotonicity are incompatible.

Dictatorial solution

The Dictatorial solutions $\mathcal{D}^i(S,d)$ and $\mathcal{D}^{*i}(S,d)$ are extreme cases of solution favoring one agent at the expense of the others. For an agent i, the Dictatorial solution chooses the alternative that maximizes agent i's payoff among those at which the remaining agents receive their disagreement payoffs.

Definition 4.8 (Dictatorial Solutions). The *Dictatorial Bargaining solution* $\mathcal{D}^i : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S,d) \in \mathcal{B}$ and $i \in N$, $\mathcal{D}^i(S,d)$ is the maximal point of S with $x_j = d_j, \ \forall\, j \in N, \ j \neq i$, i.e.,

$$\mathcal{D}^i_j(S,d) = d_j \ \text{ and } \ \mathcal{D}^i_i(S,d) \text{ is the maximal point of } S, \quad \forall\, j \in N, \ j \neq i.$$

The *Dictatorial* Bargaining solution* $\mathcal{D}^{*i}(S,d)$ is the Pareto Optimal point with maximal i^{th} coordinate.

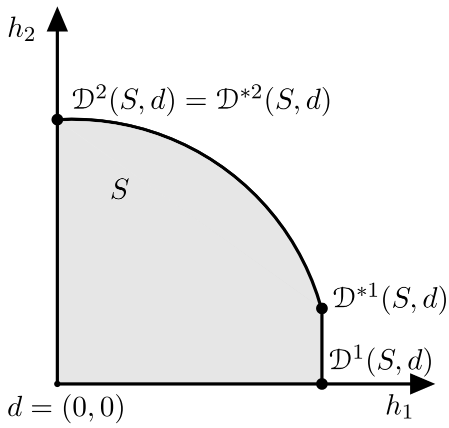

Figure 4.14: Dictatorial solutions for two agents and $d = (0,0)$

Figure 4.14 illustrates geometrically the Dictatorial solutions to the bargaining problem when $n = 2$ and $d = (0,0)$. If S is strictly comprehensive[2], $D^i(S) = D^{*i}(S)$. Moreover, if $n > 2$, the maximizer of x_i in $PO(S)$ may not be unique so, some rule needs to be formulated to break possible ties. In this case, a lexicographic operation is often suggested.

Both dictatorial solutions violate the axiom of Symmetry. However, these solutions satisfy the axioms of Independence of Equivalent Utility Representation, Independence of Irrelevant Alternatives and Strong Monotonicity. Moreover, these solutions are used as a basis for building other solutions that are later explained.

[2]The bargaining problem (S,d) is *strictly d-comprehensive* if $d \leq y \leq x$ and $x \in S$ imply $y \in S$ and $y \notin WPO(S)$.

Discrete Raiffa solution

In two versions of a paper on arbitration schemes Raiffa (1951) [29] and (1953) [30] almost simultaneously with Nash (1950) [24] and (1953) [25] proposed and analyzed four different bargaining solutions, among them the discrete Raiffa solution. However, this solution concept was unexplored for a while despite its attractive features and also being mentioned by Luce and Raiffa (1957) [19].

The discrete Raiffa solution selects the limit of the sequence $\{z^t\}$ defined as follows: z^1 is the average of the dictatorial solution outcomes; z^2 is the average of the dictatorial solution outcomes obtained when z^1 is used as disagreement point, and so on.

Definition 4.9 (**Discrete Raiffa Solution**)**.** The *Discrete Raiffa Bargaining solution* $\mathcal{R}^d : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S, d) \in \mathcal{B}$, $\mathcal{R}^d(S, d)$ is the limit point of the sequence $\{z^t\}$ defined by ${}^i x^0 = \mathcal{D}^i(S, d)$ for $i \in N$,

$$z^t = \frac{1}{n} \cdot \sum_{i=1}^{n} {}^i x^{t-1} \quad \text{for } t = 1, 2, \ldots$$

where ${}^i x^{t-1}$ is the Dictatorial solution for the problem (S, z^{t-1}) for $t > 1$, i.e., the Weakly Pareto Optimal point with ${}^i x^t_j = z^t_j$ for all $j \neq i$.

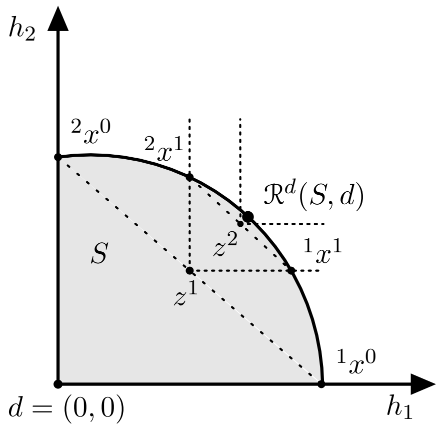

Figure 4.15: Discrete Raiffa solution for two agents and $d = (0, 0)$

Figure 4.15 illustrates geometrically the Discrete Raiffa solution to the bargaining problem when $n = 2$ and $d = (0, 0)$.

The Discrete Raiffa solution satisfies the axioms of Pareto Optimal and Independence of Equivalent Utility Representation.

Perles-Maschler solution

The two-person Perles-Maschler solution appears in Perles and Maschler (1981) [28]. A year later, Perles (1982) [27] shows that the axioms are incompatible even for 3-person bargaining games. A generalization of this solution concept for n-person games appears in Calvo and Gutiérrez (1993) [4].

We will define and discuss the Perles-Maschler solution presented by Perles and Maschler (1981) [28] for a bargaining problems (S, d) with two agents when ∂S is polygonal.

Definition 4.10 (Perles-Maschler Solution). The *Perles-Maschler Bargaining solution* $\mathcal{PM} : \mathcal{B}_2 \to \mathbb{R}^2$ is defined as follows: for every $(S, d) \in \mathcal{B}_2$, if ∂S is polygonal, $\mathcal{PM}(S, d)$ is the common limit point of the sequences $\{x^t\}$, $\{y^t\}$, defined by

$$x^0 = (x_1^0, x_2^0) = \mathcal{D}^{*1}(S, d) \quad \text{and} \quad y^0 = (y_1^0, y_2^0) = \mathcal{D}^{*2}(S, d),$$

where $\mathcal{D}^*$ is the Dictatorial* solution, and for each $t \in \mathbb{N}$, x^t, y^t are Pareto Optimal such that

(i) $x_1^t \geq y_1^t$,

(ii) the segments $[x^{t-1}, x^t]$, $[y^{t-1}, y^t]$ are Pareto Optimal and

(iii) the products $|(x_1^{t-1} - x_1^t)(x_2^{t-1} - x_2^t)|$ and $|(y_1^{t-1} - y_1^t)(y_2^{t-1} - y_2^t)|$ are equal and maximal.

For every $(S, d) \in \mathcal{B}_2$, if ∂S is not polygonal, $\mathcal{PM}(S, d)$ is defined by approximating S by a sequence of problems with polygonal border and taking the limit of associated solution outcomes.

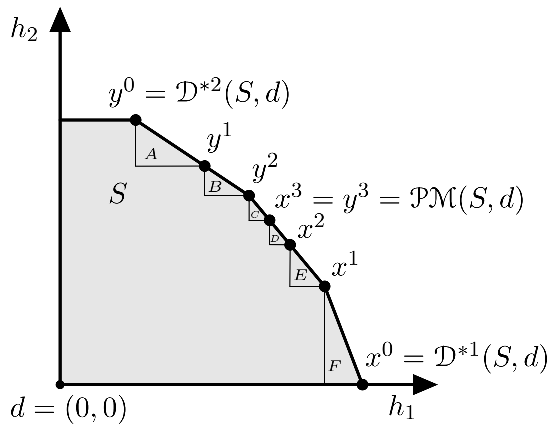

Figure 4.16: Perles-Maschler solution for two agents and $d = (0, 0)$

Figure 4.16 illustrates geometrically the Perles-Maschler solution to a polygonal bargaining problem when $n = 2$ and $d = (0, 0)$. Notice that the equality of the products implies that the triangles of Figure 4.16 are matched in pairs of equal area. So, this solution is the limit of the sequences $\{x^t\}$ and $\{y^t\}$, which are constructed in such a way that $x^0 = \mathcal{D}^{*1}(S, d)$ and $y^0 = \mathcal{D}^{*2}(S, d)$ are the two Dictatorial solutions and the areas A, B, C, D, E and F are maximal and they satisfy $A = F$, $B = E$ and $C = D$.

The Perles-Maschler solution satisfies the axioms of Pareto optimal and Independence of Equivalent Utility Representation axioms. However, it does not satisfy the Independence of Irrelevant Alternatives or Restricted Monotonicity axioms.

Equal Area solution

The Equal Area solution is analyzed in Anbarci and Bigelow (1994) [1]. This solution chooses the Pareto Optimal alternative at which the area of the set of feasible outcomes right of the vertical line and above the horizontal line are equal.

We will define the notion of the equal area solution in dimension 2. However, there are several possible generalizations for $n \geq 3$.

Definition 4.11 (**Equal Area Solution**)**.** The *Equal Area Bargaining solution* $\mathcal{A} : \mathcal{B}_2 \to \mathbb{R}^2$ is defined as follows: for every $(S, d) \in \mathcal{B}_2$, $\mathcal{A}(S, d)$ is the point $x \in PO(S)$ such that the area of S to the right of the vertical line through x is equal to the area of S above the horizontal line through x.

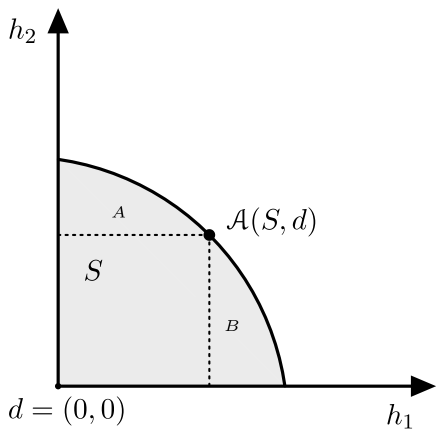

Figure 4.17: Equal Area Solution for two agents and $d = (0, 0)$

Figure 4.17 illustrates geometrically the Equal Area solution to the bargaining problem when $n = 2$ and $d = (0, 0)$. Notice that the areas A and B satisfy that $A = B$.

This solution violates the axiom of Independence of Irrelevant Alternatives but it satisfies the Independence of Equivalent Utility Representation axiom.

Utilitarian solution

The Utilitarian solution is a characterization of utilitarianism theory. Utilitarianism is a term founded at the end of the 18th century by Jeremy Bentham, who suggested that one should bring out "the greatest happiness of the greatest number". In the mid-19th century, John Stuart Mill carried on the theory of utilitarianism, he contributed to and refines this philosophical position. In particular, he argued that there are determinants of utility that should not be endorsed by the utilitarian aggregation. For instance, *Schadenfreude* (enjoyment of someone else's misfortune) is a type of enjoyment that should be excluded from the summation of utilities be maximized by society.

The Utilitarian solution dates back to the mid-19th century. This solution involves interpersonal comparisons of utilities. Each utilitarian solution selects an alternative at which the sum of payoffs is maximal among all alternatives. It moves the hyperplanes $\sum_{i=1}^n x_i = k$ where $x \in S$ and $k \in \mathbb{R}$, from the point of disagreement over S to the extreme point or points, from which it no longer has an intersection.

Definition 4.12 (**Utilitarian Solution**)**.** The *Utilitarian Bargaining solution* $\mathcal{U} : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S,d) \in \mathcal{B}$, $\mathcal{U}(S,d)$ is a maximizer in S of $\sum_{i=1}^n x_i$, i.e.,

$$\mathcal{U}(S,d) = \arg \max_{x \in I(S,d)} \sum_{i=1}^{n} x_i.$$

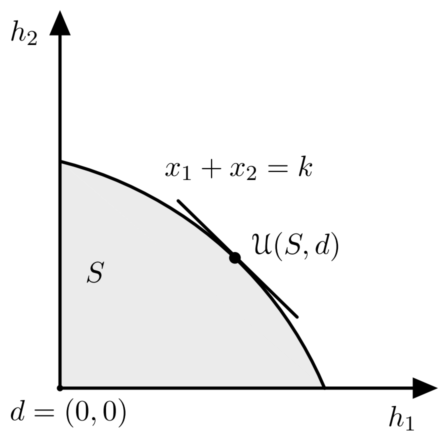

Figure 4.18: Utilitarian solution for two agents and $d = (0,0)$

Figure 4.18 illustrates geometrically the Utilitarian solution to the bargaining problem when $n = 2$ and $d = (0,0)$.

This solution presents some difficulties. On the one hand, the maximizer may not be unique. The common way to sort out this problem for $n = 2$ is to select the midpoint of the segment of maximizers. This rule can be also generalized in different ways if $n > 2$. On the other hand, the utilitarian solution to (S,d) is independent of d. A partial remedy is to search for a maximizer of $\sum_{i=1}^n x_i$ among the points of S that dominate d. In spite of these limitations, the Utilitarian solution is often defended.

The Utilitarian solution satisfies the axioms of Pareto Optimal and Independence of Irrelevant Alternatives even though it violates Restricted Monotonicity. For more information on this solution, see Myerson (1981) [22] and Thomson (1981) [35].

Yu solution

Yu Solution is analyzed by Yu (1973) [40] and Freimer and Yu (1976) [11]. They defined the Yu Solution by minimizing the p-distance from the ideal point to the feasible set.

To define the Yu solution it is necessary to explain previously the notion of p-distance. In the Euclidean space $\mathbb{R}^n$, the distance between two points is usually given by the Euclidean distance, although other distances based on others norms are sometimes used instead. In this case, the p-distance is used which is associated with the p-norm. Let $n \in \mathbb{N}$ and $p \in \mathbb{R}$, $p \geq 1$, the *p-norm* $\|\cdot\|_p$ is the norm on the real finite dimensional vector space $\mathbb{R}^n$ given by the p-th root of the sum of the p-powers of the absolute value of the components of a given vector $x \in \mathbb{R}^n$, i.e.,

$$\|x\|_p = \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p}.$$

After presenting the notion of p-distance, we can define the Yu Area solution.

Definition 4.13 (**Yu Area Solution**)**.** Given $p \in (1, \infty)$, the *Yu Area Bargaining solution* $\mathcal{Y}^p : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S, d) \in \mathcal{B}$, $\mathcal{Y}^p(S, d)$ is the point of S for which the p-distance[3] to the ideal point of the problem (S, d) is minimal, i.e.,

$$\mathcal{Y}^p(S, d) = \arg \min_{x \in I(S,d)} \left(\sum_{i=1}^{n} |a_i(S) - x_i|^p \right)^{1/p}.$$

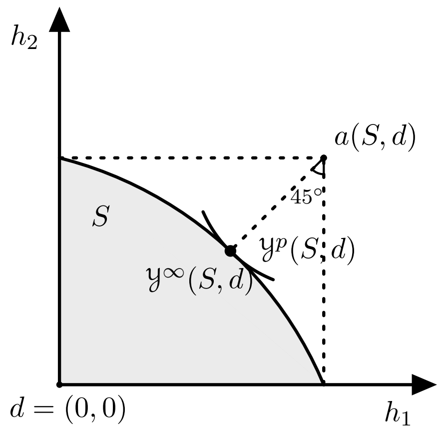

Figure 4.19: Yu Area solution for two agents and $d = (0, 0)$

[3] The 1-norm distance is called the taxicab norm, the 2-norm distance is the Euclidean distance and the infinity norm distance is the distance of the maximum.

Figure 4.19 illustrates geometrically the Yu Area solution to the bargaining problem when $n = 2$ and $d = (0, 0)$. When $p = \infty$, the Yu solution is given by the point that maximizes $min_{i=1,\ldots,n}\{|a_i(S) - x_i|\}$ in $x \in S$ but this may not yield an unique outcome except for $n = 2$.

The Yu Area solution satisfies the axioms of Pareto Optimality and Individual Monotonicity. However, this solution violates the Independence of Equivalent Utility Representation, Independence of Irrelevant Alternatives and Strong Monotonicity axioms.

Furthermore, Chun (1988) [5] proposes a solution that is a variant of the family of the solutions introduced by Yu (1973) [40]. This solution is the Equal Loss solution which equalizes across agents the losses from the ideal point.

The *Equal Loss Bargaining solution* $\mathcal{E}^* : \mathcal{B} \to \mathbb{R}^n$ is defined as follows: for every $(S, d) \in \mathcal{B}$, $\mathcal{E}^*(S, d)$ is the maximal point x of S with

$$a_i(S, d) - x_i = a_j(S, d) - x_j, \text{ for all } i, j \in N.$$

The Equal Loss solution satisfies Weak Pareto Optimality and Symmetry axioms.

To conclude, we have seen other possible solutions to the problem of bargaining that are not the best-known solution, the Nash Solution. These solutions are based on different axioms and theories. However, it is difficult to find the "perfect solution" which has not yet been found or does not exist. The bargaining theory needs to be explored in the future in order to find a solution that could better fit to our reality.

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