Lanchester Duopoly Model Revisited: Advertising Competition under Time Inconsistent Preferences

Lijue Lu[a] and Jorge Navas[a]

[a]Dept. Matemàtica econòmica, financera i actuarial and BEAT

Universitat de Barcelona, Barcelona

ABSTRACT

In this paper we study a finite time horizon advertising dynamic game under the assumption of time-inconsistent preferences. Specifically, we consider two types of discounting, heteroge-neous discounting and hyperbolic discounting. In the case of heterogeneous discounting, the relative importance of the final function will increase/decrease as the end of the planning hori-zon approaches compared with current payoffs. Whereas when agents discount future payoffs hyperbolically, their discount rates diminish rapidly in earlier stages and then slowly in the long term. We compute time-inconsistent and time-consistent feedback Nash equilibrium strategies, and compare them with those of the standard discounting case. Our results reveal that heterogeneous discounting would lead to some last-minute changes, i.e., some adapting behaviours in the last years in accordance with their increasing/decreasing valuations of the final state. Under some circumstances, the change can be so radical that the pre-commitment solution takes the contrary path of time-consistent strategies. Concerning the competition under hyperbolic discounting, the temporal evolution of advertising efforts show a quite different nature. Different strategies exhibit disparity in the beginning, and encounter in the neighbourhood in the end, which is contrary to the heterogeneous discounting. Besides, a strong commitment power might induce over investment.

KEYWORDS

Advertising; Lanchester Model; Time Inconsistency; Heterogeneous Discounting; Hyperbolic Discounting; Differential Game

1. Introduction

It is a well-known fact that a strong commitment to advertising is the key to success. This one-way communication from brands to customers is proved to be helpful in increasing the brand and product awareness, building brand images, differentiating the products from those of other companies, and so on. The importance of advertising is evidenced by the large and increasing amount of money spent by successful corporations. According to a report by Johnson (2017), Procter & Gamble, the largest global advertiser in 2016, allocated 10.5 billion U.S. dollars toward advertising activities, followed by Samsung ($9.9 billion), Nestle ($9.2 billion), Unilever ($8.6 billion) and L'Oreal ($8.3 billion).

As a consequence, a great deal of academic attention has been paid to this primary competitive marketing tool in highly competitive industries, and the tendency is still increasing. Differential game approach, drawing support from mathematical modelling and quantitative methods, successfully involves the two essential elements of the marketing problems: dynamic and strategic considerations, and has been one of the principal methodologies in marketing science.

One of the earliest and most attractive advertising market share response models is the

CONTACT Lijue Lu. Email: lijue_lu@ub.edu

Lanchester model introduced by Kimball (1957). It is characterised in depicting battles for market share in a simple and elegant way, and has been adopted in many researches of dynamic advertising competition.

In the earliest years academic attention was focused, probably for reasons of mathematical tractability, in the specific case where agents do not discount future payoffs. Case (1979) suggests that it can be seen as an approximation of small and positive discount rate. Following the same research line, a series of theoretical and empirical studies focusing on zero discounting are conducted. Erickson (1991, chap. 3) presents an analytical and numerical analysis of feedback equilibria, and compares it with that of open-loop form. Both Erickson (1992) and Chintagunta & Vilcassim (1992) empirically test the market share response function to advertising investment and examine which kind of strategies (open-loop or closed-loop) fits better the reality. The difference between these two researches derives from the data samples and the statistical procedures applied. Chintagunta & Vilcassim (1994) extends the previous work by considering multiple marketing tools such as advertising, detailing, sales promotion, and so on. The cases of zero discounting in a finite time horizon with salvage value are analysed in two empirical studies. In Wang & Wu (2001), they use the Lanchester model as a benchmark case for an extended Vidale-Wolfe model (Vidale & Wolfe, 1957) in terms of model fitting and forecast accuracy. Later on, in Wang & Wu (2007) an empirical test is run for different structures of market share response function incorporating the inflation effect. However, the zero discount case may cause problems of convergence of the objective functionals (Jørgensen & Zaccour, 2004).

The first attempt of breaking the zero discounting assumption comes from Fruchter & Kalish (1997), where they study a game of infinite time horizon and propose a new approach to obtain the so-called time-varying closed-loop strategies, which are determined by time, current states and initial states. This work is later extended to an oligopolistic competition in Fruchter (1999a,b, 2001), and Fruchter & Kalish (1998), with the incorporation of market expansion, multi-products in a growing market, and multiple marketing tools in the latter three studies. However, the equilibrium policies are not subgame perfect due to the strategy dependence on initial market share. Jarrar et al. (2004) and Breton et al. (2006) develop an alternative approach to compute the feedback Nash and Stackelberg equilibrium strategies, respectively. According to their numerical illustrations, when rates of time preferences are positive, in both modes of play (simultaneous and sequential) the advertising strategies are decreasing in the firm's own market share. This property differs from the results of zero discount rates. Another way to consider a positive discount rate is through the modification of the model structure. For instance, Sorger (1989) proposes a variant of the Lanchester model, which allows for the characterisation of feedback Nash strategies. In accordance with Jarrar et al. (2004) and Breton et al. (2006), a higher market share also implies a decrease in advertising. The empirical support from Chintagunta & Jain (1995) show that the specification made by Sorger (1989) is a good candidate for the market of pharmaceutical products, soft drinks, beers, and detergents.

Given that a positive discount rate can have a significant impact on the advertising strategies, it is natural to think, if the agents discount the future payoffs in another way rather than the standard way, in which discount rates are assumed to be constant and unchanged, will they behave differently? Besides, empirical and experimental studies show that how people discount the future payoffs depends on the time distance and the types of goods. The curiosity of exploring the time preferences' impact, as well as the impropriety of standard discounting in some decision making situations have encouraged an academic stream in the differential game literature, where alternative discounting models are applied. Although general time preferences have proven to be important in many areas such as behavioural economics (e.g., Fischer, 1999; O'Donoghue & Rabin, 2001), environmental economics (for example, Karp, 2005; Karp & Tsur, 2011), financial economics (de-Paz et al., 2013; Laibson, 1997), and so forth, such concern has never been, to the best of our knowledge, introduced into management science.

Nonetheless, any other discount function but the standard exponential one would lead to time inconsistency (Strotz, 1955). If we follow the standard approach, a decision obtained at a later time does not necessarily, and in general not, coincide with that made at an earlier time. As a consequence, the agent tends to deviate from herself constantly, and the intertemporal choice, even in an optimal control problem, can be considered as a dynamic game among the "selves" of the decision maker at different instants of time.

Hence, the purpose of this paper is to, firstly, going one step further, explore the impact of temporal discounting on advertising competition. Specifically, we confine our interest to *heterogeneous discounting* and *hyperbolic discounting*, two of the most studied alternative discount models. Secondly, we then compute different types of strategies, and compare them with the standard discounting case to analyse how firms behave under different kinds of time preferences and different commitment power.

The rest of this paper is organised as follow. In Section 2 we describe a differential game model, the determination of feedback Nash equilibria follows in Section 3. In Section 4 some numerical simulations will be run to throw light on the advertising strategies and market dynamics. Finally, in Section 5 we summarise our results, relate them to the market observations, discuss the limitations and suggest some future studies.

2. Model Formulation

2.1. Lanchester Dynamics

The Lanchester model was originally used to model military combat. It was firstly introduced into the economic world by Kimball (1957) because of the similarity between military and industrial operations, and further advanced by Case (1979) and Little (1979). This model describes a battle for the market share where the advertising is the dominant influencing factor that only affects the customers of the rival firm.

Denote by $x_i(s)$ and $u_i(s)$ the market share and the rate of advertising expenditure of firm i $(i = 1, ..., N)$ at time s, k_i the advertising effectiveness, the market share and advertising are originally related in a linear structure

$$\dot{x}_i(s) = k_i u_i(s)[1 - x_i(s)] - \sum_{\substack{j=1 \\ j \neq i}}^{N} k_j u_j(s) x_i(s) \ .$$

Specifically, in a duopolistic market (as in Chintagunta & Vilcassim, 1992; Erickson, 1985; Jarrar et al., 2004; Little, 1979), by letting $x = x_1$ and $x_2 = 1 - x$, the basic Lanchester dynamics can be simplified as

$$\dot{x}(s) = k_1 u_1(s)[1 - x(s)] - k_2 u_2(s) x(s) \ .$$

Sorger (1989) extended the Lanchester model adopting the square root structure in a Vidale-Wolfe extension proposed by Sethi (1983), and formulated the instantaneous variation of market share in the following way (specifically, $k_1 = k_2 = 1$ in Sorger's setting)

$$\dot{x}(s) = k_1 u_1(s)\sqrt{1 - x(s)} - k_2 u_2(s)\sqrt{x(s)} \ , \quad x(t) = x_t \ . \tag{1}$$

According to Sorger (1989), the two square root terms in (1) are approximation of $1-x+x(1-x)$ and $x+x(1-x)$, respectively. Therefore, a word-of-mouth communication effect is incorporated into the market share dynamics. Besides, (1) can also be explained as a joint effect of the "Lanchester-type" dynamics and the excess advertising.

This formulation is highly referenced in the literature. For instance, Prasad et al. (2009) extended it to an oligopoly setting. Prasad & Sethi (2004) enriched the discussion by introducing the decay effect of Vidale-Wolfe model as well as the stochastic setting. Bass et al. (2005a,b) analysed the situation where firms invest in brand-advertising to capture rival firm's customers and in generic-advertising to increase the primary demand. Naik et al. (2008) studied the advertising competition in an oligopoly setting with market expansion and brand confusion effect, and offered some empirical evidence. He et al. (2011) considered an advertising battle where a coalition comprised of a manufacturer and a retailer is competing against another independent retailer.

In this paper, we adopt Sorger's extension in that it offers a richer interpretation by incorporating word-of-mouth communication and excess advertising. Moreover, with the square root formulation, the equilibrium market share of firm i ($i = 1, 2$) is of S-shape, which is considered to be in better accordance with reality. Furthermore, as mentioned previously, Sorger's modification allows the computation of feedback strategies for non-zero discounting, which is critical for further discussion related to time-inconsistent discounting.

Assuming quadratic advertising costs (which give rise to diminishing effect), the two firms aim to maximise the sum of the current value of the profit stream over a finite planning interval T and the scrap value assigned to the terminal state:

$$J_1(u_1, u_2) = \int_t^T \theta_1(s-t)\left[\pi_1 x(s) - \frac{c_1}{2}(u_1(s))^2\right] ds + \theta_1(T-t)S_1 x(T) \ , \tag{2}$$

$$J_2(u_1, u_2) = \int_t^T \theta_2(s-t)\left[\pi_2\left(1 - x(s)\right) - \frac{c_2}{2}(u_2(s))^2\right] ds + \theta_2(T-t)S_2\left(1 - x(T)\right) \ , \tag{3}$$

where $\theta_i(s-t)$ ($i = 1, 2$) are discount functions and will be given in the next section.

The denotation of the variables and parameters is as follows:

π_i	= positive constant margin per unit product of firm i,
$x(t)$	= market share of firm 1 at time t (state variable),
c_i	= positive constant cost parameter of firm i,
$u_i(t)$	= rate of advertising expenditure of firm i at time t (control variable),
k_i	= positive constant advertising effect parameter of firm i,
S_i	= the valuation assigned to the final state (non-negative constant).

2.2. Discount Function

The pioneering researches into intertemporal choice mainly focused on the psychological motives that lead to *time preference*[1]. Samuelson (1937) put forward the discounted utility (DU) model to condense all the psychological motives underlying intertemporal decisions into a single parameter, the discount rate, which is assumed to be constant and invariant across time and for all kinds of goods. Its simplicity made it become the dominant theoretical framework to study intertemporal behaviours. However, numerous experimental and empirical studies conducted in the following years have shown that in some situations, people demonstrate diminishing discount rates. Furthermore, the rates of time preference vary in the types of goods and decisions. The findings of such inadequacy of constant discounting have encouraged the development of various alternative theoretical models (for an overview of this topic, see Frederick et al., 2002).

Marín-Solano & Patxot (2012) introduced a temporal bias where agents discount the utility during the planning horizon and the final function at constant but different rates. It is labelled as *heterogeneous discounting* and the corresponding discount function is given by:

$$\theta_i(s-t) = \begin{cases} e^{-\delta_i(s-t)} & \text{if} \quad s < T \ , \\ e^{-\rho_i(s-t)} & \text{if} \quad s = T \ , \ i = 1, 2. \end{cases} \tag{4}$$

The essence of heterogeneous discounting is to describe a situation where the valuation of the final function is changing over time. To better present this idea, we rewrite the discount factor at the ending point $e^{-\rho_i(T-t)}$ as $e^{-\delta_i(T-t)} \cdot e^{-(\rho_i-\delta_i)(T-t)}$ ($i = 1, 2$). We can see that if $\rho_i > \delta_i$,

[1]Defined as "*the preference for immediate utility over delayed utility*" (Frederick et al., 2002).

after omitting the discounting effect during the planning horizon, the additional term imposed on the final function $e^{-(\rho_i-\delta_i)(T-t)}$ $(i=1,2)$ is increasing in t, and *vice versa*. Therefore, we are able to model an increasing valuation of the final function by assuming $\rho_i > \delta_i$, and a decreasing valuation by $\rho_i < \delta_i$ $(i = 1, 2)$.

One typical application of this approach, as in Marín-Solano & Patxot (2012), is to discount the "hard" goods, in the sense that effort has to be made prior to the enjoyment of the benefits (some examples are sports, knowledge and human capital accumulation). It has also been applied in the field of behavioural finance, such as the consumption and investment problem (de-Paz et al., 2013), and the life insurance purchase behaviours (de-Paz et al., 2014), where the final function represents the wealth at retirement or the bequest left for her descendants. In all the cases mentioned above it appears natural to assume that the agent has an increasing concern as the time t approaches to the end of the planning horizon T.

Introducing heterogeneous discounting into the business context (corporate level) could make sense due to the following concerns: 1) as discussed in Marín-Solano & Patxot (2012), the capital accumulation of a firm can be, to some degree, regarded as a "hard" good; 2) It appears restrictive to assume the discount rate to be invariant over time. The rate of time preference is affected by social factors such as regime (Pirvu & Zhang, 2014) and state of economy (Parkin, 1988), as well as by other firm-level factors like project duration, risk and fixed cost (Chen, 2012). 3) It is also of interest to consider different ways of discounting for different things. For instance (in this model), a firm could be more concerned with the cash flow if it is required to guarantee the development. Nevertheless, the emphasis might switch to the market coverage when the company reaches a steady growth.

In addition to heterogeneous discounting, a plenty of effort has been devoted to *hyperbolic discounting*. The phenomenon that decision makers exhibit declining discount rates has been supported experimentally and empirically by many studies (e.g., Myerson & Green, 1995; Thaler, 1981, and so forth), and hyperbolic discounting is a response to such DU anomaly by relaxing the constant rate assumption. Its applications have been primarily located in the fields of macroeconomics such as consumption-saving behaviours and economic growth, behavioural economics like procrastination and addiction, and environmental economics.

We believe that it could be meaningful to incorporate hyperbolic discounting from a company's point of view. Firstly, a manager could have limited commitment like a public policy maker, in that she is not sure if the business plans made currently would be followed by the successor. Besides, as human beings, it is likely that administrators are also influenced by the temporal bias that affect personal choice when making professional decisions. Moreover, uncertainty over the hazard rate of payoff realisation or over the agents' own future discount rates would lead to hyperbolic discounting (Azfar, 1999; Dasgupta & Maskin, 2005; Farmer & Geanakoplos, 2009).

We choose a linear combination of exponential functions which is given as follows:

$$\theta_i(s-t) = \lambda e^{-\delta_i(s-t)} + (1-\lambda)e^{-\rho_i(s-t)} \; , \tag{5}$$

with the corresponding instantaneous discount rate

$$r_i(\tau) = -\frac{\theta_i'(\tau)}{\theta_i(\tau)} = \frac{\lambda\delta_i e^{-\delta_i\tau} + (1-\lambda)\rho_i e^{-\rho_i\tau}}{\lambda e^{-\delta_i\tau} + (1-\lambda)e^{-\rho_i\tau}} \; , \; i = 1, 2, \tag{6}$$

where $0 < \lambda < 1$ and $\delta_i > \rho_i$ $(i = 1, 2)$. The discount function (5) implies that the instantaneous discount rate declines relatively rapidly in the earlier stages and then more slowly in the long run. Furthermore, when the planning horizon is sufficiently large, the pure rate of time preferences will converge to ρ_i $(i = 1, 2)$. This specification is also adopted in Ekeland & Lazrak (2010) and Karp & Tsur (2011). Other functional forms of hyperbolic discounting that are frequently studied in the literature include logarithmic discounting: $\theta(s-t) = 1/[1 + k(s-t)]$ with $k > 0$; the discount factor used by Barro (1999): $\theta(s-t) = e^{-[\rho(s-t)+\phi(s-t)]}$, with ρ being a constant and $\phi(s-t)$ a continuous and twice differentiable function; and the hybrid exponential discounting by Tsoukis et al. (2017): $\theta(s-t) = e^{-\lambda(s-t)}\left[1+b(s-t)\right]^{-a/b}$, with $0 < a < 1$ and $b > 0$.

3. Determination of Feedback Nash Equilibria

We confine our interest to the feedback Nash equilibria for some reasons. It is theoretically desirable in that firstly, it is more robust than the open-loop equilibria; secondly, empirical studies show that the feedback strategies can better explain the real dynamic advertising competition (Chintagunta & Vilcassim, 1992; Erickson, 1992); thirdly, evaluation of different kinds of strategies has been made by means of estimating market share response model independently of the strategies, the results suggest that feedback strategies perform strategically better for profit maximisation (Wang & Wu, 2007). In addition, it is managerially attractive since the feedback rules, which are time and state dependent, allow the flexibility of responding to the changing market.

For the sake of completeness we introduce the definitions of some commonly used strategy concepts in dynamic inconsistency setting. A feedback equilibrium is sub-game perfect in the standard (constant discount rate) case, however it does not necessarily, and in general it does not, hold while applying any kind of non-constant discounting. This is intuitive because a decision made at time t is (normally) not optimal for the agent herself at a future time t' due to her time-varying preferences. An individual with time-inconsistent preferences may or may not be aware of that. If the agent solves the optimisation problem at the beginning of the planning horizon, and she believes that her preferences will not change in the future (and in fact they do), or she can commit herself to follow this strategy made at time 0, we call it pre-commitment solution.

Under heterogeneous discounting, the pre-commitment agents need to solve a standard game in the beginning of the planning horizon. The corresponding system of dynamic programming equations (DPEs) for feedback Nash equilibrium are given as follows (we use P to denote "pre-commitment"):

$$\delta_i V_i^P - \frac{\partial V_i^P}{\partial s} = \max_{\{u_i^P\}} \left\{ \pi_i x_i - \frac{c_i}{2}(u_i)^2 + \frac{\partial V_i^P}{\partial x}\left(k_1 u_1 \sqrt{1-x} - k_2 u_2 \sqrt{x}\right) \right\} \ , \ i = 1, 2, \tag{7}$$

with boundary conditions $V_i^P(T, x) = e^{-(\rho_i - \delta_i)T} S_i x_i(T)$.

However, the decision maker would tend to deviate from the *ex ante* policy as time goes on. If she re-optimises the problem in a future time t' according to her interest of that time and applies it, and repeats this procedure in a later time t''... As a consequence, she will end up solving the problem at every instant and applying the solution only in that particular point of time. This kind of strategy is defined as naive solution (denoted by superscript N).

If the decision makers under heterogeneous discounting act in a naive way, at every moment t they will solve

$$\delta_i V_i^t - \frac{\partial V_i^t}{\partial s} = \max_{\{u_i^N\}} \left\{ \pi_i x_i - \frac{c_i}{2}(u_i^t)^2 + \frac{\partial V_i^t}{\partial x}\left[k_1 u_1^t \sqrt{1-x} - k_2 u_2^t \sqrt{x}\right] \right\} \ , \ i = 1, 2, \tag{8}$$

together with the boundary conditions $V_i^t(T, x) = e^{-(\rho_i - \delta_i)(T-t)} S_i x_i(T)$. Moreover, they will only apply the solutions obtained from (8) at the moment $s = t$.

Note that neither the pre-commitment nor the naive solutions are time-consistent. A solution can be time-consistent (named as sophisticated solution) if the agent can anticipate and take into account her future preferences while making decisions, which implies no reason for future selves to deviate from it. Using superscript S to represent sophisticated solutions, under heterogeneous discounting, the feedback Nash equilibrium is computed by solving

$$\rho_i V_i^S + K_i - \frac{\partial V_i^S}{\partial t} = \max_{\{u_i^S\}} \left\{ \pi_i x_i - \frac{c_i}{2}(u_i^S)^2 + \frac{\partial V_i^S}{\partial x} \left[k_1 u_1^S \sqrt{1-x} - k_2 u_2^S \sqrt{x} \right] \right\} , \tag{9}$$

with

$$K_i(t,x) = (\delta_i - \rho_i) \int_t^T e^{-\delta_i(s-t)} \left[\pi_i x_i(s) - \frac{c_i}{2}(u_i^*)^2 \right] ds \ , \ i = 1,2, \tag{10}$$

where u_i^* maximises the right-hand side term of equation (9). The corresponding boundary conditions are $V_i^S(T,x) = S_i x_i(T)$, and $K_i(T,x) = 0$ $(i = 1,2)$. By differentiating (10) with respect to t, we can get a simplified version

$$\delta_i K_i - \frac{\partial K_i}{\partial t} = (\delta_i - \rho_i) \left[\pi_i x_i - \frac{c_i}{2}(u_i^*)^2 \right] + \frac{\partial K_i}{\partial x} \left[k_1 u_1^* \sqrt{1-x} - k_2 u_2^* \sqrt{x} \right] \ , \ i = 1,2. \tag{11}$$

We now proceed to compute the pre-commitment, naive and sophisticated solutions under heterogeneous discounting.

By maximising the right hand side of equations (7), (8), and (9), we get the optimal advertising strategies. We make the informed guess that the value functions are linear in the state variable $V_i^\sigma(t,x) = \eta_i^\sigma(t)x + \nu_i^\sigma(t)$. Then, the feedback Nash equilibrium advertising policies are given by

$$u_1^{\sigma *}(t,x) = \frac{k_1}{c_1}\eta_1^\sigma(t)\sqrt{1-x} \ , \qquad u_2^{\sigma *}(t,x) = \frac{k_2}{c_2}\eta_2^\sigma(t)\sqrt{x} \ , \qquad \sigma = P, N, S. \tag{12}$$

For the agent who can commit herself to following the decision taken at the beginning of planing horizon $t = 0$, she needs to solve (7). Substituting the advertising rules u_1^{P*}, u_2^{P*}, the value functions V_1^P, V_2^P and their partial derivatives into (7), after rearranging, we obtain

$$\begin{aligned} &\left[\delta_1 \eta_1^P(s) - \dot{\eta}_1^P(s) - \pi_1 + \frac{(k_1)^2}{2c_1} \left(\eta_1^P(s) \right)^2 - \frac{(k_2)^2}{c_2} \eta_1^P(s) \eta_2^P(s) \right] x \\ &= -\delta_1 \nu_1^P(s) + \dot{\nu}_1^P(s) + \frac{(k_1)^2}{2c_1} \left(\eta_1^P(s) \right)^2 \ , \end{aligned} \tag{13}$$

$$\begin{aligned} &\left[\delta_2 \eta_2^P(s) - \dot{\eta}_2^P(s) + \pi_2 - \frac{(k_2)^2}{2c_2} \left(\eta_2^P(s) \right)^2 + \frac{(k_1)^2}{c_1} \eta_1^P(s) \eta_2^P(s) \right] x \\ &= -\delta_2 \nu_2^P(s) + \dot{\nu}_2^P(s) + \pi_2 + \frac{(k_1)^2}{c_1} \eta_1^P(s) \eta_2^P(s) \ . \end{aligned} \tag{14}$$

Equations (13) and (14) hold for every x, if and only if the parameters of x are equal to zero, thus we have get a system of two Ricatti differential equations with boundary conditions

$$\eta_i^P(T) = (-1)^{i-1} e^{-(\rho_i - \delta_i)T} S_i \ , \ i = 1,2. \tag{15}$$

We can find out the naive solutions following the same pattern. By substituting u_1^{N*}, u_2^{N*}, V_1^N, V_2^N and the corresponding partial derivatives into (8), we get the same DPEs as (13) and (14), but with different boundary conditions

$$\eta_i^N(T) = (-1)^{i-1} e^{-(\rho_i - \delta_i)(T-t)} S_i \ , \ i = 1,2. \tag{16}$$

Regarding the sophisticated solutions, apart from V_i^S, we also need to make a guess of the structure of the term K_i in (9). We conjecture a linear structure, as for the value functions.

$$V_i^S(t,x) = \eta_i^S(t)x + \nu_i^S(t), \quad K_i(t,x) = \alpha_i(t)x + \beta_i(t), \; i = 1,2. \tag{17}$$

Accordingly,

$$\begin{aligned}
&\frac{\partial V_i^S}{\partial x} = \eta_i^S(t) \; , \quad \frac{\partial V_i^S}{\partial t} = \dot{\eta}_i^S(t)x + \dot{\nu}_i^S(t) \; , \\
&\frac{\partial K_i}{\partial x} = \alpha_i(t) \; , \quad \frac{\partial K_i}{\partial t} = \dot{\alpha}_i(t)x + \dot{\beta}_i(t) \; , \; i = 1,2.
\end{aligned} \tag{18}$$

We then substitute (12), (17), and (18) into (9) and (11). After rearrangement, we obtain

$$\begin{aligned}
&\left[\rho_1\eta_1^S(t) + \alpha_1(t) - \dot{\eta}_1^S(t) - \pi_1 + \frac{(k_1)^2}{2c_1}\left(\eta_1^S(t)\right)^2 - \frac{(k_2)^2}{c_2}\eta_1^S(t)\eta_2^S(t)\right]x \\
&= -\rho_1\nu_1^S(t) - \beta_1(t) + \dot{\nu}_1^S(t) + \frac{(k_1)^2}{2c_1}\left(\eta_1^S(t)\right)^2 \; ,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\left[\rho_2\eta_2^S(t) + \alpha_2(t) - \dot{\eta}_2^S(t) + \pi_2 - \frac{(k_2)^2}{2c_2}\left(\eta_1^S(t)\right)^2 + \frac{(k_1)^2)}{c_1}\eta_1^S(t)\eta_2^S(t\right]x \\
&= -\rho_2\nu_2^S(t) - \beta_2(t) + \dot{\nu}_2^S(t) + \pi_2 + \frac{(k_1)^2}{c_1}\eta_1^S(t)\eta_2^S(t) \; ,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&[\delta_1\alpha_1(t) - \dot{\alpha}_1(t) - (\delta_1 - \rho_1)\pi_1 \\
&\quad \left. - \frac{(k_1)^2(\delta_1 - \rho_1)}{2c_1}\left(\eta_1^S(t)\right)^2 + \frac{(k_1)^2}{c_1}\alpha_1(t)\eta_1^S(t) - \frac{(k_2)^2}{c_2}\alpha_1(t)\eta_2^S(t)\right]x \\
&= -\delta_1\beta_1(t) + \dot{\beta}_1(t) - \frac{(k_1)^2(\delta_1 - \rho_1)}{2c_1}\left(\eta_1^S(t)\right)^2 + \frac{(k_1)^2}{c_1}\alpha_1(t)\eta_1^S(t) \; ,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&[\delta_2\alpha_2(t) - \dot{\alpha}_2(t) + (\delta_2 - \rho_2)\pi_2 \\
&\quad \left. + \frac{(k_2)^2(\delta_2 - \rho_2)}{2c_2}\left(\eta_1^S(t)\right)^2 - \frac{(k_2)^2}{c_2}\alpha_2(t)\eta_2^S(t) + \frac{(k_1)^2}{c_1}\alpha_2(t)\eta_1^S(t)\right]x \\
&= -\delta_2\beta_2(t) + \dot{\beta}_2(t) + (\delta_2 - \rho_2)\pi_2 + \frac{(k_1)^2}{c_1}\alpha_2(t)\eta_1^S(t) \; .
\end{aligned} \tag{22}$$

Equations (19)-(22) hold for every x, if and only if the parameters of x are equal to zero. Therefore, we obtain a system of four differential equations of $\eta_i^S(t)$ and $\alpha_i(t)$, with boundary conditions $\eta_i^S(T) = S_i$ and $\alpha_i(T) = 0$ $(i = 1,2)$.

The equilibrium of the game under heterogeneous discounting is characterised in the following proposition.

Proposition 3.1. *The pre-commitment, naive and sophisticated feedback Nash equilibria solutions for the the advertising competition under heterogeneous discounting are determined by*

$$\begin{aligned}
u_i^\sigma(s,x) &= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{1 - x_i(s)} \\
&= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{x_j(s)} \; , \; \{i,j\} = \{1,2\}, \; \sigma = P,N,S.
\end{aligned} \tag{23}$$

- *For* ***pre-commitment*** *solutions,* $\eta_i^P(s)$ $(i = 1, 2)$ *are the solutions to the system of differential equations*

$$\dot{\eta}_i^P(s) = (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^P(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^P(s)\eta_j^P(s) + \delta_i\eta_i^P(s) + (-1)^i\pi_i \ , \qquad \{i,j\} = \{1,2\}, \tag{24}$$

with boundary conditions $\eta_i^P(T) = (-1)^{i-1}e^{-(\rho_i-\delta_i)T}S_i$ $(i = 1, 2)$.

- *For* ***naive*** *solutions of t-agent,* $\eta_i^N(s)$ $(i = 1, 2)$ *solve the system*

$$\dot{\eta}_i^N(s) = (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^N(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^N(s)\eta_j^N(s) + \delta_i\eta_i^N(s) + (-1)^i\pi_i \ , \qquad \{i,j\} = \{1,2\}, \tag{25}$$

with boundary conditions $\eta_i^N(T) = (-1)^{i-1}e^{-(\rho_i-\delta_i)(T-t)}S_i$ $(i = 1, 2)$.

- *For* ***sophisticated*** *solutions,* $\eta_i^S(s)$ $(i = 1, 2)$ *solve the system of differential equations*

$$\begin{aligned}\dot{\eta}_i^S(s) &= (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^S(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^S(s)\eta_j^S(s) + \rho_i\eta_i^S(s) + \alpha_i(s) + (-1)^i\pi_i \ , \\ \dot{\alpha}_i(s) &= (-1)^i\frac{(k_i)^2(\delta_i-\rho_i)}{2c_i}\left(\eta_i^S(s)\right)^2 + (-1)^{i-1}\frac{(k_i)^2}{c_i}\alpha_i(s)\eta_i^S(s) + (-1)^i\frac{(k_j)^2}{c_j}\alpha_i(s)\eta_j^S(s) \\ &\quad + \delta_i\alpha_i(s) + (-1)^i(\delta_i-\rho_i)\pi_i \ , \ \{i,j\} = \{1,2\},\end{aligned} \tag{26}$$

with boundary conditions $\eta_i^S(T) = (-1)^{i-1}S_i, \ \alpha_i(T) = 0$ $(i = 1, 2)$.

Next we derive the time-inconsistent (Pre-commitment and Naive) and time-consistent (Sophisticated) solutions for agents with hyperbolic discounting.

The system of DPEs for pre-commitment solutions are

$$r_i(s)V_i^P - \frac{\partial V_i^P}{\partial s} = \max_{\{u_i^P\}}\left\{\pi_i x_i - \frac{c_i}{2}(u_i)^2 + \frac{\partial V_i^P}{\partial x}\left[k_1u_1\sqrt{1-x} - k_2u_2\sqrt{x}\right]\right\} \ , \tag{27}$$

with boundary conditions $V_i^P(T, x) = S_i x_i(T)$ $(i = 1, 2)$.

As to the naive agents, they need to solve, at every instant t,

$$r_i(s-t)V_i^t - \frac{\partial V_i^t}{\partial s} = \max_{\{u_i^N\}}\left\{\pi_i x_i - \frac{c_i}{2}(u_i^t)^2 + \frac{\partial V_i^t}{\partial x}\left[k_1u_1^t\sqrt{1-x} - k_2u_2^t\sqrt{x}\right]\right\} \ , \tag{28}$$

together with the boundary conditions $V_i^N(T, x) = S_i x_i(T)$ $(i = 1, 2)$, and follow the solutions obtained only at the moment $s = t$.

The DPEs for sophisticated agents in a game of finite time horizon under non-constant discounting is derived in Marín-Solano & Navas (2009). Following their approach, the time-consistent equilibrium strategies can be obtained by solving the set of DPEs

$$r_i(T-t)V_i^S + K_i - \frac{\partial V_i^S}{\partial t} = \max_{\{u_i^S\}}\left\{\pi_i x_i - \frac{c_i}{2}(u_i^S)^2 + \frac{\partial V_i^S}{\partial x}\left[k_1u_1^S\sqrt{1-x} - k_2u_2^S\sqrt{x}\right]\right\} \ , \tag{29}$$

with

$$K_i(t,x) = \int_t^T \theta_i(s-t)[r_i(s-t) - r_i(T-t)]\left[\pi_i x_i(s) - \frac{c_i}{2}(u_i^S)^2\right] ds \ , \tag{30}$$

and

$$V_i^S(T,x) = S_i x_i(T) \ , \quad K_i(T,x) = 0 \ , \ i = 1,2, \tag{31}$$

where u_i^S $(i = 1,2)$ maximise the right-hand side of (29). Similarly, the term K_i $(i = 1,2)$ can be simplified by differentiating both sides with respect to t. If the discount factor is a linear combination of exponential functions given in (5), by differentiating (30) we obtain

$$\Omega_i(t)K_i - \frac{\partial K_i}{\partial t} = \Phi_i(t)\left[\pi_i x_i - \frac{c_i}{2}(u_i^S)^2\right] + \frac{\partial K_i}{\partial x}\left[k_1 u_1^S\sqrt{1-x} - k_2 u_2^S\sqrt{x}\right] \ , \tag{32}$$

with

$$\Omega_i(t) = \frac{\lambda\rho_i e^{-\delta_i(T-t)} + (1-\lambda)\delta_i e^{-\rho_i(T-t)}}{\lambda e^{-\delta_i(T-t)} + (1-\lambda)e^{-\rho_i(T-t)}} \ , \ \Phi_i(t) = \frac{\lambda(1-\lambda)(\rho_i - \delta_i)\left[e^{-\delta_i(T-t)} - e^{-\rho_i(T-t)}\right]}{\lambda e^{-\delta_i(T-t)} + (1-\lambda)e^{-\rho_i(T-t)}} \ . \tag{33}$$

Following the same procedures for the case of heterogeneous discounting, by solving equations of (27), (28), (29) and (32), we characterise the feedback Nash equilibria for the case of hyperbolic discounting, which are summarised in the following proposition.

Proposition 3.2. *The pre-commitment, naive and sophisticated feedback Nash equilibria solutions for the advertising competition under hyperbolic discounting are determined by*

$$\begin{aligned} u_i^\sigma(s,x) &= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{1-x_i(s)} \\ &= (-1)^{i-1}\frac{k_i}{c_i}\eta_i^\sigma(s)\sqrt{x_j(s)} \ , \ \{i,j\} = \{1,2\}, \ \sigma = P,N,S. \end{aligned} \tag{34}$$

- *For* ***pre-commitment*** *solutions, $\eta_i^P(s)$ $(i = 1,2)$ are the solutions to*

$$\begin{aligned} \dot{\eta}_i^P(s) = (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^P(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^P(s)\eta_j^P(s) + r_i(s)\eta_i^P(s) + (-1)^i\pi_i \ , \\ \{i,j\} = \{1,2\}, \end{aligned} \tag{35}$$

  with boundary conditions $\eta_i^P(T) = (-1)^{i-1}S_i$, $i = 1,2$.
- *For* ***naive*** *solutions of t-agent, $\eta_i^N(s)$ $(i = 1,2)$ solve the system*

$$\begin{aligned} \dot{\eta}_i^N(s) =& (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^N(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^N(s)\eta_j^N(s) \\ &+ r_i(s-t)\eta_i^N(s) + (-1)^i\pi_i \ , \ \{i,j\} = \{1,2\}, \end{aligned} \tag{36}$$

  with $r_i(\tau)$ defined in (6) *and boundary conditions $\eta_i^N(T) = (-1)^{i-1}S_i$ $(i = 1,2)$. Note that the solutions will be obtained and adopted only at the instant $s = t$.*
- *For* ***sophisticated*** *solutions, $\eta_i^S(s)$ $(i = 1,2)$ solve the system of differential equations*

$$\begin{aligned} \dot{\eta}_i^S(s) =& (-1)^{i-1}\frac{(k_i)^2}{2c_i}\left(\eta_i^S(s)\right)^2 + (-1)^i\frac{(k_j)^2}{c_j}\eta_i^S(s)\eta_j^S(s) + r_i(T-t)\eta_i^S(s) + \alpha_i(s) \\ &+ (-1)^i\pi_i, \\ \dot{\alpha}_i(s) =& (-1)^i\frac{(k_i)^2}{2c_i}\Phi_i(s)\left(\eta_i^S(s)\right)^2 + (-1)^{i-1}\frac{(k_i)^2}{c_i}\alpha_i(s)\eta_i^S(s) + (-1)^i\frac{(k_j)^2}{c_j}\alpha_i(s)\eta_j^S(s) \\ &+ \Omega_i(s)\alpha_i(s) + (-1)^i\pi_i\Phi_i(s) \ , \ \{i,j\} = \{1,2\}, \end{aligned} \tag{37}$$

with $\Omega_i(s)$ and $\Phi_i(s)$ given in (33)*, and with boundary conditions $\eta_i^S(T) = (-1)^{i-1}S_i$, $\alpha_i(T) = 0$ $(i = 1, 2)$.*

Remark 1. The feedback Nash equilibrium of the game with infinite time horizon under hyperbolic discounting can be easily obtained by setting $T \to \infty$ and $S_1 = S_2 = 0$ (for more details, see Karp, 2007). Here, we focus on the sophisticated solutions since they are time-consistent, and the corresponding equilibrium is subgame perfect. However, it is straightforward to achieve a similar analysis of pre-commitment and naive solutions.

The set of DPEs for time-consistent equilibrium with infinite horizon are

$$\rho_i \tilde{V}_i^S + \tilde{K}_i = \max_{\{\tilde{u}_i^S\}} \left\{ \pi_i x_i - \frac{c_i}{2}(\tilde{u}_i^S)^2 + \frac{\partial \tilde{V}_i^S}{\partial x} \left[k_1 \tilde{u}_1^S \sqrt{1-x} - k_2 \tilde{u}_2^S \sqrt{x} \right] \right\}, \tag{38}$$

with

$$\delta_i \tilde{K}_i = \lambda(\delta_i - \rho_i) \left[\pi_i x_i - \frac{c_i}{2}(\tilde{u}_i^S)^2 \right] + \frac{\partial \tilde{K}_i}{\partial x} \left[k_1 \tilde{u}_1^S \sqrt{1-x} - k_2 \tilde{u}_2^S \sqrt{x} \right]. \tag{39}$$

After maximising the right hand side of (38) and (39), we conjecture that both the value functions and the term $\tilde{K}_i$ have linear structures, namely, $\tilde{V}_i^S = \tilde{\eta}_i^S x + \tilde{\nu}_i^S$, $\tilde{K}_i = \tilde{\alpha}_i x + \tilde{\beta}_i$ $(i = 1, 2)$. By solving the system of DPEs, we can characterise the sophisticated solutions, determined as

$$\tilde{u}_i^S(x) = (-1)^{i-1} \frac{k_i}{c_i} \tilde{\eta}_i^S \sqrt{1 - x_i} = (-1)^{i-1} \frac{k_i}{c_i} \tilde{\eta}_i^S \sqrt{x_j}, \tag{40}$$

where $\tilde{\eta}_i^S$ $(i = 1, 2)$ solve the system of four equations

$$\begin{aligned}
&(-1)^{i-1} \frac{(k_i)^2}{2c_i} \left(\tilde{\eta}_i^S\right)^2 + (-1)^i \frac{(k_j)^2}{c_j} \tilde{\eta}_i^S \tilde{\eta}_j^S + \rho_i \tilde{\eta}_i^S + \tilde{\alpha}_i + (-1)^i \pi_i = 0, \\
&(-1)^i \frac{(k_i)^2}{2c_i} \lambda(\delta_i - \rho_i) \left(\tilde{\eta}_i^S\right)^2 + (-1)^{i-1} \frac{(k_i)^2}{c_i} \tilde{\alpha}_i \tilde{\eta}_i^S + (-1)^i \frac{(k_j)^2}{c_j} \tilde{\alpha}_i \tilde{\eta}_j^S + \delta_i \tilde{\alpha}_i + (-1)^i \lambda(\delta_i - \rho_i)\pi_i = 0, \\
&\{i, j\} = \{1, 2\}.
\end{aligned} \tag{41}$$

A graphical presentation of the policies defined above will be given in section 4.2.

4. Numerical Illustrations

Since the system of differential equations cannot be solved explicitly, we provide some numerical illustration to throw light on the impact time preferences have on firms' behaviours and the evolution of the market. Numerical solutions are calculated using *Wolfram Mathematica v11.2.*

For reasons of research interest, the two firms are assumed to be symmetric, with the exception of their time preferences and initial market share. By controlling $\pi_1 = \pi_2$, $c_1 = c_2$, and $k_1 = k_2$, we are able to concentrate on how firms' advertising investments alter in accordance with their time preferences. Furthermore, it is not impractical to assume such symmetry. For products satisfying some specific properties, it is likely that both firms have similar net profit ratio, have achieved excellence in cost control, and are of symmetric abilities in relation to media buying, quality control and some other capabilities, which implies the technical/economic symmetry. For instance, Chintagunta & Jain (1995) conducted some empirical tests using Sorger's specification, and found that the advertising effectiveness of the two duopolies in markets of pharmaceutical product, soft drink and beer are almost identical. In the following, we set $\pi_1 = \pi_2 = 300$, $c_1 = c_2 = 2$, $k_1 = k_2 = 0.3$, as what has been used in Jarrar et al. (2004).

The values of S_1 and S_2 should be carefully chosen. Sorger (1989), Wang & Wu (2001) and Wang & Wu (2007) show that how advertising efforts evolve over time is highly connected with S_1 and S_2, the parameters representing the importance of market share in the end of planning horizon for each firm. Specifically, let $\hat{u}_i(\hat{\eta}_i(s), x)$ denote the feedback Nash equilibrium strategies of both firms under standard discounting, and $\bar{\eta}_i$ be the values such that $\dot{\hat{u}}_i(s) = 0$ $(i = 1, 2)$. If the ending market shares are relatively important $(S_i > \bar{\eta}_i)$ for both firms, then once the market shares reach near the steady state, they will both increase the advertising budget over time when approaching time T, whereas the contrary happens if the final functions are relatively unimportant $(S_i < \bar{\eta}_i)$[2].

In order to mitigate these effects, here we let S_1 and S_2 be proportional to the shadow prices of market share (A_1 and A_2) for the game of infinite time horizon starting at time T with discount rate ρ_i for heterogeneous discounting, and with $\bar{\rho}_i$ defined in (42) for hyperbolic discounting $(i = 1, 2)$. Specifically, $S_i = \omega_i A_i$ $(i = 1, 2)$, where A_1 and A_2 are the solutions to the system

$$P_i A_i + (-1)^i q_i + (-1)^{i+1} \frac{(k_i)^2}{2c_i}(A_i)^2 + (-1)^i \frac{(k_j)^2}{c_j} A_i A_j = 0 \ , \ \{i, j\} = \{1, 2\},$$

where $P_i = \rho_i$ for heterogeneous discounting, and $P_i = \bar{\rho}_i$ for hyperbolic discounting $(i = 1, 2)$. It can be easily verified that A_i decrease in P_i, and that A_i coincide with $\bar{\eta}_i$ $(i = 1, 2)$. The purpose of introducing ω_i $(i = 1, 2)$ is to gain the flexibility of formulating a greater variety of situations under heterogeneous discounting, which will be explained in the next section. For standard and hyperbolic discounting, we assume that $\omega_1 = \omega_2 = 1$.

Under this setting, the current model under standard discounting and with $\omega_1 = \omega_2 = 1$ will coincide with the game of infinite time horizon.

4.1. Heterogeneous Discounting

We start by discussing the possible situations we can take into account by assigning different values of ω_i $(i = 1, 2)$. Without loss of generality, take the symmetric case under heterogeneous discounting with $\delta_i < \rho_i$ as an example. The discounted final function is given by $e^{-\rho_i(T-t)}\omega_i A_i$ $(i = 1, 2)$, with the additional term $e^{-(\rho_i - \delta_i)(T-t)}$ increasing in t (from the previous discussion in Section 2.2). Depending on the values of ω_i $(i = 1, 2)$, we can model the following cases:

- If $\omega_i = 1$, then $e^{-\rho_i(T-t)}\omega_i A_i < e^{-\delta_i(T-t)} A_i$ and $e^{-\rho_i(T-T)}\omega_i A_i < e^{-\delta_i(T-T)} A_i$.
  The values that firms ascribe to the ending market shares are relatively low. Though as time goes by, the valuations of the final states are increasing, they are always inferior to the valuations of the profits during the period t to T.
- If $1 < \omega_i < e^{(\rho_i - \delta_i)(T-t)}$, then $e^{-\rho_i(T-t)}\omega_i A_i < e^{-\delta_i(T-t)} A_i$ and $e^{-\rho_i(T-T)}\omega_i A_i > e^{-\delta_i(T-T)} A_i$.
  The assessment of scrap values is relatively lower in the beginning of the planning horizon t, then increases as firms move toward the ending point and eventually surpasses the importance of the profits before the end of the planning period.
- If $\omega_i > e^{(\rho_i - \delta_i)(T-t)}$, then $e^{-\rho_i(T-t)}\omega_i A_i > e^{-\delta_i(T-t)} A_i$ and $e^{-\rho_i(T-T)}\omega_i A_i > e^{-\delta_i(T-T)} A_i$.
  The importance of final states is higher in the beginning in comparison with the cash flow during the period, and such importance is increasing across time.

For the case of $\delta_i > \rho_i$, different situations of final functions whose importance is decreasing with the passage of time can also be modeled by letting $\omega_i = 1$ (the final state is always more important compared with the profits obtained from t to T), $e^{(\rho_i - \delta_i)(T-t)} < \omega_i < 1$ (initially more relevant but eventually less), and $\omega_i < e^{(\rho_i - \delta_i)(T-t)}$ (always less important), $i = 1, 2$.

Figures 1 and 2 illustrate the advertising strategies of both firms in a symmetric case of $\delta_1 = \delta_2 = 0.05$, $\rho_1 = \rho_2 = 0.1$, $\omega_1 = \omega_2 = 1.4$, $x_0 = 0.01$, $t = 0$, and $T = 15$. Since $\rho_i > \delta_i$

[2]For more detailed discussion, we refer to the Section 4 of Sorger (1989).

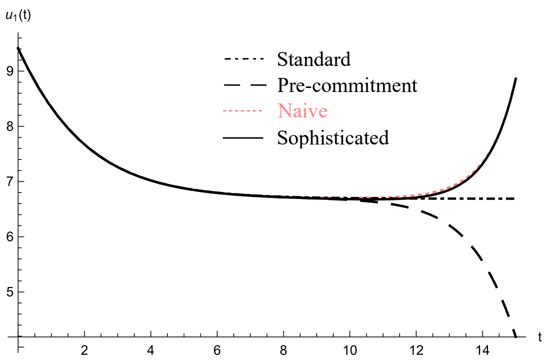

Figure 1. Advertising of Firm 1 (Heterogeneous Discounting)

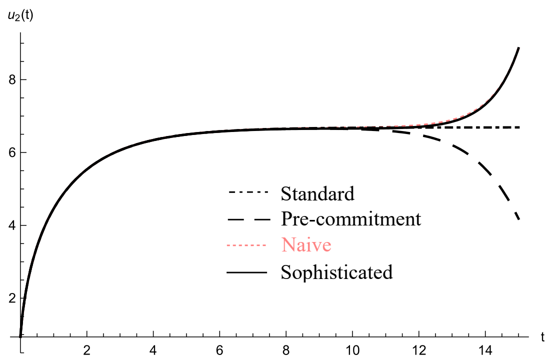

Figure 2. Advertising of Firm 2 (Heterogeneous Discounting)

$(i = 1, 2)$, both firms have increasing valuations of the ending market shares. Furthermore, as explained previously, these valuations are initially inferior to the concerns with the profits throughout the planning horizon, but eventually become dominant. We confine our interest to this special case since it can perfectly demonstrate the difference between time-inconsistent and time-consistent strategies. The standard case of $\delta_i = \rho_i = 0.05$ and $\omega_i = 1$ $(i = 1, 2)$ is also graphed to serve as a benchmark.

As shown in Figure 1, for all kinds of discounting and solution types, firm 1, which is at a disadvantage at the beginning (as $x_0 < 0.5$), pumps money into advertising in order to seize market share as soon as possible. The investment is decreasing over time, as her own market share is growing and the target market is reducing the size. On the contrary, holding a dominant market position, firm 2 invests little in the beginning and eventually increases the budget (Figure 2). Our finite-horizon model also exhibits the asymptotic properties that are discussed in Fershtman & Kamien (1990), where they borrow the terminology of "turnpike properties" in growth theory to discuss these features. The previously described battle stage lasts until the market share distribution reaches the neighbourhood of the stationary equilibrium corresponding to the game of infinite time horizon (around the year 8.5). From that moment on, each firm invests the same amount of money in advertising and holds half of the market (because of the symmetry).

If firms have standard time preferences, during the quasi-stationary period, both firms would keep the same advertising efforts until the end of planning horizon (due to the values chosen for S_1 and S_2). However, firms under heterogeneous discounting make last-minute shifts in accordance with how they discount the final market. Note that when agents commit themselves to the decision made at the beginning, they act as if they were under standard discounting, but with different boundary conditions. Here, the pre-commitment solutions are consistent with that of a standard discounting game with final function $e^{-(\rho_i - \delta_i)(T-t)}\omega_i S_i x_i$. Given $\rho_i > \delta_i$, the values that firms ascribe to the ending market share are relatively low, which implies a sharp decrease in advertising. However, as time goes by and firms approach the ending point, the relevance of the final states is increasing and at one point, it takes the priority. Anticipating such changing taste, sophisticated agents' last-minute accommodation is an increase in advertising, which is contrary to the behaviours of players with commitment power. It is worth mentioning that naive solutions and sophisticated solutions are almost identical, probably because when the last-minute change happens, the importance of final states is already dominant.

By anticipating future preferences, time-consistent strategies can help firms to act according to their true preferences. Nonetheless, sophisticated solutions do not necessarily increase or decrease the payoffs. The graphic presentation of market share dynamics is omitted because the patterns in all four cases are extremely similar. Intuitively, lower advertisement spending yields higher payoffs. We can see that agents are better off with pre-commitment than sophisticated solutions in this case.

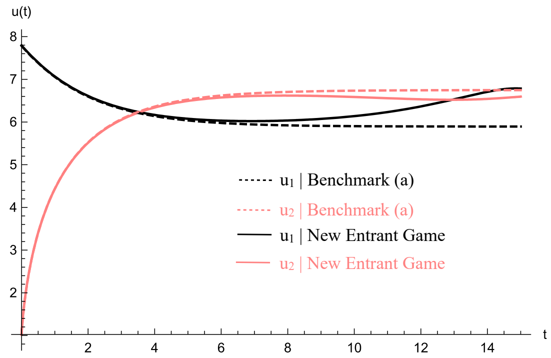

Figure 3. Advertising Strategies in New Entrant Game (a)

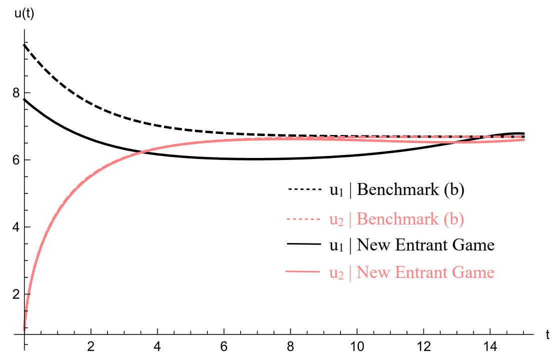

Figure 4. Advertising Strategies in New Entrant Game (b)

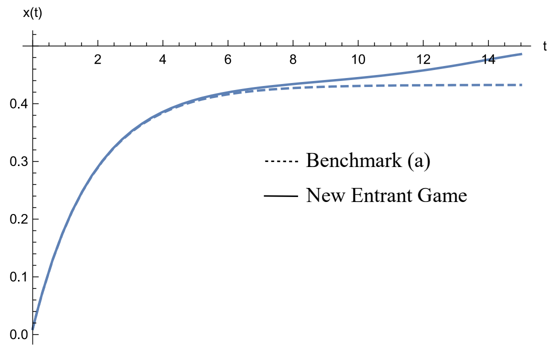

Figure 5. Market Share Dynamics in New Entrant Game (a)

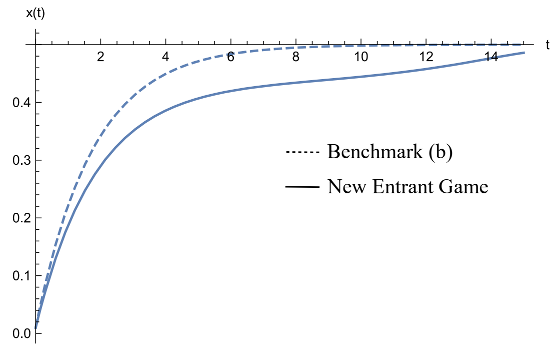

Figure 6. Market Share Dynamics in New Entrant Game (b)

Next we study another case with asymmetric discounting, as described in the following.

New Entrant Game: A new entrant in the industry is competing with the incumbent. As the new entrant could have a smaller firm size, more financial constraints, higher instantaneous crisis rate and more urgent developing necessities, she would be more impatient with the financial return, thus discounting future payoffs more heavily. However, the manager believes that after some years' developing, the firm will be less constrained and relatively more far-sighted.

We can incorporate such future belief using heterogeneous discounting. For firm 1 (the new entrant) we set $\delta_1 = 0.15$ and $\rho_1 = 0.05$, whereas firm 2 (the incumbent) uses the same constant and smaller discount rate $\delta_2 = \rho_2 = 0.05$. Since the emphasis here is not the time-variant final function, we let $\omega_1 = \omega_2 = 1$. The initial market distribution is set to be $x_0 = 0.01$, and the planning horizon is from year 0 to 15. For better interpretation, we also present graphically two benchmark cases of (a) $\delta_1 = \rho_1 = 0.15$, $\delta_2 = \rho_2 = 0.05$ and (b) $\delta_1 = \rho_1 = \delta_2 = \rho_2 = 0.05$. Figures 3 to 6 demonstrate the scenario described above. Here we focus on the sophisticated solutions, since they are theoretically more desirable and the corresponding equilibrium is subgame perfect.

As shown in Figure 3, instead of making last-minute changes as in the symmetric case, here the new entrant, the sophisticated agent under heterogeneous discounting, starts her accommodation much earlier. In the battle period both firms act similarly as in Figures 1 and 2, the initially smaller firm tries hard to steal the market share from her rival, whereas the market dominant allocates relatively little but increasing resources. In the adapting stage, the new entrant raises her advertising budget at a firstly increasing then decreasing speed. As a response to the new entrant's adjustment, firm 2 (the incumbent) chooses a lower advertising level, in comparison with the standard case, in the accommodation stage. Notice that the new entrant will end up with higher advertising spending than the incumbent, even when the difference between δ_1 and ρ_1 is extremely small. Figure 5 displays the corresponding market share evolution.

If the new entrant discounts the future in a standard way with a relatively higher discount rate compared with the incumbent, she will end up with a smaller portion of the whole market. If the manager believes that the new firm can catch up with the incumbent regarding the financial achievement, crisis management, etc., which may lead to a convergence in time preferences, the two firms will share almost equally the market in the end.

As to the benchmark (b), by comparing Figures 3 and 4, and Figures 5 and 6, one can clearly see that the new entrant game is an intermediate case between these two benchmarks.

4.2. Hyperbolic Discounting

In the following we present some numerical illustrations of advertising competition under hyperbolic discounting.

We start, like previously, with a symmetric case of $\lambda = 0.5$, $\delta_1 = \delta_2 = 0.3$, $\rho_1 = \rho_2 = 0.05$, $\omega_1 = \omega_2 = 1$, $x_0 = 0.01$, $t = 0$, and $T = 15$. The benchmark here is the standard case that shows the same overall level of impatience as the pre-commitment agent. Specifically, we apply the discount function $e^{-\bar{\rho}_i(s-t)}$, where $\bar{\rho}_i$ is the solution to

$$\int_t^T e^{-\bar{\rho}_i(s-t)} ds = \int_t^T \left(\lambda e^{-\delta_i(s-t)} + (1-\lambda)e^{-\rho_i(s-t)}\right) ds, \ i = 1, 2. \tag{42}$$

Strulik (2015) proposed this concept in the framework of infinite time horizon[3], under such restriction, both discounting methods give equivalent present value for a unitary profit stream over the planning period. Therefore, the observed difference in players' behaviours should come from the form (rather than the "amount") of temporal preferences. However, complication arises when it comes to a finite-horizon setting. It can be easily verified that the $\bar{\rho}_i$ ($i = 1, 2$) derived from (42) is dependent on t, namely, it only provides equal discounted profit stream for t-agent. One way to cope with this problem is to compute a time-varying overall impatience rate, but, as pointed out by Caliendo & Findley (2014), at the cost of losing the time-consistency property of exponential discounting. They therefore opt to adjust the hyperbolic function as a parallel approach. Nevertheless, this method changes the structure of the general discount functions in a way that lies beyond the scope of our study. Hence, we choose an equivalent overall impatience level from the perspective of the 0-agent as the benchmark case. Despite this, the comparison among pre-commitment, naive and sophisticated strategies remains valid. Besides, at the end of this section, we also present a graphical illustration of the case of infinite time horizon, where additional adjustment is not required for a fair comparison.

Note that the hyperbolic discounting is to some extent similar to heterogeneous discounting with $\delta_i > \rho_i$, in the sense that in both cases, the ending discount rate is smaller compared to that during the planning period. Therefore, we can observe some behaviours that are qualitatively consistent with those under heterogeneous discounting. For example, firm 1, who is situated at a weak market position in the beginning of time horizon, shows similar investment patterns in Figures 1 and 7. In the battle stage, she starts with heavy budget to take hold of market share, and reduces the amount over time. There also appears an adjustment during the last years based on the solution types. Specifically, the pre-commitment agent decreases the resources allocation, whereas the naive and sophisticated agents choose to do the opposite. The same kind of similarity can also be found for firm 2 in Figures 2 and 8.

Nonetheless, unlike the heterogeneous discounting case, here different types of solution show greater divergence throughout the planning horizon, especially during the quasi-stationary period. Furthermore, the sophisticated agents always apply higher advertising policies than naive agents, but in general they are quite similar. In Table 1 we summarise the strategies ranking in different periods. Note that the pre-commitment policy is equivalent to the naive strategy computed at $t = 0$, so they start with the same level at the beginning of planning horizon. Strategies made at this moment under standard discounting has the highest amount, as the corresponding

[3]The corresponding $\bar{\rho}$ is computed from $\int_t^\infty e^{-\bar{\rho}(s-t)} ds = \int_t^\infty \theta(s-t) ds$.

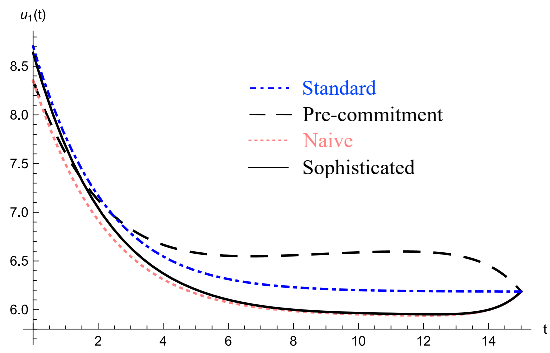

Figure 7. Advertising of Firm 1 (Hyperbolic Discounting)

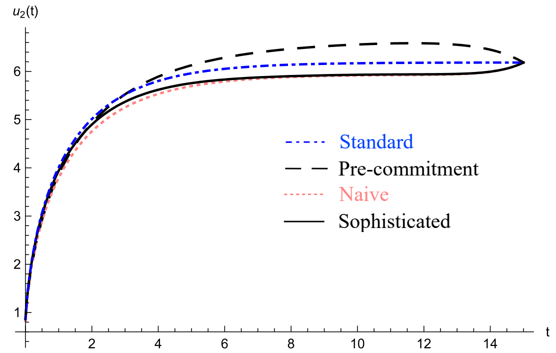

Figure 8. Advertising of Firm 2 (Hyperbolic Discounting)

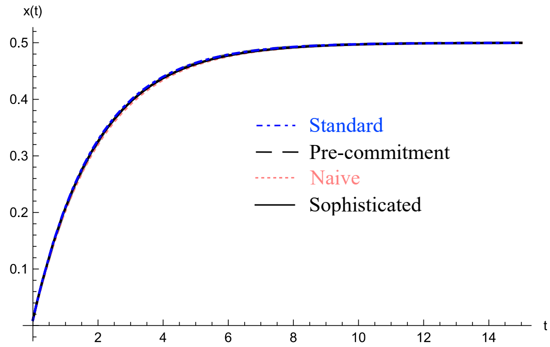

Figure 9. Market Dynamics (Hyperbolic Discounting)

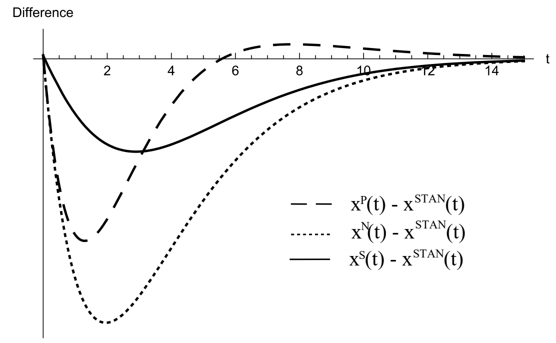

Figure 10. Difference Between the Market Shares Corresponding to each Solution and that of Standard Case

instantaneous discount rate $\bar{\rho}_i$ $(i = 1, 2)$ is lower than that of hyperbolic discounting[4]. The sophisticated decision is located between that of standard and pre-commitment/naive. From the point of view of an agent with strong commitment power, her instantaneous discount rate is decreasing over time, which translates to a relative increment in advertising with respect to other solutions. Accordingly, the ranking of pre-commitment is getting higher as time goes by.

It is worth mentioning that, in Figure 9, the market share dynamics are approximately the same under both discounting methods and for all kinds of solutions. A closer inspection can give us the ranking of $x(t)$, the market share of firm 1 in the cases where players adopt different strategies. As shown in Figure 10 and outlined in Table 1, its ranking is consistent with that of policies (although the duration of each phase differs slightly), due to the fact that a higher amount of advertising investment gives rise to a more fierce competition and a stronger fluctuation of market shares, thus firm 1 can seize the market more quickly. However, the difference in the market share evolution under distinct solutions is negligible. If firms precommit their advertising policies, or are not able to incorporate future changes in time preferences into decision making, their spending is much higher. Therefore, under hyperbolic discounting, strong commitment power leads to over-investment.

Let us reconsider the *New Entrant Game* described in Section 4.1. If both new entrant and the incumbent are under hyperbolic discounting, it is likely that the new entrant has a faster decreasing discount factor due to greater uncertainty, which can be depicted by the case of $\lambda = 0.5$, $\delta_1 = 0.6$, $\delta_2 = 0.3$, $\rho_1 = \rho_2 = 0.05$, $\omega_1 = \omega_2 = 1$, $x_0 = 0.01$, $t = 0$, and $T = 15$ (Figures 11 and 12). Notice that, in this setting, the discount rates of both firms approximately converge to the same value at time T. Firm 2 applies a lower discount rate throughout the whole

[4]In the sensitivity analysis provided in Section 4.3, we will see that a higher instantaneous discount rate implies a lower investment.

Table 1. Comparison of Different Strategies and the Corresponding State (Hyperbolic Discounting: Symmetric Case)

Phase 1	Standard	>	Sophis.	>	Pre-c.	>	Naive
Phase 2	Standard	>	Pre-c.	>	Sophis.	>	Naive
Phase 3	Pre-c.	>	Standard	>	Sophis.	>	Naive

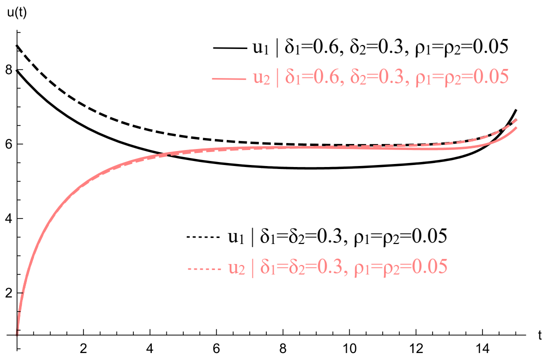

Figure 11. Advertising Strategies in New Entrant Game

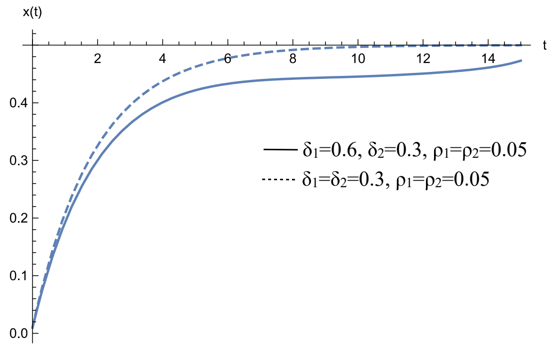

Figure 12. Market Share Dynamics in New Entrant Game

planning horizon, which leads her to implement a higher advertising rate when the equilibrium stays in the neighbourhood of stationary state (year 7 to 12). Firm 1, being more impatient, starts the accommodation stage earlier, and with a greater increasing rate than firm 2. However, this last-minute effort can not compensate completely the loss during the planning horizon, as shown in Figure 12, the market ends up with firm 1 acquiring less portion than firm 2.

Finally, we present graphically the case of infinite planning horizon when agents discount future profits hyperbolically. We confine our interest to the time-consistent solutions, which are characterised in Remark 1. As a benchmark, we also plot the standard discounting case with equivalent overall impatience level, where the discount rate is computed using (42) with $T = \infty$. We use the same set of parameters as that used in the symmetric case of finite time horizon, and the numerical simulation is given in Figure 13. The qualitative properties of players' behaviours coincide with the results obtained by Sorger (1989), where the advertising budgets decrease in market share. Furthermore, as in the finite-time-horizon case, standard discounting induces higher investment than hyperbolic discounting, even they exhibit the same overall impatience level.

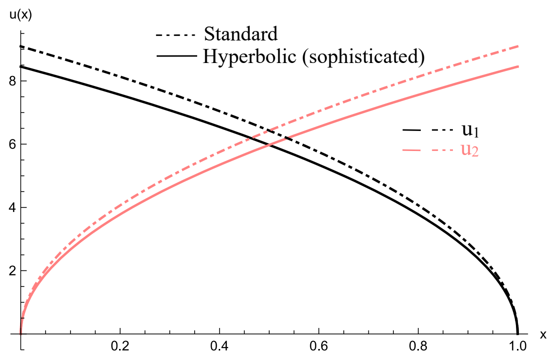

Figure 13. Advertising Strategies with Infinite Time Horizon (Hyperbolic Discounting)

Table 2. Sensitivity Analysis: Sophisticated Solutions (Finite Time Horizon)

Variable	Discounting	Parameters ($\{i,j\}=\{1,2\}$)										
		λ	δ_i	δ_j	ρ_i	ρ_j	π_i	π_j	c_i	c_j	k_i	k_j
Advertising u_i^S	Heterogeneous	/	−	+	−	+	+	−	−	+	+	−
	Hyperbolic	−	−	+	−	+						

4.3. Sensitivity Analysis

We have run multiple simulations by altering parameters and initial conditions to confirm the robustness of our results, and the sensitivity analysis is summarised in Table 2.

We explore the effect of each parameter on advertising strategies for a given market share distribution. Since larger values of δ_i, ρ_i (for both discounting methods), and λ (for hyperbolic) all yield higher instantaneous discount rate, their effects are coherent. Firms respond to this increase by lowering the advertising budget. On the contrary, a higher rival's instantaneous discount rate will imply heavier investment. These results are consistent with those of the standard case. Moreover, and intuitively, the firms will increase the advertising if the product is more profitable (higher π_i), the advertising is more effective (larger k_i) and/or less costly (lower c_i). If their rivals have these advantages, they would lower the resource allocation.

Next we explore if the discounting method has influenced the sensitivity. To do so, we focus on the case of the infinite time horizon because of the following reasons. Firstly, in the scale of the finite planning period, players' behaviours are quite time sensitive, in the sense that policies might vary over time even for a given market share. Besides, as explained previously, the overall impatience level causes complexity in this setting. However, concentrating on the scenario of infinite planning period would allow us to make a fair comparison and a neater presentation.

Hence, we perform several numerical simulations of the model under standard and hyperbolic discounting. Each time when we assign new value to one single parameter, leaving the others unchanged, the corresponding variations in advertising strategies with respect to the benchmark case are computed in percentage terms. The parameters of the benchmark case are set as: $\lambda = 0.5$, $\delta_1 = \delta_2 = 0.3$, $\rho_1 = \rho_2 = 0.05$, $\pi_1 = \pi_2 = 300$, $c_1 = c_2 = 2$, and $k_1 = k_2 = 0.3$. The instantaneous discount rate $\bar{\rho}_i$ $(i = 1, 2)$ is obtained by solving (42) with $T = \infty$.

In Table 3 we display the comparison made upon some parameters, where the greater relative changes are highlighted[5]. Not surprisingly, the signs of changes are in line with those in the finite-horizon scenario (Table 2). Furthermore, under standard discounting, firms are more sensitive to product margin π_i, advertising costliness c_i, and ρ_i $(i = 1, 2)$. On the other hand, under hyperbolic discounting, the alteration of δ_i $(i = 1, 2)$ implies a stronger fluctuation. The results seem to be rather responsive to the advertising effectiveness k_i $(i = 1, 2)$ in combination with time preferences. When one firm is slightly more effective in advertising than the other, the standard discounting induces a larger variation. However, when the asymmetry is considerably big (for instance, twice effective), the agent with advantage would increase the budget at a larger scale under hyperbolic discounting. It is also worth mentioning that δ_i and ρ_i are favoured by different discounting forms, due to the fact that, by definition, δ_i affects the instantaneous rate of hyperbolic discounting more heavily than ρ_i, and *vice versa* $(i = 1, 2)$. Accordingly, when λ (the parameter connected to δ_i) takes small value, the hyperbolic players' are more susceptible, whereas decision makers under standard discounting reacts more intensively to λ of large value.

5. Concluding Remarks

This paper has gone some way towards enhancing our understanding of horizontal advertising competition by introducing some biases in the temporal preferences. Specifically, we have introduced two alternatives to the standard exponential discounting in order to capture some

[5]Due to space constraints, we do not present the sensitivity analysis of all parameters in Table 3, but all the results will be mentioned in the text.

Table 3. Sensitivity Analysis: Standard vs. Hyperbolic Discounting (Infinite Time Horizon)

Benchmark		$\pi_1 = 300$			$k_1 = 0.03$			
New Value		500	700	900	0.04	0.05	0.06	0.07
$\Delta u_1(\%)$	Standard	**+59.23**	**+109.25**	**+152.42**	**+26.11**	**+43.19**	+54.17	+70.37
	Hyperbolic	+58.29	+108.42	+152.36	+25.37	+42.97	**+55.01**	**+74.34**
$\Delta u_2(\%)$	Standard	**−21.15**	**−33.74**	**−42.01**	**−23.69**	**−39.58**	**−50.25**	**−67.53**
	Hyperbolic	−19.18	−31.12	−39.21	−21.55	−36.82	−47.46	−65.32
Benchmark		$\delta_1 = 0.3$			$\rho_1 = 0.05$			
New Value		0.4	0.5	0.6	0.08	0.1	0.15	0.2
$\Delta u_1(\%)$	Standard	−0.61	−1.00	−1.27	**−7.38**	**−11.25**	**−18.49**	**−23.53**
	Hyperbolic	**−5.21**	**−9.32**	**−12.63**	−3.53	−5.69	−10.51	−14.63
$\Delta u_2(\%)$	Standard	+0.27	+0.43	+0.55	**+3.27**	**+5.05**	**+8.52**	**+11.03**
	Hyperbolic	**+2.08**	**+3.77**	**+5.16**	+1.40	+2.28	+4.27	+6.02

additional descriptive realism. The heterogeneous discounting describes the scene where a firm can have an increasing/decreasing valuation of the state (market share, in our case) at the end of planning horizon with the passage of time, whereas the hyperbolic discounting depicts the tendency to value more the payoffs that are closer to the present.

We have derived three different types of feedback Nash strategies, depending on how agents deal with their time-varying preferences. The pre-commitment solutions are employed by firms that are not aware of future changes or have a strong commitment power. Another option is to make decisions at every instant of time based on the corresponding instantaneous preferences, and only apply them at the very same moment (naive). The third action is to anticipate such variation and to include it into the decision making (sophisticated/time-consistent). Numerical simulations were run to illustrate some properties of different advertising paths and market dynamics corresponding to different strategies, and clear discrepancy is found.

The results reveal that firms under heterogeneous discounting act in a different manner compared to those with standard discounting (Sorger's setting). In general, the advertising strategies can be categorised in two phases. The first phase is the battle phase, in which the firm with a larger initial market share invests little at the beginning and increases the advertising effort in time, and the firm with a smaller initial market portion behaves the other way around. If the planning period is sufficiently long, they can arrive near the steady state and remain in its neighbourhood for some time. The second phase is the accommodation stage where agents raise/cut their advertising rate according to the increasing/decreasing importance they assign to the final states when approaching to the end of the planning horizon. Our numerical illustrations have demonstrated that the pre-commitment solution can show contrary adjustment direction compared with the sophisticated solution in this stage, whereas the naive solution basically coincides with the sophisticated one.

As to the advertising policies under hyperbolic discounting, a similar battle stage is also present. Another coincidence with heterogeneous discounting is the discrepancy between pre-commitment strategies and time-consistent strategies, as well as the similarity between naive and sophisticated solutions. Different from heterogeneous discounting, here strong commitment power would lead to over investment.

Our sensitivity analysis shows that firms would increase their marketing expenditure if they have advantages in profitability, advertising effectiveness, and/or costliness. On the contrary, a higher instantaneous discount rate would induce a reduction in advertisement. Moreover, we have compared the sensitivity to different parameters under different discount functions with the same overall impatience level. The tests demonstrate that agents are more sensitive to profit margin and advertising cost under standard than hyperbolic discounting. The advertising effectiveness sensitivity is also in general enhanced by exponential discounting, unless the asymmetry is significant.

The model is built based on some simplifying assumptions. First of all, we have focused on a mature market, which implies a stable market size. We have also assumed that the advertisement cannot influence the purchasing decisions of consumers who are not participating in this industry, in this sense the model could explain the alcohol and beverage industry, but might fail in explaining those industries where outsiders can be attracted by advertisement. Besides, we have not gone into detail on the advertising efficiency, which is described by a parameter. However it would be of interest to consider the factors determining the advertising efficiency apart from technical/economic ones, such as goodwill, brand loyalty and so on.

We then propose some future tasks that might be of interest. One possible extension is to consider the general time preferences in an oligopolistic market (as in Prasad et al., 2009), and/or under uncertainty. We could also relate heterogeneous discounting with market size properties. For instance, an increasing valuation of final state together with an expanding market, or *vice versa*. It would also be interesting to break the assumption of symmetric advertising efficiency by combining intangible asset of the firm like goodwill. Moreover, it would be worth introducing Lanchester dynamics into the supply chain environment, since this battle specification has been seldom applied in vertical channel.

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