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Treball final de grau

Special Solutions of the N-Body Problem: Central Configurations and Choreographies

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Abstract

The n-body problem is a classical problem in celestial mechanics which attempts to describe the motion of n bodies under their mutual gravitational attraction. The problem is only solvable for two masses, and not much is known for the general case of three or more bodies.

This work deals with some particular solutions of the n-body problem. First, using its underlying Hamiltonian structure, we state the main properties of the problem, its symmetries and first integrals. Next, we study central configurations and their relation with homothetic and relative equilibria solutions. For three bodies, the well-known Lagrange configuration provides a relative equilibria in which three shifted particles in an equilateral triangle move along a periodic orbit, known as a choreography. In the last chapter we consider the figure eight solution which is another choreography of three bodies with some particular geometrical and dynamical properties. Using an ad-hoc implementation of the Taylor method developed for the numerical integration of the n-body problem we illustrate the orbits and the properties of the particular solutions discussed in this work as well as a numerical check of the remarkable linear stability property of the figure eight.

Resum

El problema de n cossos és un problema clàssic de macànica celeste que consisteix en descriure el moviment de n cossos sota la seva atracció gravitatòria. El problema només és resoluble per dos cossos, però no es coneixen gaires resultats pels cassos generals de tres o més.

Aquest treball tracta algunes solucions del problema de n cossos. Primer, utilitzant la subjacent estructura hamiltoniana que aquest poseeix, exposem les propietats principals, les seves simetries i integrals primeres. Tot seguit, estudiem les configuracions centrals i la seva relació amb les solucions homotètiques i d'equilibri relatiu. Pel cas de tres cossos, la configuració de Lagrange proporciona un equilibri relatiu en el qual tres cossos desplaçats formant un triangle equilàter es mouen seguint una òrbita periòdica, una coreografia. En l'últim capítol, considerem la solució de la figura vuit, la qual és una coreografia de tres cossos amb certes propietats geomètriques i dinàmiques. Utilitzant una implementació ad-hoc del mètode de Taylor desenvolupat per la integració numèrica del problema, il·lustrem les òrbites i les propietats de les solucions tractades en aquest treball. A més, també proporcionem una comprovació numèrica de l'estabilitat lineal de la figura vuit.

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Agraïments

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Contents

Introduction

The main goal of this work is to present a way of finding solutions of the n-body problem. To do so, we are going to study a significant sort of configurations of masses, *central configurations*, and a special type of solutions, *choreographies*. We will introduce them in a general way and state their main properties, as well as describe the motion of these particles in the setting of the n-body problem. However, once stated, we will mainly focus in some specific examples of these special solutions.

The Newtonian n-body problem is a classical problem in celestial mechanics which consists in describing the motion of an arbitrary number of bodies, namely n ($n \geq 2$), under the influence of their mutual gravitational attraction, described by Newton's universal gravitational law.

We could generalize the statement of the problem by setting a general arbitrary force or vector field instead of the gravitational force, and describing then the motion of the particles under the influence of such force. Nonetheless, here we will discuss only the classical gravitational problem. Therefore, we are going to consider Newton's law of gravitation stated by Newton in his well known work *Principia Mathematica*, in order to describe the motions of the set of particles (we leave aside relativistic features with the masses given by General Relativity).

We are going to study the motion of the n bodies from the point of view of classical Hamiltonian mechanics. Hence, an introduction to this topic will also be done. Furthermore, we are going to use a Taylor numerical method to compute approximate solutions of the problem and throughout this work we are going to exhibit the plots of solutions obtained with this method.

We note that usually the problem is contemplated for physical spatial dimensions 2 or 3. However, we are going to present the problem and give most of the results for an arbitrary spatial dimension d, unless otherwise stated. Besides, all the plots hereby presented have been made only for solutions in the plane, for a better visualization of the geometrical properties.

The manuscript is organized as follows. In the first chapter we are going

to introduce the problem and we will provide the Hamiltonian formalism to it. Thereby we will be able to study its first integrals, which will facilitate our study and comprehension of the problem. In the last section of the chapter, we will briefly center our attention to the 2-body problem and we will relate our previous findings to the Kepler problem.

In the second chapter, we will study central configurations. Central configurations are a special kind of configurations of masses that satisfy a particular condition relating their acceleration vectors with the relative position to the other masses. We will discuss the reason of studying them for the n-body problem and we will prove some results of their existence as well. Additionally, in the last two sections of the chapter we will explain two different specific central configurations, which give rise to the only two explicit solutions known for the 3-body problem.

In the third and last chapter of this work, we will explain an extraordinary solution of the n-body problem, whose discovery resulted into the exploration of a new kind of solutions of the problem, choreographies. We will mainly focus on the figure eight solution and, although we will not present the proof of its existence, we will give an insight on its motion and characteristics. Moreover, we will numerically study the orbit, first by finding the complete initial conditions and computing the trajectory and then, by checking its distinguished attribute: linear stability.

This work also includes an appendix containing an algorithm to obtain the trajectory of the bodies, developed specifically for our problem and implemented in C. It solves numerically the ODE system of the equations of motion of the n-body problem in d dimensions, making use of the Taylor integration method.

Chapter 1

The n-body problem

We are going to study the n-body problem from the insight of Hamiltonian systems in order to analyse properly the properties of the associated system, with special emphasis on the search of first integrals of the problem. In the last section we will exclusively focus on the problem of two bodies.

1.1 Stating the problem

In this section we introduce the n-body problem, which considers the motion of n bodies under their mutual gravitational interaction. We consider this interaction given by Newton's universal gravitational law.

Consider the vector space $\mathbb{R}^d$ endorsed with an orthogonal basis, so that defines a cartesian coordinate system. Then, we denote the position of the particles with vectors $x_i \in \mathbb{R}^d$ and the masses of such bodies with scalars $m_i \in \mathbb{R}_{>0}$, for $i = 1, \dots, n$. According to Newton's universal law of gravitation, the gravitational force acting on particle i due to the presence of particle j is

$$F_{ij} = \frac{G m_i m_j}{r_{ij}^3}(x_j - x_i),$$

where G is the universal gravitational constant and $r_{ij} = |x_i - x_j|$, with $|\cdot|$ representing the Euclidian norm in $\mathbb{R}^d$. By choosing the appropiate units of mass we can arrange $G = 1$. Taking into account all the other particles of the system, the resulting force F_i acting on the particle i is the sum of the individual forces, that is,

$$F_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} F_{ij} = \sum_{j \neq i} \frac{m_i m_j}{r_{ij}^3}(x_j - x_i).$$

Then the equation of motion for the i-th particle of the system is, by Newton's second law,

$$\ddot{x}_i = \sum_{j \neq i} \frac{m_j}{r_{ij}^3}(x_j - x_i). \tag{1.1}$$

Considering the n particles, this gives a system of nd second-order (autonomous) differential equations. The study of the solutions of this equations system constitutes the *n-body problem*.

Let us denote the right-hand side of equations (1.1) by $f_i(x)$. Note that the function $f = (f_1, \ldots, f_n)$ is only defined in $\mathbb{R}^{nd} \setminus \Delta$, called the *configuration space*, where

$$\Delta = \{x \in \mathbb{R}^{nd} : x_i = x_j \text{ for some } i \neq j\}$$

is called the *collision* or *singular* set. We can express the equations system (1.1) as a system of first-order differential equations by introducing the variable v,

$$\dot{x} = v, \quad \dot{v} = f(x) \tag{1.2}$$

This differential equations system is defined in the *phase space* $\Omega = \{(x, v) : x \in \mathbb{R}^{nd} \setminus \Delta, v \in \mathbb{R}^{nd}\}$.

Let us set $y = (x, v)$ and let $\mathcal{X}_f(y) = (v, f(x))^T$ denote the vector field associated to (1.2). The solutions of the n-body problem will be of the form $y(t) = (x(t), v(t))$ such that $\dot{y} = \mathcal{X}_f(y)$. We have that the vector field $\mathcal{X}_f$ is $\mathcal{C}^r$, for any $r \geq 1$, in the open domain Ω and therefore the flow associated to the n-body problem, given by the map

$$\begin{array}{rccc} \varphi: & D \subset \mathbb{R} \times \Omega & \longrightarrow & \Omega \\ & (t; y_0) & \longmapsto & \Phi(t; 0, y_0) \end{array}$$

is also $\mathcal{C}^r$. Here Φ represents the evolution process induced by $\mathcal{X}_f$ and $D = I \times \Omega$ is an open domain, where I is the time interval where the evolution process is defined.

1.2 The Hamiltonian formalism of the n-body problem

Let us now provide a Hamiltonian formalism to the n-body problem. We define $q_i := x_i$ the position vectors and $p_i := m_i \dot{x}_i$ the momenta vectors for each particle of the system. The vectors $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n) \in \mathbb{R}^{nd}$ will be called *configuration* and *linear momentum* vectors respectively.

In this section and those that follow in this chapter, we refer to [1] for general background on the Hamiltonian formalisms.

Definition 1.1. (Hamiltonian system) *A (canonical) Hamiltonian system is a system of differential equations for which the evolution law is given by Hamilton's law*

$$\dot{z} = \mathcal{X}_H(z) = J\nabla H(z)$$

where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ *is a* $2m \times 2m$ *matrix called the standard symplectic matrix, with* I *and* 0 *representing the identity and null matrices, and* H *is called the Hamiltonian function of the system.*

Let us present a property of Hamiltonian systems which is important in our case.

Proposition 1.2. *The Hamiltonian function* $H(q,p)$ *is a first integral of the autonomous vector field*

$$\mathcal{X}_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)^T$$

Proof. Let $(q(t),p(t))^T$ be a solution of $\dot{z} = \mathcal{X}_H(z)$. Then

$$\begin{aligned}\frac{dH}{dt}(q(t),p(t)) &= \frac{\partial H}{\partial q}(q(t),p(t))\dot{q} + \frac{\partial H}{\partial p}(q(t),p(t))\dot{p} = \\ &= \frac{\partial H}{\partial q}(q(t),p(t))\frac{\partial H}{\partial p}(q(t),p(t)) - \frac{\partial H}{\partial p}(q(t),p(t))\frac{\partial H}{\partial q}(q(t),p(t)) = 0\end{aligned}$$

where in the first equality we have used the chain rule and in the second one the fact that H defines a Hamiltonian system with vector field $\mathcal{X}_H$. □

So for any autonomous Hamiltonian system we have that the Hamiltonian function itself is a first integral. Now that we have introduced Hamiltonian systems, let us implement it into the n-body problem.

Consider the function

$$H(q,p) = T(p) + V(q) \tag{1.3}$$

where $T(p)$ and $V(q)$ are known as the *kinetic energy* and the *potential energy* functions, defined by

$$T(p) = \frac{1}{2}\sum_{i=1}^{n}\frac{|p_i|^2}{m_i}, \qquad V(q) = -\sum_{i<j}\frac{m_i m_j}{r_{ij}} = -U(q). \tag{1.4}$$

Proposition 1.3. *The system of equations of the n-body problem in dimension d is Hamiltonian with Hamiltonian function*

$$\begin{aligned} H : \mathbb{R}^{nd} \times \mathbb{R}^{nd} &\longrightarrow \mathbb{R} \\ (q,p) &\longmapsto H(q,p) = T(p) + V(q) \end{aligned}$$

being T and V the kinetic and potential energies defined in (1.4).

Proof. We want to see

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q,p), \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q,p).$$

Differentiating H with respect to each variable we have

$$\frac{\partial H}{\partial p_i} = \frac{\partial T}{\partial p_i} = \frac{p_i}{m_i} = \frac{m_i \dot{q}_i}{m_i} = \dot{q}_i$$
$$\frac{\partial H}{\partial q_i} = \frac{\partial V}{\partial q_i} = \sum_{l<j} \frac{m_l m_j}{r_{lj}^2} \left(\frac{\partial r_{lj}}{\partial q_i} \right) = \sum_{i \neq j} \frac{m_i m_j}{r_{ij}^3}(q_i - q_j) = -m_i \ddot{q}_i = -\dot{p}_i$$

where the previous equality is due to

$$\frac{\partial r_{lj}}{\partial q_{ik}} = \frac{\partial}{\partial q_{ik}} \sqrt{\sum_{s=1}^{d} (q_{ls} - q_{js})^2} = \frac{1}{2 r_{ij}} 2(q_{ik} - q_{jk}) = \frac{1}{r_{ij}}(q_{ik} - q_{jk})$$

□

As a consequence, it follows from Proposition 1.2 that the Hamiltonian function (1.3) is a first integral of the n-body problem, since the associated vector field $\mathcal{X}_H$ is autonomous.

Remark 1.4. In the field of dynamical systems, when the Hamiltonian function is the sum of functions T and V we say that the system is *mechanic*. Additionally, in the case where T is a quadratic form of p we say it is a *natural* system. Thus, the Hamiltonian system of the n-body problem that defines function (1.3) is a natural Hamiltonian system.

Before we proceed further, let us clarify some notation. We shall often use z to denote the vector (q,p), and $m = nd$ to denote the dimension of the variables q and p. Hence, $z \in \mathbb{R}^{2m}$. Additionally, in Chapter 2 we will usually call Newtonian potential function to the function $U(q)$, defined in (1.4) as the negative of the potential energy.

Now, let us consider in the following example the two body problem as a particular case, and for later references to the concrete equations.

Example 1.5. Take the system of equations of motion for the 2-body problem

$$\begin{cases} m_1\ddot{q}_1 = \dfrac{m_1 m_2}{r_{12}^3}(q_2 - q_1) \\ m_2\ddot{q}_2 = \dfrac{m_1 m_2}{r_{12}^3}(q_1 - q_2) \end{cases} \tag{1.5}$$

for $q_1, q_2 \in \mathbb{R}^d$, together with the functions

$$T(p) = \frac{1}{2}\left(\frac{|p_1|^2}{m_1} + \frac{|p_2|^2}{m_2}\right) \quad \text{and} \quad U(q) = \frac{m_1 m_2}{r_{12}}.$$

Then,

$$\begin{aligned}(\nabla_1 U(q))_k &= \frac{\partial U}{\partial q_{1k}}(q) = \frac{\partial U(q)}{\partial r_{12}}\frac{\partial r_{12}}{\partial q_{1k}} = -\frac{m_1 m_2}{r_{12}}\left(\frac{1}{2r_{12}}\right)2(q_{1k} - q_{2k}) \\ &= -\frac{m_1 m_2}{r_{12}^3}(q_{1k} - q_{2k}) = m_1\ddot{q}_{1k} \\ (\nabla_2 U(q))_k &= \frac{\partial U}{\partial q_{2k}}(q) = \frac{\partial U(q)}{\partial r_{12}}\frac{\partial r_{12}}{\partial q_{2k}} = -\frac{m_1 m_2}{r_{12}}\left(\frac{1}{2r_{12}}\right)2(q_{2k} - q_{1k}) \\ &= -\frac{m_1 m_2}{r_{12}^3}(q_{2k} - q_{1k}) = m_2\ddot{q}_{2k}\end{aligned}$$

Therefore, $\dot{p}_1 = m_1\ddot{q}_1 = \nabla_1 U(q)$ and $\dot{p}_2 = m_2\ddot{q}_2 = \nabla_2 U(q)$ which is the result we were looking for.

We know that the dimension of the configuration space is the number of degrees of freedom of the Hamiltonian system. Then, the d-dimensional n-body problem has $m = nd$ degrees of freedom. A Hamiltonian system with m degrees of freedom is integrable if there exist m independent first integral functions in involution (see pg. 7 after Definition 1.9).

In the following section we look for first integrals of the n-body problem. We will see that it admits a total number of $(d+1)d/2 + 1$ first integrals. In particular, the n-body problem is not integrable for $n \geq 3$. The existence of such a set of first integrals will allow us to consider the evolution of the n-body problem on a lower dimensional manifold since motion takes place within the constant level sets of the first integrals. We remark with this argument the importance of knowing first integrals of a Hamiltonian system.

1.3 First Integrals and Symmetries

We want to seek first integrals of the vector field $\mathcal{X}_H$ in order to reduce the dimension of the problem. Before we go deeper into the subject, let us note the

following property.

Property. If the Hamiltonian function $H(q,p)$ is independent of the position vector q_i for some i, then $\dot{p}_i = \frac{\partial H}{\partial q_i} = 0$. Hence, the corresponding momentum vector p_i remains constant, i.e. p_i is a first integral of the vector field $\mathcal{X}_H$. When this property holds, the coordinate q_i is called cyclic.

This property can be seen as a particular case of a much deeper relation between the properties of invariance of the Hamiltonian function and the existence of first integrals. Symmetries of the system are then important to reveal the invariance properties otherwise difficult to detect.

Let us describe such relations in a more concrete way. To this end, we start by introducing how to extend a point transformation from $\mathbb{R}^d$ to the phase space Ω. To do so, we follow the ideas of [8]. Given

$$\begin{array}{rccc} f: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & q & \longmapsto & f(q) =: Q \end{array}$$

a diffeomorphism acting on the space configuration $\mathbb{R}^d$, we can extend f in order to induce to a symplectic map in $\mathbb{R}^{2d}$ in many different ways. These possible extensions are called *Mathieu transformations*. One possibilty is to define a Mathieu transformation M_f from f as the map $M_f(q,p) = (Q,P)$ with

$$Q = f(q), \quad P = Df(q)^{-T} \cdot p \tag{1.6}$$

Mathieu transformations are important because symmetries are translated onto transformations of the configuration space and we want to consider extensions onto the phase space that preserve the Hamiltonian structure while revealing the existence of first integrals. This is the reason why we need the transformation to be symplectic.

Definition 1.6. (Symplectic Map) *A map $f : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is symplectic if its differential map is represented by a symplectic matrix. Recall that a matrix M is symplectic if it satisfies*

$$M^T J M = J$$

where J is the corresponding $2d$-dimensional standard symplectic matrix, as in Definition 1.1.

Proposition 1.7. *The map $M_f : (q,p) \mapsto (Q,P)$ from (1.6) is symplectic.*

Proof. We need to see that the differential map of M_f is represented by a symplectic matrix. In other words, we want to see that

$$DM_f^T(q,p) \cdot J \cdot DM_f(q,p) = J \quad \text{for all } (q,p) \in \Omega$$

Since we have $M_f = \begin{pmatrix} f(q) \\ Df(q)^{-T}p \end{pmatrix}$ and $DM_f(q,p) = \begin{pmatrix} Df(q) & 0 \\ D^2f(q)^{-T}p & Df(q)^{-T} \end{pmatrix}$, then

$$\begin{pmatrix} Df(q) & 0 \\ D^2f(q)^{-T}p & Df(q)^{-T} \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} Df(q) & 0 \\ D^2f(q)^{-T}p & Df(q)^{-T} \end{pmatrix} =$$

$$\begin{pmatrix} -D^2f(q)^{-T}p & Df(q) \\ -Df(q)^{-T} & 0 \end{pmatrix} \begin{pmatrix} Df(q) & 0 \\ D^2f(q)^{-T}p & Df(q)^{-T} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

□

Various laws of conservation that exist in nature can be seen simply as particular cases of the so-called Noether Theorem. Noether Theorem relates symmetries on the phase space with first integrals of the problem. We refer to [1] for further details on the general statement. Below we present a simplified version of the theorem in the context of Hamiltonian systems.

Theorem 1.8. *Consider a Hamiltonian system $\dot{z} = J\nabla H(z) = \mathcal{X}_H(z)$ with m degrees of freedom. We say that $\dot{z} = \mathcal{X}_H(z)$ admits a one-parameter group of symplectic symmetries if there exists a Hamiltonian function $F : \mathbb{R}^{2m} \to \mathbb{R}$ such that the flow $\psi(s,z)$ associated to the system $\dot{z} = J\nabla F(z)$ satisfies*

$$H(\psi(s,z)) = H(z) + C \qquad \forall s \in \mathbb{R} \tag{1.7}$$

where $C \in \mathbb{R}$ is a constant. Then, F is a first integral of $\mathcal{X}_H$.

Before proving the theorem let us introduce the *Poisson bracket* in order to simplify the notation.

Definition 1.9. *Given two functions H and F, the Poisson bracket is defined as the binary operation $\{H,F\}(z) := DH(z)\mathcal{X}_F(z)$.*

We say two first integrals are *in involution* if their Poisson bracket vanishes identically, i.e. $\{H,F\} \equiv 0$. We recall that integrability in the sense of Liouville-Arnold requires that first integrals of motion are independent and in involution.

Furthermore, the Poisson bracket is anticommutative,

$$\begin{aligned} -\{F,H\} &= -DF \cdot \mathcal{X}_H = -DF \cdot J \cdot \nabla H = -DF \cdot J \cdot D^T H = -(DH \cdot J^T \cdot D^T F)^T \\ &= -(DH \cdot J^T \cdot D^T F) = DH \cdot J \cdot D^T F = DH \cdot \mathcal{X}_F = \{H,F\} \end{aligned}$$

where in the last equalities we have used that the product inside the parenthesis is an scalar and the property $J^T = -J$.

We proceed now to prove Theorem 1.8.

Proof. From (1.7) we have that $\frac{d}{ds}H(\psi(s,z)) = 0$. Using that $\psi(s,z)$ is the flow associated to the system $\mathcal{X}_F$, for all $(s,z) \in \mathbb{R} \times \Omega$ we have

$$\begin{aligned} \frac{d}{ds}H(\psi(s,z)) &= DH(\psi(s,z))\frac{\partial \psi}{\partial s}(s,z) = DH(\psi(s,z))J\nabla F(\psi(s,z)) \\ &= DH(\psi(s,z))\mathcal{X}_F(\psi(s,z)) = 0 \end{aligned} \tag{1.8}$$

We have obtained that $\{H,F\}(\psi(s,z)) = 0$ for all $(s,z) \in \mathbb{R} \times \Omega$, which is equivalent to $\{H,F\} \equiv 0$. Hence, H and F are in involution. Moreover, since the Poisson bracket is anticommutative we obtain $\{F,H\} \equiv 0$, or equivalently,

$$0 = DF(\psi(s,z))\mathcal{X}_H(\psi(s,z)) = DF(\psi(s,z))\frac{\partial \psi}{\partial s}(s,z) = \frac{d}{ds}F(\psi(s,z))$$

for all $(s,z) \in \mathbb{R} \times \Omega$. Therefore, F is a first integral of the vector field $\mathcal{X}_H$. □

This theorem shows how to find first integrals of the n-body problem out of its symmetries.

Corollary 1.10. *The n-body problem is translation invariant. Thus, we have that the total linear momentum is preserved.*

Proof. By linearity it is enough to see that the n-body problem is translation invariant along the different directions defined by the canonical basis vectors $e_1, \ldots, e_d$ from the spatial vector space $\mathbb{R}^d$.

For a configuration vector $q \in \mathbb{R}^m$ ($m = nd$), consider $f(q) = q + sE_k$, $s \in \mathbb{R}$, a translation in the e_k direction, i.e. $E_k = (e_k, \ldots, e_k) \in \mathbb{R}^m$. Then the Mathieu transformation is $M_f(q,p) = (q + sE_k, p)$.

We have that $M_f(q,p) = \psi(s,(q,p))$, where $\psi(s,(q,p))$ is the Hamiltonian flow associated to a function $F(q,p)$. Then $\dot{\psi}(s,(q,p)) = (E_k, 0)$ and therefore,

$$F(q,p) = \sum_{i=1}^{n} p_i \cdot e_k = p \cdot E_k$$

We can express the associated vector field as $\mathcal{X}_F = (E_k, 0)^T$. Moreover,

$$\begin{aligned} H(\psi(s,(q,p))) &= H(q + sE_k, p) = T(p) + V(q + sE_k) \\ &= T(p) + V(q) = H(q,p) \end{aligned}$$

i.e. the Newtonian problem is translation invariant. Thus, $\dot{z} = \mathcal{X}_H(z)$ admits a continuous one-parameter symmetry group spanned by F which is a first integral

of the vector field $\mathcal{X}_H$. In fact,

$$0 = \{H, F\}(\psi(s,(q,p))) = DH(\psi(s,(q,p))\mathcal{X}_F(\psi(s,(q,p))) = \begin{pmatrix} \frac{\partial H}{\partial q}(q+sE_k,p) \\ \frac{\partial H}{\partial p}(q+sE_k,p) \end{pmatrix}^T \begin{pmatrix} E_k \\ 0 \end{pmatrix} = \left(\frac{\partial H}{\partial q_i}\right)_i E_k = \sum_{i=1}^{n} \dot{p}_i \cdot e_k$$

which is equivalent to say that $\dot{p}_i = 0$ for $i = 1, \ldots, n$. Thus, the total linear momentum defined as $p_{tot} := \sum_{i=1}^{n} p_i$ is a constant quantity. □

At this point, we introduce an important element to consider when working on the n-body problem, which is the center of mass of a configuration.

Definition 1.11. *Given a configuration $q = (q_1, \ldots, q_n) \in \mathbb{R}^{nd}$ with masses $m_1, \ldots m_n$, the center of mass of the configuration q is defined as*

$$c = \frac{1}{m_0}(m_1 q_1 + \cdots + m_n q_n) \in \mathbb{R}^d$$

where $m_0 = m_1 + \cdots + m_n$ is the total mass of the configuration.

We note that

$$\dot{c} = \frac{1}{m_0} \sum_{i=1}^{n} p_i = \frac{p_{tot}}{m_0}$$

and from Corollary 1.10 we obtain

$$\ddot{c} = \frac{1}{m_0} \sum_{i=1}^{n} \dot{p}_i = 0$$

Therefore the center of mass of the system, $c(t)$, moves with uniform rectilinear motion.

Moreover, it follows from the translation invariance property that given any solution of the n-body problem, we can get another solution for which the center of mass is placed at the origin, just using simple translations of coordinates. Such solutions will be called *centered*. Actually, we can say more, from any given solution, we can get a centered one and even with zero total linear momentum.

Proposition 1.12. *Let $q(t)$ be a solution of the n-body problem with total linear momentum p_{tot} and total mass m_0. Then there exists a constant vector $c_0 \in \mathbb{R}^d$ such that the solution $\bar{q}(t) = q(t) - p_{tot}t/m_0 - c_0$ is centered and has total momentum zero.*

Proof. We know that $\bar{q}(t)$ is a solution of the n-body problem as a consequence of Corollary 1.10. We will see now that it satifies the properties stated.

First consider the configuration $\hat{q}(t) = q(t) - p_{tot}t/m_0$. Let $\hat{p}_{tot}$ and $\hat{c}$ denote the total linear momentum and center of mass of the configuration $\hat{q}$. Then,

$$\hat{p}_{tot} = \sum_{i=1}^{n} \hat{p}_i = \sum_{i=1}^{n} m_i \dot{\hat{q}}_i = \sum_{i=1}^{n} m_i(\dot{q}_i - p_{tot}/m_0) = \sum_{i=1}^{n} p_i - p_{tot} = 0$$

Hence, $\dot{\hat{c}} = \hat{p}_{tot}/m_0 = 0$, that is, $\hat{c}$ is constant. We can take $c_0 = \hat{c}$ so then $\bar{q}(t) = q(t) - p_{tot}t/m_0 - c_0$ is a configuration with center of mass at the origin and zero total linear momentum. □

Thus, we can assume from now on that all the solutions that we deal with are centered in the origin without loss of generality. We will use this result when finding solutions in Chapter 2.

Now we present another consequence of Theorem 1.8.

Corollary 1.13. *The n-body problem is invariant under rotations, hence the angular momentum of the system is preserved.*

Proof. Consider a one-parameter family of matrices $Q_s \in SO(m)$ of the form $Q_s = e^{sA}$, where A is an antisymmetric matrix (i.e. $A^T = -A$). The one-parameter group $\{Q_s\}_s$ is a subgroup of $SO(m)$.

Consider the diffeomorphism given by $f(q) = Q_s q$. Then, the Mathieu transformation is $M_f(q,p) = (Q_s q, Q_s p)$ since $(Df(q)^{-1})^T = (Q_s^T)^T$. Besides, $M_f(q,p) = \psi(s,(q,p))$, where $\psi(s,(q,p))$ is the flow associated to

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \Leftrightarrow \dot{z} = \hat{A}z = \mathcal{X}_F(z)$$

where $\hat{A} := \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ is a $2m \times 2m$ antisymmetric matrix and $F = \frac{1}{2} z^T \hat{S} z$, with $\hat{S} := -J\hat{A}$, a $2m \times 2m$ symmetric matrix.

On the other hand, $\hat{A}$ can be written in a certain basis as

$$\hat{A} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

The invariance with respect to rotations on an arbitrary plane implies that, if $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ then

$$F_{12} = \frac{1}{2} \begin{pmatrix} q_1 & q_2 & p_1 & p_2 \end{pmatrix} \hat{S}_{12} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = p_1 q_2 - p_2 q_1$$

Since the plane is arbitrary (due to the fact that Q_s is any rotation and due to the change of basis) as well as the Hamiltonian H is rotation invariant ($H(\psi(s, (q, p))) = H(Q_s q, Q_s p) = H(q, p)$), we have that F, which we define as the angular momentum, is a first integral of $\mathcal{X}_H$. □

Remark 1.14. The angular momentum is an antisymmetric $(2, 0)$-tensor. This is a consequence of the fact that two vectors determine a plane in which rotation is measured. It admits a representation as

$$L_{kl} = \sum_{i=1}^{n} q_{ik} p_{il} - q_{il} p_{ik}$$

wiht L_{kl} being the components of the tensor and q_{ik}, p_{ik} being the k-th component of the position and momentum vectors for the i-th body.

Moreover, for dimension m we have $\frac{m(m-1)}{2}$ independent planes. Focusing only in one body, we have for the cases

- $d = 2$: one independent plane, thus the angular momentum is a scalar.
- $d = 3$: three independent planes. We can define a vector $L_v = (L_x, L_y, L_z)$ where $L_x = L_{23} = p_3 q_2 - p_2 q_3$, $L_y = L_{13}$ and $L_z = L_{12}$, identifying it with the components of the angular momentum tensor. Thus, invariance with respect to

  $$R_{yz}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(s) & -\sin(s) \\ 0 & \sin(s) & \cos(s) \end{pmatrix}$$

  gives that $L_x = p_3 q_2 - p_2 q_3$ is a first integral. Analogously, choosing roations R_{xz}, R_{xy} we obtain that the other components of the vector L_v are first integrals.

Corollaries 1.10 and 1.13 provide all the first integrals of the n-body problem (1.2) [8]. As a matter of fact, we have seen that applying any simultaneous translation or rotation to a solution of the n-body problem, we obtain another solution.

This is because Newton's equations are invariant under translations and rotations of all the position and momentum vectors. This property will be important when studying a specific type of solutions in the next chapter.

1.4 The 2-body problem and the Kepler problem

We now focus on the 2-body problem in dimension 3 (see for example [1], [8] and [9]). Consider the equations of motion of the 2-body problem given in Example 1.5 for $q_1, q_2 \in \mathbb{R}^3$. We can separate the two body problem into two different problems by means of the sum and substraction of the equations. By addition, we obtain the problem of motion of the center of mass already discussed after Definition 1.11. By substraction (after simplification of the masses) we obtain,

$$\ddot{q}_2 - \ddot{q}_1 = -\frac{m_1 + m_2}{r_{12}^3}(q_2 - q_1)$$

which we can rewrite as

$$\ddot{q} = -\frac{m_0}{r^3}q, \tag{1.9}$$

that is, we obtain the equations of motion of the *Kepler problem*. Here q is the relative position vector that goes from particle 1 to particle 2 and $r := |q|$, the distance between the particles. Furthermore, the Kepler problem can be seen as the problem of the motion of a particle of position q and unit mass, attracted to a body of mass m_0 fixed at the origin.

Example 1.15. Consider the planar Kepler problem with circular solutions which move at a constant angular speed. These solutions are of the form $q(t) = r(\cos\omega t, \sin\omega t)$ where r, ω are constant. Then we have $\dot{q}(t) = r\omega(-\sin\omega t, \cos\omega t)$ and $\ddot{q}(t) = r\omega^2(-\cos\omega t, -\sin\omega t) = -\omega^2 q(t)$. From equation (1.9) we obtain the relation

$$r^3\omega^2 = m_0$$

which is the so-called Kepler's third law for this example.

Due to the fact that we derived the Kepler problem from the 2-body problem, we obtain the following result as a direct consequence of Proposition 1.3.

Corollary 1.16. *The Kepler problem (1.9) is a Hamiltonian system with Hamiltonian function*

$$H(q,p) = T(p) + V(q) = \frac{|p|^2}{2} - \frac{m_0}{r} \tag{1.10}$$

and the relation of the Hamiltonian variables is now $p = \dot{q}$.

Let us now write (1.10) in the form

$$\frac{|p|^2}{2} = H(q,p) + \frac{m_0}{r}$$

Clearly, the left-hand side of the equation is always positve. Then, if we consider the Hamiltonian H to be negative we obtain a lower limit for the second term on the left,

$$H + \frac{m_0}{r} \geq 0 \Leftrightarrow \frac{m_0}{r} \geq -H$$

or, equivalently, an upper bound for r

$$r \leq -\frac{m_0}{H}. \tag{1.11}$$

Therefore, the trajectory of the mass unit object is bounded when the Hamiltonian function is negative.

Recall that the n-body problem is invariant under rotations in the phase space (see Corollary 1.13) and that implies that the angular momentum is a first integral of the problem. Thus, we have that the angular momentum vector, as defined in Remark 1.14 for dimension 3, is also constant for the Kepler problem.

We distinguish the two different scenarios that the angular momentum vector can give.

Proposition 1.17. *Consider the Kepler problem (1.9) and the angular momentum vector given by $L = q \times p$. Then we have two different situations.*

1. *When $L = 0$, the motion is rectilinear along the radius vector towards the origin, leading to collision with the mass at the origin.*

2. *If $L \neq 0$ then the motion remains normal to the fixed constant vector L. That is, the motion takes place in the plane spanned by q and p.*

Proof.

1. Since $L = 0$, it must be $p(t) = \lambda(t)q(t)$ for some function $\lambda(t)$ and any time t. Under this assumption, we obtain

$$\frac{d}{dt}\left(\frac{q}{r}\right) = \frac{r\dot{q} - \dot{r}q}{r^2} = \frac{\dot{q}}{r} - \frac{q\cdot\dot{q}}{r^3}q = \frac{p}{r} - \frac{q\cdot p}{r^3}q = \lambda\left(\frac{q}{r} - \frac{r^2}{r^3}q\right) = 0$$

   Hence the vector $\frac{q}{r}$ is constant, i.e. $q(t) = r\vec{k}$, so the motion is rectilinear and collinear with the radius direction.

2. It follows from the definition of L. □

Therefore, the Kepler problem can be reduced to dimension 1 or 2, depending wether the angular momentum vector vanishes or not. Assume from now on that we find ourselves in the case $L \neq 0$. Then we can consider the Kepler problem in dimension 2 since the orbits will always remain in a fixed plane.

In the plane where the motion takes place, one can introduce polar coordinates (r, θ). We express the coordinate transformation as $q_1 = r\cos\theta$, $q_2 = r\sin\theta$, where here $q = (q_1, q_2)$, i.e (q_1, q_2) represent the cartesian coordinates of the position vector for the particle in motion. Regarding the canonical basis in $\mathbb{R}^2$, $\{e_1, e_2\}$, we can write $q = q_1 e_1 + q_2 e_2 = r(\cos\theta e_1 + \sin\theta e_2)$. To simplify notation, we indicate $\cos\theta e_1 + \sin\theta e_2 = e_r$.

Then, computing its derivative we obtain

$$\dot{q} = \dot{r}(\cos\theta e_1 + \sin\theta e_2) + r\dot{\theta}(-\sin\theta e_1 + \cos\theta e_2) = \dot{r}e_r + r\dot{\theta}e_\theta$$

where here, $e_\theta = -\sin\theta e_1 + \cos\theta e_2$. The set of vectors $\{e_r, e_\theta\}$ is a basis in the polar coordinate system. Furthermore, the second derivative is

$$\begin{aligned}\ddot{q} = \ddot{r}(\cos\theta e_1 + \sin\theta e_2) + \dot{r}\dot{\theta}(-\sin\theta e_1 + \cos\theta e_2) + \dot{r}\dot{\theta}(-\sin\theta e_1 + \cos\theta e_2) \\ + r\ddot{\theta}(-\sin\theta e_1 + \cos\theta e_2) - r\dot{\theta}^2(\cos\theta e_1 + \sin\theta e_2) = (\ddot{r} - r\dot{\theta}^2)e_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})e_\theta\end{aligned}$$

from where we obtain

$$\begin{cases}\ddot{r} - r\dot{\theta}^2 = -\dfrac{m_0}{r^2} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0\end{cases} \tag{1.12}$$

because in the Kepler problem the acceleration vector in the radial direction has to be the same as in (1.9) and the angular acceleration is zero. These equations represent the Kepler problem in polar coordinates.

We introduce now the *Laplace-Runge-Lenz vector* which is defined as

$$A = p \times L - \frac{m_0}{r}q. \tag{1.13}$$

As a consequence of the preservation of the angular momentum, and due to the fact that the Kepler problem is given by an inverse-square law, we have

Proposition 1.18. *The Laplace-Runge-Lenz vector is a first integral of the Kepler problem.*

Proof. We want to see that

$$\frac{dA}{dt} = \dot{p} \times L - m_0 \frac{d}{dt}\left(\frac{q}{r}\right) = 0.$$

Let us start by computing the first term using the Lagrange's formula for the triple product expansion,

$$\dot{p} \times L = \dot{p} \times (q \times p) = (\dot{p} \cdot p)q - (\dot{p} \cdot q)p$$

Now, we substitute into this expression the value for $\dot{p}$ from (1.9), $\dot{p} = \ddot{q}$, and rearranging we shall obtain

$$\dot{p} \times L = -m_0 \left(\frac{q \cdot p}{r^3} q - \frac{r^2}{r^3} p \right)$$

Finally, from Proof in Proposition 1.17, we have

$$\dot{p} \times L = -m_0 \frac{d}{dt} \left(\frac{q}{r} \right)$$

as required. □

Let us take now the dot product of A with vector q

$$A \cdot q = q \cdot (p \times L) - \frac{m_0}{r} q \cdot q = (q \times p) \cdot L - m_0 r = |L|^2 - m_0 r \tag{1.14}$$

From equation (1.14) we can obtain the expression for the radius of the orbit

$$|A| r \cos\theta + m_0 r = |L|^2 \Leftrightarrow r = \frac{|L|^2}{|A| \cos\theta + m_0}$$

and defining the *eccentricity* of the orbit as $e = \frac{|A|}{m_0}$, we arrive to

$$r = \frac{|L|^2}{m_0(1 + e\cos\theta)} \tag{1.15}$$

We observe from equation (1.15) that the orbits defined by the Kepler problem are conic sections with one focus at the origin, where for $e = 0$ we obtain a circle, for $0 < e < 1$ we have an ellipse, for $e = 1$ a parabola and for $e > 1$ one branch of a hyperbola.

Finally, we can relate these orbits to its energy given by the Hamiltonian function H. Take the dot product of A with itself and develop, then we obtain

$$\begin{aligned} A \cdot A &= \left(p \times L - m_0 \frac{q}{r} \right) \cdot \left(p \times L - m_0 \frac{q}{r} \right) \\ &= (p \times L) \cdot (p \times L) - 2\frac{m_0}{r}(p \times L) \cdot q + \frac{m_0^2}{r^2} r^2 = |p|^2 |L|^2 - 2|L|^2 \frac{m_0}{r} + m_0^2 \\ &= 2|L|^2 \left(\frac{|p|^2}{2} - \frac{m_0}{r} \right) + m_0^2 = 2|L|^2 H + m_0^2 = |A|^2 \end{aligned}$$

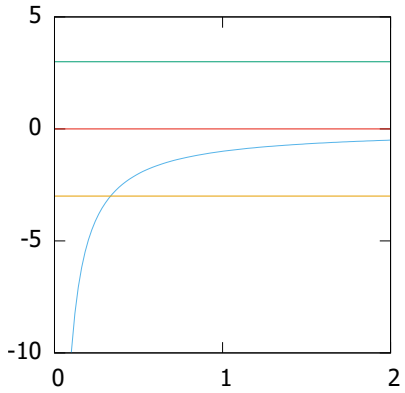

Figure 1.1: Graphic of the potential energy function $V(r)$ of the Kepler problem with total mass $m_0 = 1$ (light blue). Three level sets are plotted as well, H_+ (green), H_0 (red) and H_- (yellow).

Note that we have obtained an equation involving three integrals of the problem. Moreover, using the definition of the eccentricity $e = \frac{|A|}{m_0}$ in this equation,

$$|A|^2 = 2|L|^2 H + m_0^2 \Leftrightarrow \frac{|A|^2}{m_0} - 1 = \frac{2H|L|^2}{m_0} \Leftrightarrow e^2 - 1 = \frac{2H|L|^2}{m_0}$$

we eventually obtain the desired relation

$$e = \sqrt{1 + \frac{2H|L|^2}{m_0^2}} \tag{1.16}$$

from which we can state that $e < 1$ (resp. $e \geq 1$) corresponds to $H < 0$ (resp. $H \geq 0$), and therefore, closed orbits occur when $H < 0$, which is coherent with (1.11), whereas open orbits happen when $H \geq 0$.

We can also derive this assertion from the potential energy function as we learned in the Ordinary Differential Equations course of the degree. We consider the graphic of $V(r)$, as given in (1.10), and different level sets H_+, H_-, H_0 for when the Hamiltonian function is strictly positive, negative or zero, respectively.

We can observe from Figure 1.1 that when $H = H_- < 0$ we have periodic orbits since the radius of the orbit is bounded by the value of H_-. However, when $H = H_+, H_0$ ($H \geq 0$) we can see that the orbits are not bounded. In particular, for $H = H_0$, it must be $T = 0$ in the limit and then the body has zero escape velocity at infinity (the corresponding orbit is parabolic). On the other hand, for $H = H_+$ we have from (1.10) that T is always positive and hence the escape velocity of the orbit at infinity is positive (the orbit is hyperbolic).

Chapter 2

Central Configurations

In the previous chapter we presented the n-body problem and some of its properties. Here, we look for a way of finding explicit solutions of the n-body problem, or at least some sort of them.

To do so, we introduce a kind of configurations which provide solutions of the n-body problem, known as *central configurations*. Throughout this chapter, we state their basic properties and the relation with these solutions. After proving its existence we will end with some famous examples of central configurations.

2.1 Definition and Properties

First of all, let us start by giving their definition and stating some basic properties in order to work with them. Let c be the center of mass of a configuration $q = (q_1, \dots, q_n)$, as defined in Definition 1.11.

Definition 2.1. *A configuration of n punctual masses $q = (q_1, \dots, q_n)$ is called central configuration (CC) if there exists $\lambda > 0$ such that*

$$\ddot{q}_i = -\lambda(q_i - c) \qquad i = 1, \dots, n. \tag{2.1}$$

Note that $m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}(q)$, and then, by virtue of (1.4), the vector $\ddot{q}_i$ only depends on the relative position of the particles.

In other words, a central configuration is an arrangement of n bodies where all the acceleration vectors are pointing towards the center of mass and their magnitude is proportional to the distance from the center, with the same proportionality constant for all the masses.

The total force acting on a central configuration is 0. This follows from equations (2.1). Multiplying every equation by the corresponding mass m_i and summing all the equations for $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^{n} m_i \ddot{q}_i = -\lambda(\sum_{i=1}^{n} m_i q_i - c \sum_{i=1}^{n} m_i) = -\lambda(m_0 c - c m_0) = 0.$$

Let us see that central configurations are invariant under rotations, translations and homogeneous scalings of the configuration space.

Proposition 2.2. *Given a central configuration q, a scaling factor $k > 0$, a rotation matrix $R \in SO(d)$ and a constant vector $b \in \mathbb{R}^d$, the configuration $\bar{q}_i = kRq_i + b$ for $i = 1, \dots, n$, is a CC.*

Proof. Let us start by rotation and scaling invariance. We want to see that the configuration $\hat{q}_i = kRq_i$ is a CC for $i = 1, \dots, n$. Let c be the center of mass of q. Since q is a CC, the acceleration vector of $\hat{q}$ satisfies

$$\ddot{\hat{q}}_i = kR\ddot{q}_i = kR(-\lambda(q_i - c)) = -\lambda kRq_i + \lambda c = -\lambda(\hat{q}_i - c)$$

for every i, because the center of mass does not change under rotations and scalings. Thus, $\hat{q}$ is a central configuration.

Now consider the configuration $\bar{q}$ given by $\bar{q}_i = \hat{q}_i + b$ for $i = 1, \dots, n$. Then the center of mass of the configuration $\bar{q}$ is

$$\bar{c} = \frac{1}{m_0} \sum_{i=1}^{n} m_i \bar{q}_i = \frac{1}{m_0} \sum_{i=1}^{n} m_i \hat{q}_i + \frac{1}{m_0} \sum_{i=1}^{n} m_i b = c + b$$

Computing the acceleration vector of $\bar{q}_i$ for every i,

$$\ddot{\bar{q}}_i = \ddot{\hat{q}}_i + \ddot{b} = \ddot{\hat{q}}_i = -\lambda(\hat{q}_i - c) = -\lambda(\bar{q}_i - b - c) = -\lambda(\bar{q}_i - \bar{c})$$

we obtain that $\bar{q}$ satisfies equation (2.1) for every i. Hence, $\bar{q}$ is a central configuration. □

Remark 2.3. From the latter equalities in the proof, one can see that the translation vector b can be a linear function of time as well.

By Proposition 2.2 we shall assume, unless otherwise stated, that all central configurations which we consider are centered at the origin. For centered CC's the equations that define a central configuration (2.1) are equivalent to the equations

$$\ddot{q}_i = -\lambda q_i \qquad i = 1, \dots, n \tag{2.2}$$

As a more general consequence, it follows that we can define a relation between central configurations for which two CC's are related if we can obtain one of them from the other by applying any simultaneous translation, scaling or rotation. This is an equivalence relation in the set of central configurations and hence we can work with classes of central configurations.

When it comes to counting central configurations, we will count by equivalence classes. That means, we will count two equivalent configurations as one. Note that this implies normalizations. We have already made one by setting the center of the mass at the origin. We define the *moment of inertia* of the system as the function

$$I(q) = \frac{1}{2}\sum_{i=1}^{n} m_i |q_i|^2 \tag{2.3}$$

which measures the size of a configuration. Note then, that fixing I fixes the scale, so when counting equivalence classes of central configurations we will keep the same moment of inertia. A common normalization is taking $I = 1$.

Let us focus now on λ. If we manipulate a little bit equations (2.2),

$$\ddot{q}_i = -\lambda q_i \Leftrightarrow \dot{p}_i = -\lambda m_i q_i \Leftrightarrow \frac{\partial U}{\partial q_i} + \lambda m_i q_i = 0$$

we arrive to the equivalent equation $\nabla U(q) + \lambda M q = 0$, where M is a $nd \times nd$ matrix with d copies of each mass along the diagonal, called the *mass matrix*. In matrix form we can write the moment of inertia as $I(q) = \frac{1}{2} q^T M q$, so we have $\nabla I(q) = Mq$. Hence, the previous equation can be also written as

$$\nabla U(q) + \lambda \nabla I(q) = 0. \tag{2.4}$$

Note from this equation that λ can be considered as a Lagrange multiplier. Indeed, a central configuration can be seen as a critical point of the Newtonian potential function U restricted to a constant moment of inertia $I = I_0$. We will see the importance of this insight later on. Moreover, given a set of positions and masses on a certain level set I, there will be a different central configuration for every different value of λ, if any.

We remark that given a configuration q, every λ satisfying equations (2.2) gives a central configuration. Nonetheless, these could all be equivalent by similarity classes, and only after a normalization on the moment of inertia, the center of mass and having the mass values fixed, we have that every different value of λ leads to a different (non-equivalent) configuration.

We can explicitly state λ in terms of known functions, using *Euler's homogeneous function Theorem*.

Definition 2.4. (Homogeneous function) *A function f between two vector spaces is called homogeneous of degree $k \in \mathbb{R}$ if for all the elements x of its domain and for all real values $t > 0$, the relation $f(tx) = t^k f(x)$ holds.*

From this definition, we obtain that function U is homogeneous of degree -1 and function I is homogeneous of degree 2.

Theorem 2.5. (Euler's homogeneous function Theorem) *Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. If f is homogeneous of degree k then*

$$x \cdot \nabla f(x) = kf(x), \qquad \forall x \in \mathbb{R}^n \setminus \{0\} \tag{2.5}$$

Proof. It follows by derivating w.r.t. t the homogeneity relation and evaluating it at $t = 1$. □

Applying (2.5) to the equation (2.4), we deduce that for any configuration (since q cannot be zero) we can write $-U(q) + 2\lambda I(q) = 0$, or equivalently,

$$\lambda = \frac{U(q)}{2I(q)} > 0. \tag{2.6}$$

Before going any further, we want to present some basic examples of central configurations. Consider the case of n equal masses in $\mathbb{R}^d$, then it is a basic geometry problem to see that placing the masses at the vertices of a regular n-gon gives a central configuration, since all the acceleration vectors point towards the barycenter of the figure, i.e. the center of mass of the configuration. Then, for some λ the CC equations will hold.

This perfectly works for any n and any d, and furthermore, it is also possible to put an arbitrary mass at the center of the configuration. For example, one could think on satellital configurations of one heavy body in the center of mass and a quantity of equal masses, equally separated between them, and placed in a circle around the massive body. This type of central configurations lead to the study of the so-called $(1 + n)$-central configurations for arbitrary mass values [12].

Remark 2.6. Note that the central configurations presented above are planar configurations embedded in $\mathbb{R}^d$. Similarly, one can consider non-planar CC's for $d \geq 3$. Indeed, in dimension 3, the five regular platonic solids, the tetrahedron, cube, octahedron, dodecahedron and icosahedron, admit also central configurations with equal masses. For the polyhedrons with other numbers of vertices it is not clear anymore. Additionally, there are six kinds of regular convex four-dimensional polytopes but in higher dimensions there exist only three, namely

the generalizations of the tetrahedron, cube and octahedron. We give [10] as a reference on this matter.

2.1.1 Central Configurations of the 2-body problem

Let us now give an specific insight on central configurations for the case of two masses, which form the most trivial central configurations. In fact,

Proposition 2.7. *Any configuration of two masses is a CC.*

Proof. Consider two bodies with $q_i \in \mathbb{R}^d$ and $m_i > 0$ for $i = 1, 2$, their position vectors and masses respectively. Take reference frames so that $m_1 q_1 + m_2 q_2 = 0$, i.e. the center of mass of the configuration lies at the origin. Recall that there is no loss of generality by doing so, we only do it for simplicity.

From the equations of motion of both particles (1.5), we obtain

$$\ddot{q}_1 = \frac{m_2(q_2 - q_1)}{r_{12}^3} = \frac{m_2 q_2 - m_2 q_1}{r_{12}^3} = \frac{-m_1 q_1 - m_2 q_1}{r_{12}^3} = -\frac{m_1 + m_2}{r_{12}^3} q_1$$

and

$$\ddot{q}_2 = \frac{m_1(q_1 - q_2)}{r_{12}^3} = \frac{m_1 q_1 - m_1 q_2}{r_{12}^3} = \frac{-m_2 q_2 - m_1 q_2}{r_{12}^3} = -\frac{m_1 + m_2}{r_{12}^3} q_2$$

where the third equality in both equations is due to the fact that we chose the center of mass at the origin. Hence, we obtain that for $\lambda = m_0 / r_{12}^3$, the two masses satisfy equations (2.2). □

Note that λ from the previous result is positive and it is the same for each particle, as required from Definition 2.1. However, one can easily observe that it does depend on the distance between the particles (it determines the scale of the configuration) and it does depend on the total mass of the system.

From the fact that all the 2-body configurations are equivalent by similarity classes and due to Proposition 2.7 we have the following result.

Corollary 2.8. *Given two masses, there exists only one central configuration for the 2-body problem.*

To end this section, we check that for the configurations of two masses, λ computed from (2.6) corresponds to the one found in Proposition 2.7. In the case $n = 2$ we have

$$U(q) = \frac{m_1 m_2}{r_{12}} \quad \text{and} \quad 2I(q) = m_1 |q_1|^2 + m_2 |q_2|^2$$

so λ obtained from (2.6) gives

$$\lambda = \frac{m_1 m_2}{r_{12}(m_1|q_1|^2 + m_2|q_2|^2)}. \tag{2.7}$$

If we just consider the expression inside the parenthseses and we multiply it by $2(m_1 + m_2)$, we obtain

$$2m_1^2|q_1|^2 + 2m_1m_2|q_1|^2 + 2m_1m_2|q_2|^2 + 2m_2^2|q_2|^2$$

and now substracting two times the expression

$$m_1|q_1|(m_1|q_1| + m_2|q_2|) + m_2|q_2|(m_1|q_1| + m_2|q_2|)$$

which is 0 because we set the center of mass at the origin, we obtain

$$m_1^2(2|q_1|^2 - 2|q_1|^2) + 2m_1m_2(|q_1|^2 + |q_2|^2 - 2|q_1||q_2|) + m_2^2(2|q_2|^2 - 2|q_2|^2) = \\ = 2m_1m_2r_{12}^2$$

Hence $m_1|q_1|^2 + m_2|q_2|^2 = \frac{m_1m_2r_{12}^2}{m_1+m_2}$, and substituting this into equation (2.7), we obtain $\lambda = m_0/r_{12}^3$ as we wanted to see.

2.2 Homographic solutions

Central configurations are important because they can be used in order to construct simple, special solutions of the n-body problem, where the shape of the figure formed by the bodies will remain constant. In such solutions, the configuration changes only by simultaneous translation, scaling and rotation, so configurations $q(t)$ at different times are all similar.

Equation (2.2) gives an idea of what would happen if the bodies in a central configuration are released with zero initial velocity: all bodies would suffer an acceleration towards the origin resulting into a collision. However, if for instance the bodies are released from a CC with initial velocities normal to their position vectors and with magnitudes proportional to their distances from the origin, that is $\dot{q}_i \cdot q_i = 0$ and $|\dot{q}_i| = \mu|q_i|$, then each body would describe an elliptic orbit. In particular, they could move in circles around the center of mass.

Let us state this argumentation more rigorously. We introduce the following kinds of solutions of the n-body problem and then we will show their conncetion to central configurations. We use [10] as a reference in this topic.

Definition 2.9. *A solution of the n-body problem is called self-similar or homographic if it satisfies*

$$q(t) = r(t)Q(t)q_0 \tag{2.8}$$

where q_0 represents a constant configuration, $r(t) > 0$ a real scaling factor and $Q(t) \in SO(d)$ a rotation. There are two special cases of homographic solutions:

- *The homothetic solutions, where $q(t) = r(t)q_0$*
- *The rigid motions or relative equilibrium solutions, where $q(t) = Q(t)q_0$*

The term "self-similar" is quite descriptive because all-time configurations of these solutions will look-alike, since such solutions only change by scaling and rotation. The name to describe the second kind of homothetic solutions is also very revealing, since these solutions are equilibrium solutions in a rotating coordinate system and they move through time as a rigid body.

Let us focus now on the two simplest, yet very interesting, homographic solutions: the homothetic and the planar homographic solutions.

2.2.1 Homothetic Solutions

We will start by the simplest of them, the homothetic solutions. Let us see their relation to central configurations by providing the following result.

Proposition 2.10. *If q_0 is a central configuration with constant λ and $r(t)$ is any solution of the one-dimensional Kepler problem*

$$\ddot{r}(t) = -\frac{\lambda}{r(t)^2} \tag{2.9}$$

then $q(t) = r(t)q_0$ is a homothetic solution of the n-body problem and every homothetic solution is of this form.

Otherwise stated, this means that all homothetic solutions of the n-body problem form a central configuration at all time and therefore, we can obtain homothetic solutions by simply starting from a central configuration. Additionally, we know that its motion will be defined by the Kepler problem.

Proof. Substituting $q(t) = r(t)q_0$ into Newon's equations of motion, we get

$$\ddot{r}(t)q_{0,i} = \sum_{j \neq i} m_j \frac{r(t)(q_{0,j} - q_{0,i})}{r(t)^3 |q_{0,i} - q_{0,j}|^3} = \frac{1}{r(t)^2} \ddot{q}_{0,i}$$

This equation is satisfied if and only if there is some constant, say λ, such that $\ddot{r}(t)r(t)^2 = -\lambda$ and $\ddot{q}_{0,i} = -\lambda q_{0,i}$. □

Recall from Section 1.4 that the one-dimensional Kepler problem (2.9) describes the motion of a unit mass particle on a line, gravitationally attracted to a mass λ at the origin. The corresponding homothetic solutions mantain the shape of the central configuration q_0 while they collapse to a total collision at the center of mass and each body moves in a straight line towards the center.

Therefore, central configurations govern the motion of particles near collision, and actually, they give the limiting configuration before collapsing into a multiple collision in finite time.

An example of a configuration that follows this solutions can be any regular n-gon in the plane with equal masses.

Example 2.11. We consider indeed such an n-gon as a central configuration. Then the position of the bodies are given by the configuration vector $q_0 \in \mathbb{R}^{2n}$ as

$$q_{0,i}{}^T = r(\cos(\pi/2 + 2\pi i/n), \sin(\pi/2 + 2\pi i/n)), \quad \text{for } 1 \leq i \leq n, \tag{2.10}$$

for an n-gon of radius r. From Proposition 2.10 we know that the particles will move in linear directions towards the origin, where the center of mass is placed. Hence, we must consider initial conditions that satisfy the one-dimensional Kepler problem. We have seen in Proposition 1.17 that the angular momentum must be zero, and so we consider the initial momenta of the particles as zero.

We illustrate the configuration for $n = 5$ and $r = 1$. We propagate the initial configuration q_0 using the Taylor method [1]. The initial values of the velocities were all set to zero and we gave the same value to all the masses, with $m_0 = 1$. The result can be seen in Figure 2.1.

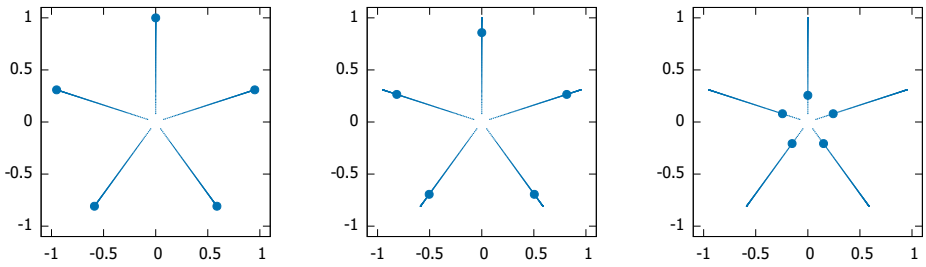

Figure 2.1: Motion of a 5-gon central configuration of radius 1, where particles are released with zero initial velocities. From left to right, pictures describe the position of the configuration at times $t = 0, 1, 2$ seconds.

[1] All the numerical illustrations of this work have been computed using the ad-hoc implementation of the Taylor mehod included in Appendix A

2.2.2 Planar Homographic Solutions

Central configurations lead to rigid motion solutions and also to more general homographic solutions. We will see this only for the case of planar configurations.

Proposition 2.12. *If q_0 is a planar central configuration and $(r(t), \theta(t))$ is any solution of the two-dimensional Kepler problem (1.12), then $q(t) = r(t)Q(\theta(t))q_0$ is a planar homographic solution of the n-body problem and every such solution is of this form.*

From this result follows that every planar homographic solution of the n-body problem is given by an initial central configuration and that the motion of its particles is given by the planar Kepler problem, hence they follow conic section orbits.

Proof. Let $d = 2$. Suppose $q_0 \in \mathbb{R}^{2n}$ is a planar CC with constant λ and let $Q(\theta) \in SO(2)$ be the rotation matrix of angle $\theta \in [0, 2\pi)$. Consider the general planar homographic solution $q(t) = r(t)Q(\theta(t))q_0$, for some functions $r(t) > 0$ and $\theta(t)$. Then,

$$\begin{aligned}\ddot{q} &= \ddot{r}Q(\theta)q_0 + 2\dot{r}DQ(\theta)\dot{\theta}q_0 + r(t)D^2Q(\theta)\dot{\theta}^2q_0 + r(t)DQ\ddot{\theta}q_0 \\ &= (\ddot{r} - r\dot{\theta}^2)Q(\theta)q_0 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})DQ(\theta)q_0\end{aligned}$$

From Newton's equations we have

$$\ddot{q}_i = \sum_{j\neq i} m_j \frac{(q_j - q_i)}{r_{ij}^3} = \sum_{j\neq i} m_j Q(\theta) \frac{r(q_{0_j} - q_{0_i})}{r^3 |q_{0_i} - q_{0_j}|^3}$$

and putting it all together we get

$$(\ddot{r} - r\dot{\theta}^2)q_0 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})Jq_0 = \frac{1}{r^2}\ddot{q}_0$$

where J is the block diagonal $2n \times 2n$ matrix with 2×2 blocks

$$Q(\theta)^{-1}DQ(\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since q_0 and Jq_0 are non-zero vectors, there exists $\lambda \in \mathbb{R}$, such that

$$\begin{cases} \ddot{r}(t) - r(t)\dot{\theta}(t)^2 = -\frac{\lambda}{r(t)^2} \\ r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) = 0 \end{cases} \tag{2.11}$$

and $\ddot{q}_0 = -\lambda q_0$, which is the equation that defines a CC. Note that equations (2.11) define the two-dimensional Kepler problem in polar coordinates as in (1.12) where λ represents the central mass value. □

In particular, if we choose a circular solution of the Kepler problem with $r(t) = 1$ we get a rigid motion solution where the initial planar central configuration just rotates at a constant angular velocity around the center of mass. Let us see it in the following example.

Example 2.13. Consider as in Example 2.11 an initial planar CC $q_0 \in \mathbb{R}^{2n}$ forming a regular n-gon with equal masses. We observe that all the consecutive mutual distances are equal. Let l denote the length of the sides, then we write $r_{ij} = l$, if $j = i+1$, for $i = 1, \ldots, n-1$ and $r_{1n} = l$.

Let us now compute the value of λ for this central configuration, from the formula given in (2.6). In our concrete setting, we can express the Newtonian potential function and the moment of inertia as

$$\begin{aligned} U(q_0) &= \sum_{i<j} \frac{m_i m_j}{r_{ij}} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{m_i m_j}{r_{ij}} = \sum_{i=1}^{n-1} \frac{m^2}{r_{i,i+1}} + \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \frac{m^2}{r_{ij}} = \\ &= \frac{nm^2}{l} + \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \frac{m^2}{r_{ij}} - \frac{m^2}{l} = \frac{nm^2}{l} + \sum_{j=3}^{n-1} \frac{m^2}{r_{1j}} + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{m^2}{r_{ij}} = \\ &= \frac{nm^2}{l} \left(1 + \frac{l}{n} \left(\sum_{j=3}^{n-1} \frac{1}{r_{1j}} + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{1}{r_{ij}} \right) \right) \\ 2I(q_0) &= \sum_{i=1}^{n} m_i |q_i|^2 = \sum_{i=1}^{n} mr^2 = nmr^2 \end{aligned} \tag{2.12}$$

Therefore, λ is given by

$$\lambda = \frac{U(q)}{2I(q)} = \frac{m}{lr^2} \left(1 + \frac{l}{n} \left(\sum_{j=3}^{n-1} \frac{1}{r_{1j}} + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{1}{r_{ij}} \right) \right). \tag{2.13}$$

Now, we shall look for the initial velocities to provide to the configuration. Since the solutions we are seeking are of the form

$$q_i(t) = r(t) \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} q_{0,i},$$

with $r(t)$ constant, the velocities are given by

$$\dot{q}_i(t) = r(t)\dot{\theta}(t) \begin{pmatrix} -\sin\theta(t) & -\cos\theta(t) \\ \cos\theta(t) & -\sin\theta(t) \end{pmatrix} q_{0,i}$$

From Proposition 2.12 we know that the function of time $\theta(t)$ must be a solution of the Kepler problem (1.12). Taking $r(t) = 1$ we have $\dot{\theta}(t)^2 = \lambda$, $\ddot{\theta}(t) = 0$, from where we obtain a constant value for the angular velocity, $\dot{\theta}(t) = \pm\sqrt{\lambda}$, with λ

being the constant of the central configuration q_0 given in (2.13). A solution of this form will have initial conditions

$$q_i(0) = q_{0,i} \qquad \dot{q}_i(0) = \pm\sqrt{\lambda}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} q_{0,i} \tag{2.14}$$

for $i = 1, \ldots, n$ and $q_{0,i}$ as in (2.10). The sign of the angular velocity indicates the rotation sense (clockwise or counterclockwise) of the particles.

The solution that can be seen in Figure 2.2 was computed with the initial conditions given by (2.14), for the case $n = 5$, $m = 0.2$, and $r = 1$. All the other necessary values follow from them.

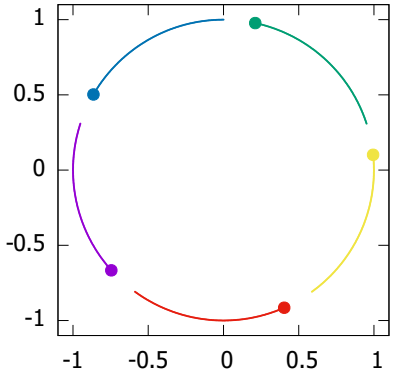

Figure 2.2: Motion followed by the bodies of a 5-gon central configuration, with initial conditions given in Example 2.13, up to a fifth of the period.

Remark 2.14. In higher spatial dimensions, the situation regarding rigid solutions and also non-homothetic solutions is more complicated, mainly due to the increased complexity of the rotation group $SO(d)$. See [10] for further details.

Before closing this section, we want to emphasize that we have given a theoretical background to find a special kind of solutions of the n-body problem. Indeed, whenever we start with a central configuration as the initial configuration and we apply any scaling and/or rotation, we will obtain homographic solutions. Furthermore, they are the only possible configurations that give place to such solutions.

2.3 Existence of Central Configurations for arbitrary masses

At this point, we have shown some interesting properties and examples of CC's and we have explained why we focus on them for the n-body problem. We have

also shown their existence imposing some pretty clear and restrictive conditions. Notice however that we don't now yet if a CC is possible for any number of masses or for any given mass values for them.

In order to study its existence, we are going to use the insight of central configurations as critical points of the Newtonian potential function, see (2.4). We will follow the steps given in [10]. Recall that a configuration vector q_0 is a CC if and only if it is a critical point of $U(q)$ subject to the constraint $I(q) = I(q_0)$.

Remark 2.15. Some authors (see [12], [15]) use a slightly different characterization to determine a CC. Instead of seeing central configurations as critical points of the Newtonian potential function U restricted to a manifold I, they see them as critical points of the homogeneous function IU^2. Both characterizations are equivalent because when substituting the λ given in (2.6) into the equation (2.4) we obtain

$$\nabla U(q) + \frac{U(q)}{2I(q)}\nabla I(q) = 0 \Leftrightarrow 2I(q)\nabla U(q) + U\nabla I(q) = 0$$

from where one concludes that for a central configuration q, $\nabla(IU^2)(q) = 0$ holds.

To work with central configurations as critical points of U, it is helpful to have a compact constraint set. Any configuration q determines a unique *normalized* configuration with $c = 0$ and $I = 1$. Recall that equation (2.2) already implies condition $c = 0$. So we only have to take into account the condition on the moment of inertia I, which gives an ellipsoid in $\mathbb{R}^{nd}$. Therefore, the manifold $\{q : I(q) = 1\}$ is diffeomorphic to the sphere $\mathcal{S}^{nd-1}$, hence, it is compact. The critical points will lie in the constraint manifold $\{q : c = 0,\ I = 1\}$.

Proposition 2.16. *A central configuration q is a critical point of the function $U(q)$ on the set $\{q : I(q) = 1\}$ if and only if q is a normalized central configuration.*

Proof. Suppose q is a normalized central configuration. Then $c = 0$, $I(q) = I_0(q) = 1$ and equation (2.4) holds for some λ.

Now suppose q is a critical point of U on $\{q : I(q) = 1\}$. Then $\nabla U(q) = -\lambda\nabla I(q)$ and $I(q) = 1$. Therefore q is a CC with $I(q) = 1$ and $c = 0$. The latter assertation is a consequence of Proposition 2.2 since all central configurations satisfying an equation equivalent to (2.2) are centered, as previously discussed. □

Now we are prepared to prove the main result of this section.

Theorem 2.17. *For every choice of masses $m_i > 0$ in the n-body problem in $\mathbb{R}^d$, there exists at least one central configuration.*

Proof. Note that the set of normalized configurations, $\mathcal{N} = \{q : c = 0,\ I(q) = 1\}$, is a compact submanifold of $\mathbb{R}^{nd}$ and that the Newtonian potential function defines a smooth function $U : \mathcal{N} \setminus \Delta \to \mathbb{R}$, where $\Delta = \{q : q_i = q_j \text{ for some } i \neq j\}$ is the *singular* set. The set $\mathcal{N} \cap \Delta$ is compact and $U(q) \to \infty$ as $q \to \Delta$. Therefore, U must achieve its minimum in $\mathcal{N} \setminus \Delta$ and thus, there exists a central configuration. □

Remark 2.18. This result is true for any number n of masses and even regardless of the choice of the mass values m_i as long as they are positve. The positiveness of the masses is important, otherwise the sign of U could change and it would not be so easy to find its critical points. In fact, it is known that for $n = 5$ there exists a continuum of central configurations if negative masses are allowed [5].

In the remainder of this chapter, we are going to focus on the most "simple" central configurations known so far, those which were analytically deduced from the direct problem. We are going to present such configurations, proving that they so are indeed, and giving some examples as well.

2.4 Collinear Central Configurations

We are going to study now a simple, yet very interesting and non trivial case of central configurations, the collinear ones.

Excluding the 2-body scenario, where obviously all configurations are collinear (and for which we have proved that they are CC's regardless of the mass values, see Proposition 2.7) it is not so simple to see that for any number n of bodies, there exist collinear central configurations, specially for arbitrary masses. For three equal masses, one could think on the example of a regular 2-gon with a mass in the middle, deducing it from the examples of CC's given in Section 2.1.

In fact, it was not until 1767 that Euler found the first non-trivial central configuration, the 3-body central configuration on a line [4]. The generalization to n bodies was proved to be a CC at the beginning of the last century, in 1910, by F.R. Moulton (see [11]), almost 150 years after the discovery of Euler.

Before getting to Moulton's Theorem, it will be interesting to see an approach of how Euler found the 3-collinear central configuration. We follow the steps of [15] to show it.

Consider the motion of three particles restricted to move on a line. If we fix the ordering of the bodies to be $q_1 < q_2 < q_3$ then this constraint is $r_{12} + r_{23} - r_{13} = 0$. Furthermore, we can normalize it by setting one of the distances to 1 and another one to x, i.e. $r_{12} = 1, r_{23} = x, r_{13} = 1 + x$.

Now, in order to find central configurations on the line, we will seek critical points of the homogeneous function $F(r_{ij}) = U(r_{ij})^2 I(r_{ij})$ with the normalized constraint, see Remark 2.15. Note that function U is already expressed in terms of the mutual distances. Let us show how to obtain such an expression for function I. Let m_0 be the total mass and assume that the center of mass is at the origin, then

$$\begin{aligned}\sum_i \sum_j m_i m_j r_{ij}^2 &= \sum_i \sum_j m_i m_j |q_i - q_j|^2 \\ &= \sum_i \sum_j m_i m_j |q_i|^2 - 2\sum_i \sum_j m_i m_j |q_i||q_j| + \sum_i \sum_j m_i m_j |q_j|^2 \\ &= 2m_0 I - 2\sum_i m_i |q_i| \sum_j m_j |q_j| + 2m_0 I = 4m_0 I,\end{aligned} \tag{2.15}$$

and we conclude

$$I = \frac{1}{4m_0} \sum_i \sum_j m_i m_j r_{ij}^2 = \frac{1}{2m_0} \sum_{i<j} m_i m_j r_{ij}^2.$$

Therefore, we seek the solutions of

$$\nabla F(r_{ij}) + \mu(-1)^{i+j+1} = 0, \qquad \text{for } 1 \le i < j \le 3,$$

where μ is the Lagrangian multiplier. This leads to the system of equations

$$\frac{U(r_{ij})}{m_0} r_{ij} - 2I(r_{ij}) r_{ij}^{-2} + (-1)^{i+j+1} \frac{\mu m_k}{U(r_{ij}) m_i m_j m_k} = 0, \qquad \text{for } 1 \le i < j \le 3.$$

We have that $\det(r_{ij}, r_{ij}^{-2}, (-1)^{i+j+1} m_k) = 0$, from where we obtain the following fifth-degree polynomial equation for x,

$$\begin{aligned}&(m_1 + m_2)x^5 + (3m_1 + 2m_2)x^4 + (3m_1 + m_2)x^3 \\ &\quad - (m_2 + 3m_3)x^2 - (2m_2 + 3m_3)x - (m_2 + m_3) = 0\end{aligned} \tag{2.16}$$

and consequently the problem reduces to determine the real positive roots of (2.16), which depend on the values of the masses.

Proposition 2.19. *Given a triplet of positive masses, there is a unique collinear central configuration for each ordering of the bodies along the line, up to rotation and scaling. That being exactly 3 different central configurations for every choice of the masses.*

Proof. Since all the masses are greater than 0, we can see that the polynomial has at least one change of sign in its positive domain. In fact, due to $(m_1 + m_2) > 0$, the polynomial is positive for large positive x; while at $x = 0$ the polynomial attains

the negative value $-(m_2 + m_3)$, thus there exists at least one real positive root. Moreover, the polynomial has one sign change in its coefficients. Then, because of Descartes's rule of signs, there is exactly one real positive root.

Additionally, we observe that interchanging the position of the bodies on the line, after normalising again the mutual distances, gives the same equation except that the masses m_i are accordingly interchanged. However, since all the masses are positive the previous reasoning holds anyway. □

Equation (2.16) is a quintic equation which can be solved using a numerical method. For example, we used the bisection method to find the positions where the three particles need to be placed in order to obtain the central configuration for masses $m_1 = 3,\ m_2 = 1,\ m_3 = 6$, with $q_1 < q_2 < q_3$. The corresponding collinear CC obtained at different times is displayed in Figure 2.3.

Figure 2.3: An example of the motion followed by 3 bodies on a line forming a collinear CC with $c = 0$. Here each color represents the configuration at three different times. The size of the points correspond to the relation beween the masses of the bodies and also help to identify the body at each time.

The motion of the bodies was not discussed in Proposition 2.19, we only covered the exitence at a certain instant of time. However, from Section 2.2 we have learned that given an initial central configuration we can find different homographic solutions of the n-body problem, concretely, homothetic solutions and rigid motion solutions. An example of an homothetic solution is shown in Figure 2.3. Let us now introduce an example of a relative equilibrium motion.

We note the fact that the result is true for any of the relation between the masses of the particles. Hence, one of them can be extremely small or the others extremely large, and the statement will still be true. For example, given an object with a certain arbitrary mass, then there exists a unique location between the Earth and the Moon such that, if we place this object there with a specific initial velocity, it will form a collinear central configuration (in rotational coordinates) together with the Earth and the Moon.

The other two collinear central configurations would be when the Earth and the Moon were in the middle of the configuration, respectively. In fact, these three

different points where a body could be placed in relation to the Earth and the Moon (or any other two primary bodies) and form a collinear central configuration, are known as the Lagrange points L_1, L_2 and L_3. These points were found by Lagrange, when studying the planar, circular, restricted 3-body problem. Indeed, in rotational coordinates, this collinear CC turn out to be three unstable fixed points.

Moulton's theorem extends collinear central configuration of 3 bodies to the case of an arbitrary number of masses. We will prove it using [10].

Theorem 2.20. *For the n-body problem with positive masses m_i, there are precisely $n!/2$ collinear central configurations. More accurately, for each way the particles can be ordered along a line, there is a unique position that causes a central configuration.*

Proof. We can consider that the bodies live in $\mathbb{R}$. Then, let $q \in \mathbb{R}^n \setminus \Delta$ be a configuration vector of n particles, and let $\mathcal{N}$ and Δ be the corresponding sets of normalized configurations and collisions respectively. The collision set Δ divides the ellipsoid $\mathcal{N}$ into $n!$ components $\mathcal{V}_k$. Let $\mathcal{V}$ represent one of them, then $\mathcal{V} = int(\mathcal{V})$ and $\partial\mathcal{V} \subseteq \Delta$. The Newtonian potential restricted to this component $U|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R}$ is smooth and $U(q) \to \infty$ as $q \to \partial\mathcal{V}$. Therefore, $U|_{\mathcal{V}}$ attains its minimum at some point $q_0 \in \mathcal{V}$ where q_0 represents a CC with a given ordering of the particles on the line, depending on the component where it lives. □

We note that for the case $n = 3$, Theorem 2.20 corresponds to the collinear central configurations that Euler had previously found, as one expects. However, we can see that the strategy we followed to prove collinear central configurations for the general case n is different to the one followed by Euler. The reason lies behind the complexity of the polynomial functions that arise from doing the same discussion as before for $n > 3$, see [12].

Additionally, let us give an insight about how F.R. Moulton proved this theorem. His goal was to find the number of possible collinear CC's for each number of masses. His argument was based on an induction about what happened every time a mass was added to the collinear configuration. He started from the Eulerian collinear configuration and proved that at each step, the number of collinear CC's increased by $(k+1)$. For further details, one can read [11].

From the discussions made for the homothetic and rigid motion solutions it follows that if we place n bodies of arbitrary masses in the same straight line with no initial velocity, their own attraction will make them collapse homotetically to the center of mass of such configuration, as it can be seen in Figure 2.3 for three masses. On the other hand, if we give some certain initial velocity to the particles, these would follow conic section orbits.

2.5 The equilateral Lagrange configuration

Only a few years after Euler's finding of the collinear central configuration for the case $n = 3$, Lagrange made another important discovery in the search for central configurations. In the year 1772, he found the equilateral triangular central configuration, that is, the configuration of three bodies lying on the vertices of an equilateral triangle [7].

In order to arrive to this result, we are going to consider the distances between the particles r_{ij}, with $1 \leq i < j \leq 3$, as we did in the previous section. If we fix the center of mass at the origin and identify two rotationally equivalent configurations, then the three variables r_{12}, r_{23}, r_{13} are local coordinates near a noncollinear configuration. Thus, by specifying the angle between a fixed line and $q_2 - q_1$, the location of the center of mass and the three mentioned variables, then the configuration of masses is uniquely specified [8].

Note that holding I fixed is the same as holding $m_0 I =: I^*$ fixed. Thus, the conditions for U to have a critical point on the set I^* in these coordinates are

$$-\frac{m_i m_j}{r_{ij}^2} + \lambda m_i m_j r_{ij} = 0, \qquad \text{for } 1 \leq i < j \leq 3, \tag{2.17}$$

from where we obtain the only real solution $r_{12} = r_{23} = r_{13} = \lambda^{-1/3}$. We observe that this solution represents an equilateral triangle independently of the values of the masses m_1, m_2, m_3 (previous normalization of the configuration).

Proposition 2.21. *For any given values of the masses, there are two and only two noncollinear central configurations for the 3-body problem, namely, the three particles are at the vertices of an equilateral triangle. The two solutions correspond to the two orientations of the triangle when the vertices are labeled by the bodies.*

This result is true for any relation between the values of the masses, opposite to what happened with the regular polygonal examples, mentioned in Section 2.1. Moreover, let us emphasize that all the five different central configurations of the 3-body problem that we have found (the three collinear configurations and these two triangular ones), happen to take place for any possible choices of positive masses. Actually, these are the only explicit solutions known for arbitrary masses.

Hence, this result holds even in extreme settings where one of the masses is infinitely smaller than the other two. For instance, consider the case where we have two massive bodies as the Earth and the Moon, and the other is just an artificial satellite or an asteroid. Following Proposition 2.21, we can affirm that there exist two different spots in the universe where we could place an object, of

any mass value (more precisely, at the vertices of the equilateral triangles drawn by setting the Earth and the Moon at the vertices of the opposite side). With the appropiate motion they would follow a trajectory forming an equilateral triangle with the Earth and the Moon. Besides, on one of these spots the object would follow the orbit of the Moon, and on the other, would lead the orbit.

In fact, from a more theoretical point of view, these two spots correspond to the Lagrangian points L_4 and L_5, which happen to be stable. Indeed, there are objects in the universe in such spots. At the beginning of the twentieth century, the German astronomer M. Wolf found for the first time an equilateral central configuration in the universe. He did it by exploring the L_4 and L_5 points of the system Sun-Jupiter, and he found asteroids at each of these points, forming two equilateral triangles. He called these asteroids *Trojans* [12].

Since they were discovered, these kind of asteroids became famous among astronomers, who tried to find them for more planets in the Solar system. At the moment, many of them have been found for various planets. In fact, since these two spots are stable, there exist two stability regions on the orbits of the planets containig a significant number of Trojan asteroids. In the case of the heaviest planet in the Solar system, Jupiter, more than 9000 Trojans have been found[2]. As a curiosity, in 2010 the first Trojan of the Earth was found, placed at the L_4 point.

Before moving on to the case of n masses, we present a couple of examples of Lagrangian configurations. As it can be seen in the plots of Figure 2.4, the bodies form equilateral triangular configurations. On the left, we have a planar homographic solution, with the bodies describing ellipses. The masses have been set to $m_i = i$. The points plotted represent the initial position of the particles, and its size represents the masses of the bodies. On the right, we can see a homothetic solution, with three different masses released with no velocity collapsing in finite time to the center of mass of the configuration.

In the general case of n masses, we have a result that shows that Proposition 2.21 can be extended for configurations with a number of bodies greater than 3. In fact, it is easy to see that for four bodies, the configuration of masses given by the tetrahedron (the natural extension of the triangle) is a CC in any dimension greater than 2. For an arbitrary value of n masses, we have that the regular $(n-1)$-simplex gives always a central configuration.

Proposition 2.22. *The regular simplex in $\mathbb{R}^d$, with $d \geq n-1$, is a central configuration for all choices of the masses.*

[2]See `https://minorplanetcenter.net//iau/lists/Trojans.html`

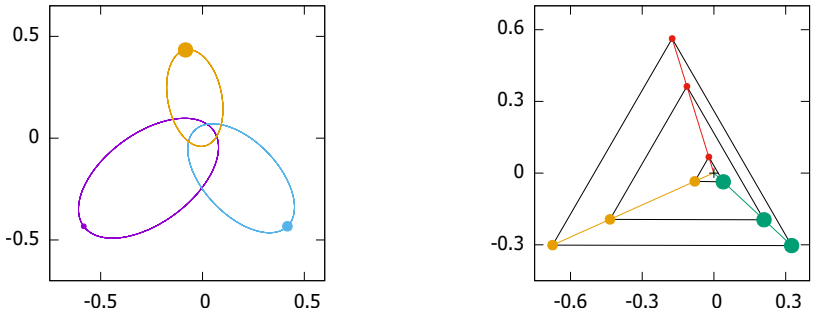

Figure 2.4: Left: relative equilibrium motion of three bodies forming an equilateral triangle. The points represent the initial position of the bodies, and their size shows the relation to their masses. Right: homothetic solution of three different masses forming an equilateral triangle collapsing to the center of mass, plotted with a cross.

Proof. Let q be a configuration of n masses in dimension $d \geq n-1$. We already know that a central configuration can be found as a critical point of the Newtonian potential function subject to a constraint on the moment of inertia. Consider the function $F(r_{ij}) = U(r_{ij}) + \lambda I(r_{ij})$ with mutual distance coordinates r_{ij}, $1 \leq i < j \leq n$. Then F is expressed in such coordinates as

$$F(r_{ij}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}} + \frac{\lambda}{2m_0} \sum_{i<j} m_i m_j r_{ij}^2$$

following the computations that we carried out in (2.15). Now computing the partial derivatives w.r.t. the mutual distance coordinates, we obtain

$$\frac{\partial F}{\partial r_{ij}} = m_i m_j \left(-\frac{1}{r_{ij}^2} + \frac{\lambda}{m_0} r_{ij} \right)$$

So the condition for a configuration with mutual distances r_{ij} to be a CC for a certain λ would be written as

$$m_i m_j \left(-\frac{1}{r_{ij}^2} + \frac{\lambda}{m_0} r_{ij} \right) = 0 \quad \text{for} \quad 1 \leq i < j \leq n \tag{2.18}$$

Solving for every r_{ij} in (2.18) we obtain that the mutual distances are all $(m_0/\lambda)^{1/3}$, condition that gives a regular simplex. □

Chapter 3

The figure eight solution

In this chapter we present a remarkable solution of the 3-body problem, the *figure eight solution*. It was found in December 1999 by A. Chenciner and R. Montgomery (see [3]).

3.1 Introduction to the figure 8. Choreographies

The figure eight solution consists of three bodies of equal masses moving periodically along the same eight-shaped curve on the plane, and with a constant time shift between the particles of a third of the period. It belongs to a kind of solutions of the n-body problem called *choreographies*.

Definition 3.1. *A choreography is a solution of the n-body problem for which all the bodies move periodically along the same fixed curve, without colliding, and with a constant phase shift.*

The name was coined by C. Simó in [2] and refers to the dance-like motion that follow the bodies. The figure eight was the second choreography found, after Lagrange's equilateral triangular solution discussed in Section 2.5.

Let us now set a mathematical background for the figure eight solution. We focus on the 3-body problem in the plane, i.e. $n = 3$ and $d = 2$. Consider three bodies of unit mass in the Euclidean plane $\mathbb{R}^2$ under the influence of the Newtonian gravitational field. Let $q_i(t) \in \mathbb{R}^2$ for $i = 1, 2, 3$, denote the positions of the three particles at time t. A parametrization of the figure-eight solution of period T, will be a map $\gamma : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$ such that

$$q_{j+1}(t) = \gamma(t + jT/3) \qquad \text{for } j = 0, 1, 2.$$

The existence of this solution is given by the following theorem, proved by A. Chenciner and R. Montgomery in [3]. In order to announce it, let us define first the actions of the Klein group[1] $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}/T\mathbb{Z}$ and $\mathbb{R}^2$, where T is any positive real number. Given a pair of generators $\langle \sigma, \tau \rangle$ of the Klein group,

$$\sigma \cdot t = t + \frac{T}{2}, \quad \tau \cdot t = -t + \frac{T}{2}, \qquad \sigma \cdot (x, y) = (-x, y), \quad \tau \cdot (x, y) = (x, -y)$$

for all $t \in \mathbb{R}/T\mathbb{Z}$ and $(x, y) \in \mathbb{R}^2$.

Theorem 3.2. *There exists an eight-shaped planar loop $\gamma : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$ with $\gamma(0) = 0$, satisfying:*

(i) for every t,

$$\gamma(t) + \gamma(t + T/3) + \gamma(t + 2T/3) = 0$$

(ii) γ is equivariant with respect to the actions of the Klein group on $\mathbb{R}/T\mathbb{Z}$ and $\mathbb{R}^2$ defined above,

$$\gamma(\sigma \cdot t) = \sigma \cdot \gamma(t) \quad \text{and} \quad \gamma(\tau \cdot t) = \tau \cdot \gamma(t)$$

(iii) the loop $q : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^6 \setminus \Delta$ defined by

$$q(t) = (\gamma(t), \gamma(t + T/3), \gamma(t + 2T/3))$$

is a zero angular momentum T-periodic solution of the planar 3-body problem with equal masses.

The proof of this theorem requires a variational argument and because of its complexity we will not give it here. We refer to the literature for its examination.

Let us note that when refering to γ, t is the parameter of the curve, but when refering to the bodies, it is the time of the solution. We observe that property (*iii*) reflects that the position of the particles follow a collision-free solution of the 3-body problem in which they are shifted by one third of the period. Property (*i*) means that the figure is always centered at the origin and lastly, we illustrate in Figure 3.1 the symmetries stated by property (*ii*).

It can be seen from the motion of the bodies that in this solution the three possible Eulerian collinear configurations take place twice in a full period. Such configurations happen on the lines $y = \pm ax$ with $a \approx 0.2248$, and each line is visited alternatively every third of the period.

[1] The Klein group is a four-order group that is not cyclic.

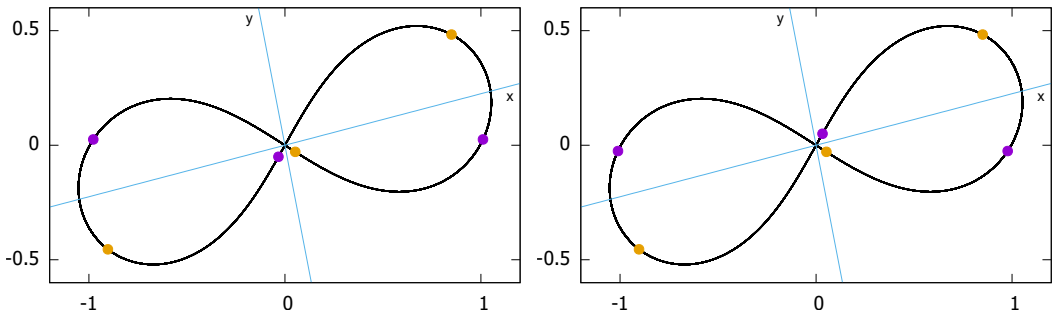

Figure 3.1: The action of σ and τ on the figure eight solution given by property (ii) of Theorem 3.2. In yellow the bodies on the curve at time t, in purple at times $\sigma(t)$ and $\tau(t)$. On the left, σ gives a symmetry with respect to the y-axis. On the right, we observe the symmetry with respect to the x-axis given by the action of τ.

The most important property discovered for the figure eight solution is its stability. The solution is indeed linearly stable, and to remark its significance, let us mention that it is actually the only choreography known so far, which fulfills this property. Due to the complexity of the stability study by analytical means, this feature has only been proven numerically. We will check the linear stability of the figure eight in Section 3.2.2.

Another important property to mention is that the figure eight solution lives on the zero angular momentum level. We refer to [13] to other properties of this solution.

Remark 3.3. The discovery of the figure eight led to the question about the existence of other choreographies of the n-body problem. For $n = 3$, there exist relative and satellite choreographies related to the eight. That means, they are a direct consequence of the existence of this solution and they can be found from continuous perturbations of the angular momentum and using rotation frames. Furthermore, choreographies not related to the eight have been found as well. Further details are given in [14].

Moreover, some studies were carried out in order to generalize the study of choreographies to n bodies for $n > 3$, and in fact, many choreographies have been found for different values of n. In [2] and [13] one can find many examples of different choreographies for different numbers of masses.

3.2 Numerical analysis of the figure 8 solution

In this section we give details on the numerical computations we have performed on the figure eight.

3.2.1 Computing the orbit

First of all, we are going to explain how we obtained the orbit numerically. We used the Taylor method adapted to this problem, as explained in Appendix A, to compute the orbit of the figure eight as a solution of the planar 3-body problem with the specific initial conditions given in [13].

We denote the position of the bodies by $q_i = (x_i, y_i) \in \mathbb{R}^2$, for $1 \leq i \leq 3$, where the subscript denotes the particle. For the velocities of each particle we will set $\dot{q}_i = (\dot{x}_i, \dot{y}_i)$. For explainig the evolution of the masses along the figure eight, we label the particles as q_1 (resp. q_2) the one in the left (resp. right) and q_3 as the particle in the middle of the configuration at the initial time, as in [13].

The initial conditions given in this paper correspond to a collinear configuration along the axis $y = 0$ (note this is not the symmetry axis from Figure 3.1, hence in the (x, y) coordinates used for the computation the figure is rotated). Concretely, the initial conditions given are the velocity $\dot{q}_3$, and the positions of q_1 and q_2 on the x-axis. The velocities of these two bodies are considered so that $\dot{q}_1 = \dot{q}_2$, as a consequence of the angular momentum being zero. Additionally, the masses of the bodies are all equal to 1 and the period of the orbit is set to 2π.

The conditions not explicitly stated in the paper can be found analytically from the ones given, by using the first integrals of the center of mass which is said to be placed at the origin. Concretely,

- Since $c = 0$, we get $x_3 = 0$, hence q_3 is located at the origin.
- Taking time derivatives on the relation $c = 0$ and using $\dot{q}_1 = \dot{q}_2$, one obtains the relation $\dot{q}_1 = \dot{q}_2 = -\frac{1}{2}\dot{q}_3$.

From these considerations we obtain an initial condition in $\mathbb{R}^{12}$ which we propagate numerically to obtain the figure eight. Note that we have not used the reduced problem given by constant values of the first integrals, instead we propagate in $\mathbb{R}^{12}$ and we use the constant values of the first integrals for performing checkings of propagation of the error along the orbit.

The numerical values with the exact decimal numbers obtained from the previous calculations and thus, used for the computation of the orbit, are given in Table 3.1. With these data we computed the solution of the figure eight orbit using

$x_1 = -0.995492$	$y_1 = 0$	$\dot{x}_1 = -0.347902$	$\dot{y}_1 = -0.533930$
$x_2 = 0.995492$	$y_2 = 0$	$\dot{x}_2 = -0.347902$	$\dot{y}_2 = -0.533930$
$x_3 = 0$	$y_3 = 0$	$\dot{x}_3 = 0.695804$	$\dot{y}_3 = 1.067860$

Table 3.1: Initial conditions data to plot the figure eight solution.

the Taylor integration method implemented in C (see Appendix A). We display the obtained orbit, together with the initial positions, in Figure 3.2.

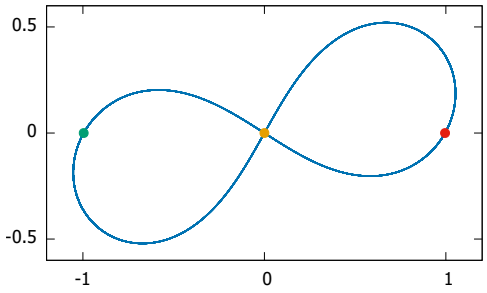

Figure 3.2: Figure eight solution obtained with the Taylor integration method. See the text for details on the computation. The three coloured dots correspond to the initial positions of the three bodies.

We emphasize that rotating the figure in the plot with the appropiate angle, we could easily see that the figure has the symmetries stated in Theorem 3.2.

We have computed the error of the solution when it returns to the starting points. This error is of the order of 10^{-6}, which is the same order of precision of the initial conditions we took from [13]. However, after two periods, we can observe that the error has increased up to an order of 10^{-5}, and it remains similar after 10 revolutions. The error respect to the initial conditions after 100 revolutions becomes of the order of 10^{-4}.

3.2.2 Stability analysis

Here we study the stability property of the figure eight which we mentioned before in Section 3.1 . We are going to present the numerical methods we employed to investigate on this property and the results achieved. We used [14] for a reference on this matter.

In the first place, let us set a theorical background on the subject. In dynamical systems, an orbit is usually said to be stable if all the points near enough to the orbit remain close to it after a long period of time. This was checked by numerical

explorations in [14]. The first step is to guarantee that the orbit is linearly stable, meaning that the normal behaviour is elliptic. To investigate the linear stability of the orbit it is very natural to make use of the Poincaré (or first return) map. Let us recall the following theorem from the Ordinary Differential Equations degree course, for concreteness here we omit the proof.

Theorem 3.4. *Let $\mathcal{X} : \Omega \to \mathbb{R}^m$ be a $\mathcal{C}^r$ vector field and let $\varphi : D \to \Omega$ be the associated flow. Consider a codimension 1 surface Σ transversal to the flow φ in $z_0 \in \Omega$ and $z_1 = \varphi(\tau(z_0), z_0)$. Then, there exists an open neighbourhood $U \subset \Sigma$ of z_0 such that there is a map $\tau : U \to \mathbb{R}$ and a diffeomorphism $P : U \to P(U) \subset \Sigma$, both $\mathcal{C}^r$, for which*

$$\varphi(\tau(z_0), z_0) = P(z_0).$$

□

The surface Σ is called a Poincaré section and the map P is the Poincaré map, or also first return map to the section Σ, because it gives the point of the trajectory of z_0 where the orbit returns to the section. Moreover, $\tau(z_0)$ represents the time it takes to return to the section.

The reason why we are interested in working with the Poincaré map is because it converts, locally, the study of the dynamics in the phase space given by the flow, to the study of a codimension-one discrete dynamical system in Σ. As an example of this relation, we can see that the search of periodic orbits is reduced to the search of fixed points, a more studied and well-known problem.

In particular, the figure eight solution corresponds to a fixed point of a suitable Poincaré map. Hence, in order to study the linear stability of the solution we shall compute the Jacobian matrix at the fixed point of such a Poincaré map. To pursue this goal, there are different ways to compute a numerical approximation. We will do it using a numercial differentiation scheme, based on the usage of centered finite differences combined with Richardson extrapolation, as learned in the first course of Numerical Methods of the degree.

We focus specifically in our problem. As a suitable Poincaré section we define

$$\Sigma = \{q \in \mathbb{R}^{12} \mid q \text{ is a collinear central configuration}\}$$

Note that Σ defines a transversal section to the flow around the figure eight. Otherwise, the bodies would move along the collinear configuration and that would lead to collision, as we showed in Section 2.4. We can then consider the Poincaré map $\hat{P} : \Sigma \to \Sigma$. In particular, the figure eight orbit is a fixed point of $P = \hat{P}^6$, due to the fact that a collinear configuration happens every sixth of the period, see Section 3.1. Then for z_0 as such a fixed point, we have that

$DP(z_0) = \left(\frac{\partial P}{\partial z_1}(z_0) \ \dots \ \frac{\partial P}{\partial z_{11}}(z_0)\right)$, where $\frac{\partial}{\partial z_k}$ represents the partial derivative w.r.t. the k-th element of a normalized basis $\{u_1, \dots, u_{11}\}$ of $T_{z_0}\Sigma$. Now, using finite differences we can compute these twelve partial derivatives as

$$\frac{\partial P}{\partial z_k}(z_0) \approx \frac{P(z_0 + hu_k) - P(z_0 - hu_k)}{2h} \qquad \text{for } h \approx 0, \quad 1 \leq k \leq 11. \tag{3.1}$$

From (3.1) it follows that we need to compute the images of the Poincaré map P for different points, concretely the images of the points $\tilde{z}_{0,k_\pm}(h) = z_0 \pm hu_k$, $k = 1, \dots, 11$, for a fixed h. In our case, since z_0 is a point of the orbit we recall that $P(z_0) = \varphi(\tau(z_0), z_0) = \varphi(T, z_0)$, where T denotes the period of the orbit. However, for any of the approximated points $\tilde{z}_{0,k_\pm}$, close to z_0, we do not know its return time $\tau(\tilde{z}_{0,k_\pm})$ because these points do not belong to the orbit. We avoid the problem by computing the Poincaré return map to Σ.

We will describe now the steps followed to numerically compute $\hat{P}$. Basically, our goal is to obtain the points at which the three particles return to the next collinear configuration, which correspond to a point in Σ, and then we compute $P = \hat{P}^6$.

We start with the initial conditions stated in [14] for which the bodies are placed at Σ. These are different from the ones shown in Table 3.1 (in this paper the figure eight is rescaled regarding the one in Figure 3.2 and the period also changes accordingly). Then, at each step of the Taylor integration method used to compute the orbit, we want to detect when the particles cross the collinear configuration section. To do so, we take the vector that goes from particle 3 to particle 2, $v_{32}(t) = (q_2(t) - q_3(t))$, and the orthogonal vector that goes from particle 3 to particle 1, ${v_{31}}^\perp(t) = (q_1(t) - q_3(t))^\perp$. After the bodies depart from the collinear section Σ, these two vectors will form an angle α with either, $0 < \alpha < \frac{\pi}{2}$ or $\frac{\pi}{2} < \alpha < \pi$. Being in the same line means $\alpha = \frac{\pi}{2}$.

Hence, we look for changes in sign of the function $g(t) = {v_{31}}^\perp(t) \cdot v_{32}(t)$. One checks that $T_{z_0}\Sigma = \langle (e_2 + e_4 - 2e_6)^\perp \rangle$, where $\{e_i\}$ denotes the canonical basis of $\mathbb{R}^{12}$. Note that Σ is defined by the condition $g(t) = 0$. Therefore, at every time step we look for a change of sign of g and, once detected we use the last computed value where the change has been achieved as the initial approximation for the Newton method in order to find the zero of g.

Since the Taylor method that we developed computes the Taylor polynomial expansion of the solution near the time given in the previous step (see Appendix A), when employing the Newton method we compute a new time guess and then we obtain the new point of the solution at that time by simply evaluating the Taylor polynomial.

Once we have found the intersection of the trajectory with Σ, we continue computing the orbit of the solution in order to proceed to the next collinear configuration. At the sixth time we repeat this process, we will obtain the return value of the initial condition, that is, $P(z) = \varphi(\tau(z), z)$. Let us remark that with this method, not only we obtain the images of P, but also we obtain, from the Taylor integration method, $\tau(z)$ (the time that the body takes to return to the section).

Hence, now we are able to compute all the values $P(\tilde{z}_{0,k_\pm})$, and then, following (3.1) we could obtain the differential of the Poincaré map. As said, in order to improve the numerical precision of the computations for the derivatives, we use Richardson extrapolation applied to centered finite differences. Let us denote by $D_1(h)$ the approximation to first order of the derivative with difference h, of a real-valued function f. Then we can express $f'(z_0) = D_1(h) - \sum_{s>0} h^{2s} M_s$, where $M_s = \frac{f^{(2s+1)}(z_0)}{(2s+1)!}$. Now we consider the approximation of $f'(z_0)$ for a difference of $h/2$. What we obtain is $f'(z_0) = D_1(\frac{h}{2}) - \sum_{s>0} \frac{h^{2s}}{4^s} M_s$. Multiplying the last expression by 4 and substracting the first one, we obtain

$$f'(z_0) = D_1(\tfrac{h}{2}) - \frac{D_1(\frac{h}{2}) - D_1(h)}{3} - \sum_{s>1} \frac{1 - 4^{s-1}}{3 \cdot 4^{s-1}} h^{2s} M_s \tag{3.2}$$

and now we set $D_2(h) = D_1(\frac{h}{2}) - \frac{D_1(\frac{h}{2}) - D_1(h)}{3}$. Let us prove by induction that, in fact, we can express the first derivative of f at z_0 as

$$f'(z_0) = D_k(h) + \mathcal{O}(h^{2k}), \quad \text{where} \quad D_k(h) = D_{k-1}(\tfrac{h}{2}) + \frac{D_{k-1}(\frac{h}{2}) - D_{k-1}(h)}{4^{k-1} - 1} \tag{3.3}$$

for $k > 1$.

We can see in (3.2) that the initial case $k = 2$ satisfies (3.3). Now, consider for $k > 2$ the expression in (3.3) for a difference of $\frac{h}{2}$. We then have,

$$f'(z_0) = D_k(\tfrac{h}{2}) + \mathcal{O}((\tfrac{h}{2})^{2k}) = D_k(\tfrac{h}{2}) + \frac{1}{4^k}\mathcal{O}(h^{2k})$$

By multypling this expression by the factor 4^k and substracting the expression from (3.3), we obtain

$$(4^k - 1) f'(z_0) = 4^k D_k(\tfrac{h}{2}) - D_k(h) + \mathcal{O}(h^{2(k+1)})$$

from where we deduce

$$f'(z_0) = D_{k+1}(h) + \mathcal{O}(h^{2(k+1)}), \quad \text{with} \quad D_{k+1}(h) = D_k(\tfrac{h}{2}) + \frac{D_k(\frac{h}{2}) - D_k(h)}{4^k - 1}, \tag{3.4}$$

attaining the same expression as in (3.3).

We obtain the following method when translating the formula into an algorithm to calculate the derivative up to an r-th step. First we start by computing the approximation $D_1(h)$. For simplicty, let us drop the h and let us write $D_{k,j} = D_k(\frac{h}{2^j})$. We observe from (3.4) that at the step $k+1$ we only need to compute $D_{k,1}$, since we already have $D_{k,0}$ from the previous step.

By a thorough inspection, we notice that in order to compute $D_{k,1}$, we need to have $D_{k-1,1}$ and $D_{k-1,2}$, but it turns out that we had already computed $D_{k-1,1}$ for the computation of $D_{k,0}$. Again by induction, one can see that in fact, the only derivative we need to calculate to apply the formula in (3.4) is $D_{1,k-1}$.

Note also that, by the recurrence in the formula, it is only necessary to have the values of the functions D_k from the current and the previous step. Thus, we will denote $D_j^{(1)}$ to the values D_j computed at the current step and by $D_j^{(0)}$ to the ones computed in the previous step, to simplify a little bit more the notation, but specially, to emphasize that we only need those values.

Therefore, at the $(k+1)$-th step, we want to find $D_{k+1,0}$. We only need to do three things. First, we start by computing

$$D_1^{(1)} = D_{1,k} = D_1(\tfrac{h}{2^k}) = \frac{2^{k-1}}{h}(f(z_0 + \tfrac{h}{2^k}) - f(z_0 - \tfrac{h}{2^k})).$$

Then, for $j = 2, \ldots, k+1$, we compute $D_j^{(1)} = D_{j-1}^{(1)} + \dfrac{D_{j-1}^{(1)} - D_{j-1}^{(0)}}{4^{j-1} - 1}$. Finally, we set the values $D_j^{(1)}$ in the variables $D_j^{(0)}$. Note that the desired derivative at the $(k+1)$-th step is $D_{k+1}^{(1)}$. We repeat the process for $k > 2$ until the difference between two consecutive derivatives is lower than the precision desired. Using this algorithm we are able to accurately compute each of the partial derivatives $\frac{\partial P_i}{\partial z_k}(z_0)$ that compose the differential matrix of the Poincaré map, $DP(z_0) = \left(\frac{\partial P_i}{\partial z_k}(z_0)\right)_{1 \le i,k \le 11}$.

Once the Jacobian matrix is obtained we proceed to computing its eigenvalues using PARI/GP [16] and we obtain 7 eigenvalues equal to 1 (with an error of 10^{-4}) and 4 complex eigenvalues $\lambda = \exp(\pm 2\pi i \nu_j)$, with $\nu_1 \approx 0.008422721636461862$, and $\nu_2 \approx 0.2980925294189548$, which coincide on eight/nine digits, with the ones given in [14]. The reason why we have obtained less digits might be due to the fact that we have used double precision and only one Poincaré section for our computations, while C. Simó used quadruple precision and a parallel shooting strategy.

The results obtained regarding the eigenvalues can be heuristically justified as follows. The planar three-body problem is a hamiltonian system with $m = 3 \cdot 2 = 6$

degrees of freedom. We know from Section 1.3 that there are two first integrals given by the linear momentum and one given by the angular momentum. Thus, the problem can be reduced to three degrees of freedom. First integrals are functions of the original coordinates of the system which are constant along the solutions of the problem. Hence the gradient vectors of these functions become eigenvectors of eigenvalue 1. The vectors associated to the conjugated coordinates are also eigenvectors of eigenvalue 1, due to the Hamiltonian structure (the eigenvalues come by pairs, see [8]). Hence, we must have six real eigenvalues equal to 1, two for each integral. The additional eigenvalue equal to 1 is due to the preservation of the Hamiltonian function. Finally, the four remaining eigenvalues correspond to two pairs of conjugated eigenvalues that provide the elliptic linear behaviour of the figure eight orbit.

We conclude that the figure eight orbit is linearly stable due to the fact that all the eigenvalues related to normal directions of the orbit have modulus 1. Note that DP is a square matrix of dimenision 11. If we remove the conjugated direction to the tangent direction, then DP can be considered as a matrix of dimension 10 and it must be symplectic (it has the same complex eigenvalues and six ones).

The previous discussions and computations show the linear stability of the figure eight solution. As we mentioned in Section 3.1, the figure eight is indeed stable for long times and it is the only choreography known having this property. This conclusion introduces a completely new and very relevant physical possibility: universes of three stellar objects following a figure eight orbit without a central star can exist.

Conclusions

We have investigated different properties of the n-body problem and their relation with its Hamiltonian structure. We also discussed about central configurations and their relation with simple (explicit) solutions of the n-body problem.

In particular, the Lagrange configuration of three bodies leads to a choreography, a fact that motivated the study of the so-called figure eight solution. The latter is also a choreography, which has several geometric and dynamical properties. Concretely, we investigated the linear stability of this solution using the Taylor method adapted to this problem for numerical integration of the equations of motion.

During this time, not only have we discovered many interesting results, but also open questions and unsolved problems on this field of study that can lead to a further exploration in the future and might also be an starting point for investigating several related questions. We highlight some of the most relevant:

- *Existence of central configurations in higher dimensions.* As mentioned in Section 2.1, the regular platonic solids lead to central configurations. However, for higher dimensions there are fewer of such configurations. Moreover, it is not known in general if there are other CC's with equal masses not related with regular platonic solids. The geometrical approach in [10] to the n-body problem for higher dimensions might help in investigating such CC's.

- *Other stable choreographies related, or not, to the figure eight.* In [14] it is discussed the existence of many choreographies that follow different paths but resemble the figure eight solution, as well as other many choreographies following completely different trajectories. It is stated that all of them are linearly unstable, so one can wonder wether there are other stable choreographies or not.

- *Central configurations, relative equilibria and choreographies in other related problems.* One can consider the existence of similar particular solutions for other related problems like the n-body problem with different potential, the n-

body problem in curvature spaces, etc. Several partial results can be found in the literature, see for example [13].

- *Applications to astrodynamics.* The particular solutions can be potentially interesting for applications to astrodynamics for mission design purposes, station keeping, etc. Those solutions that are linearly stable can be of particular interest to this end. Certainly, this is an interesting subject to be studied in the future that we had no time to consider in this work.

As a final comment we remark that the combination of the theoretical and numerical techniques used in this work helped to investigate and to understand several properties of the problem. Developing our own implementation of the Taylor method for the n-body problem turned out to be very useful to this end.

Appendix A

Taylor Integration Method

A.1 Applying the method to the n-body problem

In this appendix we describe the ad-hoc implementation of the Taylor method used in this work to numerically solve the n-body problem for an aribitrary number of masses. We mainly use [6] as a reference.

We want to solve the initial value problem

$$\begin{cases} \dot{z}(t) = \mathcal{X}_H(z(t)) \\ z(0) = z_0 \end{cases} \qquad \text{where } \mathcal{X}_H(z) = \left(\frac{\partial H}{\partial p}(z), -\frac{\partial H}{\partial q}(z) \right)^T \tag{A.1}$$

for $z = (q, p)$ and $H = T + V$ the Hamiltonian function defined in (1.3). Solving this problem means finding a function $z : [0, t_{max}] \longrightarrow \mathbb{R}^{2m}$ such that $z(t)$ is a solution of (A.1) for all $t \in [0, t_{max}]$.

Let us consider some intermediate times $t_s \in [0, t_{max}]$. These will be computed from the initial time value $t_0 = 0$ and a succession of time steps denoted by h_s, $s > 0$ (we explain further on how we compute them). Thus, at the s-th time step we will have $t_s = t_{s-1} + h_s$. We will calculate the approximation of the function z at each time t_s and we will represent these approximations by z_s.

Suppose we finished to compute one of the steps and we find ourselves at the beginning of the s-th step. Then, we compute the approximation of $z(t_s)$ from the Taylor polynomial of $z(t)$ at $t = t_{s-1}$,

$$z_s = z_{s-1} + \dot{z}(t_{s-1})h_s + \frac{\ddot{z}(t_{s-1})}{2!}h_s^2 + \cdots + \frac{z^{(r)}(t_{s-1})}{r!}h_s^r \tag{A.2}$$

So, the only thing we need to do to obtain this approximation is computing the derivatives $z^{(k)}$. To obtain them, we note the following.

Recall we have $z = (q, p)$ for $q, p \in \mathbb{R}^{nd}$, then $\dot{z} = (\dot{q}, \dot{p})$ and for $j = 1, \ldots, n$,

$$\begin{cases} \dot{q}_j = \dfrac{\partial H}{\partial p_j} = \dfrac{p_j}{m_j} \\ \dot{p}_j = -\dfrac{\partial H}{\partial q_j} = \displaystyle\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{m_i m_j}{r_{ij}^3}(q_i - q_j) \end{cases} \tag{A.3}$$

Therefore, to compute the derivative $z^{(k)}$ we need the derivatives $(q^{(k)}, p^{(k)})$. Note that in order to have $z^{(k)}$ it is sufficient to calculate the derivatives for only one of the Hamiltonian variables, because we can obtain one from the other using the relation

$$q_j{}^{(k)} = \frac{p_j{}^{(k)}}{m_j} \tag{A.4}$$

In order to symplify the discussion below, we introduce the following notation. If $x : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ denotes a smooth function, we call its normalized k-th derivative to the value

$$x^{[k]}(t) = \frac{x^{(k)}(t)}{k!} \tag{A.5}$$

Then, $z(t) = \displaystyle\sum_{k=0}^{r} z^{[k]}(t) h^k$ represents the Taylor polynomial for z at time t.

Let us see now how we compute the derivatives $q_j{}^{[k]}$ and $p_j{}^{[k]}$. Given the initial values $q_1{}^{[0]}, \ldots, q_n{}^{[0]}, p_1{}^{[0]}, \ldots, q_n{}^{[0]}$, we have for $j = 1, \ldots, n$

$$\begin{aligned} q_j{}^{[1]} &= \dot{q}_j = \frac{p_j{}^{[0]}}{m_j} \\ p_j{}^{[1]} &= \dot{p}_j = \sum_{i \neq j} \frac{m_i m_j}{r_{ij}^3}(q_i{}^{[0]} - q_j{}^{[0]}) \end{aligned} \tag{A.6}$$

Let us call f_j to the function $f_j = \sum_{i \neq j} \frac{m_i m_j}{r_{ij}^3}(q_i - q_j)$ to simplify notation as well. Then, to compute the k-th normalized derivatives for each j we have

$$\begin{aligned} q_j{}^{[k]} &= \frac{1}{k m_j} p_j{}^{[k-1]} \\ p_j{}^{[k]} &= \frac{1}{k} f_j{}^{[k-1]} \qquad\qquad \text{for } k = 1, \ldots, r \end{aligned} \tag{A.7}$$

In order to compute the derivatives $f_j{}^{[k]}$ we will use *automatic differentiation*. Automatic differentiation (AD) is a recursive procedure used to calculate the values of the derivatives of a certain function at a given point.

To apply AD we write the function f_j as a combination of intermiadate scalar/vector functions x^k, $k = 1, \dots, 7$. To those which correspond, we write $x^k = (x^k_l)_{1 \le l \le d}$. Thus, we have for $i, j = 1, \dots, n$, $i \neq j$,

$$\begin{aligned}
x^1_{ij,l} &= q_{il} - q_{jl} \\
x^2_{ij,l} &= (q_{il} - q_{jl})^2 = x^1_{ij,l} \cdot x^1_{ij,l} \\
x^3_{ij} &= \sum_{l=1}^{d} (q_{il} - q_{jl})^2 = \sum_{l=1}^{d} x^2_{ij,l} \\
x^4_{ij} &= r_{ij}^{-3} = (x^3_{ij})^{-3/2} = \exp(-\frac{3}{2} \log(x^3_{ij})) \\
x^5_{ij,l} &= \frac{q_{il} - q_{jl}}{r^3_{ij}} = x^4_{ij} \cdot x^1_{ij,l} \\
x^6_{ij,l} &= \frac{m_i}{r^3_{ij}} (q_{il} - q_{jl}) = m_i x^5_{ij,l} \\
x^7_{j,l} &= \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{m_i}{r^3_{ij}} (q_{il} - q_{jl}) = \sum_{\substack{i=1 \\ i \neq j}}^{n} x^6_{ij,l} \\
f_{j,l} &= m_j \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{m_i}{r^3_{ij}} (q_{il} - q_{jl}) = m_j x^7_{j,l}
\end{aligned} \tag{A.8}$$

Let us clarify the notation used. The superscript indicates the ordinal number of the elementary function. The subscripts before the comma indicate the dependence of the operation to the particles i and j. Lastly, the subscript after the comma notes that the corresponding function x is a vector on the d dimensional components.

The use of intermediate functions allows us to compute the k-th normalized derivatives of q and p. To this end, we use the following properties relating AD and the operations used above to define the intermediate functions x^k in (A.8) [6].

Properties. Given functions b and c of class $\mathcal{C}^k$ and $\alpha \in \mathbb{R} \setminus \{0\}$, we have for a function a of b and c,

1. If $a(t) = b(t) \pm c(t)$, then $a^{[k]}(t) = b^{[k]}(t) \pm c^{[k]}(t)$
2. If $a(t) = b(t)c(t)$, then $a^{[k]}(t) = \sum_{s=0}^{k} b^{[k-s]}(t) c^{[s]}(t)$
3. If $a(t) = b(t)^\alpha$, then $a^{[k]}(t) = \frac{1}{k b^{[0]}(t)} \sum_{s=0}^{k-1} (k\alpha - s(\alpha + 1)) b^{[k-s]}(t) a^{[s]}(t)$

The first property follows from the linearity property of derivatives, and the second one follows from Leibniz formula for the derivatives of a product,

$$a^{[k]}(t) = \frac{1}{k!}a^{(k)}(t) = \frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}b^{(k-s)}(t)c^{(s)}(t) = \sum_{s=0}^{k} b^{[k-s]}(t)c^{[s]}(t)$$

For Property 3, we give some more details. Taking logarithms on both sides of $a(t) = b(t)^{\alpha}$ we get $\log a(t) = \alpha \log b(t)$. Now derivating w.r.t. t we acquire $\dot{a}(t)b(t) = \alpha a(t)\dot{b}(t)$. Applying item 2 on both sides up to the $(k-1)$-th derivative we shall obtain

$$\sum_{s=0}^{k-1}(s+1)b^{[k-1-s]}(t)a^{[s+1]}(t) = \alpha\sum_{s=0}^{k-1}(k-s)b^{[k-s]}(t)a^{[s]} \iff$$

$$ka^{[k]}(t)b^{[0]}(t) + \sum_{s=0}^{k-2}(s+1)b^{[k-1-s]}(t)a^{[s+1]}(t) = \alpha\sum_{s=0}^{k-1}(k-s)b^{[k-s]}(t)a^{[s]} \iff$$

$$ka^{[k]}(t)b^{[0]}(t) = \sum_{s=0}^{k-1}\alpha(k-s)b^{[k-s]}(t)a^{[s]} - \sum_{s=0}^{k-1}sb^{[k-s]}(t)a^{[s]}(t) \iff$$

$$a^{[k]}(t) = \frac{1}{kb^{[0]}}\sum_{s=0}^{k-1}(k\alpha - s(\alpha+1))b^{[k-s]}(t)a^{[s]}(t)$$

Now we are ready to compute ${f_j}^{[k-1]}$, applying the previous properties to the equations in (A.8) and setting $\alpha = -3/2$:

$$\begin{aligned}
\left(x^1_{ij,l}\right)^{[k-1]} &= {q_{il}}^{[k-1]} - {q_{jl}}^{[k-1]} \\
\left(x^2_{ij,l}\right)^{[k-1]} &= \sum_{s=0}^{k-1}\left(x^1_{ij,l}\right)^{[k-1-s]}\left(x^1_{ij,l}\right)^{[s]} \\
\left(x^3_{ij}\right)^{[k-1]} &= \sum_{l=1}^{d}\left(x^2_{ij,l}\right)^{[k-1]} \\
\left(x^4_{ij}\right)^{[k-1]} &= \frac{1}{k\left(x^3_{ij}\right)^{[0]}}\sum_{s=0}^{k-2}(k\alpha - s(\alpha+1))\left(x^3_{ij}\right)^{[k-1-s]}\left(x^4_{ij}\right)^{[s]} \\
\left(x^5_{ij,l}\right)^{[k-1]} &= \sum_{s=0}^{k-1}\left(x^4_{ij}\right)^{[k-1-s]}\left(x^1_{ij,l}\right)^{[s]} \\
\left(x^6_{ij,l}\right)^{[k-1]} &= m_i\left(x^5_{ij,l}\right)^{[k-1]}
\end{aligned} \tag{A.9}$$

for $i, j = 1, \ldots, n$, $i \neq j$, and then

$$\begin{aligned}
\left(x^7_{j,l}\right)^{[k-1]} &= \sum_{\substack{i=1\\ i\neq j}}^{n}\left(x^6_{ij,l}\right)^{[k-1]} \\
{f_j, l}^{[k-1]} &= m_j\left(x^7_{j,l}\right)^{[k-1]}
\end{aligned}$$

and we finally obtain $p_j^{[k]} = f_j^{[k-1]}/k$. Doing so for $j = 1, \ldots, n$, we eventually get $z^{[k]}(t)$.

Summarising, in order to obtain an approximation of $z(t)$, we need to compute at different times t_s the Taylor approximation of z at the previous time value. That involves computing derivatives of z (as a vector of q and p) up to r-th order. For each derivative to compute, we use AD and we follow the steps given by (A.7) and (A.9).

A.2 Coding the Taylor method

Here we add further comments on the C implementation of the previous scheme. Throughout this section we shall refer to the code in Section A.3. First of all, we created a function which manages the computation of the approximation z_s from z_{s-1}. This function is called `integ_taylor`.

Before proceeding further, let us clarify the variables we use in the code. Variable `z` is a 3-array, where the first component indicates the k-th order derivative. The second one indicates the position or momentum of the particle, that is, for $i = 0, \ldots n-1$ we have the positions of the n bodies, and for $i = n, \ldots, 2n-1$ the momentums. At last, the third component denotes the spatial coordinate.

Moreover, `m` is an array containing the n mass values, `t` is a pointer representing the time t_s at each step and `tmax` is a double variable representing the end point of the time interval. The pointer `h` will be the corresponding time step h_s and `hmin` and `hmax` are the bounds we set for the time steps h_s so as to ensure a control on the size of the steps. The integer variables `ordre`, `n` and `dim` give the values for the maximum order of the derivatives, the number of bodies of the problem and the space dimension respectively. Lastly, `tol` indicates the local tolerance required at each step of the taylor method.

All these variables are declared in the main function. Let us detail now the variables we used only for the function `integ_taylor`.

On computing the derivatives of z, we distinguish two cases. The initial case, for which we calculate the first derivatives of z strictly following (A.6) and (A.8); and the general k-th order case, where we compute the derivatives of z following (A.7) and (A.9). In order to compute steps in (A.8) and (A.9) we declared the following variables representing the elementary operations into which we broke f down.

Let us remark that the variables we used and the way we decided to save

memory are justified over the need of that value for further computations on that time step.

In the first place, we declared `difq` as a 4-array to represent $(x^1)^{[k]}$, its dependence on the particles i and j and the fact that x^1 is a vector of d components as well. Then, we represented $(x^2)^{[k]}$ with `difq2` (since it is basically x^1 squared) as a vector of d components. Note that we just need this value for that concrete derivative and not any other, so it acts as sort of an slack variable for each derivative step.

We declared `sumdifq2` and `norma` as 3-arrays representing $(x^3)^{[k]}$ and $(x^4)^{[k]}$ and its dependence on the particles i and j with which are calculated. Finally, to represent $(x^5)^{[k]}$ we used the 2-array variable `prod` depending only on the particle i and the spatial coordinate l. That is because on the implementation of (A.9) it is only used once for each derivative and although it depends on the particle j for which is calculated it is not necessary to keep this value for any other j.

The last intermediate variables x^6 and x^7 are not declared as variables on the code, but are calculated on the corresponding momentum vector following the relation given in (A.7) and (A.9) with such variables, the final function value ${f_j}^{[k-1]}$ and ${p_j}^{[k]}$.

As we already explained, we start computing the derivatives of q and p. First for the initial case, following (A.6) and (A.8) (lines 55-105), and afterwards for the general k-th order case following (A.7) and (A.9) (lines 107-171).

Once we have calculated the $r =$ `ordre` derivatives, we proceed to compute the s-th time step h_s (lines 173-187). To determine the step size we require the last terms of the series to be of the order of the local error demanded. This is heuristically justified since one expects the error between the true solution $z(t_s + h_s)$ and the truncated solution at order r to be of the order of the first ignored term, by Taylor theorem. Then one requires

$$|z^{[r+1]}(t_s)h_s^{r+1}|_\infty < \texttt{tol}$$

from where we obtain

$$|h_s|_\infty < \left(\frac{\texttt{tol}}{z^{[r+1]}}\right)^{\frac{1}{r+1}}.$$

To avoid possible cancellations (due to symmerties of the problem) we consider the two last computed terms to estimate h_s, hence we choose

$$h_s = \min\left(\left(\frac{\texttt{tol}}{|z^{[r]}(t_s)|_\infty}\right)^{\frac{1}{r}}, \left(\frac{\texttt{tol}}{|z^{[r-1]}(t_s)|_\infty}\right)^{\frac{1}{r-1}}\right) \tag{A.10}$$

as the s-th time step to guarantee a local error of order `tol`[1].

At last, we verify that the step we computed is inside the bounds given as `hmin` and `hmax`. Once we have the definitive value for the time step, we compute the new time value to which the approximation will correspond. In the case when the new time overcomes the bound of the time interval, we take this bound as the new time value and we notify it to the main function by returning the value `flag` $= 1$, which indicates that we have reached the last step to computing the approximation of the function $z(t)$ (lines 188-199).

Finally, we are ready to compute the new approximation of the solution for the new time value. In order to do so, we evaluate (A.2) using Horner's method (lines 201-209).

A.3 Code sample in C

We include in this section the code in C for the implementation of the Taylor time-stepper function `integ_taylor`.

```
1  #include <stdio.h>
2  #include <stdlib.h>
3  #include <math.h>
4
5
6  double min(double var1, double var2);
7  int integ_taylor(double ***z, double *m, double *t, double tmax, double *h,
       double hmin, double hmax, int ordre, int n, int dim, double tol);
8  /* defineixo:
9           - z          vector del camp ( z = (q,p); (ordre+1)*2n*dim )
10          - q          vector de posicions ((ordre+1)*n*dim) [der][num][coord]
11          - p          vector de moments ((ordre+1)*n*dim) [der][num][coord]
12          - m          vector de masses (n)
13          - t          instant de temps
14          - tmax       temps maxim de l'interval temporal de definicio
15          - h          pas de Taylor
16          - hmin       cota inferior pel pas de Taylor
17          - hmax       cota superior pel pas de Taylor
18          - ordre      ordre del polinomi de Taylor
19          - n          nombre de cossos
20          - dim        dimensio espacial del problema
21          - tol        tolerancia                                          */
22
23
24
```

[1] A more restrictive strategy used for some of the computations consists in dividing the step from (A.10) by the norm of $z(t_s)$.

```
25 int integ_taylor(double ***z, double *m, double *t, double tmax, double *h,
       double hmin, double hmax, int ordre, int n, int dim, double tol) {
26     /* el vector z =(q,p) ve donat per q[0][][] = z[0][0->n-1][] i p[0][][] =
           z[0][n->2n-1] i m ve donat per m[0->n-1]  */
27
28     /* declaracio de variables */
29     int i,j,k,l,s;
30     int flag;
31     double alfa = -1.5, suma, max1, max2, h1, h2;
32     double ****difq, *difq2, ***sumdifq2, ***norma, **prod;
33
34     difq    = (double ****)malloc((ordre+1)*sizeof(double***));
35     difq2   = (double *)malloc(dim*sizeof(double));
36     sumdifq2= (double ***)malloc((ordre+1)*sizeof(double**));
37     norma   = (double ***)malloc((ordre+1)*sizeof(double**));
38     prod    = (double **)malloc(n*sizeof(double*));
39     for (k = 0; k <= ordre; k++) {
40         difq[k]     = (double ***)malloc(n*sizeof(double*));
41         sumdifq2[k] = (double **)malloc(n*sizeof(double));
42         norma[k]    = (double **)malloc(n*sizeof(double));
43         for (j = 0; j < n; j++) {
44             difq[k][j]      = (double **)malloc(n*sizeof(double*));
45             sumdifq2[k][j]  = (double *)malloc(n*sizeof(double));
46             norma[k][j]     = (double *)malloc(n*sizeof(double));
47             for (i = 0; i < n; i++)
48                 difq[k][j][i] = (double *)malloc(dim*sizeof(double));
49         }
50     }
51     for (i = 0; i < n; i++)
52         prod[i] = (double *)malloc(dim*sizeof(double));
53
54
55     /* Derivada ordre 1 */
56     /* Calculem la derivada q[1] */
57     for (j = 0; j < n; j++) {
58         for (l = 0; l < dim; l++)
59             z[1][j][l] = z[0][j+n][l]/m[j];
60     }
61     /* Calculem la derivada p[1] */
62     for (j = 0; j < n; j++) {
63         for (i = 0; i < j; i++) {
64             /* calculem diferencia entre les q */
65             for (l = 0; l < dim; l++)
66                 difq[0][j][i][l] = z[0][i][l] - z[0][j][l];
67             /* calculem el quadrat de les diferencies */
68             for (l = 0; l < dim; l++)
69                 difq2[l] = difq[0][j][i][l]*difq[0][j][i][l];
70             /* calculem la suma de les diferencies al quadrat */
71             sumdifq2[0][j][i] = 0;
72             for (l = 0; l < dim; l++)
73                 sumdifq2[0][j][i] += difq2[l];
74             if (fabs(sumdifq2[0][j][i]) <= tol)
75                 return -1;
76             /* calculem la norma del denominador */
77             norma[0][j][i] = exp(alfa*log(sumdifq2[0][j][i]));
78             /* calculem el quocient */
```

```
79              for (l = 0; l < dim; l++)
80                  prod[i][l] = difq[0][j][i][l]*norma[0][j][i];
81          }
82          for (i = j+1; i < n; i++) {
83              for (l = 0; l < dim; l++)
84                  difq[0][j][i][l] = z[0][i][l] - z[0][j][l];
85              for (l = 0; l < dim; l++)
86                  difq2[l] = difq[0][j][i][l]*difq[0][j][i][l];
87              sumdifq2[0][j][i] = 0;
88              for (l = 0; l < dim; l++)
89                  sumdifq2[0][j][i] += difq2[l];
90              if (fabs(sumdifq2[0][j][i]) <= tol)
91                  return -1;
92              norma[0][j][i] = exp(alfa*log(sumdifq2[0][j][i]));
93              for (l = 0; l < dim; l++)
94                  prod[i][l] = difq[0][j][i][l]*norma[0][j][i];
95          }
96          /* per a cada cos j, calculem el seu moment component a component,
                sumant sobre les i */
97          for (l = 0; l < dim; l++) {
98              z[1][j+n][l] = 0;
99              for (i = 0; i < j; i++)
100                 z[1][j+n][l] += m[i]*prod[i][l];
101             for (i = j+1; i < n; i++)
102                 z[1][j+n][l] += m[i]*prod[i][l];
103             z[1][j+n][l]*= m[j];
104         }
105     }
106
107     /* Derivada ordre k > 1 */
108     for (k = 2; k <= ordre; k++) {
109         for (j = 0; j < n; j++) {
110             /* Calculem les derivades q[k] */
111             for (l = 0; l < dim; l++)
112                 z[k][j][l] = z[k-1][j+n][l]/(k*m[j]);
113             /* Calculem les derivades p[k] */
114             for (i = 0; i < j; i++) {
115                 /* calculem diferencia entre les q */
116                 for (l = 0; l < dim; l++)
117                     difq[k-1][j][i][l] = z[k-1][i][l] - z[k-1][j][l];
118                 /* calculem el quadrat de les diferencies */
119                 for (l = 0; l < dim; l++) {
120                     difq2[l] = 0;
121                     for (s = 0; s <= k-1; s++)
122                         difq2[l] += difq[k-1-s][j][i][l]*difq[s][j][i][l];
123                 }
124                 /* calculem la suma de les diferencies al quadrat */
125                 sumdifq2[k-1][j][i] = 0;
126                 for (l = 0; l < dim; l++)
127                     sumdifq2[k-1][j][i] += difq2[l];
128                 /* calculem la norma, considerem alfa=-1.5 i multipliquem */
129                 suma = 0;
130                 for (s = 0; s <= k-2; s++)
131                     suma += ((k-1)*alfa - s*(alfa+1))*sumdifq2[k-1-s][j][i]*
                            norma[s][j][i];
132                 norma[k-1][j][i] = suma/((k-1)*sumdifq2[0][j][i]);
```

```
133                 /* calculem el quocient */
134                 for (l = 0; l < dim; l++) {
135                     prod[i][l] = 0;
136                     for (s = 0; s <= k-1; s++)
137                         prod[i][l] += difq[s][j][i][l]*norma[k-1-s][j][i];
138                 }
139             }
140             for (i = j+1; i < n; i++) {
141                 for (l = 0; l < dim; l++)
142                     difq[k-1][j][i][l] = z[k-1][i][l] - z[k-1][j][l];
143                 for (l = 0; l < dim; l++) {
144                     difq2[l] = 0;
145                     for (s = 0; s <= k-1; s++)
146                         difq2[l] += difq[k-1-s][j][i][l]*difq[s][j][i][l];
147                 }
148                 sumdifq2[k-1][j][i] = 0;
149                 for (l = 0; l < dim; l++)
150                     sumdifq2[k-1][j][i] += difq2[l];
151                 suma = 0;
152                 for (s = 0; s <= k-2; s++)
153                     suma += ((k-1)*alfa - s*(alfa+1))*sumdifq2[k-1-s][j][i]*
                        norma[s][j][i];
154                 norma[k-1][j][i] = suma/((k-1)*sumdifq2[0][j][i]);
155                 for (l = 0; l < dim; l++) {
156                     prod[i][l] = 0;
157                     for (s = 0; s <= k-1; s++)
158                         prod[i][l] += difq[s][j][i][l]*norma[k-1-s][j][i];
159                 }
160             }
161             /* per a cada cos j, calculem el seu moment component a component,
                    sumant sobre les i */
162             for (l = 0; l < dim; l++) {
163                 suma = 0;
164                 for (i = 0; i < j; i++)
165                     suma += m[i]*prod[i][l];
166                 for (i = j+1; i < n; i++)
167                     suma += m[i]*prod[i][l];
168                 z[k][j+n][l] = m[j]*suma/k;
169             }
170         }
171     }
172
173     /* triem pas h */
174     max1 = tol; max2 = tol;
175     for (i = 0; i < 2*n; i++) {
176         for (l = 0; l < dim; l++) {
177             /* norma subinfinit de z per la derivada d'ordre k=ordre-1 */
178             if (fabs(z[ordre-1][i][l]) > max1)
179                 max1 = fabs(z[ordre-1][i][l]);
180             /* norma subinfinit de z per la derivada d'ordre k=ordre */
181             if (fabs(z[ordre][i][l]) > max2)
182                 max2 = fabs(z[ordre][i][l]);
183         }
184     }
185     h1 = pow(tol/max1,1./(ordre-1));
186     h2 = pow(tol/max2,1./ordre);
```

```
187     *h = min(h1,h2);
188     /* Miro que h estigui en l'interval correcte */
189     if(*h > hmax) *h = hmax;
190     else if(*h < hmin) {
191         *h = hmin;
192         flag = 2;
193     }
194     /* mirem no passar-nos de temps*/
195     if(*t + *h > tmax) {
196         *h = tmax - *t;
197         flag = 1;
198     }
199     *t = *t + *h; /* nou temps */
200 
201     /* sumem fent Horner pel polinomi de Taylor (component a component) */
202     for (j = 0; j < 2*n; j++) {
203         for (l = 0; l < dim; l++) {
204             suma = z[ordre][j][l];
205             for (k = ordre-1; k >= 0; k--)
206                 suma = suma*(*h) + z[k][j][l];
207             z[0][j][l] = suma;
208         }
209     }
210 
211     /* alliberem memoria */
212     for (k = 0; k <= ordre; k++) {
213         for (j = 0; j < n; j++) {
214             for (i = 0; i < n; i++)
215                 free(difq[k][j][i]);
216             free(difq[k][j]); free(sumdifq2[k][j]); free(norma[k][j]);
217         }
218         free(difq[k]); free(sumdifq2[k]); free(norma[k]);
219     }
220     for (i = 0; i < n; i++)
221         free(prod[i]);
222     free(difq); free(difq2); free(sumdifq2); free(norma); free(prod);
223 
224     return flag;
225 }
```

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