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Treball final de grau

Minimal surfaces

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Abstract

In the present work we define the concept of Minimal Surface and prove some important results related to it. To begin with, we review some elementary definitions and results of differential geometry. Then, we study normal variations of curves and surfaces and solve some optimisation problems as examples of this techniques. Afterwards, we define Minimal Surface and prove a theorem relating Minimal Surfaces and normal variations of surfaces. The next section is dedicated to graph surfaces and in it we prove Jörgen's Theorem and Bernstein's Theorem. Finally, we extend the definitions introduced to a higher number of dimensions, study the cone in three and four dimensions and give a brief account of the history of Bernstein's theorem and its generalisation to higher dimensions.

Resum

En el present treball definim el concepte de Superfície Minimal i demostrem alguns resultats importants en relació amb aquest. Per començar, repassem algunes definicions elementals i alguns resultats de geometria diferencial. Aleshores, estudiem variacions normals de corbes i superfícies i resolem alguns problemes d'optimització com a exemples d'aquestes tècniques. Seguidament, definim Superfície Minimal i demostrem un teorema que relaciona les Superfícies Minimals amb les variacions normals de superfícies. La següent secció està dedicada a les superfícies formades per una gràfica i en aquesta demostrem el teorema de Jörgen i el teorema de Bernstein. Finalment, estenem les definicions introduïdes a un nombre gran de dimensions, estudiem el con en tres i quatre dimensions i donem un breu resum sobre la història del teorema de Bernstein i la seva generalització a dimensions altes.

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Contents

"Then came they to these lands where now thine eyes
Behold yon walls and yonder citadel
Of newly rising Carthage. For a price
They measured round so much of Afric soil
As one bull's hide encircles, and the spot
Received its name, the Byrsa."
- Virgil
Æneid (between 29 and 19 BC) ([20])

Introduction

The theory of Minimal Surfaces is a very broad topic in Mathematics dating back to the nineteenth century. The study of this kind of surfaces has motivated the development of numerous concepts and techniques in the calculus of variations which later have been applied to solving a great quantity of problems in very diverse areas ranging from the study of partial differential equations to topology and even to settling conjectures in relativity.

The aim of this project is to define the concept of Minimal Surface and to prove some important results related to it. It is divided in four chapters.

The first chapter is a review of some elementary definitions and results of differential geometry. Definitions such as parametrized curve, regular surface or curvature are given as well as some propositions and theorems without proof.

In the second chapter, we first study normal variations of curves. We define what a normal variation is and use it to solve an optimisation problem as an example, namely finding the shortest path between two points. Then we study briefly the regularity of the solutions we can find using the tools introduced and comment on their limitations. To finish the chapter, we define Minimal Surfaces, which are surfaces whose mean curvature vanishes everywhere, define normal variation of a surface and the variation of area. Then we show that the catenoid is an example of such surfaces and introduce and prove a theorem relating minimal surfaces and normal variations.

The next chapter is dedicated to proving Bernstein's Theorem, which states that if a surface S is given by the graph of a function mapping $\mathbb{R}^2$ to $\mathbb{R}$, and is also minimal, then S is a plane. In the first place we study graph surfaces with a focus on those that are minimal. Then, we present some well known results without giving any proof and use them to prove Jörgen's theorem. The chapter concludes with a proof of Bernstein's theorem for surfaces in three dimensions.

In the fourth and last chapter we begin by extending some of the definitions introduced to higher dimensions. Then we study the cone in three and four dimensions. To conclude, in the last section we present a brief account of the history of Bernstein's theorem and its generalisations to higher dimensions.

Notation

- n represents a natural number (including 0).
- Differentiable will be used to mean infinitely differentiable.
- Given a function on 1 variable f, we indicate its derivative as f'.
- Given a function f on variables $x_1, ..., x_n$ and $1 \leq i,\ j \leq n$, we indicate its partial derivative with respect to x_i evaluated at a point a, $\frac{\partial f}{\partial x_i}(a)$, as $\partial_{\mathrm{x_i}} f(a)$. And successive partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}(a)$ as $\partial_{\mathrm{x_i x_j}} f(a)$.
- Given functions $f_1,\ f_2,\ ...\ , f_n$,

$$\frac{\partial(f_1,\ f_2,\ ...\ ,\ f_n)}{\partial(x_1,\ x_2,\ ...\ ,\ x_n)}\ (p) := \begin{pmatrix} \partial_{\mathrm{x_1}} f_1(p) & \partial_{\mathrm{x_2}} f_1(p) & ... & \partial_{\mathrm{x_n}} f_1(p) \\ \partial_{\mathrm{x_1}} f_2(p) & \partial_{\mathrm{x_2}} f_2(p) & ... & \partial_{\mathrm{x_n}} f_2(p) \\ ... & & & \\ \partial_{\mathrm{x_1}} f_n(p) & \partial_{\mathrm{x_2}} f_n(p) & ... & \partial_{\mathrm{x_n}} f_n(p) \end{pmatrix}$$

- $\mathbb{S}^{n-1} = \{(x_1,\ ...\ ,\ x_n) \in \mathbb{R}^n : x_1^2 + ... + x_n^2 = 1\}$ is the (n-1)-dimensional sphere of radius 1 in $\mathbb{R}^n$.
- $\mathscr{C}^n\,(A)$ is the set of functions defined on A that are n times continuously differentiable.
- Let D be a set. We denote by ∂D the boundary of D and by $\overline{D} = D \cup \partial D$ the closure of D.
- For any $x = (x_1,\ x_2,\ ...\ ,\ x_n), y = (y_1,\ y_2,\ ...\ ,\ y_n) \in \mathbb{R}^n$, we denote the Euclidean distance between x, y as

$$d\,(x, y) = \sqrt{\langle x - y, x - y\rangle} = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

- We write the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$ as $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\}$

Chapter 1

Review of basic concepts in differential geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$

In this chapter we are going to recall some of the basic concepts and results we are going to use throughout this thesis. They can be found in any book on differential geometry such as [9], [13] and [17].

1.1 Curves

Definition 1.1. Given an open interval of real numbers $I \subseteq \mathbb{R}$, we define a parametrized curve as a differentiable function $\alpha : I \to \mathbb{R}^n$.

We will often call differentiable curves just curves and, unless specified otherwise, I, J will be open intervals of $\mathbb{R}$.

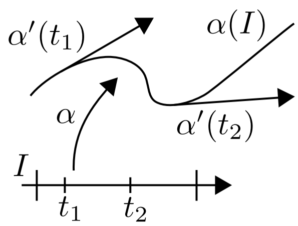

Figure 1.1: The curve $\alpha : I \to \mathbb{R}^n$ and its tangent vectors $\alpha'(t_1)$, $\alpha'(t_2)$.

Definition 1.2. Let $\alpha : I \to \mathbb{R}^n$ be a curve. We define the tangent vector to α at $t \in I$ as $\alpha'(t)$.

Definition 1.3. Let $I, J \subseteq \mathbb{R}$. A change of parameters from I to J is a differentiable function $h : I \to J$ such that h is bijective and $\forall t \in I,\ h'(t) \neq 0$.

Remark 1.4. The inverse of h, $h^{-1} : J \to I$, is also differentiable.

Definition 1.5. Given two curves $\alpha : I \to \mathbb{R}^n$ and $\beta : J \to \mathbb{R}^n$, we say β is a reparametrization of α if there exists a change of parameters $h : I \to J$ such that $\alpha = \beta \circ h$.

Definition 1.6. Let $\alpha : I \to \mathbb{R}^n$ be a curve and let $t \in I$. We say that t is a:

- regular point of α if $\alpha'(t) \neq 0$
- singular point of α if $\alpha'(t) = 0$

We say that α is 1-regular if t is a regular point of α $\forall t \in I$.

Definition 1.7. Let $t \in I$ be a regular point of a curve $\alpha : I \to \mathbb{R}^n$. We define the tangent line to α at t as the straight line $\{\alpha(t) + k\alpha'(t) \in \mathbb{R}^n : k \in \mathbb{R}\}$.

Definition 1.8. Let $\alpha : I \to \mathbb{R}^n$ be a curve. Given two points $t_0, t_1 \in I,\ t_0 \leq t_1$ we define the arc length of α from t_0 to t_1 as:

$$\operatorname{len}(\alpha, t_0, t_1) := \int_{t_0}^{t_1} \|\alpha'(t)\|\, dt$$

Remark 1.9. Using Weiesrstrass extreme value theorem we see that this integral exists since $[t_0, t_1]$ is a compact set and $\|\alpha'(t)\|$ is continuous on I.

Remark 1.10. We can approximate a curve using straight line segments. The length of the approximation is defined as the sum of the lengths of each segment. In the limit as the length of each segment is smaller and the number of segments is larger, the approximated length and the length defined previously coincide.

Proposition 1.11. *Arc length is invariant under change of parameters in the sense that if $\alpha : I \to \mathbb{R}^n$, $\beta : J \to \mathbb{R}^n$ are curves and $h : I \to J$ is a change of parameters such that $\forall t \in I,\ h'(t) > 0$ and $\alpha = \beta \circ h$, then $\forall t_0, t_1 \in I,\ t_0\ \leq t_1$*

$$\operatorname{len}(\alpha, t_0, t_1) = \operatorname{len}(\beta, h(t_0), h(t_1))$$

Definition 1.12. A curve $\alpha : I \to \mathbb{R}^n$ is said to be parametrized by arc length if $\forall t \in I,\ \|\alpha'(t)\| = 1$.

Proposition 1.13. *Let $\alpha : I \to \mathbb{R}^n$ be a curve. There exists a change of parameters $h : J \to I$ such that the curve $\beta := \alpha \circ h$ is parametrized by arc length if, and only if, α is 1-regular.*

Definition 1.14. Let $\alpha : I \to \mathbb{R}^n$ be a curve parametrized by arc length. The curvature of α is defined as the function

$$\begin{aligned} \kappa_\alpha : I &\to \mathbb{R} \\ t &\mapsto \kappa_\alpha(t) := \|\alpha''(t)\| \end{aligned}$$

Definition 1.15. Let $\alpha : I \to \mathbb{R}^n$ be a curve parametrized by arc length. The curvature vector of α at a point $t \in I$ is defined as $\alpha''(t)$.

Remark 1.16. We can think of the curvature vector of α at t as the derivative of the unit tangent vector to α at t.

Definition 1.17. Let $\alpha : I \to \mathbb{R}^n$ be a curve parametrized by arc length. A vector $N(t)$ is said to be normal to α at t if it is orthogonal to $\alpha'(t)$, that is, $N(t) \perp \alpha'(t)$.

Remark 1.18. If α is parametrized by arc length, then at every point its tangent vector and curvature vector are orthogonal:

$$\begin{aligned} \|\alpha'(t)\| = 1 \implies \|\alpha'(t)\|^2 = \langle \alpha'(t),\, \alpha'(t) \rangle = 1 \implies \frac{d}{dt}\langle \alpha'(t),\, \alpha'(t) \rangle = 0 \\ \implies \langle \alpha'(t),\, \alpha''(t) \rangle = 0 \end{aligned}$$

Definition 1.19. If $\alpha : I \to \mathbb{R}^n$ is a curve parametrized by arc length and $\|\alpha''(t)\| \neq 0$ for some $t \in I$, the unit vector $N(t) := \frac{\alpha''(t)}{\|\alpha''(t)\|}$ is normal to α at t and is called the unit normal vector to α at t.

Remark 1.20. In this case, $N(t) = \frac{\alpha''(t)}{\kappa_\alpha(t)}$.

Remark 1.21. If α is 2-regular, that is, $\forall t \in I,\ \alpha''(t) \neq 0$, the function $N(t) := \frac{\alpha''(t)}{\|\alpha''(t)\|}$ defined by the unit normal vector at each point $t \in I$ is differentiable.

Remark 1.22. If $\alpha = (\alpha_1, \alpha_2) : I \to \mathbb{R}^2$ is a curve on the plane, the vector $(-\alpha_2'(t), \alpha_1'(t))$ is a normal vector to α at $t \in I$ and it is differentiable.

1.2 Surfaces

Unless specified otherwise, Ω will denote an open set in $\mathbb{R}^2$.

Definition 1.23. A parametrized surface is defined as a differentiable function $\varphi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

Definition 1.24. A parametrized surface $\varphi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is called regular at a point $q \in \Omega$ if vectors $\partial_{x_1}\varphi(q)$ and $\partial_{x_2}\varphi(q)$ are linearly independent. We say that φ is regular if it is regular at every point $q \in \Omega$.

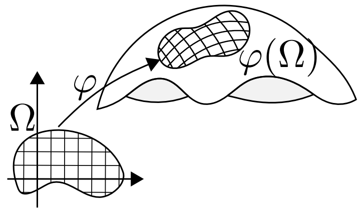

Figure 1.2: The surface $\varphi : \Omega \to \mathbb{R}^3$

Proposition 1.25. *Let $\varphi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface. Then, for each point $q \in \Omega$, the following statements are equivalent:*

- *φ is regular at q.*
- *The jacobian matrix $\begin{pmatrix} \partial_{x_1}\varphi(q) & \partial_{x_2}\varphi(q) \end{pmatrix}$ has rank 2.*
- *$\exists i, j \in \mathbb{N}$ such that $i < j$ and $\frac{\partial(\varphi_i,\varphi_j)}{\partial(x_1,x_2)}(q) \neq 0$.*
- *$\partial_{x_1}\varphi(q) \times \partial_{x_2}\varphi(q) \neq 0$.*

Remark 1.26. Given a differentiable function $f : \Omega \to \mathbb{R}$, we can define a parametrized surface as $\varphi : \Omega \to \mathbb{R}^3$, $\varphi(x_1,\ x_2) := (x_1,\ x_2,\ f(x_1,\ x_2))$. This surface is regular because for every point $q \in \Omega$, $\partial_{x_1}\varphi(q) = (1,\ 0,\ \partial_{x_1}f(q))$ and $\partial_{x_2}\varphi(q) = (0,\ 1,\ \partial_{x_2}f(q))$ which are linearly independent. Such parametrized surfaces are called graph surfaces.

Definition 1.27. Let $S \subseteq \mathbb{R}^3$. A local regular parametrization of S is defined as a differentiable function $\varphi : \Omega \subseteq \mathbb{R}^2 \to S$ such that:

- φ is a regular parametrized surface.
- $\varphi(\Omega) \subseteq S$ is an open set in S.
- $\varphi : \Omega \to \varphi(\Omega)$ is a homeomorfism.

Definition 1.28. A set $S \subseteq \mathbb{R}^3$ is called a regular surface at a point $p \in S$ if there exist a local regular parametrization of S $\varphi : \Omega \subseteq \mathbb{R}^2 \to S$ and a point $q \in \Omega$ such that $\varphi(q) = p$. S is said to be regular if it is regular at every point $p \in S$.

Proposition 1.29. *Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $\varphi : \Omega \to S$, $\psi : \Delta \to S$ be local regular parametrizations of S where Ω, $\Delta \subseteq \mathbb{R}^2$ are open sets. Suppose also that $W := \varphi(\Omega) \bigcap \psi(\Delta) \neq \emptyset$. Then, $h := \psi^{-1} \circ \varphi : \Omega \to \Delta$ is a diffeomorphism, that is, h is differentiable and invertible, and its inverse is also differentiable.*

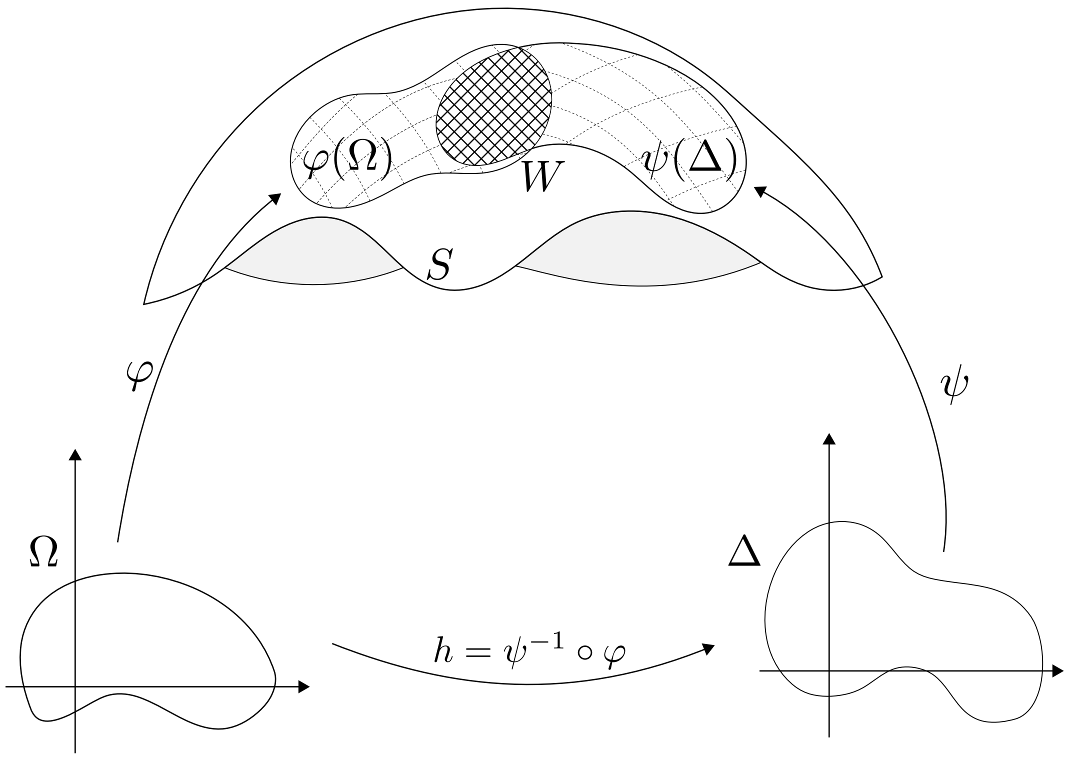

Figure 1.3: Given two local regular parametrizations φ and ψ of a regular surface S, the function $h = \psi^{-1} \circ \varphi$ is a diffeomorphism from Ω to Δ as seen in Proposition 1.29.

Definition 1.30. Let $S \subseteq \mathbb{R}^3$ be a regular surface and let p be a point in S. Let $\varphi : \Omega \subseteq \mathbb{R}^2 \to S$ be a local regular parametrization of S such that for some $q \in \Omega$, $p = \varphi(q)$. The tangent plane to S at p is defined as:

$$T_pS := \{ \lambda\, \partial_{x_1}\varphi(q) + \mu\, \partial_{x_2}\varphi(q) : \lambda,\ \mu \in \mathbb{R} \}$$

Remark 1.31. This definition does not depend on the choice of φ.

Remark 1.32. The tangent plane to S at a point is a 2-dimensional vector space.

Definition 1.33. Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $p \in S$ be a point in S. We say a function $F : S \to \mathbb{R}^n$ is differentiable at p if there exist a local regular parametrization of S $\varphi : \Omega \subseteq \mathbb{R}^2 \to S$ and a point $q \in \Omega$ such that $p = \varphi(q)$ and $F \circ \varphi : \Omega \to \mathbb{R}^n$ is differentiable at q.

Remark 1.34. Using Proposition 1.29 it can be seen that this definition does not depend on the choice of φ.

Definition 1.35. We say a curve $\alpha : I \to \mathbb{R}^3$ lies on a parametrized surface $\varphi : \Omega \to \mathbb{R}^3$ if $\alpha(I) \subseteq \varphi(\Omega)$.

Proposition 1.36. *Let $\varphi : \Omega \to \mathbb{R}^3$ be a local regular parametrization of a surface S such that for a point $q \in \Omega$, $\varphi(q) = p$ is a regular point in S. Then, for any curve $\alpha : I \to \mathbb{R}^3$ lying on S such that for some $t \in I$, $\alpha(t) = p$, the tangent vector to α at t lies on the tangent plane to S at p, that is $\alpha'(t) \in T_pS$.*

Remark 1.37. Using this proposition one can show that for any regular point p in a surface S and for any vector $w \in T_pS$ there exist a curve $\alpha : I \to \mathbb{R}^3$ and a point $t \in I$ such that, in a neighbourhood of p, α lies on S, $\alpha(q) = p$ and $\alpha'(q) = w$.

Definition 1.38. Let $S \subseteq \mathbb{R}^3$ be a regular surface, let $p \in S$ be a point in S and let $F : S \to \mathbb{R}^n$ be a differentiable function at p. Let $w \in T_pS$ and let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a curve on S such that $\alpha(0) = p$ and $\alpha'(0) = w$. Consider the curve $\beta = F \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$.

We define the differential of F at p as a function $d_pF : T_pS \to \mathbb{R}^n$ that acts on vectors $w \in T_pS$ in the following manner: $d_pF(w) := \beta'(0)$.

Proposition 1.39. *The definition of d_pF does not depend on the choice of α and d_pF is linear.*

Definition 1.40. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$ a point in S. We define the normal space to S at p as the orthogonal complement of T_pS in $\mathbb{R}^3$ and denote it by $T_pS^\perp$.

A vector $N \in T_pS^\perp$ is called a normal vector to S at p.

Definition 1.41. [Gauss map] Let $S \subseteq \mathbb{R}^3$ be a regular surface. A Gauss map on S is defined as a continuous function $N : S \to \mathbb{S}^2$ such that at every point $p \in S$, $N(p)^\perp = T_pS$. Equivalently, $N(p)$ is a normal vector to S at every point p in S.

Definition 1.42. A regular surface S is said to be orientable if there exist Gauss maps on S. An orientation of S is a certain Gauss map on S. We say S is oriented if we have chosen a certain orientation of S.

Remark 1.43. There are only two possible orientations of a regular surface S.

Remark 1.44. [Normal map to a regular surface] All regular surfaces in $\mathbb{R}^3$ are locally orientable. Indeed, if $\varphi : \Omega \to S$ is a local regular parametrization of a regular surface S, the map

$$\begin{aligned} N : \varphi(\Omega) &\to \mathbb{S}^2 \\ p \mapsto N(p) &:= \frac{\partial_{x_1}\varphi(\varphi^{-1}(p)) \times \partial_{x_2}\varphi(\varphi^{-1}(p))}{\|\partial_{x_1}\varphi(\varphi^{-1}(p)) \times \partial_{x_2}\varphi(\varphi^{-1}(p))\|} \end{aligned}$$

is differentiable and satisfies $N(p)^{\perp} = T_pS$ for all p in $\varphi(\Omega)$.
We can also define the map

$$\begin{aligned} \widetilde{N} : \Omega &\to \mathbb{S}^2 \\ (x_1, x_2) \mapsto \widetilde{N}(x_1, x_2) &:= \frac{\partial_{x_1}\varphi(x_1, x_2) \times \partial_{x_2}\varphi(x_1, x_2)}{\|\partial_{x_1}\varphi(x_1, x_2) \times \partial_{x_2}\varphi(x_1, x_2)\|} \end{aligned}$$

Notice that $\widetilde{N} = N \circ \varphi$. We will denote both of this maps with the letter N and call them the unit normal map to S as the context is usually enough to determine which one we are referring to.

Definition 1.45. Let $S \subseteq \mathbb{R}^3$ be a regular surface, let p be a point in S and let $N : S \to \mathbb{S}^2$ be a Gauss map on S. Since $T_{N(p)}\mathbb{S}^2 = N(p)^{\perp} = T_pS$, the differential $d_pN : T_pS \to T_{N(p)}\mathbb{S}^2$ can be understood as an endomorphism of T_pS. We call $d_pN : T_pS \to T_pS$ Weingarten's endomorphism.

Definition 1.46. [First fundamental form] Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and denote the usual dot product in $\mathbb{R}^3$ as $\langle \cdot, \cdot \rangle$. The first fundamental form of the surface S at p is defined as:

$$\begin{aligned} I_p : T_pS \times T_pS &\to \mathbb{R} \\ (w_1, w_2) &\mapsto I_p(w_1, w_2) := \langle w_1, w_2 \rangle \end{aligned}$$

Remark 1.47. Since the first fundamental form at a point p on a surface S is the standard dot product in $\mathbb{R}^3$ restricted to the tangent plane to S at p, it is a positive-definite bilinear simmetric function. Therefore, we can express it as a matrix $g(p) := \big(g_{ij}(p)\big)$, where $1 \leq i, j \leq 2$, in the basis $\{\partial_{x_1}\varphi(q),\ \partial_{x_2}\varphi(q)\}$ of T_pS where $\varphi : \Omega \to \mathbb{R}^3$ is a local regular parametrization of S such that $\varphi(q) = p$ for some $q \in \Omega$. We have

$$\begin{aligned} g_{11} &= \langle \partial_{x_1}\varphi,\ \partial_{x_1}\varphi \rangle \\ g_{21} = g_{12} &= \langle \partial_{x_1}\varphi,\ \partial_{x_2}\varphi \rangle = \langle \partial_{x_2}\varphi,\ \partial_{x_1}\varphi \rangle \\ g_{22} &= \langle \partial_{x_2}\varphi,\ \partial_{x_2}\varphi \rangle \end{aligned} \tag{1.1}$$

where all g_{ij} are evaluated at p and all $\partial_{x_i}\varphi$ are evaluated at q.

Note that the matrix $g(p)$ depends on the parametrization φ but I_p doesn't.

By varying p over S we obtain functions $g_{ij} : \varphi(\Omega) \to \mathbb{R}$ which are differentiable.

Proposition 1.48. *Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $\varphi : \Omega \to \mathbb{R}^3$ be a local regular parametrization of S. Then, $\forall (x_1, x_2) \in \Omega$*

$$\|\partial_{x_1}\varphi(x_1, x_2) \times \partial_{x_2}\varphi(x_1, x_2)\| = \sqrt{\det g(p)} = \sqrt{g_{11}(p)g_{22}(p) - g_{12}(p)^2}$$

where $p = \varphi(x_1, x_2)$

Corollary 1.49. *A parametrized surface $\varphi : \Omega \to \mathbb{R}^3$ is regular at $p = \varphi(x_1, x_2) \in \varphi(\Omega)$ if, and only if, $g_{11}(p)g_{22}(p) - g_{12}(p)^2 \neq 0$.*

Definition 1.50. [Second fundamental form] Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and let $N : S \to \mathbb{S}^2$ be a Gauss map on S. The second fundamental form of the surface S at p is defined as:

$$\begin{aligned} II_p : T_pS \times T_pS &\to \mathbb{R} \\ (w_1, w_2) &\mapsto II_p(w_1, w_2) := -I_p(d_pN(w_1), w_2) \end{aligned}$$

Proposition 1.51. *Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and let $N : S \to \mathbb{S}^2$ be a Gauss map on S. Weingarten's endomorphism d_pN is a self-adjoint linear map for all p in S. That is $\forall w_1, w_2 \in T_pS$, $\langle d_pN(w_1),\ w_2\rangle = \langle w_1,\ d_pN(w_2)\rangle$.*

Remark 1.52. Since the first fundamental form and Weingarten's endomorphism are linear, the second fundamental form is bilinear. Therefore, similarly to the first fundamental form, we can express the second fundamental form as a matrix $h(p) := \big(h_{ij}(p)\big)$, where $1 \leq i, j \leq 2$, in the basis $\{\partial_{x_1}\varphi(q),\ \partial_{x_2}\varphi(q)\}$ of T_pS where $\varphi : \Omega \to \mathbb{R}^3$ is a local regular parametrization of a regular surface S such that $\varphi(q) = p$ for some $p \in S$ and $q \in \Omega$.

What's more, it can be shown that the coefficients of the matrix associated to the second fundamental form can be calculated as

$$\begin{aligned} h_{11} &= \frac{\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi,\ \partial_{x_1x_1}\varphi\rangle}{\|\partial_{x_1}\varphi \times \partial_{x_2}\varphi\|} \\ h_{21} = h_{12} &= \frac{\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi,\ \partial_{x_1x_2}\varphi\rangle}{\|\partial_{x_1}\varphi \times \partial_{x_2}\varphi\|} \\ h_{22} &= \frac{\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi,\ \partial_{x_2x_2}\varphi\rangle}{\|\partial_{x_1}\varphi \times \partial_{x_2}\varphi\|} \end{aligned} \tag{1.2}$$

where all h_{ij} are evaluated at p and all $\partial_{x_i}\varphi$ are evaluated at q.

Note that the matrix $h(p)$ depends on the parametrization φ but II_p doesn't. By varying p over S we obtain functions $h_{ij} : \varphi(\Omega) \to \mathbb{R}$ which are differentiable.

In what follows, to simplify the notation, φ, N and their derivatives are considered to be evaluated at $q \in \Omega$, and g_{ij}, h_{ij} at $p = \varphi(q)$.

Proposition 1.53. *Let $\varphi : \Omega \to \mathbb{R}^3$ be a regular parametrized surface and N be a unit normal map to $\varphi(\Omega)$. Then*

$$\begin{aligned} h_{11} &= -\langle \partial_{x_1}\varphi, \partial_{x_1}N\rangle \\ h_{12} &= h_{21} = -\langle \partial_{x_2}\varphi, \partial_{x_1}N\rangle = -\langle \partial_{x_1}\varphi, \partial_{x_2}N\rangle \\ h_{22} &= -\langle \partial_{x_2}\varphi, \partial_{x_2}N\rangle \end{aligned} \tag{1.3}$$

where φ and N are evaluated at $q \in \Omega$ and h_{ij} are evaluated at $p = \varphi(q)$, $i, j = 1, 2$.

Proof. Let $\alpha, \beta \in T_p\varphi(\Omega)$ be vectors on the tangent plane to $\varphi(\Omega)$ at a point $p \in \varphi(\Omega)$. Since φ is regular, vectors $\{\partial_{x_1}\varphi, \partial_{x_2}\varphi\}$ are linearly independent and form a basis of $T_p\varphi(\Omega)$. Therefore, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $\alpha = \alpha_1\,\partial_{x_1}\varphi + \alpha_2\,\partial_{x_2}\varphi$ and $\beta = \beta_1\,\partial_{x_1}\varphi + \beta_2\,\partial_{x_2}\varphi$. Denote the coordinates of α and β on this basis as $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. On one hand,

$$\begin{aligned} II_p(\alpha, \beta) &= -\langle d_pN(\alpha), \beta\rangle \\ &= -\langle \partial_{x_1}N\,\alpha_1 + \partial_{x_2}N\,\alpha_2, \partial_{x_1}\varphi\,\beta_1 + \partial_{x_2}\varphi\,\beta_2\rangle \\ &= -\big(\langle \partial_{x_1}N, \partial_{x_1}\varphi\rangle\alpha_1\beta_1 + \langle \partial_{x_1}N, \partial_{x_2}\varphi\rangle\alpha_1\beta_2 \\ &\qquad + \langle \partial_{x_2}N, \partial_{x_1}\varphi\rangle\alpha_2\beta_1 + \langle \partial_{x_2}N, \partial_{x_2}\varphi\rangle\alpha_2\beta_2\big) \end{aligned}$$

On the other hand,

$$\begin{aligned} II_p(\alpha, \beta) &= \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\ &= h_{11}\alpha_1\beta_1 + h_{12}\alpha_1\beta_2 + h_{21}\alpha_2\beta_1 + h_{22}\alpha_2\beta_2 \end{aligned}$$

Comparing these expressions we obtain the equalities stated. □

Definition 1.54. [Normal curvature] Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and let $\alpha : I \to S$ be a curve on S such that $\alpha(t) = p$ and $\alpha'(t) = w \in T_pS$ for some $t \in I$. Let $N \in T_pS$ be a normal vector to S at p. The normal curvature of S at p in the direction w with respect to the normal N is defined as

$$\kappa_n(p, N, w) := \alpha''(t) \cdot N$$

Remark 1.55. This definition is independent of α.

Definition 1.56. [Principal curvatures] Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and let $w \in T_pS$. Let $N \in T_pS$ be a normal vector to S at p. Fixing N we can consider

$$\kappa_1(p, N) := \max_{w \in T_pS} \kappa_n(p, N, w), \quad \kappa_2(p, N) := \min_{w \in T_pS} \kappa_n(p, N, w)$$

The quantities $\kappa_1(p, N)$ and $\kappa_2(p, N)$ are called the principal curvatures of S at a point p with respect to the normal N.

Remark 1.57. It can be shown that the principal curvatures at some point $p \in S$ $\kappa_1(p, N)$, $\kappa_2(p, N)$ are the eigenvalues of d_pN. By Proposition 1.51, d_pN has 2 real eigenvalues with a corresponding eigenvector each, and these are orthogonal. The directions defined by these eigenvectors are called principal directions of curvature of S at p.

Definition 1.58. [Gauss curvature] Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and let $N \in T_pS$ be a normal vector to S at p. The Gauss curvature of S at a point p is defined as

$$\kappa(p, N) := \kappa_1(p, N) \cdot \kappa_2(p, N)$$

Definition 1.59. [Mean curvature] Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S and let $N \in T_pS$ be a normal vector to S at p. The mean curvature of S at a point p is defined as

$$H(p, N) := \frac{\kappa_1(p, N) + \kappa_2(p, N)}{2}$$

Remark 1.60. As a consequence of Remark 1.57 we have the following:

$$\kappa(p, N) = \det d_pN \tag{1.4}$$

$$H(p, N) = \frac{1}{2} \operatorname{tr} d_pN \tag{1.5}$$

Sometimes, we might want to choose a parametrization of S $\varphi : \Omega \to \mathbb{R}^3$ containing p at its image to calculate $\kappa(p, N)$ and $H(p, N)$. Note that $\kappa(p, N)$ and $H(p, N)$ are independent of the parametrization φ.

Now we restrict ourselves to working with oriented surfaces in $\mathbb{R}^3$ and, unless specified otherwise, given a local regular parametrization of a surface $S \subseteq \mathbb{R}^3$ $\varphi : \Omega \to \mathbb{R}^3$ we are going to consider the orientation chosen to be the one defined in Remark 1.44. The next propositions show us how to calculate the Gauss and mean curvatures in terms of the fundamental forms.

Proposition 1.61. *Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S, $\varphi : \Omega \to \mathbb{R}^3$ a local regular parametrization of S such that, for some $q \in \Omega$, $p = \varphi(q)$ and let N be the chosen Gauss map on S. Then,*

$$\kappa(p, N(p)) = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

where all g_{ij} and h_{ij} are evaluated at q.

Proposition 1.62. *Let $S \subseteq \mathbb{R}^3$ be a regular surface, $p \in S$ a point in S, $\varphi : \Omega \to \mathbb{R}^3$ a local regular parametrization of S such that, for some $q \in \Omega$, $p = \varphi(q)$ and let N be the chosen Gauss map on S. Then,*

$$H(p, N(p)) = \frac{1}{2}\frac{h_{11}g_{22} - h_{12}g_{12} + h_{22}g_{11}}{g_{11}g_{22} - g_{12}^2} \tag{1.6}$$

where all g_{ij} and h_{ij} are evaluated at q.

Remark 1.63. Since g_{ij} and h_{ij} are continuous, κ and H are continuous where $g_{11}g_{22} - g_{12}^2 \neq 0$.

Definition 1.64. [Area] Let $\varphi : \Omega \to \mathbb{R}^3$ be a parametrized surface, $D \subseteq \Omega$ be a bounded set with closure $\overline{D} \subseteq \Omega$ and write $p = \varphi(x_1, x_2)$. The area of $\varphi(\overline{D})$ is defined as

$$\begin{aligned}\operatorname{area}(\varphi(\overline{D})) &:= \int_{\overline{D}} \|\partial_{x_1}\varphi(x_1, x_2) \times \partial_{x_2}\varphi(x_1, x_2)\| \, dx_1 dx_2 \\ &= \int_{\overline{D}} \sqrt{g_{11}(p)g_{22}(p) - g_{12}(p)^2} \, dx_1 dx_2\end{aligned}$$

Proposition 1.65. *Let $S \subseteq \mathbb{R}^3$ be a surface and $R \subseteq S$ be a subset of S. Let $\varphi : \Omega \to S$ and $\psi : \Delta \to S$ be parametrizations of S such that $R \subseteq \varphi(\Omega)$ and $R \subseteq \psi(\Delta)$. Let $D = \varphi^{-1}(R)$ and $Q = \psi^{-1}(R)$ be bounded and satisfy $\overline{D} \subseteq \Omega$, $\overline{Q} \subseteq \Omega$. Then,* $\operatorname{area}(\varphi(\overline{D})) = \operatorname{area}(\psi(\overline{Q}))$. *That is, the area of a region R of a surface S is independent of the parametrization.*

Chapter 2

Minimal surfaces

One of the first problems which motivated the study of minimal surfaces is the so called Plateau's problem, in honour of the Belgian physicist J. Plateau (1801-1883). This problem consists in finding the surface which has the smallest area among all surfaces having a certain curve as their boundary. Plateau is known for having performed several experiments with soap films and, experimentally, having determined a number of properties of soap films and soap bubbles. While Plateau's work is commendable, we seek to provide a more mathematical study of the problem and of the properties of minimal surfaces. In this section we are going to introduce the notion of normal variations of curves and surfaces, we are going to define what a minimal surface is and we are going to study some of their properties.

2.1 Normal variations of curves

There are many problems in which there is a quantity whose value depends on a curve and the goal is to find the curve which minimizes this quantity. For instance, the problem of finding the path an object subjected to gravity has to follow to go from a higher point to a lower point such that it minimizes the time it takes the object to travel between the points.[1] Or the problem of finding the path a particle will take when one or more forces are acting on it, which can be formulated in terms of finding the path that minimizes the action integral.[2] In many of these problems it is infeasible to consider all possible curves so we restrict ourselves to a set of curves that neighbour a given curve α. This set of curves is called the set of normal variations of α.

[1] This is a problem first posed by Galileo Galilei in 1638. The solution is the cycloid. See [11], p. 42, *3. The brachistochrone problem*.

[2] Also see [11].

Definition 2.1. Let $\alpha : I \to \mathbb{R}^n$ be a 2-regular curve. Consider the differentiable function $N : I \to \mathbb{R}^n$ where $N(t)$ is the unit normal vector to $\alpha(I)$ at each point $\alpha(t) \in I$ and choose an arbitrary differentiable function $h : \overline{D} \to \mathbb{R}$ where $D \subseteq I$ is a bounded open set and $\overline{D}$ is the closure of D. Let $\varepsilon > 0$. The normal variation of α determined by h is defined as the family of curves

$$\begin{aligned}\overline{\alpha} : \overline{D} \times (-\varepsilon,\ \varepsilon) &\to \mathbb{R}^n \\ (t,\ \delta) &\mapsto \overline{\alpha}(t,\ \delta) := \alpha(t) + \delta h(t) N(t)\end{aligned}$$

Remark 2.2. The function $\alpha_\delta = \overline{\alpha}(\cdot,\ \delta) : \overline{D} \to \mathbb{R}^n$, $\alpha_\delta(t) := \alpha(t) + \delta h(t) N(t)$ is a differentiable curve for all $\delta \in (-\varepsilon,\ \varepsilon)$.

Now, if we have a quantity that depends on a curve, we can consider this quantity evaluated at curves α_δ and attempt to find the curve which minimizes it by minimizing the quantity with respect to δ. Let's see some examples.

2.1.1 Shortest path between two points

Let's try to find the shortest path between two points on a plane. It is well known that it is a straight line and this can be shown by much simpler methods than those we will present. However, by means of this example we will introduce ideas and techniques which will be very useful later on, especially Lemma 2.3.

Let $\alpha : I = [a,\ b] \to \mathbb{R}^2$ be a 2-regular curve that joins the points $\alpha(a),\ \alpha(b)$. Let N and h be the corresponding functions in the previous definition but now impose

$$h(a) = h(b) = 0 \tag{2.1}$$

so that we only consider curves with endpoints $\alpha(a),\ \alpha(b)$, that is, $\alpha_\delta(a) = \alpha(a)$, $\alpha_\delta(b) = \alpha(b)$, $\forall \delta \in (-\varepsilon,\ \varepsilon)$. The length of α_δ is

$$\operatorname{len}(\alpha_\delta, a, b) = \int_a^b \|\alpha_\delta'(t)\|\, dt$$

As $\alpha,\ h$ and N are differentiable on $[a, b]$, they are continuous on the same interval and this integral exists. Since the length is invariant under change of parameters, assume α is parametrized by arc length, that is $\forall t \in [a, b],\ \|\alpha'(t)\| = 1$.

If $\alpha = \alpha_{\delta=0}$ is the shortest curve among the ones we are considering, it is a minimum of length for all h fulfilling (2.1) so it has to satisfy $\frac{d}{d\delta} \operatorname{len}(\alpha_\delta, a, b)\big|_{\delta=0} = 0$ for all such h. Let's find the necessary condition α has to fulfill in order to be such minimum. We have

$$\frac{d}{d\delta} \operatorname{len}(\alpha_\delta, a, b) = \frac{d}{d\delta} \int_a^b \|\alpha_\delta'(t)\|\, dt = \int_a^b \frac{\partial}{\partial \delta} \|\alpha_\delta'(t)\|\, dt$$

On what follows we assume all functions are evaluated at t and κ is the curvature of α. From the definition of α_δ it follows that

$$\begin{aligned}
\frac{\partial}{\partial\delta}\|\alpha'_\delta\| &= \frac{\partial}{\partial\delta}\|\alpha' + \delta(hN)'\| \\
&= \frac{\partial}{\partial\delta}\langle \alpha' + \delta(hN)', \alpha' + \delta(hN)'\rangle^{1/2} \\
&= \frac{1}{\|\alpha'_\delta\|}\langle \alpha' + \delta(hN)', (hN)'\rangle \\
&= \frac{1}{\|\alpha'_\delta\|}\left(\langle \alpha', (hN)'\rangle + \delta\langle (hN)', (hN)'\rangle\right) \qquad (2.2)
\end{aligned}$$

Now,

$$\langle \alpha', (hN)'\rangle = \langle \alpha', h'N + hN'\rangle = \langle \alpha', h'N\rangle + \langle \alpha', hN'\rangle$$

Since α is parametrized by arc length, N is orthogonal to α' and the first term in the sum is zero. To calculate the second term, notice that for this same reason α' and N form a basis of the plane. Therefore, N' can be expressed as a linear combination of α' and N. As $\langle N, N\rangle = 1$, by differentiating we see that $\langle N, N'\rangle = 0$, so N' is proportional to α'. Once again, using the fact that $\langle \alpha', N\rangle = 0$ and differentiating we arrive at $\langle \alpha', N'\rangle = -\langle \alpha'', N\rangle = -\langle \kappa N, N\rangle = -\kappa\langle N, N\rangle = -\kappa$. Thus, $N' = -\kappa\alpha'$ and $\langle \alpha', (hN)'\rangle = -h\kappa$.

Let's calculate the second term at (2.2):

$$\langle (hN)', (hN)'\rangle = \langle h'N, h'N\rangle + 2\langle h'N, hN'\rangle + \langle hN', hN'\rangle$$

We have seen that α' and N' are parallel so N and N' are orthogonal and the middle term of this summation is null. The first term is

$$\langle h'N, h'N\rangle = (h')^2\langle N, N\rangle = (h')^2$$

And the third term is

$$\langle hN', hN'\rangle = h^2\|N'\|^2 = h^2\|-\kappa\alpha'\|^2 = h^2|\kappa|^2\|\alpha'\|^2 = h^2|\kappa|^2$$

Then $\langle (hN)', (hN)'\rangle = (h')^2 + h^2|\kappa|^2$.

Putting it all together we get

$$\begin{aligned}
\frac{\partial}{\partial\delta}\|\alpha'_\delta(t)\| &= \frac{1}{\|\alpha'_\delta\|}\left(\langle \alpha', (hN)'\rangle + \delta\langle (hN)', (hN)'\rangle\right) \\
&= \frac{1}{\|\alpha'_\delta\|}\left(-h\kappa + \delta\left((h')^2 + h^2|\kappa|^2\right)\right)
\end{aligned}$$

Finally,

$$\frac{d}{d\delta}\operatorname{len}(\alpha_\delta, a, b) = \int_a^b \frac{-h\kappa + \delta\left((h')^2 + h^2|\kappa|^2\right)}{\|\alpha'_\delta\|}\, dt$$

Therefore, if $\alpha = \alpha_{\delta=0}$ is the shortest path between $\alpha(a)$, $\alpha(b)$ for all possible choices of h satisfying (2.1), then, remembering $\|\alpha'_{\delta=0}\| = \|\alpha'\| = 1$ we have

$$\frac{d}{d\delta}\operatorname{len}(\alpha_\delta, a, b)\bigg|_{\delta=0} = \int_a^b -h(t)\kappa(t)\,dt = 0 \tag{2.3}$$

for all such h. Intuitively we might think that since this integral is zero for all h fulfilling (2.1), the curvature must be zero. The following lemma tells us this is indeed the case.

Lemma 2.3. *Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a continuous function. If for all differentiable functions $h : \overline{U} \to \mathbb{R}$ such that $h = 0$ on ∂U it is satisfied $\int_U fh = 0$, then f is zero everywhere on U.*

Proof. We prove by contradiction that f is null on U. Assume $\int_U fh = 0$ for all differentiable functions $h : \overline{U} \to \mathbb{R}$ such that $h = 0$ on ∂U and suppose there exists $\overline{x} \in U$ such that $f(\overline{x}) \neq 0$. Suppose also $f(\overline{x}) > 0$, the case $f(\overline{x}) < 0$ is proved analogously. Since f is continuous, there exists $\varepsilon > 0$ such that $f(x) > 0$ for all $x \in B_\varepsilon(\overline{x})$ and since U is open, $\overline{x}$ is not isolated and $B_\varepsilon(\overline{x})$ has nonzero measure. Consider a function $h : \overline{U} \to \mathbb{R}$ such that $h(x) > 0$ for all $x \in B_\varepsilon(\overline{x})$ and $h = 0$ elsewhere. Then,

$$\int_U fh = \int_{B_\varepsilon(\overline{x})} fh > 0$$

because f and h are strictly positive on $B_\varepsilon(\overline{x})$, which is a contradiction. □

Hence, we have shown that if α is the curve with minimum length for all of its normal variations, then its curvature κ must be zero everywhere.

Now, if κ is identically zero, (2.3) implies $\delta = 0$ is a critical point of length independently of the choice of h. Therefore α is not necessarily a minimum of length for a given normal variation. However, α having zero curvature everywhere means it is contained in a straight line, so all normal variations of α are longer than it is and α is the curve with minimum length.

2.1.2 Regularity

While the techniques we have just presented are very powerful, they have some limitations. One such limitation is that when working with normal variations of a given curve, we are only considering a subset of the set of all possible curves; namely, the set of curves that can be obtained by varying the given curve in the way we have described. An example of this is the proof we have given of the fact that the shortest path between two points is a curve with null curvature everywhere. It is not hard to come up with a proof that the shortest path between two points is a straight line which does not make use of variations and is a more general proof. This proof can include curves which are not necessarily twice continuously differentiable, curves which are self intersecting and others.

Also, the hypotheses we are considering are quite restrictive and do not allow us to work with more general but still easy to deal with objects like curves which are not 1-regular. For instance, consider the problem of finding the following infimum

$$\inf\left\{ \int_0^1 \left(1 - \left(f'(x)\right)^2\right)^2 dx \ : \ f(0) = f(1) = 0,\ f \in \mathscr{C}^1\left([0,1]\right) \right\} \tag{2.4}$$

That is, we want to find a continuously differentiable function $f : [0,1] \to \mathbb{R}$ satisfying $f(0) = f(1) = 0$ which minimizes the integral

$$\int_0^1 \left(1 - \left(f'(x)\right)^2\right)^2 dx \tag{2.5}$$

Thinking of the graph of f as the curve $\alpha : [0,1] \to \mathbb{R}^2$, $\alpha(x) := (x,\ f(x))$, we could be tempted of applying the techiques presented in the previous sections to find such f. However, this approach results unsatisfactory as there does not exist any function f such that α is 1-regular and f is a minimizer of the integral above. We can convince ourselves of the truth of this statement by looking at Figure 2.1. For details see [7] p.51.

However, if we allow ourselves to consider piecewise continuously differentiable functions, the function $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) := \frac{1}{2} - \left|x - \frac{1}{2}\right| = \begin{cases} x & \text{if } x \leq \frac{1}{2} \\ 1 - x & \text{if } x \geq \frac{1}{2} \end{cases}$$

satisfies $\left(f'(x)\right)^2 = 1$ for all $x \in [0,1] \setminus \{\frac{1}{2}\}$. Therefore integral (2.5) evaluates to 0 and, since the integrand is nonnegative, the infimum at (2.4) is 0. Nevertheless, f is not a minimizer because it is not $\mathscr{C}^1\left([0,1]\right)$.

In conclusion, the variational techniques introduced in this section are quite powerful and allow us to tackle many problems. However, they come with certain limitations, some of which can be surmounted by general and more advanced techniques like those presented in [1].

2.2 Minimal surfaces

In this section we are going to define minimal surface and present some results as they appear in [9]. Then, we will present related theorems and results that can be found in [6], [17] and [19].

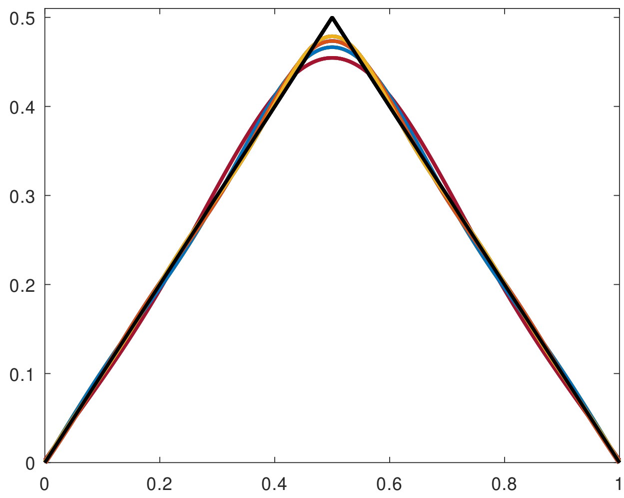

Figure 2.1: In black, f, a solution to the problem of minimizing integral (2.5). In other colours, various 1-regular curves which are not solutions of that problem. We can see at $x = 0,5$ f is not differentiable and therefore its graph is not 1-regular.

Definition 2.4. [Minimal surface] A regular parametrized surface $\varphi : \Omega \to \mathbb{R}^3$ is called minimal if its mean curvature is null everywhere.

Definition 2.5. Let $S \subseteq \mathbb{R}^3$ be a regular surface. We say S is a minimal surface if for every point p in S and every local regular parametrization of S $\varphi : \Omega \to S$ such that $p \in \varphi(\Omega)$, φ is a minimal surface.

Example 2.6. [Catenoid] A classic example of a minimal surface is the catenoid, the surface of revolution generated by rotating a catenary around a fixed axis. Let a be a positive real number. The catenary with equation $x_2 = a \cosh \frac{x_3}{a}$ when rotated around the x_3 axis generates the catenoid given by

$$\varphi(x_1, x_2) = (a \cosh x_2 \cos x_1,\ a \cosh x_2 \sin x_1,\ a x_2)$$

where $0 < x_1 < 2\pi$, $-\infty < x_2 < \infty$. Let's calculate its mean curvature and see it vanishes everywhere.

Its partial derivatives are:

$$\partial_{x_1} \varphi(x_1, x_2) = (-a \cosh x_2 \sin x_1,\ a \cosh x_2 \cos x_1,\ 0)$$
$$\partial_{x_2} \varphi(x_1, x_2) = (a \sinh x_2 \cos x_1,\ a \sinh x_2 \sin x_1,\ a)$$

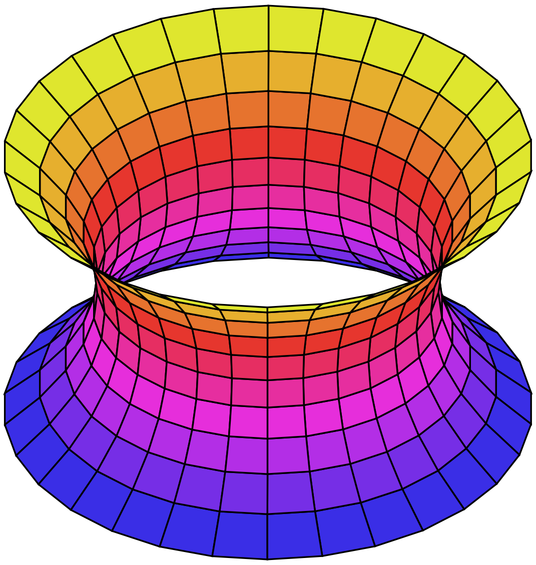

Figure 2.2: Catenoid from [21]

Then the matrix coefficients of the first fundamental form evaluated at $\varphi(x_1, x_2)$ are easily calculated using (1.1):

$$g_{11} = a^2 \cosh(x_2)^2$$
$$g_{21} = g_{12} = 0$$
$$g_{22} = a^2 \left(1 + \sinh(x_2)^2\right) = a^2 \cosh(x_2)^2$$

Now to calculate the coefficients of the second fundamental form matrix at $\varphi(x_1, x_2)$ we first find

$$\partial_{x_1x_1}\varphi = (-a\cosh x_2 \cos x_1,\ -a\cosh x_2 \sin x_1,\ 0)$$
$$\partial_{x_1x_2}\varphi = (-a\sinh x_2 \sin x_1,\ a\sinh x_2 \cos x_1,\ 0)$$
$$\partial_{x_2x_2}\varphi = (a\cosh x_2 \cos x_1,\ a\cosh x_2 \sin x_1,\ 0)$$
$$\partial_{x_1}\varphi \times \partial_{x_2}\varphi = a^2 (\cosh x_2 \cos x_1,\ \cosh x_2 \sin x_1,\ -\cosh x_2 \sinh x_2)$$
$$\|\partial_{x_1}\varphi \times \partial_{x_2}\varphi\| = a^2 \cosh(x_2)^2$$

and then,

$$\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi,\ \partial_{x_1x_1}\varphi \rangle = -a^3 \cosh(x_2)^2$$
$$\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi,\ \partial_{x_1x_2}\varphi \rangle = 0$$
$$\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi,\ \partial_{x_2x_2}\varphi \rangle = a^3 \cosh(x_2)^2$$

where all derivatives are evaluated at (x_1, x_2).

Finally, using (1.2), at $\varphi(x_1, x_2)$ we find:

$$\begin{aligned} h_{11} &= -a \\ h_{12} &= h_{21} = 0 \\ h_{22} &= a \end{aligned}$$

According to (1.6) the mean curvature at every point is then

$$H = \frac{1}{2} \frac{h_{11}g_{22} - h_{12}g_{12} + h_{22}g_{11}}{g_{11}g_{22} - g_{12}^2} = \frac{1}{2} \frac{-a \cdot a^2 \cosh(x_2)^2 + a \cdot a^2 \cosh(x_2)^2}{(a^2 \cosh(x_2))^2} = 0$$

so the catenoid is a minimal surface.

Definition 2.7. [Normal variation of a surface] Let $\varphi : \Omega \to \mathbb{R}^3$ be a regular parametrized surface and let $D \subseteq \Omega$ be a bounded domain. Let $h : \overline{D} \to \mathbb{R}$ be a differentiable function and $N : \Omega \to \mathbb{S}^2$ be a unit normal map to $\varphi(\Omega)$. The normal variation of $\varphi\left(\overline{D}\right)$ determined by h is defined as the function

$$\begin{aligned} \overline{\varphi} : \overline{D} \times (-\varepsilon, \varepsilon) &\to \mathbb{R}^3 \\ (x_1, x_2, t) &\mapsto \overline{\varphi}(x_1, x_2, t) := \varphi(x_1, x_2) + th(x_1, x_2)N(x_1, x_2) \end{aligned}$$

Proposition 2.8. *Let $\varphi : \Omega \to \mathbb{R}^3$ be a regular parametrized surface, $D \subseteq \overline{D} \subseteq \Omega$ a bounded domain, $h : \overline{D} \to \mathbb{R}$ differentiable and N a unit normal map to $\varphi(\Omega)$. Consider the function $\varphi_t : D \to \mathbb{R}^3$, $\varphi_t(x_1, x_2) := \overline{\varphi}(x_1, x_2, t)$ and denote $\left(g_{ij}^t\right)$, $i, j = 1, 2$ the matrix of the first fundamental form of φ_t. Then,*

$$g_{11}^t g_{22}^t - \left(g_{12}^t\right)^2 = \left(g_{11}g_{22} - g_{12}^2\right)\left(1 - 4thH + R(t)\right) \tag{2.6}$$

for a function $R = R(t) = \mathcal{O}(t^2)$.

Proof. From the definition of $\overline{\varphi}$ we have

$$\begin{aligned} \partial_{x_1}\varphi_t(q) &= \partial_{x_1}\varphi + t\,\partial_{x_1}hN + th\,\partial_{x_1}N \\ \partial_{x_2}\varphi_t(q) &= \partial_{x_2}\varphi + t\,\partial_{x_2}hN + th\,\partial_{x_2}N \end{aligned} \tag{2.7}$$

A simple calculation shows that (1.1) is in this case:

$$\begin{aligned} g_{11}^t &= g_{11} + 2th\langle\partial_{x_1}\varphi, \partial_{x_1}N\rangle + t^2h^2\langle\partial_{x_1}N, \partial_{x_1}N\rangle + t^2(\partial_{x_1}h)^2 \\ g_{12}^t &= g_{12} + th\left(\langle\partial_{x_1}\varphi, \partial_{x_2}N\rangle + \langle\partial_{x_2}\varphi, \partial_{x_1}N\rangle\right) + t^2h^2\langle\partial_{x_1}N, \partial_{x_2}N\rangle + t^2\,\partial_{x_1}h\,\partial_{x_2}h \\ g_{22}^t &= g_{22} + 2th\langle\partial_{x_2}\varphi, \partial_{x_2}N\rangle + t^2h^2\langle\partial_{x_2}N, \partial_{x_2}N\rangle + t^2(\partial_{x_2}h)^2 \end{aligned}$$

Therefore, using (1.3) we have

$$\begin{aligned} g_{11}^t g_{22}^t &= g_{11}g_{22} - 2thg_{11}h_{22} - 2thg_{22}h_{11} + R_1(t) \\ \left(g_{12}^t\right)^2 &= g_{12}^2 - 4thg_{12}h_{12} + R_2(t) \end{aligned}$$

where R_1 and R_2 are functions such that $R_1(t) = \mathcal{O}(t^2)$ and $R_2(t) = \mathcal{O}(t^2)$.

Since φ is regular, $g_{11}g_{22} - g_{12}^2$ is nonzero. Hence, we can write $R(t) := (R_1(t) - R_2(t)) \,/\, \left(g_{11}g_{22} - g_{12}^2\right)$. We have $R(t) = \mathcal{O}(t^2)$ and

$$\begin{aligned} g_{11}^t g_{22}^t - \left(g_{12}^t\right)^2 &= g_{11}g_{22} - g_{12}^2 - 2th\left(g_{11}h_{22} - 2h_{12}g_{12} + g_{22}h_{11}\right) + R_1(t) - R_2(t) \\ &= \left(g_{11}g_{22} - g_{12}^2\right)\left(1 - 2th\frac{g_{11}h_{22} - 2h_{12}g_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2} + R(t)\right) \\ &= \left(g_{11}g_{22} - g_{12}^2\right)\left(1 - 4thH + R(t)\right) \end{aligned} \tag{2.8}$$

where in the last equality we have used (1.6):

$$H = \frac{1}{2}\frac{h_{11}g_{22} - h_{12}g_{12} + h_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}$$

□

Corollary 2.9. *Assume the same hypothesis than the previous proposition. Then, for $\varepsilon > 0$ small enough, for every $t \in (-\varepsilon, \varepsilon)$, the function $\varphi_t : D \to \mathbb{R}^3$ is a regular parametrized surface.*

Proof. Clearly from (2.6), for t small enough $g_{11}^t g_{22}^t - \left(g_{12}^t\right)^2 \neq 0$. So, by choosing an appropriate ε, Corollary 1.49 implies φ_t is regular. □

Remark 2.10. This can also be proven more easily in the following manner: note that at equation (2.7) for $t = 0$

$$\begin{aligned} \partial_{x_1}\varphi_{t=0}(q) &= \partial_{x_1}\varphi(q) \\ \partial_{x_2}\varphi_{t=0}(q) &= \partial_{x_2}\varphi(q) \end{aligned}$$

which are linearly independent because φ is regular. Therefore, since both $\partial_{x_1}\varphi_t(q)$ and $\partial_{x_2}\varphi_t(q)$ are continuous with respect to t, there exists a neighbourhood of $t = 0$ in which they are linearly independent.

Let's see why surfaces with null mean curvature are called minimal. Consider a regular parametrized surface $\varphi : \Omega \to \mathbb{R}^3$ and let $D \subseteq \overline{D} \subseteq \Omega$ be a bounded domain, $h : \overline{D} \to \mathbb{R}$ be differentiable and N be a unit normal map to $\varphi(\Omega)$. Using

(2.8) and choosing an adequate ε we can write the area of each $\varphi_t\left(\overline{D}\right)$ defined by the normal variation of φ determined by h as

$$\begin{aligned} A(t) := \text{area}\left(\varphi_t\left(\overline{D}\right)\right) &= \int_{\overline{D}} \sqrt{g_{11}^t g_{22}^t - \left(g_{12}^t\right)^2}\, dx_1 dx_2 \\ &= \int_{\overline{D}} \sqrt{g_{11}g_{22} - g_{12}^2}\sqrt{1 - 4thH + R(t)}\, dx_1 dx_2 \end{aligned}$$

Differentiating this expression yields

$$A'(t) = -\int_{\overline{D}} \sqrt{g_{11}g_{22} - g_{12}^2}\frac{2hH - R'(t)/2}{\sqrt{1 - 4thH + R(t)}}\, dx_1 dx_2$$

and therefore

$$A'(0) = -\int_{\overline{D}} 2hH\sqrt{g_{11}g_{22} - g_{12}^2}\, dx_1 dx_2$$

Thus, if $H = 0$ everywhere, $A'(0) = 0$ and $\varphi = \varphi_{t=0}$ is the surface among all normal variations of φ for which the area of $\varphi_t\left(\overline{D}\right)$ is a critical point. This critical point might not be a minimum, however, due to historical reasons, we say the surface is minimal. The following theorem states what we have seen in a more precise manner and also shows that the reciprocal is true as well.

Theorem 2.11. *Let $\varphi : \Omega \to \mathbb{R}^3$ be a regular parametrized surface and let $D \subseteq \overline{D} \subseteq \Omega$ be a bounded domain. Then, φ is minimal if, and only if, $A'(0) = 0$ for all sets D and all normal variations of φ.*

Proof. We have already proved that if φ is minimal then $A'(0) = 0$. Let's now prove the reciprocal. Assume for all normal variations h of φ it is satisfied

$$A'(0) = -\int_{\overline{D}} 2hH\sqrt{g_{11}g_{22} - g_{12}^2}\, dx_1 dx_2 = 0$$

Since this is satisfied for all h, in particular it is also satisfied for those h such that $h = 0$ on ∂D. Thus, from Lemma 2.3 with $f = 2H\sqrt{g_{11}g_{22} - g_{12}^2}$ and $h = h$ restricted on $\overline{D}$, it follows that $2H\sqrt{g_{11}g_{22} - g_{12}^2}$ is identically zero on $\overline{D}$. Since φ is regular, $\sqrt{g_{11}g_{22} - g_{12}^2} \neq 0$ and therefore H must be zero everywhere on $\overline{D}$. □

Remark 2.12. All surfaces which minimize area are also minimal and they are called minimizing minimal surfaces.

Chapter 3

Bernstein's theorem

An interesting kind of surfaces are those which are the graph of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. This chapter is devoted to studying some of the properties of these surfaces and to proving an important result concerning them which is Bernstein's theorem. We are going to follow [17] in the first part. The second part is focused on Bernstein's theorem and has [19] as its reference.

3.1 Graph surfaces

Definition 3.1. Let $f : \Omega \to \mathbb{R}$ be a differentiable function. The graph of f is defined as: $\operatorname{graph} f := \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in \Omega\}$

Definition 3.2. Let $f : \Omega \to \mathbb{R}$ be a differentiable function. The set $\{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in \Omega\} \subseteq \mathbb{R}^3$ is a surface. Surfaces defined in this way are called graph surfaces.

The function $\varphi : \Omega \to \mathbb{R}^3$ defined by $\varphi(x_1, x_2) := (x_1, x_2, f(x_1, x_2))$ is a parametrized surface fulfilling $\varphi(\Omega) = \operatorname{graph} f$.

Proposition 3.3. *The surface defined by the graph of a differentiable function $f : \Omega \to \mathbb{R}$ is regular.*

Proof. Consider the surface $\varphi : \Omega \to \mathbb{R}^3$, $\varphi(x_1, x_2) := (x_1, x_2, f(x_1, x_2))$. Differentiating at (x_1, x_2) we get

$$\partial_{x_1}\varphi = (1, 0, f_{x_1})$$
$$\partial_{x_2}\varphi = (0, 1, f_{x_2})$$

Clearly these two vectors are independent and φ is regular. $\square$

We now recall the inverse function theorem as it is used to prove the proposition following it.

Theorem 3.4 (Inverse function theorem). *Let $A \subseteq \mathbb{R}^n$ be an open set and $T : A \to \mathbb{R}^n$ be a $\mathscr{C}^k(A)$ function for some integer $k \geq 1$. If at some point $a \in A$ $\det D_a T \neq 0$ then there exists an open set $U \subseteq A$ such that*

- *$V := T(U)$ is an open set in $\mathbb{R}^n$*
- *$T : U \to V$ is bijective*
- *The inverse function T^{-1} is $\mathscr{C}^k(V)$ and $D_{T(a)}(T^{-1}) = (D_a T)^{-1}$*

Proposition 3.5. *Let $S \subseteq \mathbb{R}^3$ be a surface regular at a point $p \in S$. Then, there exists a neighbourhood $\Sigma \subseteq S$ of p such that Σ is a graph surface.*

Proof. Let $\varphi : \Omega \to \mathbb{R}^3$ be a local regular parametrization of S such that $p = \varphi(q)$ for some $q \in \Omega$. Write $\varphi(q) = (\varphi_1(q), \varphi_2(q), \varphi_3(q))$. Since $\partial_{x_1}\varphi(q)$, $\partial_{x_2}\varphi(q)$ are linearly independent, by Proposition 1.25 there exist $i, j \in \{1, 2, 3\}$, $i \neq j$, such that

$$\frac{\partial(\varphi_i,\ \varphi_j)}{\partial(x_1,\ x_2)}(q) \neq 0$$

Using the inverse function theorem, there exist a neighbourhood $\Delta \subseteq \Omega$ of q and a neighbourhood $A \subseteq \{(\varphi_i(r), \varphi_j(r)) : r \in \Omega\}$ of $(\varphi_i,\ \varphi_j)(q)$ such that the function $g := (\varphi_i,\ \varphi_j) : \Delta \to A$ is a diffeomorphism. Therefore, if $k \in \{1, 2, 3\}$, $k \neq i, j$, the composition of functions $f := \varphi_k \circ g^{-1} : A \to \mathbb{R}$ is a differentiable function such that its graph is a graph surface containing p. Finally, $\Sigma = \varphi\left(g^{-1}(A)\right)$. $\square$

Remark 3.6. This proposition tells us that regular surfaces can be seen locally as graphs of functions.

In what follows, to simplify the notation we write $f_{x_i} = \partial_{x_i} f$, $f_{x_i x_j} = \partial^2_{x_i x_j} f$.

Lemma 3.7. *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$, $\varphi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$ be the associated graph surface. Then,*

$$\begin{aligned} g_{11} &= 1 + (f_{x_1})^2 \\ g_{12} &= f_{x_1} f_{x_2} \\ g_{22} &= 1 + (f_{x_2})^2 \end{aligned}$$

Proof. For $(x_1, x_2) \in \Omega$, $\varphi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$. Then $\partial_{x_1}\varphi = (1, 0, f_{x_1})$ and $\partial_{x_2}\varphi = (0, 1, f_{x_2})$. A simple calculation then shows

$$
\begin{aligned}
g_{11} &= \langle \partial_{x_1}\varphi,\ \partial_{x_1}\varphi \rangle = 1 + (f_{x_1})^2 \\
g_{21} &= g_{12} = \langle \partial_{x_1}\varphi,\ \partial_{x_2}\varphi \rangle = \langle \partial_{x_2}\varphi,\ \partial_{x_1}\varphi \rangle = f_{x_1} f_{x_2} \\
g_{22} &= \langle \partial_{x_2}\varphi,\ \partial_{x_2}\varphi \rangle = 1 + (f_{x_2})^2
\end{aligned}
$$

□

Lemma 3.8. *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$, $\varphi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$ be the associated graph surface. Then,*

$$
\begin{aligned}
h_{11} &= \frac{f_{x_1x_1}}{\sqrt{1 + (f_{x_1})^2 + (f_{x_2})^2}} \\
h_{12} &= \frac{f_{x_1x_2}}{\sqrt{1 + (f_{x_1})^2 + (f_{x_2})^2}} \\
h_{22} &= \frac{f_{x_2x_2}}{\sqrt{1 + (f_{x_1})^2 + (f_{x_2})^2}}
\end{aligned}
$$

Proof. For $(x_1, x_2) \in \Omega$, $\varphi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$. Then

$$
\begin{aligned}
\partial_{x_1}\varphi &= (1, 0, f_{x_1}) \\
\partial_{x_2}\varphi &= (0, 1, f_{x_2}) \\
\partial_{x_1x_1}\varphi &= (0, 0, f_{x_1x_1}) \\
\partial_{x_1x_2}\varphi &= (0, 0, f_{x_1x_2}) \\
\partial_{x_2x_2}\varphi &= (0, 0, f_{x_2x_2})
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi, \partial_{x_1x_1}\varphi \rangle &= \begin{vmatrix} 1 & 0 & f_{x_1} \\ 0 & 1 & f_{x_2} \\ 0 & 0 & f_{x_1x_1} \end{vmatrix} = f_{x_1x_1} \\
\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi, \partial_{x_1x_2}\varphi \rangle &= \begin{vmatrix} 1 & 0 & f_{x_1} \\ 0 & 1 & f_{x_2} \\ 0 & 0 & f_{x_1x_2} \end{vmatrix} = f_{x_1x_2} \\
\langle \partial_{x_1}\varphi \times \partial_{x_2}\varphi, \partial_{x_2x_2}\varphi \rangle &= \begin{vmatrix} 1 & 0 & f_{x_1} \\ 0 & 1 & f_{x_2} \\ 0 & 0 & f_{x_2x_2} \end{vmatrix} = f_{x_2x_2}
\end{aligned}
$$

Using Lemma 3.7 and Proposition 1.48 we get

$$\|\partial_{x_1}\varphi \times \partial_{x_2}\varphi\| = \sqrt{g_{11}g_{22} - (g_{12})^2} = \sqrt{1 + (f_{x_1})^2 + (f_{x_2})^2}$$

Substituting at (1.2) we get the expressions stated. □

Proposition 3.9. *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$ be the associated graph surface. Then,*

$$H = \frac{1}{2}\frac{\left(1 + (f_{x_1})^2\right) f_{x_2x_2} - 2f_{x_1}\, f_{x_2}\, f_{x_1x_2} + \left(1 + (f_{x_2})^2\right) f_{x_1x_1}}{\left(1 + (f_{x_1})^2 + (f_{x_2})^2\right)^{3/2}} \tag{3.1}$$

Proof. For $(x_1, x_2) \in \Omega$, $\varphi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$. Substituting Lemmas 3.7 and 3.8 at (1.6) we readily arrive at the expression shown. □

Proposition 3.10. *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$ be the associated graph surface. Then, the mean curvature of φ is*

$$H = \frac{1}{2}\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right) \tag{3.2}$$

Proof. We have

$$\vec{\nabla} f = (f_{x_1}, f_{x_2})$$

and

$$\left(\vec{\nabla} f\right)^2 = (f_{x_1})^2 + (f_{x_2})^2$$

Therefore

$$\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right) = \frac{\partial}{\partial x_1}\left(\frac{f_{x_1}}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{f_{x_2}}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right)$$

The first term in this expression is

$$\begin{aligned}\frac{\partial}{\partial x_1}\left(\frac{f_{x_1}}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right) &= \frac{1}{1 + \left(\vec{\nabla} f\right)^2}\left(f_{x_1x_1}\sqrt{1 + \left(\vec{\nabla} f\right)^2} - \frac{(f_{x_1})^2 f_{x_1x_1} + f_{x_1}f_{x_2}f_{x_1x_2}}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right)\\ &= \frac{f_{x_1x_1}}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}} - \frac{(f_{x_1})^2 f_{x_1x_1} + f_{x_1}f_{x_2}f_{x_1x_2}}{\left(1 + \left(\vec{\nabla} f\right)^2\right)^{3/2}}\end{aligned}$$

Similarly, the second term is

$$\frac{\partial}{\partial x_2}\left(\frac{f_{x_2}}{\sqrt{1+\left(\vec{\nabla} f\right)^2}}\right) = \frac{1}{1+\left(\vec{\nabla} f\right)^2}\left(f_{x_2x_2}\sqrt{1+\left(\vec{\nabla} f\right)^2} - \frac{(f_{x_2})^2 f_{x_2x_2} + f_{x_1}f_{x_2}f_{x_1x_2}}{\sqrt{1+\left(\vec{\nabla} f\right)^2}}\right)$$
$$= \frac{f_{x_2x_2}}{\sqrt{1+\left(\vec{\nabla} f\right)^2}} - \frac{(f_{x_2})^2 f_{x_2x_2} + f_{x_1}f_{x_2}f_{x_1x_2}}{\left(1+\left(\vec{\nabla} f\right)^2\right)^{3/2}}$$

Therefore,

$$\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1+\left(\vec{\nabla} f\right)^2}}\right) = \frac{f_{x_1x_1}+f_{x_2x_2}}{\sqrt{1+\left(\vec{\nabla} f\right)^2}} - \frac{(f_{x_1})^2 f_{x_1x_1} + (f_{x_2})^2 f_{x_2x_2} + 2f_{x_1}f_{x_2}f_{x_1x_2}}{\left(1+\left(\vec{\nabla} f\right)^2\right)^{3/2}}$$
$$= \frac{(f_{x_1x_1}+f_{x_2x_2})\left(1+(f_{x_1})^2+(f_{x_2})^2\right) - (f_{x_1})^2 f_{x_1x_1} - (f_{x_2})^2 f_{x_2x_2} - 2f_{x_1}f_{x_2}f_{x_1x_2}}{\left(1+\left(\vec{\nabla} f\right)^2\right)^{3/2}}$$
$$= \frac{\left(1+(f_{x_1})^2\right) f_{x_2x_2} - 2f_{x_1}\, f_{x_2}\, f_{x_1x_2} + \left(1+(f_{x_2})^2\right) f_{x_1x_1}}{\left(1+(f_{x_1})^2+(f_{x_2})^2\right)^{3/2}}$$
$$= 2H$$

□

Proposition 3.11 (Minimal surface equation)**.** *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$ be the associated graph surface. Then, φ is minimal if, and only if, at all points it satisfies*

$$\left(1+(f_{x_1})^2\right) f_{x_2x_2} - 2f_{x_1}\, f_{x_2}\, f_{x_1x_2} + \left(1+(f_{x_2})^2\right) f_{x_1x_1} = 0$$

Proof. A surface is minimal if, and only if, its mean curvature H vanishes everywhere. Therefore, Proposition 3.9 implies this result. □

Corollary 3.12. *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$ be the associated graph surface. Then, φ is minimal if, and only if, at all points it satisfies*

$$\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1+\left(\vec{\nabla} f\right)^2}}\right) = 0$$

Proof. Immediate from (3.2) and Proposition 3.11.

For an alternative proof, see [12]. □

3.2 Jörgen's theorem

In this section our goal is to prove Jörgen's theorem, a result we will use in the next section to prove Bernstein's theorem.

First we introduce some definitions and results which will prove useful later.

Definition 3.13. Let $T : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a function. We say T is distance-increasing if for all $a, b \in A$

$$d(a, b) \leq d(T(a), T(b))$$

Proposition 3.14. *Let $T : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a function. If T is distance-increasing then it is injective.*

Proof. Suppose there exist $a, b \in A$ such that $T(a) = T(b)$. Then,

$$d(a, b) \leq d(T(a), T(b)) = 0$$

Therefore $a = b$ which concludes the proof. □

Lemma 3.15. *Let $T : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function. If T is distance-increasing then the image of T is a closed set.*

Proof. Consider a convergent sequence of points in the image of T $\{T(x_i, y_i)\} \subseteq T(A)$ with limit $\alpha \in \mathbb{R}^2$. This sequence is also a Cauchy sequence and therefore $\{(x_i, y_i)\} \subseteq \mathbb{R}^2$ is a Cauchy sequence because T is distance increasing. Since $\mathbb{R}^2$ is a complete space, $\{(x_i, y_i)\}$ is convergent with limit $\beta \in \mathbb{R}^2$. As T is continuous, $T(\beta) = \alpha$ so α belongs to the image of T and it is closed. □

We now recall the following well known theorems that will be instrumental in proving Jörgen's theorem.

Theorem 3.16 (Invariance of domain, [15]). *Let $T : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function on an open set U. If T is injective, then T is a homeomorphism between U and $T(U)$.*

Definition 3.17. [Cauchy-Riemann equations] Let $F = (u, v) : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function on an open set A. If F fulfills the following equations

$$u_{x_1} = v_{x_2}$$
$$u_{x_2} = -v_{x_1}$$

we say F fulfills the Cauchy-Riemann equations.

Proposition 3.18. *Let $F = (u, v) : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function on an open set A. Consider the complex function $F : A \subseteq \mathbb{C} \to \mathbb{C}$ given by $F(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2)$. If, and only if, F satisfies the Cauchy-Riemann equations at a point $a \in A$, then F is complex-differentiable at a.*

Theorem 3.19 (Liouville)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be a function complex-differentiable on the whole complex plane. If F is bounded, then it is constant.*

Theorem 3.20 (Jörgen's theorem)**.** *Let $\Xi : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathscr{C}^2$ function on the whole plane satisfying*

$$\det D^2\Xi = \Xi_{x_1x_1}\Xi_{x_2x_2} - \Xi^2_{x_1x_2} = 1 \tag{3.3}$$

Then Ξ is a quadratic polynomial in $x_1,\ x_2$.

This theorem is very simple to state and, at first glance, its hypothesis does not seem restrictive enough to imply the conclusion. If one tries to find counterexamples (obviously none will be found), one realizes quickly this hypothesis puts stricter limits on Ξ than were apparent initially.

Example 3.21. For instance, if we replace 1 by -1 in (3.3) so that it reads

$$\det D^2\Xi = \Xi_{x_1x_1}\Xi_{x_2x_2} - \Xi^2_{x_1x_2} = -1 \tag{3.4}$$

Jörgen's theorem is not true anymore. To see this, consider functions $g = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x_1, x_2) := (x_2, x_1 + G(x_2))$ for some differentiable function $G : \mathbb{R} \to \mathbb{R}$. If $g = D\Xi$, (3.4) becomes

$$\partial_{x_1} g_1\, \partial_{x_2} g_2 - \partial_{x_2} g_1{}^2 = -1 \tag{3.5}$$

Since $\partial_{x_1} g_1 = 0$ and $\partial_{x_2} g_1 = 1$, this equation is satisfied. It is easy now to find examples of functions G such that $D\Xi = g$, Ξ is defined on the whole plane but Ξ is not a polynomial. For instance, taking $G(x) = \sin x$ we have $g(x_1, x_2) = (x_2, x_1 + \sin x_2)$ and $\Xi(x_1, x_2) = x_1x_2 - \cos x_2$.

Example 3.22. Another natural attempt would be trying $\partial_{x_2} g_1 = 0$ so (3.3) becomes

$$\partial_{x_1} g_1\, \partial_{x_2} g_2 - \partial_{x_2} g_1{}^2 = \partial_{x_1} g_1\, \partial_{x_2} g_2 = 1 \tag{3.6}$$

From $\partial_{x_2} g_1 = 0$, $g_1(x_1, x_2) = c(x_1)$ for some differentiable function $c : \mathbb{R} \to \mathbb{R}$. Therefore, $\partial_{x_2} g_2 = 1/\, \partial_{x_1} g_1 = 1/c'(x_1)$ and consequently

$$g_2(x_1, x_2) = \frac{x_2}{c'(x_1)}$$

On the other hand, since $g = D\Xi$, $\Xi_{x_1}(x_1, x_2) = g_1(x_1, x_2) = c(x_1)$ and thus

$$\Xi(x_1, x_2) = C(x_1) + M(x_2)$$

where $C'(x_1) = c(x_1)$ and $M : \mathbb{R} \to \mathbb{R}$ is some differentiable function. It also has to be fulfilled that $g_2(x_1, x_2) = \Xi_{x_2}(x_1, x_2)$ so, writing $m(x_2) = M_{x_2}(x_2)$, we arrive at

$$g_2(x_1, x_2) = m(x_2)$$

Comparing the two expressions for g_2 we see it has to be satisfied that $c'(x_1)$ is a nonzero constant α and $m(x_2) = x_2$. Therefore, $C(x_1) = \frac{\alpha}{2}x_1^2$ and $M(x_2) = \frac{\beta}{2}x_2^2$. Finally,

$$\Xi(x_1, x_2) = \frac{\alpha}{2}x_1^2 + \frac{\beta}{2}x_2^2$$

So in this case it is inevitable that Ξ is a polynomial of degree 2 on x_1, x_2, as in the general case stated on the theorem.

Note we have omitted some constants for the sake of simplicity but the argument remains valid.

Let's now prove Jörgen's theorem.

Proof. The hypothesis

$$\Xi_{x_1x_1}\Xi_{x_2x_2} - \Xi^2_{x_1x_2} = 1$$

implies that $\Xi_{x_1x_1}\Xi_{x_2x_2} > 0$ so $\Xi_{x_1x_1}$ and $\Xi_{x_2x_2}$ have the same sign. Without loss of generality, we can assume

$$\Xi_{x_1x_1} > 0 \quad \text{and} \quad \Xi_{x_2x_2} > 0 \tag{3.7}$$

everywhere because if it it not the case, we can replace Ξ with $-\Xi$. Take two points on $\mathbb{R}^2$ $(\overline{x}_0, \overline{y}_0)$ and $(\overline{x}_1, \overline{y}_1)$ and define the function $h : \mathbb{R} \to \mathbb{R}^2$ as

$$h(t) = \Xi\left(\overline{x}_0 + t(\overline{x}_1 - \overline{x}_0), \overline{y}_0 + t(\overline{y}_1 - \overline{y}_0)\right)$$

Differentiating with respect to t yields

$$\begin{aligned} h'(t) &= (\overline{x}_1 - \overline{x}_0)\Xi_{x_1} + (\overline{y}_1 - \overline{y}_0)\Xi_{x_2} \\ h''(t) &= (\overline{x}_1 - \overline{x}_0)^2\Xi_{x_1x_1} + 2(\overline{x}_1 - \overline{x}_0)(\overline{y}_1 - \overline{y}_0)\Xi_{x_1x_2} + (\overline{y}_1 - \overline{y}_0)^2\Xi_{x_2x_2} \end{aligned}$$

where all derivatives of Ξ are evaluated at $(\overline{x}_0 + t(\overline{x}_1 - \overline{x}_0), \overline{y}_0 + t(\overline{y}_1 - \overline{y}_0))$. If $\overline{x}_1 = \overline{x}_0$, $h''(t) = (\overline{y}_1 - \overline{y}_0)^2\Xi_{x_2x_2} \geq 0$. If $\overline{x}_1 \neq \overline{x}_0$, then

$$h''(t) = (\overline{x}_1 - \overline{x}_0)^2\left(\Xi_{x_1x_1} - 2\Xi_{x_1x_2}\left(\frac{\overline{y}_1 - \overline{y}_0}{\overline{x}_1 - \overline{x}_0}\right) + \Xi_{x_2x_2}\left(\frac{\overline{y}_1 - \overline{y}_0}{\overline{x}_1 - \overline{x}_0}\right)^2\right)$$

The second factor is a quadratic polynomial in $(\overline{y}_1 - \overline{y}_0)/(\overline{x}_1 - \overline{x}_0)$ with discriminant

$$4\Xi_{x_1x_2}^2 - 4\Xi_{x_1x_1}\Xi_{x_2x_2} = 4\Xi_{x_1x_2}^2 - 4(1+\Xi_{x_1x_2}^2) = -4 < 0$$

by (3.3), so it never vanishes. We note now that this polynomial evaluated at $(\overline{y}_1 - \overline{y}_0)/(\overline{x}_1 - \overline{x}_0) = 0$ equals $\Xi_{x_1x_1} > 0$ which is always strictly positive by (3.7). Hence, since the polynomial is never zero and at a point it is strictly positive, it is strictly positive everywhere. Therefore, in any case $h''(t) \geq 0$ and consequently

$$h'(1) \geq h'(0)$$

Writing $p_i = \Xi_{x_1}(\overline{x}_i, \overline{y}_i)$ and $q_i = \Xi_{x_2}(\overline{x}_i, \overline{y}_i)$ for $i = 0, 1$, this is equivalent to

$$(\overline{x}_1 - \overline{x}_0)p_1 + (\overline{y}_1 - \overline{y}_0)q_1 \geq (\overline{x}_1 - \overline{x}_0)p_0 + (\overline{y}_1 - \overline{y}_0)q_0$$

which implies

$$(\overline{x}_1 - \overline{x}_0)(p_1 - p_0) + (\overline{y}_1 - \overline{y}_0)(q_1 - q_0) \geq 0 \tag{3.8}$$

Consider the so called transformation of Lewy $T = (\xi, \eta) : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{aligned} T(x_1, x_2) &:= (\xi(x_1, x_2), \eta(x_1, x_2)) \\ &:= (x_1 + \Xi_{x_1}(x_1, x_2), x_2 + \Xi_{x_2}(x_1, x_2)) \end{aligned}$$

Write now $\xi_i = \xi(\overline{x}_i, \overline{y}_i)$ and $\eta_i = \eta(\overline{x}_i, \overline{y}_i)$ for $i = 0, 1$. From equation (3.8) we deduce the following:

$$\begin{aligned} d\left(T(\overline{x}_0, \overline{y}_0), T(\overline{x}_1, \overline{y}_1)\right)^2 &= (\xi_1 - \xi_0)^2 + (\eta_1 - \eta_0)^2 \\ &= (\overline{x}_1 - \overline{x}_0 + p_1 - p_0)^2 + (\overline{y}_1 - \overline{y}_0 + q_1 - q_0)^2 \\ &= (\overline{x}_1 - \overline{x}_0)^2 + (p_1 - p_0)^2 + 2(\overline{x}_1 - \overline{x}_0)(p_1 - p_0) \\ &\quad + (\overline{y}_1 - \overline{y}_0)^2 + (q_1 - q_0)^2 + 2(\overline{y}_1 - \overline{y}_0)(q_1 - q_0) \\ &\geq (\overline{x}_1 - \overline{x}_0)^2 + (\overline{y}_1 - \overline{y}_0)^2 \\ &= d\left((\overline{x}_0, \overline{y}_0), (\overline{x}_1, \overline{y}_1)\right)^2 \end{aligned}$$

Thus, T is distance-increasing and, in particular, injective by Proposition 3.14.

The Jacobian of T is

$$\begin{pmatrix} \xi_{x_1} & \xi_{x_2} \\ \eta_{x_1} & \eta_{x_2} \end{pmatrix} = \begin{pmatrix} 1 + \Xi_{x_1x_1} & \Xi_{x_1x_2} \\ \Xi_{x_1x_2} & 1 + \Xi_{x_2x_2} \end{pmatrix}$$

and has determinant

$$\begin{aligned} (1 + \Xi_{x_1x_1})(1 + \Xi_{x_2x_2}) - \Xi_{x_1x_2}^2 &= 1 + \Xi_{x_1x_1} + \Xi_{x_2x_2} + \Xi_{x_1x_1}\Xi_{x_2x_2} - \Xi_{x_1x_2}^2 \\ &= 2 + \Xi_{x_1x_1} + \Xi_{x_2x_2} \geq 2 \end{aligned}$$

where we have used (3.3) in the last equality and (3.7) in the inequality. Therefore the differential of T is injective everywhere. Also, since T is injective and continuous, by the invariance of domain theorem (Theorem 3.16) its image is open. At the same time, however, according to Lemma 3.15 the image of T is closed. Hence, the image of T is the whole $\mathbb{R}^2$ and the inverse function theorem (Theorem 3.4) implies T is a diffeomorphism of $\mathbb{R}^2$ onto itself. Abusing of notation we write the inverse of T as $T^{-1}(\xi,\eta) = (x_1(\xi,\eta), x_2(\xi,\eta))$. It has Jacobian

$$\begin{aligned}
\begin{pmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_1}{\partial \eta} \\ \frac{\partial x_2}{\partial \xi} & \frac{\partial x_2}{\partial \eta} \end{pmatrix} &= \begin{pmatrix} 1+\Xi_{x_1x_1} & \Xi_{x_1x_2} \\ \Xi_{x_1x_2} & 1+\Xi_{x_2x_2} \end{pmatrix}^{-1} \\
&= \frac{1}{2+\Xi_{x_1x_1}+\Xi_{x_2x_2}} \begin{pmatrix} 1+\Xi_{x_2x_2} & -\Xi_{x_1x_2} \\ -\Xi_{x_1x_2} & 1+\Xi_{x_1x_1} \end{pmatrix}
\end{aligned}$$

Define now a function $F = (U,V) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned}
F(\xi,\eta) &= (U(\xi,\eta), V(\xi,\eta)) \\
&:= (x_1(\xi,\eta) - \Xi_{x_1}(x_1(\xi,\eta), x_2(\xi,\eta)), -x_2(\xi,\eta) + \Xi_{x_2}(x_1(\xi,\eta), x_2(\xi,\eta))) \\
&= (x_1 - \Xi_{x_1}, -x_2 + \Xi_{x_2})
\end{aligned}$$

Using the Jacobian of T^{-1} we have calculated and remembering (3.3) it is easy to see that

$$\begin{aligned}
\frac{\partial U}{\partial \xi} &= \frac{\partial x_1}{\partial \xi} - \frac{\partial \Xi_{x_1}}{x_1}\frac{\partial x_1}{\partial \xi} - \frac{\partial \Xi_{x_1}}{\partial x_2}\frac{\partial x_2}{\partial \xi} \\
&= \frac{(1+\Xi_{x_2x_2} - \Xi_{x_1x_1}(1+\Xi_{x_2x_2}) - \Xi_{x_1x_2}(-\Xi_{x_1x_2}))}{2+\Xi_{x_1x_1}+\Xi_{x_2x_2}} \\
&= \frac{\Xi_{x_2x_2} - \Xi_{x_1x_1}}{2+\Xi_{x_1x_1}+\Xi_{x_2x_2}} = \frac{\partial V}{\partial \eta}
\end{aligned}$$

and

$$\frac{\partial U}{\partial \eta} = -\frac{2\Xi_{x_1x_2}}{2+\Xi_{x_1x_1}+\Xi_{x_2x_2}} = -\frac{\partial V}{\partial \xi}$$

Therefore U and V satisfy the Cauchy-Riemann equations so the function $F : \mathbb{C} \to \mathbb{C}$ defined by

$$\begin{aligned}
F(\xi + i\eta) &= U(\xi,\eta) + iV(\xi,\eta) \\
&= x_1 - \Xi_{x_1} + i(-x_2 + \Xi_{x_2})
\end{aligned}$$

is complex-differentiable on the whole complex plane. Its complex derivative F' is

$$F'(\xi + i\eta) = \frac{\partial U}{\partial \xi} + i\frac{\partial V}{\partial \xi} = \frac{\Xi_{x_2x_2} - \Xi_{x_1x_1} + i2\Xi_{x_1x_2}}{2+\Xi_{x_1x_1}+\Xi_{x_2x_2}} \tag{3.9}$$

with modulus

$$
\begin{aligned}
|F'(\xi + i\eta)|^2 &= \frac{(\Xi_{x_2x_2} - \Xi_{x_1x_1})^2 + 4\Xi_{x_1x_2}^2}{(2 + \Xi_{x_1x_1} + \Xi_{x_2x_2})^2} \\
&= \frac{(\Xi_{x_2x_2} - \Xi_{x_1x_1})^2 + 4\Xi_{x_1x_1}\Xi_{x_2x_2} - 4}{(2 + \Xi_{x_1x_1} + \Xi_{x_2x_2})^2} \\
&= \frac{(\Xi_{x_1x_1} + \Xi_{x_2x_2})^2 - 4}{(2 + \Xi_{x_1x_1} + \Xi_{x_2x_2})^2} \\
&= \frac{\Xi_{x_1x_1} + \Xi_{x_2x_2} - 2}{\Xi_{x_1x_1} + \Xi_{x_2x_2} + 2} \\
&= \frac{\Xi_{x_1x_1} + \Xi_{x_2x_2} + 2}{\Xi_{x_1x_1} + \Xi_{x_2x_2} + 2} - \frac{4}{\Xi_{x_1x_1} + \Xi_{x_2x_2} + 2} \\
&= 1 - \frac{4}{\Xi_{x_1x_1} + \Xi_{x_2x_2} + 2}
\end{aligned} \tag{3.10}
$$

where we have used the hypothesis (3.3) at the second equality. From (3.7), $4/\left(\Xi_{x_1x_1} + \Xi_{x_2x_2} + 2\right) > 0$, so

$$|F'(\xi + i\eta)|^2 < 1$$

Hence F' is bounded and, by Liouville's theorem (Theorem 3.19), it is constant. Rearranging (3.10) we get

$$2 + \Xi_{x_1x_1} + \Xi_{x_2x_2} = \frac{4}{1 - |F'|^2} \tag{3.11}$$

Writing *Re* and *Im* to represent the real and imaginary parts of a complex quantity, respectively, we can now solve for $\Xi_{x_1x_2}$. From (3.9) we get

$$ReF' = \frac{\Xi_{x_2x_2} - \Xi_{x_1x_1}}{2 + \Xi_{x_1x_1} + \Xi_{x_2x_2}} \tag{3.12}$$

$$ImF' = \frac{2\Xi_{x_1x_2}}{2 + \Xi_{x_1x_1} + \Xi_{x_2x_2}} \tag{3.13}$$

From this and using (3.11)

$$\Xi_{x_1x_2} = \frac{2 + \Xi_{x_1x_1} + \Xi_{x_2x_2}}{2} ImF' = \frac{2ImF'}{1 - |F'|^2} \tag{3.14}$$

To solve for $\Xi_{x_2x_2} - \Xi_{x_1x_1}$ we use (3.12) and (3.11) to get

$$\Xi_{x_2x_2} - \Xi_{x_1x_1} = \frac{4ReF'}{1 - |F'|^2}$$

We can solve for $\Xi_{x_2x_2} + \Xi_{x_1x_1}$ using (3.14) and we find

$$\Xi_{x_2x_2} + \Xi_{x_1x_1} = \frac{4}{1 - |F'|^2} - 2$$

Adding and subtracting the last two expressions we finally obtain

$$\Xi_{x_1x_1} = \frac{1}{2}\left(\frac{4}{1-|F'|^2} - \frac{4ReF'}{1-|F'|^2} - 2\right)$$
$$\Xi_{x_2x_2} = \frac{1}{2}\left(\frac{4ReF'}{1-|F'|^2} + \frac{4}{1-|F'|^2} - 2\right)$$

To conclude, since F' is constant, $\Xi_{x_1x_1}$, $\Xi_{x_1x_2}$ and $\Xi_{x_2x_2}$ are constants as well so Ξ is a quadratic polynomial in $x_1,\ x_2$. □

3.3 Bernstein's theorem

Minimal surfaces can be characterised by a partial differential equation. This kind of equations are notoriously hard to solve and oftentimes the solution is only defined in a subset of $\mathbb{R}^n$. A somewhat natural question one might ask then is what *shape* does a minimal surface defined on the whole plane have. We are now ready to prove Bernstein's theorem which gives the answer to this question when this minimal surface is a graph.

Lemma 3.23 (Poincaré's lemma, [14])**.** *Let $h = (h_1, ..., h_n) : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function defined on a open star-shaped set A. If h is closed, meaning*

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i} \qquad \forall i, j \in \{1, ..., n\}$$

then h is conservative, that is, there exists a function $U : A \to \mathbb{R}$ such that $U \in \mathscr{C}^2(A)$ and $h = \vec{\nabla} U$.

Proof. Without loss of generality, assume $0 \in A$. As A is star-shaped, we can also suppose 0 is its center and then we can define the function U for every $x \in A$ as

$$U(x) := \int_0^1 \langle h(tx), x\rangle \, dt$$

Taking the partial derivative of U with respect to any of its variables x_j we obtain

$$\frac{\partial U}{\partial x_j} = \int_0^1 \frac{\partial}{\partial x_j}\langle h(tx), x\rangle \, dt$$

Calculating the integrand yields

$$\begin{aligned}\frac{\partial}{\partial x_j}\langle h(tx), x\rangle &= \frac{\partial}{\partial x_j}\sum_{i=1}^n h_i(tx)x_i = h_j(tx) + \sum_{i=1}^n \frac{\partial h_i}{\partial x_j}(tx)tx_i \\ &= h_j(tx) + \sum_{i=1}^n \frac{\partial h_j}{\partial x_i}(tx)tx_i = \frac{d}{dt}\left[th_j(tx)\right]\end{aligned}$$

where in the second to last equality we have used that h is closed.

Therefore,

$$\frac{\partial U}{\partial x_j} = \int_0^1 \frac{d}{dt}\left[th_j(tx)\right]\, dt = \left[th_j(tx)\right]_{t=0}^{t=1} = h_j(tx)$$

Since this is true for all variables x_j of U, we finally arrive at $h = \vec{\nabla} U$. □

Remark 3.24. The converse of this lemma is easily proved: let $U : A \to \mathbb{R}$ such that $U \in \mathscr{C}^2(A)$ and $h = \vec{\nabla} U$. Then, using Schwarz's theorem we have

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial^2 U}{\partial x_j \partial x_i} = \frac{\partial^2 U}{\partial x_i \partial x_j} = \frac{\partial h_j}{\partial x_i}$$

Theorem 3.25 (Bernstein's theorem). *Planes are the only minimal surfaces in $\mathbb{R}^3$ which are the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$.*

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^3$ be the associated graph surface. If φ is minimal, according to Proposition 3.11 it satisfies:

$$\left(1 + (f_{x_1})^2\right) f_{x_2x_2} - 2 f_{x_1} f_{x_2} f_{x_1x_2} + \left(1 + (f_{x_2})^2\right) f_{x_1x_1} = 0 \tag{3.15}$$

Let $W = \sqrt{1 + (f_{x_1})^2 + (f_{x_2})^2}$. The following equations hold:

$$\frac{\partial}{\partial x_1}\left(\frac{1 + (f_{x_2})^2}{W}\right) - \frac{\partial}{\partial x_2}\left(\frac{f_{x_1} f_{x_2}}{W}\right) = 0 \tag{3.16}$$

$$\frac{\partial}{\partial x_1}\left(\frac{f_{x_1} f_{x_2}}{W}\right) - \frac{\partial}{\partial x_2}\left(\frac{1 + (f_{x_1})^2}{W}\right) = 0 \tag{3.17}$$

Let's prove the first one, the second one is done analogously. First observe that

$$\begin{aligned} \frac{\partial W}{\partial x_1} &= \frac{f_{x_1} f_{x_1x_1} + f_{x_2} f_{x_1x_2}}{W} \\ \frac{\partial W}{\partial x_2} &= \frac{f_{x_1} f_{x_1x_2} + f_{x_2} f_{x_2x_2}}{W} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x_1}\left(\frac{1 + (f_{x_2})^2}{W}\right) &= \frac{2 f_{x_2} f_{x_1x_2} W - (1 + (f_{x_2})^2)(f_{x_1} f_{x_1x_1} + f_{x_2} f_{x_1x_2})/W}{W^2} \\ &= \frac{1}{W^3}\Big(2 f_{x_2} f_{x_1x_2} W^2 - (1 + (f_{x_2})^2)(f_{x_1} f_{x_1x_1} + f_{x_2} f_{x_1x_2})\Big) \end{aligned}$$

and

$$\frac{\partial}{\partial x_2}\left(\frac{f_{x_1}f_{x_2}}{W}\right) = \frac{(f_{x_1x_2}f_{x_2} + f_{x_1}f_{x_2x_2})W - f_{x_1}f_{x_2}(f_{x_1}f_{x_1x_2} + f_{x_2}f_{x_2x_2})/W}{W^2}$$
$$= \frac{1}{W^3}\Big((f_{x_1x_2}f_{x_2} + f_{x_1}f_{x_2x_2})W^2 - f_{x_1}f_{x_2}(f_{x_1}f_{x_1x_2} + f_{x_2}f_{x_2x_2})\Big)$$

Hence,

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\left(\frac{1+(f_{x_2})^2}{W}\right) - \frac{\partial}{\partial x_2}\left(\frac{f_{x_1}f_{x_2}}{W}\right) = \\
&= \frac{1}{W^3}\Big(2f_{x_2}f_{x_1x_2}W^2 - (1+(f_{x_2})^2)(f_{x_1}f_{x_1x_1} + f_{x_2}f_{x_1x_2})\Big) \\
&\qquad - \frac{1}{W^3}\Big((f_{x_1x_2}f_{x_2} + f_{x_1}f_{x_2x_2})W^2 - f_{x_1}f_{x_2}(f_{x_1}f_{x_1x_2} + f_{x_2}f_{x_2x_2})\Big) \\
&= \frac{1}{W^3}\Big(2f_{x_2}f_{x_1x_2}(1+(f_{x_1})^2 + (f_{x_2})^2) \\
&\qquad - (1+(f_{x_2})^2)(f_{x_1}f_{x_1x_1} + f_{x_2}f_{x_1x_2}) \\
&\qquad - (f_{x_1x_2}f_{x_2} + f_{x_1}f_{x_2x_2})(1+(f_{x_1})^2 + (f_{x_2})^2) \\
&\qquad + f_{x_1}f_{x_2}(f_{x_1}f_{x_1x_2} + f_{x_2}f_{x_2x_2})\Big) \\
&= \frac{1}{W^3}\Big(2f_{x_2}f_{x_1x_2} + 2(f_{x_1})^2f_{x_2}f_{x_1x_2} + 2(f_{x_2})^3f_{x_1x_2} - f_{x_1}f_{x_1x_1} - f_{x_2}f_{x_1x_2} \\
&\qquad - f_{x_1}(f_{x_2})^2f_{x_1x_1} - (f_{x_2})^3f_{x_1x_2} - f_{x_2}f_{x_1x_2} - (f_{x_1})^2f_{x_2}f_{x_1x_2} - (f_{x_2})^3f_{x_1x_2} \\
&\qquad - f_{x_1}f_{x_2x_2} - (f_{x_1})^3f_{x_2x_2} - f_{x_1}(f_{x_2})^2f_{x_2x_2} + (f_{x_1})^2f_{x_2}f_{x_1x_2} + f_{x_1}(f_{x_2})^2f_{x_2x_2}\Big) \\
&= \frac{1}{W^3}\Big(2(f_{x_1})^2f_{x_2}f_{x_1x_2} - f_{x_1}f_{x_1x_1} - f_{x_1}(f_{x_2})^2f_{x_1x_1} - f_{x_1}f_{x_2x_2} - (f_{x_1})^3f_{x_2x_2}\Big) \\
&= -\frac{f_{x_1}}{W^3}\Big(-2f_{x_1}f_{x_2}f_{x_1x_2} + f_{x_1x_1} + (f_{x_2})^2f_{x_1x_1} + f_{x_2x_2} + (f_{x_1})^2f_{x_2x_2}\Big)
\end{aligned}$$

$$= -\frac{f_{x_1}}{W^3}\Big((1+(f_{x_1})^2)f_{x_2x_2} - 2f_{x_1}f_{x_2}f_{x_1x_2} + (1+(f_{x_2})^2)f_{x_1x_1}\Big) = 0$$

where the last expression equals zero because of (3.15)

Now, defining $h : \mathbb{R}^2 \to \mathbb{R}^2$ as $h := \left(\frac{1+(f_{x_1})^2}{W}, \frac{f_{x_1}f_{x_2}}{W}\right)$, since h satisfies (3.17) and $\mathbb{R}^2$ is convex, Poincaré's lemma implies there exists a function $\alpha : \mathbb{R}^2 \to \mathbb{R}$,

$\alpha \in \mathscr{C}^2(A)$ such that

$$\alpha_{x_1} = \frac{1 + (f_{x_1})^2}{W} \tag{3.18}$$

$$\alpha_{x_2} = \frac{f_{x_1} f_{x_2}}{W} \tag{3.19}$$

Similarly, using (3.16) we see there exists $\beta : \mathbb{R}^2 \to \mathbb{R}$, $\beta \in \mathscr{C}^2(A)$ such that

$$\beta_{x_1} = \frac{f_{x_1} f_{x_2}}{W} \tag{3.20}$$

$$\beta_{x_2} = \frac{1 + (f_{x_2})^2}{W} \tag{3.21}$$

Repeating this argument once more but now taking the function h in Poincaré's lemma as $h = (\alpha, \beta)$ and considering equations (3.19) and (3.20), this lemma implies there exists a function $\Xi : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\begin{aligned} \Xi_{x_1} &= \alpha \\ \Xi_{x_2} &= \beta \end{aligned}$$

Thus, from equations (3.18) and (3.21) we get

$$\Xi_{x_1 x_1} = \alpha_{x_1} = \frac{1 + (f_{x_1})^2}{W} \tag{3.22}$$

$$\Xi_{x_1 x_2} = \alpha_{x_2} = \beta_{x_1} = \frac{f_{x_1} f_{x_2}}{W} \tag{3.23}$$

$$\Xi_{x_2 x_2} = \beta_{x_2} = \frac{1 + (f_{x_2})^2}{W} \tag{3.24}$$

which implies that

$$\begin{aligned} \Xi_{x_1 x_1} \Xi_{x_2 x_2} - \Xi_{x_1 x_2}^2 &= \frac{\left(1 + (f_{x_1})^2\right)\left(1 + (f_{x_2})^2\right)}{W^2} - \frac{(f_{x_1} f_{x_2})^2}{W^2} \\ &= \frac{1 + (f_{x_1})^2 + (f_{x_2})^2 + (f_{x_1} f_{x_2})^2 - (f_{x_1} f_{x_2})^2}{1 + (f_{x_1})^2 + (f_{x_2})^2} \\ &= 1 \end{aligned}$$

Finally, Jörgen's theorem implies that $\Xi_{x_1 x_1}$, $\Xi_{x_1 x_2}$, $\Xi_{x_2 x_2}$ are constants.

Our goal now is to prove that f_{x_1}, f_{x_2} are constants.

First, since (3.22) and (3.24) are constant, their addition is also constant for some constant we call $c \in \mathbb{R}$. Therefore

$$c = \frac{1 + (f_{x_1})^2}{W} + \frac{1 + (f_{x_2})^2}{W} = \frac{1 + (f_{x_1})^2 + 1 + (f_{x_2})^2}{W} = \frac{W^2 + 1}{W} = W + \frac{1}{W}$$

From this, $W^2 - cW + 1 = 0$, therefore $W = \left(c \pm \sqrt{c^2 - 4}\right)/2$ is constant (W exists by definition so we do not have to worry about the existence of real solutions to the square root in this last expression).

Second, since $W = \sqrt{1 + (f_{x_1})^2 + (f_{x_2})^2} = c$, then $W^2 = 1 + (f_{x_1})^2 + (f_{x_2})^2 = c^2$ so

$$(f_{x_2})^2 = c^2 - 1 - (f_{x_1})^2 \tag{3.25}$$

Now, (3.23) and W being constant, imply $(f_{x_1})^2(f_{x_2})^2 = A$ is constant for some number $A \in \mathbb{R}$. Therefore, using (3.25) we get

$$\begin{aligned} A =&(f_{x_1})^2(f_{x_2})^2 = (f_{x_1})^2(c^2 - 1 - (f_{x_1})^2) = -(f_{x_1})^4 + \left(c^2 - 1\right)(f_{x_1})^2 \\ \implies& (f_{x_1})^4 + \left(1 - c^2\right)(f_{x_1})^2 + A = 0 \\ \implies& (f_{x_1})^2 = \frac{-(1 - c^2) \pm \sqrt{(1 - c^2)^2 - 4A}}{2} \end{aligned}$$

Consequently, $(f_{x_1})^2$ is constant and so is f_{x_1}. From (3.25) we deduce f_{x_2} is constant.

Lastly, f_{x_1} and f_{x_2} being constant implies $f(x_1, x_2) = C_1x_1 + C_2x_2 + C_3$ for some $C_1, C_2, C_3 \in \mathbb{R}$ so the graph of f, the surface φ, is a plane.

$\square$

Chapter 4

Higher dimensions

In this chapter we are going to discuss how one can generalize the concepts introduced to higher dimensions. We are also going to show an important example of Minimal Surface in higher dimensions, namely the cone.

4.1 Basic concepts

Let's begin by extending some of the concepts reviewed in the first chapter to arbitrary dimensions. Now we denote by Ω an open set in $\mathbb{R}^{n-1}$.

Definition 4.1. A parametrized surface is defined as a differentiable function $\varphi : \Omega \to \mathbb{R}^n$.

Definition 4.2. A parametrized surface $\varphi : \Omega \to \mathbb{R}^n$ is called regular at a point $q = (x_1, ..., x_{n-1}) \in \Omega$ if vectors $\{\partial_{\mathrm{x_i}}\varphi(q) : i = 1, ..., n-1\}$ are linearly independent. We say that φ is regular if it is regular at every point $q \in \Omega$.

Remark 4.3. [Graph surfaces] Given a differentiable function $f : \Omega \to \mathbb{R}$, we can define a parametrized surface as $\varphi : \Omega \to \mathbb{R}^n$,

$$\varphi(x_1, .., x_{n-1}) := (x_1, ..., x_{n-1},\ f(x_1, ..., x_{n-1}))$$

This surface is regular because $\forall q \in \Omega$ and $\forall i = 1, ..., n-1$, vectors $\partial_{\mathrm{x_i}}\varphi(q) = (0, ..., 0\ , 1\ , 0, ..., 0,\ \partial_{\mathrm{x_i}} f(q))$, where the 1 is at position i, are linearly independent. Such parametrized surfaces are called graph surfaces.

Definition 4.4. Let $S \subseteq \mathbb{R}^n$. A local regular parametrization of S is defined as a differentiable function $\varphi : \Omega \to S$ such that:

- φ is a regular parametrized surface.

- $\varphi(\Omega) \subseteq S$ is an open set in S.
- $\varphi : \Omega \to \varphi(\Omega)$ is a homeomorfism.

Definition 4.5. A set $S \subseteq \mathbb{R}^n$ is called a regular surface at a point $p \in S$ if there exist a local regular parametrization of S $\varphi : \Omega \to S$ and a point $q \in \Omega$ such that $\varphi(q) = p$. S is said to be regular if it is regular at every point $p \in S$.

Definition 4.6. Let $S \subseteq \mathbb{R}^n$ be a regular surface and let p be a point in S. Let $\varphi : \Omega \to S$ be a local regular parametrization of S such that for some $q \in \Omega$, $p = \varphi(q)$. The tangent space to S at p is defined as:

$$T_pS := \{ \lambda_1 \partial_{\mathrm{x_1}} \varphi(q) + ... + \lambda_{n-1} \partial_{\mathrm{x_{n-1}}} \varphi(q) : \lambda_1, ..., \lambda_{n-1} \in \mathbb{R} \}$$

Definition 4.7. Let $S \subseteq \mathbb{R}^n$ be a regular surface and $p \in S$ a point in S. We define the normal space to S at p as the orthogonal complement of T_pS in $\mathbb{R}^n$ and denote it by $T_pS^\perp$.

A unit vector $N \in T_pS^\perp$ is called a normal vector to S at p.

Definition 4.8. [Gauss map] Let $S \subseteq \mathbb{R}^n$ be a regular surface. A Gauss map on S is defined as a continuous function $N : S \to \mathbb{S}^{n-1}$ such that at every point $p \in S$, $N(p)^\perp = T_pS$. Equivalently, $N(p)$ is a normal vector to S at every point p in S.

Remark 4.9. The differential of the Gauss map N on a point p on a surface S, $d_pN : T_pS \to T_pS$, is defined analogously to the case $n = 3$.

Definition 4.10. [First fundamental form] Let $S \subseteq \mathbb{R}^n$ be a regular surface, $p \in S$ a point in S and denote the usual dot product in $\mathbb{R}^n$ as $\langle \cdot, \cdot \rangle$. The first fundamental form of the surface S at p is defined as:

$$\begin{aligned} I_p : T_pS \times T_pS &\to \mathbb{R} \\ (w_1, w_2) &\mapsto I_p(w_1, w_2) := \langle w_1, w_2 \rangle \end{aligned}$$

Remark 4.11. Similarly to Remark 1.47, the first fundamental form at a point p on a surface S is a positive-definite bilinear symmetric function. Therefore, we can express it as a matrix $g(p) := \big(g_{ij}(p)\big)$, where $1 \leq i, j \leq n-1$, in the basis $\{\partial_{\mathrm{x_i}} \varphi(q) : i = 1, ..., n-1\}$ where $\varphi : \Omega \to S$ is a local regular parametrization of S such that $\varphi(q) = p$ for some $q \in \Omega$.

Definition 4.12. [Second fundamental form] Let $S \subseteq \mathbb{R}^n$ be a regular surface, $p \in S$ a point in S and let $N : S \to \mathbb{S}^{n-1}$ be a Gauss map on S. The second fundamental form of the surface S at p is defined as:

$$\begin{aligned} II_p : T_pS \times T_pS &\to \mathbb{R} \\ (w_1, w_2) &\mapsto II_p(w_1, w_2) := -I_p(d_pN(w_1), w_2) \end{aligned}$$

Remark 4.13. Since the first fundamental form and d_pN are linear, the second fundamental form is bilinear. Therefore, we can express the second fundamental form as a matrix $h(p) := (h_{ij}(p))$, where $1 \leq i, j \leq n-1$, in the basis of S $\{\partial_{x_i}\varphi(q) : i = 1, ..., n-1\}$ where $\varphi : \Omega \to S$ is a local regular parametrization of S such that $p = \varphi(q)$ for some $q \in \Omega$.

Proposition 4.14. *Let $S \subseteq \mathbb{R}^n$ be a regular surface, $p \in S$ a point in S and let $N : S \to \mathbb{S}^{n-1}$ be a Gauss map on S. The differential of the Gauss map d_pN is a self-adjoint linear map for all p in S. That is $\forall w_1, w_2 \in T_pS$, $\langle d_pN(w_1),\ w_2\rangle = \langle w_1,\ d_pN(w_2)\rangle$.*

Remark 4.15. This implies that there exist $n-1$ real eigenvalues of d_pN, each with a corresponding eigenvector. This eigenvectors are orthogonal and are called principal directions of curvature of S at p.

Definition 4.16. [Mean curvature] Let $S \subseteq \mathbb{R}^n$ be a regular surface, $p \in S$ a point in S and let $N : S \to \mathbb{S}^{n-1}$ be a Gauss map on S. The mean curvature of S at p is defined as

$$H(p, N) := \frac{1}{n-1} \operatorname{tr} d_pN \tag{4.1}$$

Proposition 4.17 (See for instance [5]). *Let $f : \Omega \to \mathbb{R}$ be a differentiable function and $\varphi : \Omega \to \mathbb{R}^n$ be the associated graph surface. Then, the mean curvature of φ is*

$$H = \frac{1}{n-1} \operatorname{div} \left(\frac{\vec{\nabla} f}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}} \right) \tag{4.2}$$

Definition 4.18. [Area] Let $\varphi : \Omega \to \mathbb{R}^n$ be a parametrized surface, $D \subseteq \Omega$ be a bounded set with closure $\overline{D} \subseteq \Omega$ and write $p = \varphi(x_1, ..., x_{n-1})$. The area of $\varphi(\overline{D})$ is defined as

$$\operatorname{area}(\varphi(\overline{D})) := \int_{\overline{D}} \sqrt{\det\left(g(p)\right)}\, dx_1 ... dx_{n-1}$$

where $g(p)$ is the matrix of the first fundamental form of $\varphi(\Omega)$ at p in the basis $\{\partial_{x_i}\varphi(x_1, ..., x_{n-1}) : i = 1, ..., n-1\}$

Remark 4.19. The definition of area does not depend on the parametrization φ.

Plateau's problem of finding the surface of smallest area among the surfaces having a certain curve as its perimeter can be extended to n dimensional space with the following definition:

Definition 4.20. [Area minimizing surface] Let $S \subseteq \mathbb{R}^n$ be a regular surface. We say S is area minimizing if for all open bounded sets $U \subseteq \mathbb{R}^n$ and for all regular surfaces $R \subseteq \mathbb{R}^n$ such that $S = R$ in $\mathbb{R}^n \setminus U$, then

$$\operatorname{area}(R \cap U) \geq \operatorname{area}(S \cap U)$$

Remark 4.21. As in the three dimensional case, surfaces in $\mathbb{R}^n$ which are area minimizing are minimal (i.e. $H = 0$ everywhere). The converse is not true, exactly as in three dimensions.

4.2 The cone

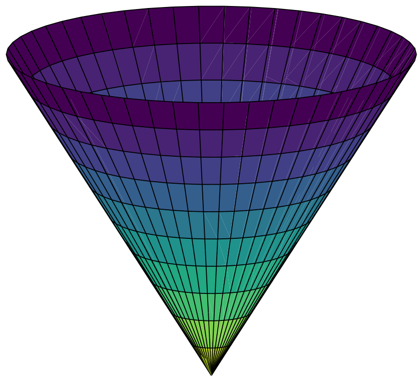

Figure 4.1: Upper half of a cone in 3 dimensions

Example 4.22. Let's calculate the mean curvature of a cone in three dimensional space. The cone is a surface given by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2\} \setminus \{(0,0,0)\}$. We have removed the point $(0,0,0)$ so that this surface is regular. Define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ as $f(x_1, x_2) := \sqrt{x_1^2 + x_2^2}$. The graph of f is a graph surface corresponding to the upper half of the cone. To calculate the mean curvature of the cone we use Proposition 3.10:

$$H = \frac{1}{2} \operatorname{div} \left(\frac{\vec{\nabla} f}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}} \right)$$

Writing $S = x_1^2 + x_2^2 = f(x_1, x_2)^2$ we have

$$\vec{\nabla} f = \left(\frac{x_1}{\sqrt{S}}, \frac{x_2}{\sqrt{S}} \right)$$

and

$$\left(\vec{\nabla} f\right)^2 = \frac{x_1^2}{S} + \frac{x_2^2}{S} = 1$$

Therefore $\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1+\left(\vec{\nabla} f\right)^2}}\right) = \operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \operatorname{div}\left(\vec{\nabla} f\right)$ and

$$\begin{aligned}\operatorname{div}\left(\vec{\nabla} f\right) &= \frac{\partial}{\partial x_1}\left(\frac{x_1}{\sqrt{S}}\right) + \frac{\partial}{\partial x_2}\left(\frac{x_2}{\sqrt{S}}\right) = \frac{1}{S}\left(\sqrt{S} - x_1 \frac{x_1}{\sqrt{S}}\right) + \frac{1}{S}\left(\sqrt{S} - x_2 \frac{x_2}{\sqrt{S}}\right) \\ &= \frac{S - x_1^2}{S^{3/2}} + \frac{S - x_2^2}{S^{3/2}} = \frac{x_1^2 + x_2^2}{(x_1^2 + x_2^2)^{3/2}} = \frac{1}{\sqrt{x_1^2 + x_2^2}} \neq 0\end{aligned}$$

Hence $H \neq 0$ and the cone in three dimensions is not a minimal surface.

For the lower half of the cone the calculations are done analogously picking $f(x_1, x_2) := -\sqrt{x_1^2 + x_2^2}$.

Example 4.23. Similarly to the previous example, consider now a cone in four dimensional space $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2\} \setminus \{(0,0,0,0)\}$. Define $f : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ as $f(x_1, x_2, x_3) := \sqrt{x_1^2 + x_2^2 - x_3^2}$. Again, the graph of f is a parametrization of a portion of the cone and its mean curvature is given by (4.2). Writing $S = x_1^2 + x_2^2 - x_3^2 = f(x_1, x_2, x_3)^2$ and $W = x_1^2 + x_2^2 + x_3^2$, we have

$$\vec{\nabla} f = \left(\frac{x_1}{\sqrt{S}}, \frac{x_2}{\sqrt{S}}, \frac{-x_3}{\sqrt{S}}\right)$$

and

$$\left(\vec{\nabla} f\right)^2 = \frac{x_1^2}{S} + \frac{x_2^2}{S} + \frac{x_3^2}{S} = \frac{W}{S}$$

Therefore

$$\sqrt{1 + \left(\vec{\nabla} f\right)^2} = \sqrt{1 + \frac{W}{S}} = \frac{1}{\sqrt{S}}\sqrt{S + W} = \frac{1}{\sqrt{S}}\sqrt{2}\sqrt{x_1^2 + x_2^2}$$

Consequently

$$\begin{aligned}\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1 + \left(\vec{\nabla} f\right)^2}}\right) &= \operatorname{div}\left(\frac{1}{\frac{1}{\sqrt{S}}\sqrt{2}\sqrt{x_1^2 + x_2^2}}\left(\frac{x_1}{\sqrt{S}}, \frac{x_2}{\sqrt{S}}, \frac{-x_3}{\sqrt{S}}\right)\right) \qquad (4.3) \\ &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1}\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right) - \frac{\partial}{\partial x_3}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2}}\right)\right)\end{aligned}$$

We have

$$\frac{\partial}{\partial x_1}\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right) = \frac{x_2^2}{\left(x_1^2+x_2^2\right)^{3/2}}$$
$$\frac{\partial}{\partial x_2}\left(\frac{x_2}{\sqrt{x_1^2+x_2^2}}\right) = \frac{x_1^2}{\left(x_1^2+x_2^2\right)^{3/2}}$$
$$\frac{\partial}{\partial x_3}\left(\frac{x_3}{\sqrt{x_1^2+x_2^2}}\right) = \frac{1}{\sqrt{x_1^2+x_2^2}}$$

So (4.3) becomes

$$\begin{aligned}\operatorname{div}\left(\frac{\vec{\nabla} f}{\sqrt{1+\left(\vec{\nabla} f\right)^2}}\right) &= \frac{1}{\sqrt{2}}\left(\frac{x_2^2}{\left(x_1^2+x_2^2\right)^{3/2}} + \frac{x_1^2}{\left(x_1^2+x_2^2\right)^{3/2}} - \frac{1}{\sqrt{x_1^2+x_2^2}}\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{x_1^2+x_2^2}{\left(x_1^2+x_2^2\right)^{3/2}} - \frac{x_1^2+x_2^2}{\left(x_1^2+x_2^2\right)^{3/2}}\right) \\ &= 0\end{aligned}$$

Thus, from (4.2), $H = 0$ everywhere and the cone in four dimensions is a minimal surface.

Remark 4.24. This is an interesting example because while the cone in $\mathbb{R}^3$ is neither minimal nor area minimizing, the cone in $\mathbb{R}^4$ is minimal but it is not area minimizing either (this is much harder to prove and it is beyond the scope of this project, see [2] for a proof).

4.3 Bernstein's theorem in higher dimensions

A natural question is whether Bernstein's theorem still holds for a number of dimensions other than 3, that is, if a graph surface defined by a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is minimal in $\mathbb{R}^n$, is f a linear function? In this section we will give a brief account on the history of Bernstein's theorem according mainly to [4], [8] and [22] and in the end we will answer this question.

In [3] (1910) Bernstein aims to generalise Liouville's lemma (Lemma 3.19) to the case of minimal graph surfaces in $\mathbb{R}^3$ defined on the whole plane and he proves the theorem named after him we presented in Theorem 3.25. Note, nevertheless, that the proof we have given is not the original one, it is based on [16] and [19].

Some years later, in 1965, extending Fleming's results presented in [10], De Giorgi in [8] proved that if Bernstein's theorem is false for surfaces in $\mathbb{R}^n$, then there exists an area minimizing cone in $\mathbb{R}^{n-1}$. Since the cone in $\mathbb{R}^3$ is not area minimizing, De Giorgi's result implies Bernstein's theorem holds for surfaces in $\mathbb{R}^4$.

The case $n = 5$ was solved by Almgren in [2] (1966). In that paper he showed some results about three dimensional surfaces in $\mathbb{R}^4$ and he proved that area minimizing cones in $\mathbb{R}^4$ do not exist. From this it is deduced that Bernstein's theorem also is true in $\mathbb{R}^5$.

Simons in 1968 in [18] made a remarkable leap in the study of Bernstein's problem by generalising Almgren's result to $\mathbb{R}^k$ for any $k \leq 7$. As a consequence, it is deduced that Bernstein's theorem is true for all $n \leq 8$. He noted as well the existence of a special class of cones in $\mathbb{R}^{2m}$ for every $m \geq 4$, but he did not prove whether this cones are globally area minimizing.

Finally, just one year later, Bombieri, De Giorgi and Giusti in [4] showed that Simons' cones are indeed area minimizing. Starting from this fact, they also deduced that for all $n \geq 9$ there exist graph surfaces defined everywhere which are minimal yet are not the graph of a linear function in $\mathbb{R}^n$. Thus, they showed Bernstein's theorem does not hold for $n \geq 9$.

In summary, a complete solution to Bernstein's problem is given by Simons' proof of its veracity for $n \leq 8$ and by Bombieri, de Giorgi and Giusti's proof of its falseness for $n \geq 9$.

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