Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

The generalised Gauss-Bonnet-Chern theorem as an instance in the theory of characteristic classes

Javier Maravall López

Director: Dr. Ignasi Mundet Riera
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 20 de juny de 2021

Contents

Abstract

The Gauss-Bonnet theorem is one of the earliest classical results in differential geometry. It provides a link between the topology and the geometry of a smooth surface (that is, a smooth 2-manifold). A well-known, highly non-trivial generalisation of this to arbitrary (finite) dimension exists, which was first proven intrinsically (in other words, without recourse to the existence of an embedding of the manifold into an Euclidean space) by Shiing-Shen Chern in 1944. The aim of this work is to provide a full proof of a slightly more general result, which is valid for arbitrary vector bundles over a differential manifold, that gives as a direct corollary the Gauss-Bonnet-Chern theorem when considering the tangent bundle.

2020 Mathematics Subject Classification. 58-01, 58A10, 58D99

Acknowledgments

Agraiexo al Dr. Ignasi Mundet haver acceptat la tutorització d'aquest treball, i el seu impecable seguiment i ajut en aquest projecte, així com la seva insistència en els detalls per garantir una assimilació profunda dels conceptes, i la seva comprensió en els moments més delicats del mateix.

Ich danke den Professoren Marc Kegel und Klaus Mohnke an der HU Berlin für die Einführung in die tiefe Schönheit der algebraischen Topologie und Differentialgeometrie.

A les meves amistats, per tot.

A mis padres y a mi hermana, por todo.

Chapter 1

Introduction

The Gauss-Bonnet theorem is one of the earliest classical results in differential geometry. It provides a link between the topology and the geometry of a smooth surface (that is, a smooth 2-manifold). In particular:

Theorem 1.0.1. (Gauss-Bonnet). *Let M be a compact Riemannian surface (without boundary). Then,*

$$\chi(M) = \frac{1}{2\pi} \int_M K dA, \tag{1.1}$$

where K is the Gaussian curvature and dA is the element of area of the surface.

Proof. See ([40], p. 4).

□

A well-known, highly non-trivial generalisation of this to arbitrary (finite) dimension exists, which was first proven intrinsically (in other words, without recourse to the existence of an embedding of the manifold into an Euclidean space) by Shiing-Shen Chern in 1944, while on a stay at the Institute for Advanced Study, in a cardinal paper [14] now become classic. Naturally, such a proof is most faithful to the spirit of the original result, which involves intrinsic magnitudes (recall that the Gaussian curvature is intrinsic by virtue of the celebrated Theorema Egretium).

The aim of this work is to provide a full proof of a slightly more general result, which is valid for arbitrary vector bundles over a differential manifold, that gives as a direct corollary the Gauss-Bonnet-Chern theorem when considering the tangent bundle.

To be able to do this, we will need to plunge into the universe of modern differential geometry, on the way acquiring some necessary tools coming from abstract algebra or algebraic topology. In Chapters 2 and 3, we lay out the basic framework of *fibre bundles* over manifolds and the basics of calculus on manifolds (the *exterior derivative*, *integration of differential forms* and *connections* playing the lead role). Some elementary *Riemannian geometry* is reviewed in Chapter 4. Chapter 5 is the topological core of the text, where we expose the *De Rham cohomology* and some elements of *intersection theory* (including the relevant *Thom isomorphism theorem*). We will also compute the integral of the *topological Euler class* of a vector bundle over a manifold. In Chapter 6 -in some sense the central chapter of the text- we give a first glimpse into the fundamental tool of *characteristic classes*, in the process presenting some important examples. One of them, the *geometrical Euler class* of an oriented, real vector bundle over a smooth manifold, will be the key element generalising the right-hand side of Equation (1.1). In the final Chapter 7, we will at last arrive at our main result. We will show that the two definitions of the Euler class, in the topological and the geometrical sense, in fact coincide, from which an extension of Theorem 1.0.1 to arbitrary finite dimension is immediately derived. We have also collected some final thoughts in Chapter 8, outlining some possible further developments of this work.

In the effort to make this work as self-contained as possible, we have included, in Appendix I, some preliminary definitions and results from linear algebra and smooth manifolds. Furthermore, on account of space restrictions a second Appendix II made itself necessary, containing some lengthy proofs.

We advise the reader that we have not followed Chern's original approach via Cartan's formalism: so-called *moving frames*, of which the well-know Frenet trihedron in low-dimensional differential geometry is a particular case. In the present work, we have favoured a presentation in terms of the more usual modern differential geometric language, but a very natural transition from one to the other is of course possible. For more details on this matter we refer the reader to Yin Li's exhaustive paper ([32]). The necessary background may be found in the book by Cartan himself [13].

Note: in this work, we will extensively make use of *Einstein's summation convention* without explicitly acknowledging it, whereby one sums over indices appearing twice, once as a *subscript*, and once as a *superscript*.

Chapter 2

Natural constructions on manifolds

In this section, we introduce the main constructions that make up the basis for analysis on manifolds. The notion of a *bundle* over a manifold will be of pivotal importance, since, as Liviu Nicolaescu points out:

> Chern had the remarkable insight that the Gauss-Bonnet formula is not just a statement about a Riemann manifold: it is a statement about an oriented vector bundle (the tangent bundle) together with a special connection on it (the Levi-Civita connection). The shift of emphasis from the manifold to the vector bundle is fundamental. ([40], p.9).

Indeed, the fundamental role that bundles play in the general framework of our pursued theorem will become clear as the machinery we are developing unfolds, and it will be definitely settled once we reach Chapter 6, where a general method for producing bundle-invariant cohomological classes -one of which, the Euler class, is key to proving our main theorem- is explored.

We mainly follow ([41], Chapter 2) throughout.

2.1 The tangent bundle

2.1.1 The tangent space at a point

The tangent space of a manifold at a point, a first-order linear approximation of the manifold near that point, is a key concept through which the linear-algebraic notions laid out in Section A of Appendix I can be transferred to the manifold. It turns out that these collection of spaces can be endowed with a manifold structure and can be organized in a very particular way (that of a *bundle*) that may be shared by other, less immediate structures on manifolds.

([41], section 2.1), chooses the construction of the tangent space at a point in terms of equivalence classes of curves. This, however, has the drawback that its vector-space character is less straightforward. For this reason, we follow, with ([17], Ch. 2), the construction in terms of derivations, and we briefly show the equivalence of these two perspectives at the end.

Definition 2.1.1. *(a) Let M be an m-dimensional smooth manifold. A map $\delta_p : C^\infty(M) \to \mathbb{R}$ is called a* derivation at a point $p \in M$ *if it verifies the following properties:*

(i) $\delta_p(f_1 + f_2) = \delta_p(f_1) + \delta_p(f_2)$

(ii) $\delta_p(\lambda f) = \lambda \delta_p(f), \forall \lambda \in \mathbb{R}$

(iii) $\delta_p(f_1 \cdot f_2) = f_1(p) \cdot \delta_p(f_2) + f_2(p) \cdot \delta_p(f_1)$, *(Leibniz's rule)*
where $\cdot$ in $C^\infty(M)$ is pointwise defined as the natural product in $\mathbb{R}$.
Note that, as a direct consequence of (iii), $\delta_p(\lambda) = 0, \forall \lambda \in \mathbb{R}$.

(b) We denote by $D_p(M, \mathbb{R})$ the space of derivations of $C^\infty(M)$ at a point $p \in M$. *It clearly has a vector-space structure (notice we are simply considering a closed restriction on the set of linear maps $C^\infty(M) \to \mathbb{R}$).*

Definition 2.1.2. *We denote by T_pM the* tangent space of M at $p \in M$, *and we define it to be the vector space $D_p(M, \mathbb{R})$.*

Lemma 2.1.3. *Let $f : M \to N$ be a diffeomorphim. Then, f induces an isomorphism $f_\# : D_p(M, \mathbb{R}) \to D_{f(p)}(N, \mathbb{R})$.*

Proof. Let $\delta_p \in D_p(M, \mathbb{R})$. We define $f_\#(\delta_p)$ by setting $f_\#(\delta_p)(h) \coloneqq \delta_p(h \circ f)$. As f is a diffeomorphism, it is immediate to check, using f^{-1}, that $f_\#$ is an isomorphism.

□

Corollary 2.1.4. *Let (U, ϕ) be a chart of M such that $p \in U$. Then, $\phi_\# : D_p(U, \mathbb{R}) \to D_{\phi(p)}(\phi(U), \mathbb{R})$ is an isomorphism.*

Proof. It is a direct application of the above lemma. One needs only point out that $\phi : U \to \phi(U) \subset \mathbb{R}^m$ is a diffeomorphism, considering the smooth structures of U as an open set of M and of $\phi(U)$ as an open set of $\mathbb{R}^m$ (see Remark B.2.3).

□

The above results imply that the task of studying any tangent space at a point reduces to studying the space of derivations at a point of $\mathbb{R}^m$. In particular:

Proposition 2.1.5. $D_0(\mathbb{R}^m, \mathbb{R}) \cong \mathbb{R}^m$.

Proof. Let $\mathbf{x} = (x^1, \ldots, x^m)$ be coordinates in $\mathbb{R}^m$. We can interpret them as a smooth family $\{x^i \in C^\infty(\mathbb{R}^m)\}$ with $x^i(p) = x^i_p \in \mathbb{R}$.

We denote by $\frac{\partial}{\partial x^i}|_0$ the derivation along any differential curve through 0, with initial vector e_i.

Let $\delta_0 \in D_0(\mathbb{R}^m, \mathbb{R})$ with $\delta_0(x^i) = \lambda^i \in \mathbb{R}$. We consider the derivation at 0, $\eta_0 = \sum_{i=1}^m \lambda^i (\frac{\partial}{\partial x^i}|_0)$, and we will show that it coincides with δ_0.

Let $f \in C^\infty(\mathbb{R}^m)$, $f(q) = f(0) + \int_0^1 \frac{d}{dt}(f(tq))dt$.

By the chain rule, we have:

$$f(q) = f(0) + \int_0^1 (\sum_{i=1}^m x^i(q) \frac{\partial f}{\partial x^i}(tq))dt = f(0) + \sum_{i=1}^m x^i(q)(\int_0^1 \frac{\partial f}{\partial x^i}(tq))dt),$$

from which we deduce:

$$\delta_0(f) = \sum_{i=1}^m \delta_0(x^i)\big(\int_0^1 \frac{\partial f}{\partial x^i}(0)dt\big) + \sum_{i=1}^m x^i(0) \cdot \delta_0\big(\int_0^1 \frac{\partial f}{\partial x^i}(tq)dt\big) = \sum_{i=1}^m \lambda^i \frac{\partial}{\partial x^i}|_0(f)$$

It is easy to see that the derivations $\{\frac{\partial}{\partial x^i}|_0\}_{i=1,\ldots,m}$ are linearly independent, so that we infer that they form a basis of $D_0(\mathbb{R}^m, \mathbb{R})$ and it has dimension m. Furthermore, the i-th component of a derivation δ_0 expressed in this basis is $\delta_0(x^i)$.

□

Consequently,

Corollary 2.1.6. $T_pM \cong \mathbb{R}^m$.

Remark 2.1.7. It is now easy to explicitly describe a basis of T_pM. Let $p \in M$ and (U, ϕ) a chart centered at 0 (i.e., $\phi(p) = 0 \in \mathbb{R}^m$). We can describe T_pU using the isomorphism $\phi_\# : D_p(U, \mathbb{R}) \to D_0(\phi(U), \mathbb{R})$ from Corollary 2.1.4. Indeed, the elements $\{\phi_\#^{-1}(\frac{\partial}{\partial x^i}|_0)\}_{i=1,\ldots,m}$ form a basis of T_pU, that we denote by $\{\frac{\partial}{\partial x^i}|_p)\}_{i=1,\ldots,m}$.

Now, if $f \in C^\infty(U)$, we have that

$$\frac{\partial}{\partial x^i}|_p(f) = \frac{\partial}{\partial x^i}|_0(f \circ \phi^{-1})$$

Finally, bearing in mind that $i : U \hookrightarrow M$ induces an isomorphism $i_\# : T_pU \to T_pM$ (cfr. ([17], p.26)), we have that the images of the above basis by $i_\#$ are a basis of T_pM. We use the same notation for them.

We now provide the construction of the tangent space at a point in terms of equivalence classes of paths and show its correspondence with the above notions.

Definition 2.1.8. *(a) Let M be a smooth m-dimensional manifold and p_0 a point in M. Two smooth paths $\alpha, \beta : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = \beta(0) = p_0$ are said to have a* first order contact *at p_0 if there exist local coordinates $\mathbf{x} = (x^1, \ldots, x^m)$ centered at p_0 such that the expression of the derivative of α and β in $\mathbf{x}$ coincides at 0, i.e., $\dot{\mathbf{x}}_\alpha(0) = \dot{\mathbf{x}}_\beta(0)$.*

We write $\alpha \sim \beta$. It can be shown that $\sim$ is an equivalence relation.

(b) A tangent vector *to M at p is a first-order-contact equivalence class of curves through p. The set of these equivalence clases is denoted by $\tilde{T}_p M$.*

Remark 2.1.9. Let γ be a path representing a tangent vector to M at the point $\gamma(0)$. Clearly, $\dot{\gamma}(0)(f) = \frac{d(f \circ \gamma(t))}{dt}|_{t=0}$ defines a derivation at $\gamma(0)$. It is easy to check that equivalent curves define equivalent derivations.

Proposition 2.1.10. *Every element $\delta_p \in T_p M$ is equal to $\dot{\gamma}(0)$ for some path γ, with $\gamma(0) = p$.*
In particular, $\tilde{T}_p M \cong T_p M$.

Proof. Let (U, ϕ) be a chart centered at p, with coordinates (x^i). We have shown that the element $\delta_p \in T_p M$ can be written in the form $\delta_p = \sum_{i=1}^m \lambda^i \frac{\partial}{\partial x^i}|_p$.

If we pick the path $\gamma(t) = \phi^{-1}(t\lambda^i)$, it is easy to check that $\dot{\gamma}(0) = \delta_p$.

□

Remark 2.1.11. This motivates the use of the term *tangent vector of M at p* when referring to the elements of $T_p M$.

2.1.2 The tangent bundle

We have learnt that we can naturally associate to an arbitrary point p in a manifold M a vector space $T_p M$. We will now show how to coherently organise the family $\{T_p M\}_{p \in M}$. Concretely, we will formalise the intuitive idea that the dependence of $T_p M$ on p is smooth.

Definition 2.1.12. *Consider the disjoint union of all tangent spaces of a smooth manifold M, $TM := \bigsqcup_{p \in M} T_p M$. There is a natural surjection $\pi : TM \to M$, $\pi(v) = p \;\Leftrightarrow v \in T_p M$.*

Now, we know that any local coordinate system $\mathbf{x} = (x^1, \ldots, x^m)$ on an open set $U \subset M$ induces a natural basis $\{\frac{\partial}{\partial x^i}|_p)\}_{i=1,\ldots,m} = \{\partial_{x^i}(p)\}_{i=1,\ldots,m}$ of $T_p M$, for any $p \in U$. Thus, an element $v \in TU = \bigsqcup_{p \in U} T_p M$ is completely determined by two pieces of information:

(1) The point $p = \pi(v) \in M$, i.e., the tangent spaces it belongs to.

(2) The coordinates $X^i(v)$ of v in the basis $\{\partial_{x^i}(p)\}_{i=1,\ldots,m}$.

Therefore, we have a bijection $\Psi_x : TU \to U^x \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m$, where U^x is the image in $\mathbb{R}^m$ of U by the coordinates $\mathbf{x} = (x^i)$. We can use Ψ_x to transfer the topology of $\mathbb{R}^m \times \mathbb{R}^m$ to TU.

The natural topology of TM is obtained by patching together the topologies of $\{TU_\alpha\}_{\alpha \in \Lambda}$, where $(U_\alpha, \phi_\alpha)_{\alpha \in \Lambda}$ is a countable *atlas of M (see B.2.3, (a)). A set $D \subset TM$ is open if its intersection with any TU_α is open in TU_α. Thus, TM is a smooth manifold with $(TU_\alpha, \Psi_\alpha)_{\alpha \in \Lambda}$ a defining atlas (with Ψ_α defines in an analogous manner to Ψ_x above). Furthermore, the natural projection $\pi : TM \to M$ is a smooth map.*

The above-described smooth manifold TM is called the tangent bundle.

For this definition to be valid, we need to ascertain that the topology is independent of the choice of local coordinates. This is indeed the case:

Proposition 2.1.13. *The topology of TU induced by Ψ_x is independent of the chosen coordinate system $\mathbf{x} = (x^i)$.*

Proof. Let us pick an alternative coordinate system $\mathbf{y} = (y^i)$ on U. It suffices to show that the transition map $\Psi_y \circ \Psi_x^{-1} : U^x \times \mathbb{R}^m \to TU \to U^y \times \mathbb{R}^m$ is smooth.

Let $A := (\bar{x}, X) \in U^x \times \mathbb{R}^m$. Then $\Psi_x^{-1}(A) = (p, \dot{\alpha}(0))$, where $x(p) = \bar{x}$, and $\alpha : (-\epsilon, \epsilon) \to U$ is a path through p given in the $\mathbf{x}$ coordinates as $\alpha(t) = \bar{x} + tX$.

Denote by $F : U^x \to U^y$ the transition map $\mathbf{x} \mapsto \mathbf{y}$. Then, $\Psi_y \circ \Psi_x^{-1}(A) = (y(\bar{x}); Y^1, \ldots, Y^m)$, where $\dot{\alpha}(0) = (\dot{y}^j_\alpha(0)) = \sum_j Y^j \partial_{y^j}(p)$, and $(y_\alpha(t))$ is the description of the path $\alpha(t)$ in the coordinates $\mathbf{y}$.

Applying the chain rule, we deduce:

$$Y^j = \dot{y}^j_\alpha(0) = \sum_i \frac{\partial y^j}{\partial x^i}\dot{x}^i(0) = \sum_i \frac{\partial y^j}{\partial x^i} X^i, \tag{2.1}$$

which finishes the proof.

□

Definition 2.1.14. *Let M, N be smooth manifolds and $f : M \to N$ a smooth map.*

We define the differential of f at a point $p \in M$ *to be the map $d_pf : T_pM \to T_{f(p)}N$ uniquely determined by $(d_pf(v_p))(g) = v_p(g \circ f)$, for any differential function g defined in a neighbourhood of $f(p)$.*

Remark 2.1.15. (a) The map above is well-defined, i.e., $d_pf(v_p)$ is a derivation at the point $f(p)$. The linearity is straightforward. As for Leibniz's rule, observe that

$$\begin{aligned}(d_pf(v_p))(g\cdot h) &= v_p((g\cdot h)\circ f) = v_p((g\circ f)\cdot(h\circ f))\\ &= v_p(g\circ f)\cdot(h\circ f)(p) + (g\circ f)(p)\cdot v_p(h\circ f)\\ &= d_pf(v_p)(g)\cdot h(f(p)) + g(f(p))\cdot d_pf(v_p)(h)\end{aligned}$$

(b) A geometrical interpretation of the differential at a point can be derived from the description of T_pM in terms of tangent vectors. In particular:

Recall that $v_p = \dot\gamma(0)$ for a certain path representative γ. If we compute

$$d_pf(\dot\gamma(0))(g) = \dot\gamma(0)(g\circ f) = \frac{d}{dt}\big((g\circ f)(\gamma(t))\big)|_{t=0} = \frac{d}{dt}\big((g\circ(f\circ\gamma))(t)\big)|_{t=0},$$

we see that $d_pf(\dot\gamma(0))$ is the tangent vector at $f(p)$ to the curve $f\circ\gamma$.

(c) It is easy to see that, chosen local coordinates $\mathbf{x} = (x^i)$ in M around p and local coordinates $\mathbf{y} = (y^j)$ in N around $f(p)$, d_pf is given with respect to the natural bases of T_pM and $T_{f(p)}N$ by the matrix $(\frac{\partial y^j}{\partial x^i})_{1\le j\le n, 1\le i\le m}$ (see Remark B.2.3 (c)).

This implies that f induces a smooth map $df : TM \to TN$ such that $\forall p \in M, df(T_pM) \subset T_{f(p)}N$.

Definition 2.1.16. *(a) A smooth map $f : M \to N$ is called* immersion *(resp.* submersion*) if for every $p \in M$ the differential $d_pf : T_pM \to T_{f(p)}N$ is injective (resp. surjective).*

(b) A smooth map $f : M \to N$ is called an embedding if it is an injective immersion.

2.2 Vector bundles

It turns out that the tangent bundle TM of a manifold M has a special structure over M that happens to be shared by a more general type of objects over manifolds, *vector bundles*. Formally:

Definition 2.2.1. *(a) A* vector bundle over a smooth manifold M *is a quadruple (E, π, M, F), where E is a smooth manifold, $\pi : E \to M$ is a surjective submersion and F is a vector space of dimension k over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$, such that there exists a* trivialising cover of M*, i.e., an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of M together with diffeomorphisms $\Psi_\alpha : F \times U_\alpha \to E|_{U_\alpha} := \pi^{-1}(U_\alpha)$ that make the below diagram commutative:*

$$\begin{tikzcd} F\times U_\alpha \arrow[rr, "\Psi_\alpha"] \arrow[dr, "p"'] & & E|_{U_\alpha} \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

where p is the natural Cartesian projection.

(b) The manifold E is called the total space *and M is called the* base space*. The vector space F is called the* standard fibre *and its dimension over $\mathbb{K}$ is called the* rank of the bundle*. A* line bundle *is a vector bundle of rank one.*

We will frequently use the notation $E \xrightarrow{\pi} M$ to denote a vector bundle E over M.

(c) *The definition above implies that if $U, V \in \mathcal{U}$ are two trivialising neighbourhoods with non-empty overlap $U \cap V$, then for any $p \in U \cap V$, the* transition map $\Psi_{UV} := \Psi_U^{-1} \circ \Psi_V : F \times U \cap V \to F \times U \cap V$ *is well-defined and satisfies*

$$\Psi_{UV}(v, p) = (g_{UV}(p)(v), p),$$

for some smooth map $g_{UV} : U \cap V \to Aut(F) = GL(k, \mathbb{R})$.

Remark 2.2.2. Note that the definition above implies, in particular, that for any $p \in M$, $E_p := \pi^{-1}(p)$ has the structure of F (i.e., it is a $\mathbb{K}$-vector space of dimension k).

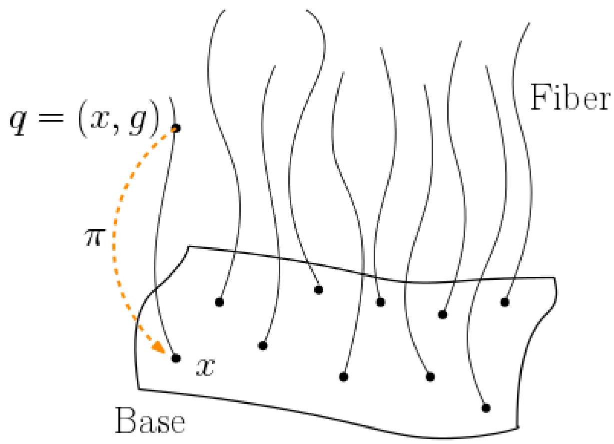

Figure 2.1: Visualisation of a vector bundle. Extracted from [28].

Remark 2.2.3. As in the case of manifolds (see Remark B.2.2 (a)), the reflection on examples is the shortest path to an adequate grasp of this notion. We refer to ([41], subsection 2.1.5) for a collection.

Remark 2.2.4. We saw in the previous section that, given a coordinate chart of M, (U, ϕ) with local coordinates $\mathbf{x} = (x^1, \ldots, x^m)$ around $p \in M$, the tangent bundle can be (locally) described as $U \times \mathbb{R}^m$, where the second space encodes the coordinates of a given $v \in T_pM$ in the base $\{\partial_{x^i}\}_{i=1,\ldots,m}$ and the first one encodes the point p. This defines a *local trivialisation* of the tangent bundle.

The vector-bundle formalisation captures the intuitive idea that TM *locally looks like* $U \times \mathbb{R}^m$.

Remark 2.2.5. The transition maps from Definition 2.2.1 allow one to give an alternative, equivalent definition of a vector bundle. In particular, it can be shown that given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$, the smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(F)$ (which we will also refer to as *transition functions*) satisfy the *cocycle condition*:

(i) $g_{\alpha\alpha} = \mathbb{1}_F$

(ii) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathbb{1}_F$ on $U_\alpha \cap U_\beta \cap U_\gamma$

Conversely, given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of M and a collection of smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(F)$ satisfying the cocycle condition, we can construct a vector bundle by gluing the product bundles $E_\alpha = F \times U_\alpha$ on the overlaps $U_\alpha \cap U_\beta$ according to the gluing rules prescribed by $g_{\alpha\beta}$. In detail:

$$E_\alpha \ni (v, p) \sim (g_{\beta\alpha}(p)v, p) \in E_\beta, \forall p \in U_\alpha \cap U_\beta$$

We will refer to such a collection of maps $\{g_{\alpha\beta}\}$ satisfying the cocycle condition as a *gluing cocycle.*

Remark 2.2.6. In the case of the tangent bundle, the transition functions are explicitly described by the matrix expressed in Equation (2.1) (that is, the change of coordinates in the tangent bundle).

Definition 2.2.7. (a) *A* section of a vector bundle $E \xrightarrow{\pi} M$, *defined over the open subset $U \subset M$ is a smooth map $s : U \to E$ such that $s(p) \in E_p$ $\forall p \in U$, which is equialent to $\pi \circ s = \mathbb{1}_U$.*

The space of smooth sections of E over U will be denoted by $\Gamma(U, E)$ or $C^\infty(U, E)$. Note that the vector-space character of E_p, $\forall p \in U$, implies that $\Gamma(U, E)$ is naturally a vector space.

We will use $C^\infty(E)$ of $\Gamma(E)$ when referring to the space of sections of E over M.

(b) *A section of the tangent bundle of a smooth manifold M is called a* vector field of M*. The space of vector fields over an open subset U of a smooth manifold is denoted by Vect(U).*

Example 2.2.8. Consider the trivial vector bundle $\underline{\mathbb{R}}^n_M \to M$ over the smooth manifold M (that is, we assign to each point of M a copy of $\mathbb{R}^n$, so that $\underline{\mathbb{R}}^n_M = \mathbb{R}^n \times M$).

A section of this vector bundle can be interpreted as a smooth map $s : M \to \mathbb{R}^n$. We can think of s as a smooth family of vectors $\{s(p) \in \mathbb{R}^n\}_{p\in M}$.

Definition 2.2.9. (a) *Let $E^i \xrightarrow{\pi_i} M_i$ be two smooth vector bundles. A* vector bundle map *consists of a pair of smooth maps $f : M_1 \to M_2$ and $F : E^1 \to E^2$ satisfying the following properties:*

(i) *The map F* covers f*, i.e., $F(E^1_p) \subset E^2_{f(p)}$ $\forall p \in M_1$. In other words, the following diagram is commutative:*

$$\begin{tikzcd} E^1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E^2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

(ii) *The induced map $F : E^1_p \to E^2_{f(p)}$ is linear.*

The composition and the identity morphism are defined in the obvious manner, so that a natural notion of bundle isomorphism *can be defined.*

(b) *If E and F are two vector bundles over the same manifold, then we denote by $\underline{Hom}(E, F)$ the space of bundle maps $E \to F$ that* cover the identity $\mathbb{1}_M$*. Such bundle maps are called* bundle morphisms*.*

Example 2.2.10. The differential df of a smooth map $f : M \to N$ is a bundle map $df : TM \to TN$ covering f.

Definition 2.2.11. *Let $f : X \to M$ be a smooth map between manifolds, and E a vector bundle over M defined by an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ and gluing cocycle $\{g_{\alpha\beta}\}$.*

The pullback of E by f *is the vector bundle f^*E over X defined by the open cover $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ and the gluing cocycle $\{g_{\alpha\beta} \circ f\}$.*

Remark 2.2.12. It is easy to see that the isomorphism class of the pullback of a vector budle E is independent of the choice of gluing cocycle describing E. The pullback operation defines a linear map between the spaces of sections of E and the space of sections of f^*E.

In detail, if $s \in \Gamma(E)$ is defined by the open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and the collection of smooth maps $\{s_\alpha\}_{\alpha\in\Lambda}$, then the *pullback* f^*s is defined by the open cover $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ and the smooth maps $\{(s_\alpha \circ f)\}_{\alpha\in\Lambda}$. Again, the independence of the various choices is proven easily.

Definition 2.2.13. (a) *Given a rank-r vector bundle $E \xrightarrow{\pi} M$ over $\mathbb{K}$, a* frame at a point $p \in M$ *is an ordered basis of the vector space E_p (notice that this can be seen, alternatively, as an isomorphism $\mathbb{K}^r \to E_p$).*

(b) *A rank-r vector bundle $E \xrightarrow{\pi} M$ (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$) is said to be* trivial *or* trivialisable *if there exists a bundle isomorphism $E \cong \underline{\mathbb{K}}^r_M$.*

A bundle isomorphism $E \to \underline{\mathbb{K}}^r_M$ is called a trivialisation of E*, whereas an isomorphism $\underline{\mathbb{K}}^r_M \to E$ is called a* framing of E*.*

Remark 2.2.14. Let us explain the intuition behind the term *framing*. Consider a bundle isomorphism $\varphi : \underline{\mathbb{K}}^r_M \to E$ and denote by $\{e_1, \ldots, e_r\}$ the canonical basis of $\mathbb{K}^r$. We can regard the vectors e_i as constant maps $M \to \mathbb{K}^r$, i.e., as particular sections of $\underline{\mathbb{K}}^r_M$. The isomorphism φ determines sections $f_i = \varphi(e_i)$ of E with the property that for every $p \in M$, the collection $\{f_1(p), \ldots, f_r(p)\}$ is a frame of the fibre E_p.

In other words, this observation shows that we can interpret any framing of a rank-r vector bundle as a collection of r sections $\{s_1, \ldots, s_r\}$ that are *pointwise* linearly independent.

Example 2.2.15. Let G be a Lie group (see Definition B.2.8). Then, its tangent bundle TG is trivial.

To see this, we set $\dim G = n$ and consider $\{e_1, \ldots, e_n\}$, the basis of the tangent space at the origin $T_{\mathbb{1}}G$. We denote by R_g the *right translation by g in the group*, defined by $R_g : x \mapsto x \cdot g$ $\forall x \in G$, where $\cdot$ denotes the group operation. This is a diffeomorphism with inverse $R_{g^{-1}}$, so that the differential dR_g defines a linear isomorphism $dR_g : T_{\mathbb{1}}G \to T_gG$. We set $E_i(g) = dR_g(e_i) \in T_gG$, for $i = 1, \ldots, n$.

Since the multiplication $G \times G \to G$ is a smooth map, we deduce that the vectors $E_i(g)$ define *smooth* vector fields over G. Furthermore, for every $g \in G$, the collection $\{E_1(g), \dots, E_n(g)\}$ is a basis of T_gG. Therefore, there exists a well-defined map: $\Phi : \mathbb{R}^n_G \to TG$

$$(g; X^1, ..., X^n) \mapsto (g; \sum_i X^i E_i(g))$$

It is immediate to verify that Φ is a vector-bundle isomorphism, which proves the claim.

Remark 2.2.16. (a) We see then that the tangent bundle TM of an m-dimensional manifold M is trivial if, and only if, there exist vector fields $\{X_1, \dots, X_m\}$ such that, for each $p \in M$, $\{X_1(p), \dots, X_m(p)\}$ spans T_pM. This suggests the more general problem of computing $v(M)$, the maximum number of pointwise linearly independent vector fields over M. For Michael Atiyah's approach through the theory of elliptic operators on manifolds, see ([6]).

(b) Another approach to measuring the triviality of an arbitrary vector bundle is given by the so-called theory of characteristic classes, that we expose in Chapter 6. It does hold that trivial bundles have vanishing characteristic classes, as we will see. However, counterexamples to the opposite statement exist (for details, see ([24], p.75)).

Remark 2.2.17. Let E, F be two vector bundles over a smooth manifold M with standard fibres V_E and respectively V_F, given by a common open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ and gluing cocycles $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V_E)$ and respectively $h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V_F)$.

Then one can show that the collections

$$g_{\alpha\beta} \oplus h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V_E \oplus V_F),\ g_{\alpha\beta} \otimes h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V_E \otimes V_F)$$

$$(g^\dagger_{\alpha\beta})^{-1} : U_\alpha \cap U_\beta \to GL(V_E^*),\ \Lambda^r g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\Lambda^r V_E),$$

where † denotes the dual of a linear map, satisfy the cocycle condition, and therefore define vector bundles which we denote by $E \oplus F$, $E \otimes F$, E^* and $\Lambda^r V_E$ respectively.

In particular, if E is of rank r, then $\Lambda^r V_E$ has rank 1. It is called the *determinant line bundle of* E and it is denoted by $\det E$.

Given the adjunction isomorphism $V_E^* \otimes V_F \cong \mathrm{Hom}(V_E, V_F)$, we set $\mathrm{Hom}(E, F) := E^* \otimes F$.

Definition 2.2.18. *Let $E \xrightarrow{\pi} M$ be a $\mathbb{K}$-vector bundle over M ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). A* metric *on E is a section h of $E^* \otimes_{\mathbb{K}} \overline{E^*}$ ($\overline{E} = E$ when $\mathbb{K} = \mathbb{R}$) such that, for any $m \in M$, $h(m)$ defines a metric on E_m (Euclidean if $\mathbb{K} = \mathbb{R}$ or Hermitian if $\mathbb{K} = \mathbb{R}$) (see Definition A.5.2).*

2.2.1 Tensor fields

In this subsection we apply the above operations of vector bundles to the particular case of the tangent bundle, thereby exposing the main objects of study of tensor calculus. This was a very important development, as, in the words of Chern:

> In our subject of differential geometry, where you talk about manifolds, one difficulty is that the geometry is described by coordinates, but the coordinates do not have meaning. They are allowed to undergo [arbitrary] transformations. And in order to handle this kind of situation, an important tool is the so-called tensor analysis, or Ricci calculus, which was new to mathematicians ([27], p.861).

The modern approach to said difficulty, pioneered by the influential Élie Cartan, is via so-called *differential forms*. Intuitively, one of the motivations for defining differential forms on $\mathbb{R}^n$ is the pathological implication of the elementary change-of-variable integration formula: namely, that integration is not a coordinate-free operation. In particular, as Daniel Litt asserts:

> Diffeomorphic distortions of the coordinate system (that is, continuously differentiable and invertible maps, whose inverse is also continuously differentiable) change the integrals of maps, even though no information is added or lost ([33], p.19).

This is undesirable, as there seems to be no canonical choice of a given coordinate system over any other. In a more down-to-earth spirit, Jerrold Marsden points out that when defined on arbitrary manifolds:

[The main idea behind differential forms is to] generalize the basic operations of vector calculus, *div, grad, curl*, and the integral theorems of Green, Gauss, and Stokes to manifolds of arbitrary dimension ([34], p.1).

Definition 2.2.19. *(a) The* cotangent bundle *of a smooth manifold M is $T^*M := (TM)^*$*

(b) The tensor bundles *of M are $\mathcal{T}^r_s(M) := \mathcal{T}^r_s(TM) = (TM)^{\otimes r} \otimes (T^*M^{\otimes s})$.*

Definition 2.2.20. *Let M be a smooth manifold.*

(a) A tensor field of type (r, s) *over the open set $U \subset M$ is a section of $\mathcal{T}^r_s(M)$ over U.*

(b) A degree-r differential form *(or r-form for brevity) is a section of $\Lambda^r(T^*M)$. The space of smooth r-forms over M is denoted by $\Omega^r(M)$. We define*

$$\Omega^\bullet(M) := \bigoplus_{r\leq 0} \Omega^r(M)$$

(c) A Riemannian metric on M *is a metric on the tangent bundle. In other words, it is a symmetric $(0, 2)$-tensor field g, such that for every $p \in M$, the bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$ defines a Euclidean metric on T_pM.*

Remark 2.2.21. If we interpret the tangent bundle as a smooth family of vector spaces, then a tensor field can be interpreted as a smooth selection of a tensor in each of the tangent spaces. In particular, a Riemann metric defines a smoothly-varying procedure of measuring lengths of vectors in tangent spaces.

Remark 2.2.22. As in Remark A.1.9, it is useful to have a local description of these objects. Let $\mathbf{x} = (x^1, \ldots, x^m)$ be local coordinates on an open set U of a smooth manifold M. Then, we have seen that the vector fields $\{\partial_{x^1}, \ldots, \partial_{x^m}\}$ trivialise $TM|_U$. We can form a dual framing of $T^*M|_U$ using the 1-forms $\{dx^1, \ldots, dx^m\}$ uniquely defined by the duality conditions

$$dx^i(\partial_{x^j}) = \delta^j_i \ \forall i, j \in \{1, \ldots, m\},$$

where δ^j_i is the usual Kronecker symbol.

This means that a basis of $\mathcal{T}^r_s(T_xM)$ is given by

$$\{\partial_{x^{i_1}} \otimes \cdots \otimes \partial_{x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} : 1 \leq i_1, \ldots, i_r \leq m, 1 \leq j_1, \ldots, j_s \leq m$$

and that any tensor field $T \in \mathcal{T}^r_s(TM)$ has a local description

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \partial_{x^{i_1}} \otimes \cdots \otimes \partial_{x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

so that any r-form ω has the local description:

$$\omega = \sum_{1\leq i_1<\cdots<i_r\leq m} \omega_{i_1\ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \text{ where } \omega_{i_1\ldots i_r} = \omega(\partial_{x^{i_1}}, \ldots, \partial_{x^{i_r}}),$$

while a Riemannian metric g has the local description:

$$g = \sum_{i,j} dx^i \otimes dx^j, \text{ where } g_{ij} = g_{ji} = g(\partial_{x^{i_1}}, \partial_{x^{i_j}})$$

Remark 2.2.23. (a) A covariant tensor field (i.e., a $(0, s)$-tensor field S) naturally defines a $C^\infty(M)$-multilinear map

$$S : \bigoplus_1^s \mathrm{Vect}(M) \to C^\infty(M)$$

$$(X_1, \ldots, X_s) \mapsto \Big(p \mapsto S_p\big(X_1(p), \ldots, X_s(p)\big)\Big) \in C^\infty(M)$$

Conversely, any such map uniquely defines a $(0, s)$-tensor field. In particular, an r-form η can be identified with a skew-symmetric $C^\infty(M)$-multilinear map $\eta : \bigoplus_1^r \mathrm{Vect}(M) \to C^\infty(M)$.

(b) Let $f \in C^\infty(M)$. Its differential $df : TM \to T\mathbb{R} \cong \mathbb{R}_\mathbb{R}$ is naturally a 1-form. Indeed, we get a smooth $C^\infty(M)$-multilinear map $df : \mathrm{Vect}(M) \to C^\infty(M)$ defined by $(df X)_p := d_{f(p)}f(X) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}, \forall p \in M$.

Definition 2.2.24. *Any smooth map between manifolds $f : M \to N$ defines a linear map $f^* : \mathcal{T}^0_s(N) \to \mathcal{T}^0_s(M)$ called the* pullback by f*.*

Explicitly, if S is a covariant tensor field of N defined by a $C^\infty(N)$-multilinear map $S : \bigoplus_1^s \mathit{Vect}(N) \to C^\infty(N)$, then f^*S is the covariant tensor field of M defined by

$$(f^*S)_p(X_1(p), \dots, X_s(p)) \coloneqq S_{f(p)}(d_pf(X_1), \dots, d_pf(X_s)),$$

for any $X_1, \dots, X_s \in \mathit{Vect}(M)$ and $p \in M$.

Remark 2.2.25. (a) A Riemann metric g on a manifold M induces metrics in all the associated tensor bundles $\mathcal{T}^r_s(M)$. In detail, let us choose local coordinates $\mathbf{x} = (x^1, \dots, x^m)$ on an open set U of M. Using the local descriptions explained in Remark 2.2.22 and denoting by (g^{ij}) the inverse of the matrix (g_{ij}), then, for every point $p \in U$ the length of $T(p) \in \mathcal{T}^r_s(T_pM)$ is $|T(p)|_g \in \mathbb{R}$, defined by

$$|T(p)|_g = g_{i_1k_1} \dots g_{i_rk_r} g^{j_1l_1} \dots g^{j_sl_s} T^{i_1\dots i_r}_{j_1\dots j_s} T^{k_1\dots k_r}_{l_1\dots l_s}$$

(b) The exterior product defines an exterior product on the space of smooth differential forms: $\wedge : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \Omega^\bullet(M)$. Thus, $(\Omega^\bullet(M), +, \wedge)$ is an associative algebra.

It is then easy to see that:

Proposition 2.2.26. *Let $f : M \to N$ be a smooth map between manifolds.*

The pullback by f defines a morphism of associative algebras $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$.

2.3 Fibre bundles

In the preceding section, we learnt that the tangent bundle of a manifold is a particular case of the more general vector bundles. In turn, these are special instances of *fibre bundles*, where the standard fibre no longer needs to be a vector space, but an arbitrary smooth manifold. The mathematical theory of fibre bundles has a remarkable conceptual specularity with physical gauge theories ([15], p.348).

We will also introduce the notion of *principal bundles*, which are fibre bundles with fibre a Lie group G that satisfy a further compatibility condition with an action by G. A special case of these bundles, the *orthonormal frame bundle* of a vector bundle over a manifold, will play an important role in our main proof. Principal bundles can be thought of as the mathematical formalisation of what in physics is called "symmetry". For more details on these sort of correspondences, which we already mentioned above, between differential geometry and modern physics, see [25].

Definition 2.3.1. *In an analogous fashion to Definition 2.2.1:*

(a) A fibre bundle *over a smooth manifold B is a quadruple (E, π, B, F), where E, B and F are smooth manifolds and $\pi : E \to M$ is a surjective submersion such that there exists a* trivialising cover *of the base B, i.e., an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of B together with diffeomorphisms $\Psi_\alpha : F \times U_\alpha \to E|_{U_\alpha}$ that make the below diagram commutative:*

$$\begin{tikzcd}
F \times U_\alpha \arrow[rr, "\Psi_\alpha"] \arrow[dr, "p"'] & & E|_{U_\alpha} \arrow[dl, "\pi"] \\
& U_\alpha &
\end{tikzcd}$$

where p is the natural Cartesian projection.

(b) Again, the definition above implies that the transition maps *$\Psi_{\alpha\beta} \coloneqq \Psi_\alpha^{-1} \circ \Psi_\beta : F \times U_{\alpha\beta} \to F \times U_{\alpha\beta}$, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, satisfy $\Psi_{\alpha\beta}(f, b) = (T_{\alpha\beta}(b)(f), b)$, where $\{T_{\alpha\beta}(b)\}_{b\in U_{\alpha\beta}}$ is a smooth family of diffeomorphisms of F.*

Example 2.3.2. Let E be a rank-k vector bundle over a smooth manifold M. Any metric h on E (either Hermitian or Euclidean) defines a submanifold $S(E) \subset E$ by $S(E) = \{v \in E : |v|_h = 1|\}$.

It is easy to see that $S(E)$ is a fibre bundle over M with standard fibre S^{k-1}. It is called the *(unitary) sphere bundle*.

Definition 2.3.3. *A* section *of a fibre bundle* $E \xrightarrow{\pi} B$ *is a smooth map* $s : B \to E$ *such that* $\pi \circ s = \mathbb{1}_B$ *(i.e.,* $s(b) \in E_b \ \forall b \in B$*).*

Definition 2.3.4. *(a) Let* M *be a smooth manifold and* G *a Lie group. We say that* the group G acts on M *from the left (respectively right) if there exists a smooth map* $\Phi : G \times M \to M$, $(g, m) \mapsto T_g M$, *such that*

(i) $T_1 \equiv \mathbb{1}_M$

(ii) $T_g(T_h m) = T_{gh} m$ *(respectively* $T_g(T_h m) = T_{hg} m$*)* $\forall g, h \in G, m \in M$.
In particular, we infer that T_g *is a diffeomorphism of* M $\forall g \in G$. *For any* $m \in M$, *the set* $G \cdot m = \{T_g m : g \in G\}$ *is called the* orbit of the action through m.

(b) Let G *act on* M. *The action is called* effective *if,* $\forall g \in G \setminus \{1\}$, $T_g \neq \mathbb{1}_M$.
The action is called free *if,* $\forall g \in G \setminus \{1\}$ *and* $\forall m \in M$, $T_g m \neq m$.

Definition 2.3.5. *Let* G *be a Lie group. A* linear representation of G *on a vector space* V *is a left action of* G *on* V *such that each* T_g *is a linear map.*

Example 2.3.6. The tautological linear action of $SO(n)$ on $\mathbb{R}^n$ defines a linear representation of $SO(n)$.

Definition 2.3.7. *Let* G *be a Lie group and* (E, π, B, F) *a fibre bundle. Then, it is called a* G-fibre bundle *if it satisfies the following additional conditions:*

(i) There exists an effective left action of the Lie group G *on* F, $G \times F \to F$, $(g, f) \mapsto g \cdot f = T_g f$.
G *is called the* symmetry group of the bundle.

(ii) There exist smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ *satisfying the cocycle condition (see Remark 2.2.5) and such that,* $\forall b \in B$, $T_{\alpha\beta}(b) = T_{g_{\alpha\beta}(b)}$.
Intuitively, this formalises the requirement that the above action on the fibre be compatible with the transition functions of E.

Example 2.3.8. A rank-r vector bundle over $\mathbb{K} = \mathbb{C}, \mathbb{R}$ is a $\mathrm{GL}(r, \mathbb{K})$-fibre bundle with standard fibre $\mathbb{K}^r$ and where the symmetry group $\mathrm{GL}(r, \mathbb{K})$ (the group of linear endomorphisms of $\mathbb{K}^r$) acts on $\mathbb{K}^r$ in the natural way.

2.3.1 Principal and associated bundles

Definition 2.3.9. *Let* G *be a Lie group. A* principal G*-bundle is a* G*-bundle* $P \xrightarrow{\pi} B$ *with fibre* G*, where* G *acts on itself by left translations.*

Remark 2.3.10. Let $P \xrightarrow{\pi} B$ be a principal G-bundle. Consider a trivialising cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$.

Then, two collection of gluing maps on $\mathcal{U}$, $h_{\alpha\beta}, g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ define *isomorphic principal bundles* (in a sense completely analogous to that of Definition 2.2.9) if, and only if, there exists a family of smooth maps $T_\alpha : U_\alpha \to G$ such that

$$h_{\alpha\beta}(p) = T_\beta(p) g_{\alpha\beta} T_\alpha^{-1}(p), \ \forall \alpha, \beta \in \Lambda, \ \forall p \in U_\alpha \cap U_\beta$$

We say in this case that the cocycles $h_{\alpha\beta}, g_{\alpha\beta}$ are *cohomologous*.

Example 2.3.11. The canonical example of a principal bundle is the frame bundle of a vector bundle.

Recall that if $E \xrightarrow{\pi} M$ is a rank-k $\mathbb{K}$-vector bundle over a smooth manifold M ($\mathbb{K} = \mathbb{C}, \mathbb{R}$), a frame at any $p \in M$ can be seen as a linear isomorphism $\mathbb{K}^r \to E_p$ (see Definition 2.2.13). Thus, we can identify the set of all frames at a point, F_p, with the linear group $GL(k, \mathbb{K})$, which acts naturally on itself by left composition. As it is well-known, there is a unique non-singular linear map sending one given basis of a vector space onto another, which means that this action is both free and transitive. For details, see Section 6.1.

The intuitive relation between a vector bundle and its frame bundle is formalised in the following definitions.

Definition 2.3.12. *Let* $P \xrightarrow{\pi} B$ *be a principal* G*-bundle. Consider a trivialising cover* $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ *and collection on* $\mathcal{U}$ *of gluing maps* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

Suppose that G *acts on the left on a smooth manifold* F, $\tau : G \times F \to F$, $(g, f) \mapsto \tau(g) f$.

The collection $\tau_{\alpha\beta} = \tau(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathit{Diffeo}(F)$ *satisfies the cocycle condition and can be used to define a* G*-fibre bundle with fibre* F *in a manner analogous to that of Remark 2.2.5. This bundle is independent of the choice of cover and gluing maps.*

It is called the fibre bundle associated to P via τ *and it is denoted by* $P \times_\tau F$.

Definition 2.3.13. *Let G be a Lie group and $E \xrightarrow{\pi} M$ a vector bundle over a smooth manifold M, with standard fibre a vector space V. A G-*structure *on E is defined by the following data:*

(1) A representation $\rho : G \to GL(V)$.

(2) A principal G-bundle P over M such that E is associated to P via ρ.

In other words, there exist an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of M and a gluing cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that the vector bundle E can be defined by the cocycle $\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to GL(V)$.

We denote a G-structure by the pair (P, ρ).

Remark 2.3.14. In Section 6.1 we will further explore the structures defined above in the particular cases where $G = U(r), O(r), SO(r)$ (that is, the unitary, orthogonal and special orthogonal groups).

2.3.2 Orientation

The notions introduced in Section A.4, together with the existence of the tangent space at a point allow us to talk about orientation on manifolds. In particular:

Definition 2.3.15. *Let M be a smooth m-manifold.*

(a) We define a pointwise orientation *on M to be a choice of orientation of each tangent space T_pM.*

Of course, we would like these collection of orientations to be coherently related to each other, i.e., to have some link to the smooth structure. This asks for some additional conditions.

(b) Let M be endowed with a pointwise orientation. Let $\mathbf{E} = (E_1, \dots, E_m)$ be a local frame of TM in the open subset U of M, i.e., $\mathbf{E}_p = (E_1(p), \dots, E_m(p))$ is a frame of T_pM for every $p \in U$.

We say that $\mathbf{E}$ is positively oriented *if $\mathbf{E}_p$ is a positively oriented basis of T_pM for every $p \in M$. A* negatively oriented *local frame is analogously defined.*

(c) A pointwise orientation on M is said to be continuous *if every point of M is in the domain of an oriented local frame.*

(d) An orientation *of M is a continuous pointwise orientation. We say that M is* orientable *if there exists an orientation on it.*

Definition 2.3.16. *Let M be a smooth manifold.*

(a) Let $\mathbf{x} = (x^1, \dots, x^m)$ be local coordinates on an open subset $U \subset M$ provided by a chart (U, φ) of M. The chart is said to be positively oriented *if the coordinate frame (∂_{x^i}) is positively oriented, and* negatively oriented *analogously.*

(b) A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ of M is said to be consistently oriented *if for each $\alpha, \beta \in \Lambda$ the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_{\alpha\beta})$.*

We can now define orientation on fibre bundles:

Definition 2.3.17. *Let $E \xrightarrow{\pi} B$ be a smooth fibre bundle with standard fibre F. The bundle E is said to be* orientable *if the following conditions hold:*

(i) The manifold F is orientable.

(ii) There exists an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of B with trivialisations $\{\Psi_\alpha\}_{\alpha\in\Lambda}$ such that the gluing maps $\Psi_{\alpha\beta}$ are fibrewise orientation-preserving, i.e., for each $p \in U_{\alpha\beta}$ the diffeomorphism of F, $f \mapsto \Psi_{\alpha\beta}(f, p)$ preserves any orientation on F.

Remark 2.3.18. (a) It can be shown that if the base B of an orientable bundle $\pi : E \to B$ is orientable, then so is the total space (as an abstract smooth manifold).

(b) If $\pi : E \to B$ is an orientable bundle with oriented basis B, then the *natural orientation of the total space E* is defined as follows:

If $E = F \times B$, then the orientation of the tangent space $T_{(f,b)}E$ is given by $\omega_F \times \omega_B$, where $\omega_F \in \det T_f F$ (respectively $\omega_B \in \det T_b B$) defines the orientation of $T_f F$ (respectively $T_b B$) (see Remark A.4.6).

The general case reduces to this one, since any bundle is locally a product and the gluing maps are fibrewise orientation-preserving. This will be our chosen default orientation on fibre bundles.

Chapter 3

Calculus on manifolds

Once established the basic scaffolding of modern differential geometry in the preceding chapter, it is time to learn how to operate on these objects. We present some basic definitions and results of global analysis fundamental to the generalisation of the Gauss-Bonnet theorem. In particular, as announced in the introduction, the notion of a *connection* will be of utmost importance.

We will also present the basic *exterior derivative* as well as integration of differential forms over manifolds and bundles defined on them, exposing the fundamental Stokes's theorem, which completes the programme indicated by Marsden in Chapter 2 (see 2.2.1).

We mainly follow ([41], Ch. 3) throughout; in section 3.2, ([31], Ch. 16); and in section 3.4, ([41], subsection 8.1.1).

3.1 The exterior derivative

For a smooth manifold M, several derivations of $\Omega^\bullet(M)$, such as the Lie derivative or the contraction along a vector field, can be defined. However, for our purposes we can restrict ourselves to the study of the *exterior derivative.*

Proposition 3.1.1. *Let M be a smooth manifold. There exists a linear map $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ uniquely characterised by the following conditions:*

(i) For any differential function $f \in \Omega^0(M) = C^\infty(M)$, df coincides with the differential of f.

(ii) $d^2 = 0$

(iii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p(\omega \wedge d\eta)$, where ω is a p-form.

In other words, d is an antiderivation *of degree 1 on the algebra $\Omega^\bullet(M)$.*

Proof. See ([41], p.90).

□

Definition 3.1.2. *The above conditions determine a local description of the operator d.*

In particular, if $\mathbf{x} = (x^1, \dots, x^m)$ are local coordinates on an open set U of a smooth manifold M, we know from Remark 2.2.22 that any r-form ω has the local expression:

$$\omega = \sum_{1\leq i_1<\cdots<i_r\leq m} \omega_{i_1\dots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

Then, we have that

$$d\omega = \sum_{1\leq i_1<\cdots<i_r\leq m} (\sum_{i=1}^{m} \frac{\partial \omega_{i_1\dots i_r}}{\partial x^i}) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_r})$$

Note that that any addend where $i = i_j$ for some $i = 1, \dots, m$; $j = 1, \dots, r$ vanishes.

The operator d is called the exterior derivative.

Remark 3.1.3. Condition *(iii)* in Proposition 3.1.1 can be substituted by the equivalent one: d is natural, i.e., for any smooth map between manifolds $\phi : N \to M$ and any form ω on M, we have $d\phi^*\omega = \phi^* d\omega$.

Example 3.1.4. Let us look at some examples of the exterior derivative in $\mathbb{R}^3$:

(a) Let $f \in C^\infty(\mathbb{R}^3)$. Then, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ looks like the gradient of f.

(b) Let $\omega \in \Omega^1(\mathbb{R}^3), \omega = Pdx + Qdy + Rdz$. Then, $d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz$

$$= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy + (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dy \wedge dz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dz \wedge dx$$

Thus, $d\omega$ looks like a curl.

(c) Let $\omega \in \Omega^2(\mathbb{R}^3), \omega = P(dy \wedge dz) + Q(dz \wedge dx) + R(dx \wedge dy)$. Then, $d\omega = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz$, which ressembles a divergence.

In other words, with the introduction of the exterior derivative we have already acquired some of the generalisations announced by Marsden (see 2.2.1). The second part of his statement, that concerns integration, will become clear in the following section.

3.2 Integration on manifolds

The purpose of this section is to generalise integrals of $\mathbb{R}^n$ to arbitrary manifolds. Differential forms turn out to be the objects that permit an intrinsic definition of integration. We will see that this is invariant under orientation-preserving diffeomorphisms, thereby solving the undesired consequences indicated by Litt (see 2.2.1).

After presenting the basics of integration on differential forms, we present the salient Stokes' theorem, which has far-reaching implications and contains as particular cases both the fundamental theorem of calculus and the fundamental theorem for line integrals, as well as the three great classical theorems of multivariable calculus: Green's, (the classical) Stokes' and the divergence theorem.

Finally, we extend integration to vector bundles over a manifold, introducing the integration-along-fibres operator. The constructions and results we present here are all very standard, so that for limitations of space we will only provide references to the proofs.

3.2.1 Integration of differential forms on manifolds

In this subsection, we consider all manifolds to be with or without boundary (cfr. Subsection B.2.1).

We begin by defining integrals of n-forms over appropriate subsets of $\mathbb{R}^n$:

Definition 3.2.1. *(a) A* domain of integration *in $\mathbb{R}^n$ is a bounded subset whose boundary has measure zero.*

(b) Let $D \subseteq \mathbb{R}^n$ be a domain of integration and let ω be an n-form on $\overline{D}$. We know that $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some smooth $f : \overline{D} \to \mathbb{R}$.

We define the integral of ω over D *to be: $\int_D \omega := \int_D f dx^1 \dots dx^n$.*

(c) Somewhat more generally, let U be an open subset of $\mathbb{R}^n$ and suppose ω is a compactly supported n-form on U (i.e., an n-form on U with compact support, see Definition B.2.11).

We define $\int_U \omega := \int_D \omega$, where $D \subseteq \mathbb{R}^n$ is any domain of integration containing $\operatorname{supp} \omega$, and ω is extended to be zero on the complement of its support.

It is easy to see that this definition does not depend on the domain D chosen.

Proposition 3.2.2. *Let U, V be open subsets of $\mathbb{R}^n$ and $g : U \to V$ an either orientation-preserving or orientation-reversing diffeomorphism.*

If ω is a compactly-supported n-form on V, then $\int_V \omega = \pm \int_U g^* w$,

where the sign is positive if g is orientation-preserving and it is negative in the other case. g^* denotes the pullback by g (see Definition 2.2.24).

Proof. See ([31], p.404).

□

We can now define the integral of a differential form over an *oriented* manifold.

Definition 3.2.3. *Let M be an oriented smooth manifold of dimension m and let ω be an m-form on M.*

Suppose that ω is compactly supported in the domain of a chart (U, φ) that is either positively or negatively oriented.

We define the integral of ω over M *to be: $\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega$, with the positive sign for a positively oriented chart, and the negative sign otherwise.*

Note that, since $(\varphi^{-1})^*\omega$ is a compactly-supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$, its integral is defined as above.

Proposition 3.2.4. *The above-defined integral $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains* suppω*.*

Proof. See ([31], p.405).

□

Now, to integrate over an entire manifold, we will use a partition of unity (see subsection B.2.3) to patch up the above local definition:

Definition 3.2.5. *Let M be an oriented smooth manifold of dimension m and let ω be a compactly supported m-form on M.*

Let $\{U_i\}$ be a finite open cover of supp ω *(recall it is a compact) by domains of positively or negatively oriented smooth charts, and let $\{f_\beta\}_{\beta \in \mathcal{B}} \subset C^\infty(M)$ be a smooth partition of unity subordinated to this cover.*

The integral of ω over M *is defined to be*

$$\int_M \omega = \sum_i \int_M f_i \omega$$

Proposition 3.2.6. *The above-defined integral $\int_M \omega$ does not depend on the choice of open cover or partition of unity.*

Proof. See ([31], p.405).

□

Remark 3.2.7. If $S \subset M$ is an oriented immersed k-dimensional manifold and ω is a k-form on M whose restriction to S is compactly supported, we set $\int_S \omega$ to mean $\int_S \iota^* \omega$, where $\iota : S \hookrightarrow M$ is the inclusion.

Proposition 3.2.8. (Properties of integrals of forms). *Suppose M and N are non-empty smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.*

Then, the following properties hold:

(i) *(*Linearity*)* $\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta \ \forall a, b \in \mathbb{R}$

(ii) *(*Orientation reversal*) If $-M$ denotes M with the opposite orientation, then $\int_{-M} \omega = - \int_M \omega$.*

(iii) *(*Diffeomorphism invariance*) If $f : N \to M$ is an orientation-preserving or orientation-reversing diffeomorphism, then:*

$$\int_M \omega = \begin{cases} \int_N f^* \omega & \text{if } f \text{ is orientation-preserving} \\ -\int_N f^* \omega & \text{if } f \text{ is orientation-reversing} \end{cases}$$

Proof. See ([31], p.408).

□

Remark 3.2.9. It is possible to define integrals of a more general class of objects that allow for the compact-support requirement to be dropped, called *densities.* In particular, the 1-densities of an oriented smooth n-manifold M can be identified with the n-forms of M. For details, see ([41], subsection 3.4.1).

3.2.2 Stokes' theorem

Recall Definition B.2.9.

Theorem 3.2.10. (Stokes' theorem). *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M.*

Then,

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. See ([31], p.412). □

Remark 3.2.11. In the statement above, d is the exterior derivative, ∂M is understood to have the induced orientation and $\int_{\partial M} \omega$ is interpreted, as in Remark 3.2.7, to be $\int_{\partial M} \iota^* \omega$.

Remark 3.2.12. Let us list some consequences of this result that are useful (particularly for cohomology theory, see Ch. 5):

(a) If M is a compact oriented smooth manifold without boundary, then the integral of any differential form η of M, such that $\eta = d\omega$ for some other form ω, vanishes.

(b) Suppose M is a compact oriented smooth manifold with boundary. If ω is a form on M such that $d\omega = 0$, then the integral of ω over ∂M vanishes.

(c) In particular, the facts above imply that if M is a smooth manifold with or without boundary, $S \subseteq M$ is an oriented compact smooth k-dimensional submanifold without boundary and ω is a k-form on M as in (b) such that $\int_S \omega \neq 0$, then the following hold:

  (i) ω is not of the form $\omega = d\eta$ for any other form η on M.

  (ii) S is not the boundary of an oriented compact smooth submanifold with boundary in M.

Remark 3.2.13. We can finally lay out some of the generalisations promised in 2.2.1:

(a) Let M be a smooth manifold and suppose that $\gamma : [a, b] \to M$ is a smooth embedding, so that $S = \gamma([a, b])$ is an embedded 1-submanifold with boundary in M.

If we choose the orientation of S such that γ is orientation-preserving, then for any smooth function $f \in C^\infty(M)$, Stokes' theorem guarantees that

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a))$$

Thus, Stokes' theorem reduces to the fundamental theorem of line integrals. In particular, choosing $\gamma : [a, b] \to \mathbb{R}$ to be the inclusion map, then Stokes' theorem is simply the ordinary fundamental theorem of calculus.

(b) Recall the statement of Green's theorem: Suppose D is a compact regular domain in $\mathbb{R}^2$, and P, Q are smooth real-valued functions on D. Then,

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} Pdx + Qdy$$

We now realise that this is just Stokes' theorem applied to the 1-form $Pdx + Qdy$.

3.2.3 Fibered calculus

In this subsection, we will extend the integration operation to orientable (see 2.3.2) smooth fibre bundles $E \xrightarrow{\pi} B$. We can informally regard this operation as the integration of a family of forms indexed on the base B.

We will work with *split coordinates* $(\mathbf{x}, \mathbf{y}) = (x^i; y^j)$, where $\mathbf{x} = (x^i)$ are local coordinates on the standard fibre F and $\mathbf{y} = (y^j)$ are local coordinates on the base B.

Proposition 3.2.14. *Let $E \xrightarrow{\pi} B$ be an orientable smooth fibre bundle with standard fibre F of dimension r and base B of dimension m. We denote by $\Omega^{\bullet}_{cpt}(\cdot)$ the algebra of compactly-supported differential forms.*

Then, there exists a linear operator

$$\pi_* = \int_{E/B} : \Omega^{\bullet}_{cpt}(E) \to \Omega^{\bullet - r}_{cpt}(B)$$

uniquely defined by its action on forms supported on domains D of split coordinates, $D \cong \mathbb{R}^r \times \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^m$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{y}$.

If ω is a form on E such that $\omega = f dx^I \wedge dx^J$, $f \in C^{\infty}_{cpt}(\mathbb{R}^{r+m})$, with respect to such a domain, then

$$\int_{E/B} = \begin{cases} 0 & |I| \neq r \\ (\int_{\mathbb{R}^r} f dx^I) dy^J & |I| = r \end{cases}$$

The operator $\int_{E/B}$ is called the integration-along-fibres *operator.*

Proof. The idea is completely analogous to that of integration on manifolds. We use a partition of unity to show that these local definitions patch up to a well-defined operator.

□

Proposition 3.2.15. *Let $E \xrightarrow{\pi} B$ be an orientable smooth fibre bundle with r-dimensional standard fibre F.*

Then, for any $\omega \in \Omega^{\bullet}_{cpt}(E)$ and $\eta \in \Omega^{\bullet}_{cpt}(B)$ such that $\mathrm{deg}\omega + \mathrm{deg}\eta = \mathrm{dim}E$, we have:

$$\int_{E/B} d_E \omega = (-1)^r d_B \int_{E/B} \omega$$

If additionally B is oriented, then:

*(i) (*Fubini*) $\int_E (\omega \wedge \pi^* \eta) = \int_B (\int_{E/B} \omega) \wedge \eta$.*

*(ii) (*Projection formula*) $\int_{E/B} (\omega \wedge \pi^* \eta) = (\int_{E/B} \omega) \wedge \eta$.*

Proof. See ([40], p.135).

□

3.3 Connections on vector bundles

We now come to the fundamental notion of a *connection*. As Chris Wendl points out, the need for such a notion arises when trying to answer a question like the following:

> If M is a manifold, X is a vector field and γ is a smooth path in M, how can we judge whether X is constant *along* γ? ([52], p.1).

Of course, this questions would have a clear-cut answer if we had a canonical way to measure lengths of tangent vectors and angles between them in an abstract smooth manifold M. As this is not the case (it calls for the additional structure of a Riemannian metric -see Definition 2.2.20 (c)-),

> Intuitively, one would think that one should call X constant along γ if its derivative in directions tangent to γ is always zero. ([52], p.1).

However, if we try to compute such a derivative with respect to a choice of local coordinates, we hit the snag of it being *non-invariant* under chart transformations. A *connection*, then, defines a notion of *constant* vector fields along paths. This, in turn, allows us to define a notion of *parallel transport* of a tangent vector along a path. We will elaborate a bit more on the ideas behind the latter later on. We strongly recommend ([52], Ch. 1) for further motivations of the concept at hand.

More generally, given E a $\mathbb{K}$-vector bundle over a smooth manifold M ($\mathbb{K} = \mathbb{C}, \mathbb{R}$), we would like to establish a procedure of measuring the rate of change of a section u of E along a direction described by a vector field X. The properties one should expect from such an operation could we listed as:

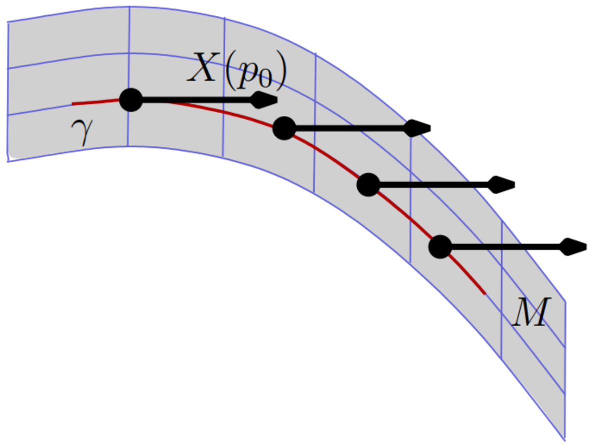

Figure 3.1: Visualisation of the notion of *parallel transport*. Extracted from ([52], p.2).

(i) It should be an operator

$$\nabla : \mathrm{Vect}(M) \times C^\infty(E) \to C^\infty(E)$$
$$(X, u) \mapsto \nabla_X u$$

(ii) If we think of the usual directional derivative, we should expect that the rescalation of the direction X implies rescalation of the derivative along X by the same factor. In other words:

$$\nabla_{f\cdot X} u = f \cdot \nabla_X u, \; \forall f \in C^\infty(M),$$

where the multiplication of a smooth function by a section is pointwise defined in the natural way, as scalar multiplication in the fibres.

(iii) Since ∇ is to be some sort of derivative, it should satisfy some sort of Leibniz rule. The only product that is defined in an abstract vector bundle is the above-mentioned multiplication of a section with a smooth function. Therefore, we expect

$$\nabla_X(fu) = (Xf)u + f\nabla_X u, \; \forall f \in C^\infty(M), u \in C^\infty(E),$$

where $\Big(Xf : p \mapsto X_p(f)\Big) \in C^\infty(M)$, thinking of X_p as a derivation (see Definition 2.1.1), and $\big(X : f \mapsto Xf\big)$ is the derivation of the algebra $C^\infty(M)$ induced by the vector field X (for details, see ([17], p. 52)).

Remark 3.3.1. With Xf defined as above, we can introduce the *Lie bracket* of two vector fields over a smooth manifold, the vector field $[X, Y]$ uniquely defined by its action on smooth maps: $[X, Y]f \coloneqq X(Yf) - Y(Xf)$.

We can finally give the following definition:

Definition 3.3.2. *Let $E \xrightarrow{\pi} M$ be a smooth $\mathbb{K}$-vector bundle over a smooth manifold M ($\mathbb{K} = \mathbb{C}, \mathbb{R}$). A* covariant derivative *(or* linear connection*) on E is a $\mathbb{K}$-linear map $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E) \cong \underline{Hom}(TM, E)$ such that*

$$\nabla(fu) = df \otimes u + f\nabla_u, \; \forall f \in C^\infty(M), u \in C^\infty(E)$$

(Note: for clarification on the above isomorphism, cfr. Definition 2.2.9 and Remark 2.2.17).

Example 3.3.3. Let $\underline{\mathbb{K}}^r_M \cong \mathbb{K}^r \times M$ be the rank-r trivial vector bundle over M. The space $C^\infty(\underline{\mathbb{K}}^r_M)$ of smooth sections coincides with the space $C^\infty(M, \mathbb{K}^r)$ of $\mathbb{K}^r$-valued smooth functions on M. We can define

$$\nabla^0 : C^\infty(M, \mathbb{K}^r) \to C^\infty(M, T^*M \otimes \mathbb{K}^r)$$
$$(f_1, \ldots, f_r) \mapsto (df_1, \ldots, df_r)$$

It is easy to check that ∇^0 is a covariant derivative, the so-called *trivial connection*.

Remark 3.3.4. Let ∇_0, ∇_1 be two connections on a vector bundle $E \to M$. Note that for any $\alpha \in C^\infty(M)$, the map $\nabla = \alpha\nabla_1 + (1-\alpha)\nabla_0 : C^\infty(E) \to C^\infty(T^*M \otimes E)$ is again a connection.

Definition 3.3.5. *For any vector bundle E over a smooth manifold M, we set $\Omega^k(E) := C^\infty(\Lambda^k T^*M \otimes E)$, and we will refer to these sections as* differential k-forms with coefficients in the vector bundle E.

We can work locally with connections via the so-called connection 1-forms:

Definition 3.3.6. *Let ∇ be a connection on a $\mathbb{K}$-vector bundle $E \xrightarrow{\pi} M$ of rank k over a smooth manifold M. Let (U, Ψ) be a trivialisation of E.*

Then, the connection 1-form of ∇ with respect to the trivialisation Ψ *is defined to be the $k \times k$-matrix-valued 1-form*

$$A := \Psi \circ \nabla \circ \Psi^{-1} - d \in \Omega^1\big(U, M(k, \mathbb{R})\big),$$

where d is the exterior derivative.

In other words, given a section $s : U \to E$, $\Psi(\nabla s)) = d(\Psi \circ s) + A(\Psi \circ s)$.

Lemma 3.3.7. *Let Ψ be as above and (V, Φ) be another trivialisation of E. Let $g : U \cap V \to End(\mathbb{K}^k)$ such that the associated transition function verifies $\Phi \circ \Psi^{-1}(p, v) = \big(p, g(p)(v)\big)$.*

Then, for the connection 1-forms A_Ψ and A_Φ, we have: $A_\Psi = g^{-1}dg + g^{-1}A_\Phi g$.

Proof. See ([38], p.28).

□

Conversely, one can show that:

Proposition 3.3.8. *Let $E \xrightarrow{\pi} M$ be a $\mathbb{K}$-vector bundle of rank k over a smooth manifold M. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be a trivialising cover with transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{K})$.*

Then, any collection of matrix-valued 1-forms $\Gamma_\alpha \in \Omega^1(\text{End}\ \underline{\mathbb{K}}^k_{U_\alpha})$ satisfying

$$\Gamma_\beta = g^{-1}_{\alpha\beta} dg_{\alpha\beta} + g^{-1}_{\alpha\beta}\Gamma_\alpha g_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta \tag{3.1}$$

uniquely defines a covariant derivative on E.

This result allows us to define the pullback of a connection:

Definition 3.3.9. *Let $f : M \to N$ be a smooth map between manifolds and E a $\mathbb{K}$-vector bundle of rank k over M defined by the trivialising cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ with transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{K})$.*

Let ∇ be a connection on E defined by a collection of matrix-valued 1-forms $\Gamma_\alpha \in \Omega^1(\text{End}\ \underline{\mathbb{K}}^k_{U_\alpha})$ satisfying the gluing conditions expressed in Equation (3.1).

Then, the pullback of ∇ by f *is the connection $f^*\nabla$ on f^*E described by the open cover $\{f^{-1}(U_\alpha)\}_{\alpha \in \Lambda}$, transition maps $g_{\alpha\beta} \circ f$ and 1-forms $f^*\Gamma_\alpha$.*

One can check that this connection is independent of the various choices made.

To finish off, we can now give some intuition of how a connection defines a notion of *parallel transport* on a manifold M. The main idea to bear in mind is that it provides a way of identifying different fibres. Indeed:

Remark 3.3.10. Let $E \xrightarrow{\pi} M$ be a rank-r vector bundle over a smooth m-manifold M and let ∇ be a covariant derivative on E. For any smooth path $\gamma : [0, 1] \to M$, we will define a *linear* isomorphism $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ called the *parallel transport along* γ. In fact, we will construct an entire family of linear isomorphisms $T_t : E_{\gamma(0)} \to E_{\gamma(t)}$.

As stated, we will think of T_t as identifying different fibres. In particular, if $u_0 \in E_{\gamma(0)}$, then the path $t \mapsto u_t := T_t(u_0) \in E_{\gamma(t)}$ should be a "constant path". Of course, we now have a rigorous way of expressing this requirement: as usual, an object is constant if its derivative is identically 0.

The only derivative we have at hand is ∇. In other words, we should require u_t to satisfy

$$\nabla_{\frac{d}{dt}} u_t = 0, \text{where } \frac{d}{dt} = \dot{\gamma}$$

The above equation suggests a way of defining T_t. For any $u_0 \in E_{\gamma(0)}$, and any $t \in [0, 1]$, define $T_t u_0$ to be the value at t of the solution of the initial value problem:

$$\begin{cases} \nabla_{\frac{d}{dt}} u(t) & = & 0 \\ u(0) & = & u_0 \end{cases}$$

Choosing local coordinates $\mathbf{x} = (x^1, \ldots, x^m)$ and using the above-described connection 1-forms, this equation can be seen to be a system of linear ordinary differential equations.

3.3.1 The curvature of a connection

Proposition 3.3.11. *Let $E \xrightarrow{\pi} M$ be a rank-r vector bundle over a smooth m-manifold M and let ∇ be a covariant derivative on E. Note that ∇ can be thought of as an operator $\nabla : \Omega^0(E) = C^\infty(E) \to \Omega^1(E)$.*

The connection ∇ has a natural extension to an operator $d^\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ uniquely defined by the requirements:

(i) $d^\nabla|_{\Omega^0(E)} = \nabla$

(ii) $\forall \omega \in \Omega^k(M), \eta \in \Omega^s(E)$, $d^\nabla(\omega \wedge \eta) := d\omega \wedge \eta + (-1)^k \omega \wedge d^\nabla \eta$.

This is usually called the exterior derivative associated to ∇.

Proof. See ([41], p.102).

□

Example 3.3.12. The extension d^{∇^0} of the trivial connection from Example 3.3.3 is the ordinary exterior derivative.

Lemma 3.3.13. *For any smooth function $f \in C^\infty(M)$ and any $\omega \in \Omega^k(E)$ we have: $(d^\nabla)^2(f \cdot \omega) = f \cdot \big((d^\nabla)^2(\omega)\big)$.*

In other words, $(d^\nabla)^2$ is a bundle morphism $\Lambda^r T^*M \otimes E \to \Lambda^{r+2} T^*M \otimes E$.

Proof. It is a straightforward computation:

$$(d^\nabla)^2(f \cdot \omega) = d^\nabla(df \wedge \omega + f d^\nabla \omega)$$

$$= -df \wedge d^\nabla \omega + df \wedge d^\nabla \omega + f(d^\nabla)^2 \omega$$

$$= f(d^\nabla)^2 \omega$$

□

Remark 3.3.14. As a map $\Omega^0(E) \to \Omega^2(E)$, the operator $(d^\nabla)^2$ can be identified with a section of

$$\mathrm{Hom}_{\mathbb{K}}(E, \Lambda^2 T^*M \otimes_{\mathbb{R}} E) \cong E^* \otimes \Lambda^2 T^*M \otimes_{\mathbb{R}} E \cong \Lambda^2 T^*M \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{K}}(E)$$

Thus, $(d^\nabla)^2$ is an $\mathrm{End}_{\mathbb{K}}(E)$-valued 2-form.

Definition 3.3.15. *For any connection ∇ on a smooth vector bundle $E \to M$, the object $(d^\nabla)^2 \in \Omega^2\big(\mathrm{End}_{\mathbb{K}}(E)\big)$ is called the* curvature of ∇, *and it is denoted by $F(\nabla)$.*

Example 3.3.16. Consider the trivial bundle $\underline{\mathbb{K}}^r_M$. The sections of this bundle can be seen as smooth $\mathbb{K}^r$-valued functions on M. We know (cfr. Example 3.3.3) that the exterior derivative d defines the trivial connection on $\underline{\mathbb{K}}^r_M$, and any other connection differs from d by a $M(\mathbb{K}, r)$-valued 1-form on M.

If A is such a form, then the curvature of the connection $d + A$ is the 2-form $F(A)$ defined by

$$F(A)s = (d + A)^2 s = (dA + A \wedge A)s, \; \forall s \in C^\infty(M, \mathbb{K}^r),$$

where $\wedge$ above is the operation defined for any vector bundle E as the bilinear map

$$\wedge : \Omega^j(\mathrm{End}(E)) \times \Omega^j(\mathrm{End}(E)) \to \Omega^{j+k}(\mathrm{End}(E))$$

$$(\omega^j \otimes f) \wedge (\eta^s \otimes g) = \omega^j \wedge \eta^s \otimes fg, \; \forall f, g \in C^\infty(\mathrm{End}(E))$$

The curvature can be seen as indicating the failing, in general, of the associated exterior derivative $d^\nabla : \Omega^r \to \Omega^{r+1}$ satisfying the usual $(d^\nabla)^2 = 0$. We have, however, the following result:

Proposition 3.3.17. (Bianchi identity). *Consider a connection ∇ on a smooth vector bundle $E \to M$. This induces a connection in $E^* \otimes E \cong \mathrm{End}(E)$ that we denote by $\nabla^{\mathrm{End}(E)}$. This extends to an associated exterior derivative $D^E = d^{\nabla^{\mathrm{End}(E)}} : \Omega^p(\mathrm{End}(E)) \to \Omega^{p+1}(\mathrm{End}(E))$.*

Then, $D^E F(\nabla^E) = 0$.

Proof. See ([41], p.108).

□

Example 3.3.18. Let $\underline{\mathbb{K}}$ be a trivial line bundle over a smooth manifold M. As we remarked in Example 3.3.16, any connection on $\underline{\mathbb{K}}$ has the form $\nabla^\omega = d + \omega$, where d is the trivial connection and ω is a $\mathbb{K}$-valued 1-form on M.

The curvature of this connection is $F(\omega) = dw$. The Bianchi identity in this case is precisely the equality $d^2\omega = 0$.

3.4 Connections on principal bundles

We will now extend the notion of a connection to principal G-bundles. We will assume all the appearing Lie groups to be *matrix Lie groups*. This is not a severe restriction, since, as a corollary of the Peter-Weyl theorem, any compact Lie group is isomorphic to a matix Lie group (see [51] for details).

Definition 3.4.1. *The Lie algebra $\mathfrak{g}$ of a matrix Lie group G is a Lie algebra of matrices in which the bracket is the usual commutator. The Lie group G operates on its Lie algebra $\mathfrak{g}$ via the* adjoint action $Ad: G \times \mathfrak{g} \to \mathfrak{g}$

$$(g, X) \mapsto Ad_g(X) := gXg^{-1}$$

We denote by $\mathrm{Ad}(P)$ *the vector bundle with standard fibre $\mathfrak{g}$ associated to P via the adjoint representation (see Definition 2.3.12).*

Now, using Proposition 3.3.8 as motivation, we can introduce the following definitions:

Definition 3.4.2. *Consider a principal G-bundle $P \xrightarrow{\pi} M$ over a smooth manifold M, defined by an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and gluing cocycles $g_{\alpha\beta} : U_{\alpha\beta} \to G$, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$.*

(a) A connection A on P *is a collection $\{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$, satisfying the transition rules*

$$A_\beta(p) = g_{\alpha\beta}^{-1}(p)dg_{\alpha\beta}(p) + g_{\alpha\beta}^{-1}(p)A_\alpha(p)g_{\alpha\beta}(p) \text{ for any } p \in U_{\alpha\beta}$$

(b) The curvature of a connection A *on the principal bundle P is the collection $\{F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] \in \Omega^2(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$.*

We have the following results:

Proposition 3.4.3. *For any connection A, on a principal G-bundle $P \xrightarrow{\pi} M$ over a smooth manifold M, defined as in 3.4.2, the collection $\{F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] \in \Omega^2(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$ defines a global $\mathfrak{g}$-valued 2-form.*

We will denote it by $F(A)$ and we will refer to it as the curvature of the connection A.

Proof. See ([41], p. 313).

□

Remark 3.4.4. In particular, Proposition 3.4.3 tells us that a collection $\{F_\alpha\}_{\alpha\in\Lambda}$ as in Definition 3.4.2 satisfies, on each $U_{\alpha\beta}$, the transition rule $F_\beta = g_{\alpha\beta}F_\alpha g_{\alpha\beta}^{-1}$, so that the locally defined curvature 2-forms patch up to a global form on account of their compatibility on the overlaps.

Proposition 3.4.5. (Bianchi identity). *Consider a connection A on a principal G-bundle $P \xrightarrow{\pi} M$ over a smooth manifold M, as in Definition 3.4.2. Let $\{F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] \in \Omega^2(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$ be its curvature.*

Then, it holds:

$$dF_\alpha + [A_\alpha, F_\alpha] = 0, \; \forall \alpha \in \Lambda$$

Proof. See ([41], p. 314).

□

Remark 3.4.6. It is common practice to think of the connection and curvature forms of a principal fibre bundle as matrices of $\mathbb{K}$-valued forms, assuming we are working with a $\mathbb{K}$-matrix Lie group. We thought it useful to provide the reader here with a brief, explicit description of this convention.

Let $P \xrightarrow{\pi} M$ be a principal G-bundle over a smooth n-manifold M as in Definition 3.4.2, where G is a matrix Lie group. Consider the element $\omega \coloneqq A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$. This is a $\mathfrak{g}$-valued 1-form, which means that $\omega_p \coloneqq \omega(p) : T_pM \to \mathfrak{g}$, for any $p \in U_\alpha$.

Provided that $\mathfrak{g}$ is a matrix Lie subalgebra of $M(n, \mathbb{K})$, ($\mathbb{K} = \mathbb{C}, \mathbb{R}$), we have that $\omega_p(v)$ is a $\mathbb{K}$-valued $n \times n$ matrix. We set then $\omega_p(v) = (a_{ij})$. If we now vary v along T_pM, we get $\mathbb{K}$-valued 1-forms $a_{ij}(p) : T_pM \to \mathbb{K}$. Thus, we can think of ω as a matrix of such forms and write $\omega = [\omega_{ij}]$.

Analogously, let Ω be the curvature form associated to a connection on P. Then, $\forall p \in M$, $\Omega(p) : T_pM \times T_pM \to \mathfrak{g}$, so, as above, we have a matrix $\Omega(p)(v_1, v_2) = [b_{ij}]$ for each pair $v_1, v_2 \in T_pM$. We can therefore write $\Omega = [\Omega_{ij}]$, with $\Omega_{ij} : M \to \Lambda^2 T^*M$.

Chapter 4

Elements of Riemannian geometry

In the two previous chapters, we have already hinted at the notion of a Riemannian metric, which allows one to define on an arbitrary smooth manifold such classical geometric notions as the length of a curve or the angle at an intersection of curves. A manifold with an additional such structure is called a *Riemannian manifold*. Intuitively, the only transformations that such a manifold is allowed to undergo are those which preserve the length of paths on it.

The setting of the original proof of the Gauss-Bonnet-Chern theorem was in fact that of a Riemannian manifold. Therefore, it is only natural we expose here some basic definitions and results concerning these objects.

We mainly follow ([41], Ch. 4) throughout.

4.1 Connections on tangent bundles

We start off by making some remarks on the special case when we have a connection on the *tangent bundle* of a manifold.

Definition 4.1.1. *Let ∇ be a connection on the tangent bundle TM of a smooth m-manifold M.*

Note that a choice of local coordinates $\mathbf{x} = (x^1, \ldots, x^m)$ naturally defines a local frame $\mathbf{d} = (\partial_{x^1}, \ldots, \partial_{x^1})$. Furthermore, recall that the algebra of vector fields is invariant under covariant derivation along a vector field. In other words, for any $i, j = 1, \ldots, m$, the result of the covariant derivation of the vector field ∂_{x^j} along the vector field ∂_{x^i} can again be expressed in the frame $\mathbf{d}$:

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \sum_{k=1}^{m} \Gamma_{ij}^k \partial_{x^k}$$

The coefficients Γ_{ij}^k are known as the Christoffel symbols of the connection.

In the case where $E = TM$, the description of a connection in terms of these coefficients is equivalent to a description in terms of the 1-forms described in Definition 3.3.6.

Definition 4.1.2. *In this case, we can explicitly describe the* curvature tensor *field in the following way. Given two vector fields X, Y over the smooth manifold M and a connection ∇ on TM:*

$$F(X, Y) \coloneqq [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in C^\infty(End(TM)),$$

where $[\cdot, \cdot]$ is the Lie bracket (see Remark 3.3.1).

However, in this case there is an additional tensor field naturally associated to the connection. Indeed, it is easy to see that:

Lemma 4.1.3. *For a smooth manifold M, a connection ∇ on TM and $X, Y \in$ Vect(M), consider*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in Vect(M)$$

Then, for any $f \in C^\infty(M)$, $T(fX, Y) = T(X, fY) = fT(X, Y)$, so that $T(\cdot, \cdot) \in \Omega^2(TM)$, i.e., a 2-form whose coefficients are vector fields on M (recall Definition 3.3.5).

The tensor field T is called the torsion *of the connection ∇.*

Definition 4.1.4. *A connection ∇ on TM is said to be* symmetric *if $T = 0$.*

4.2 Metric properties

We have already pointed out (see Definition 2.2.20 (c)) that to define a Riemannian metric on a manifold, one needs only endow it with precisely what we pointed out was lacking for a canonical notion of *parallel transport* to be defined on an arbitrary manifold (see section 3.3): a canonical way to measure lengths of tangent vectors. More precisely:

Definition 4.2.1. *(a) A* Riemannian manifold *is a pair* (M, g) *consisting of a smooth manifold* M *and a real metric* g *on the tangent bundle, i.e., a smooth, symmetric, positive definite* $(0, 2)$*-tensor field on* M*, called a* Riemann metric on M*.*

(b) Two Riemann manifolds (M_i, g_i) $(i = 1, 2)$ *are said to be* isometric *if there exists a diffeomorphism* $\phi : M_1 \to M_2$ *such that* $\phi^* g_2 = g_1$*.*

Remark 4.2.2. If (M, g) is a Riemann manifold then, for any $p \in M$, the restriction $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product on the tangent space T_pM.

(a) The *length of a tangent vector* $v \in T_pM$ is defined as usual: $|v|_p := g_p(v, v)^{\frac{1}{2}}$.

(b) The *length of a piecewise smooth path* $\gamma : [a, b] \to M$ is defined as $l(\gamma) := \int_a^b |\dot\gamma(t)|_{\gamma(t)} dt$.

Example 4.2.3. The space $\mathbb{R}^n$ has a natural Riemann metric $g_0 = (dx^1)^2 + \cdots + (dx^n)^2$. The geometry of $(\mathbb{R}^n, g_0)$ is the classical Euclidean geometry.

Remark 4.2.4. Let (M, g) be a Riemann manifold and $S \subset M$ a submanifold. If $\iota : S \to M$ denotes the natural inclusion, then we obtain by pullback a metric on S: $g_s = \iota^* g = g|_S$.

Proposition 4.2.5. *Let* M *be a smooth* m*-manifold. Then, there exists a Riemannian metric on* M*.*

Proof. Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ be an atlas of M and let $\mathbf{x}_\alpha = (x^i_\alpha)$ be corresponding local coordinates on U_α. Using these local coordinates we can construct the metric g_α on U_α by

$$g_\alpha = (dx^1_\alpha)^2 + \cdots + (dx^m_\alpha)^2$$

Now, pick a partition of unity $\{f_\beta\}_{\beta \in \mathcal{B}} \subset C^\infty(M)$ subordinated to the cover $\mathcal{U}$. This means that there exists some $\phi : \mathcal{B} \to \Lambda$ such that $\text{supp} f_\beta \subset U_{\phi(\beta)}$. Then define

$$g = \sum_{\beta \in \mathcal{B}} f_\beta g_{\phi(\beta)}$$

It is easy to check that g is well defined and it is a Riemann metric on M.

□

4.2.1 The Levi-Civita connection

Definition 4.2.6. *Let* (M, g) *be a smooth Riemannian manifold. A connection* ∇ *on a vector bundle* $E \xrightarrow{\pi} M$ *is said to be* compatible with the metric g *on* M *if for any vector fields* X, Y, Z *on* M*:*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

where the vector field X *operates on the left-hand side, pointwise, as a derivation on the smooth function* $g(Y, Z) : p \mapsto g_p\big(Y(p), Z(p)\big)$*.*

Proposition 4.2.7. *Consider a Riemann manifold* (M, g)*. Then, there exists a* unique symmetric *connection* ∇ *on* TM *compatible with the metric* g*.*

The connection ∇ *is called the* Levi-Civita connection associated to the metric g.

Proof. See ([41], p.142).

□

Definition 4.2.8. *(a) A neighborhood $\mathcal{N}$ of $\{0\} \times M$ in $\mathbb{R} \times M$ is called* balanced *if, $\forall m \in M$, there exists $r \in (0, \infty]$ such that*

$$(\mathbb{R} \times \{m\}) \cap \mathcal{N} = (-r, r) \times \{m\}$$

(b) A local flow *is a smooth map $\Phi : \mathcal{N} \to M$, $(t, m) \mapsto \Phi^t(m)$, where $\mathcal{N}$ is a balanced neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that:*

(i) $\Phi^0(m) = m$, $\forall m \in M$

(ii) $\Phi^t(\Phi^s(m)) = \Phi^{t+s}(m)$ *for all* $s, t \in \mathbb{R}$, $m \in M$ *such that*

$$(s, m), (s + t, m), (t, \Phi^s(m)) \in \mathcal{N}$$

When $\mathcal{N} = \mathbb{R} \times M$, Φ is called a flow.

The conditions (i) and (ii) above mean that a flow is simply a left action of the additive Lie group $(\mathbb{R}, +)$ on M.

Now, observe that a line segment of $\mathbb{R}^3$ can be thought of as a smooth path $\gamma : [0, 1] \to \mathbb{R}^3$ satisfying $\ddot{\gamma}(t) = 0$. This motivates the following definition, that tries to formalise the notion of a "straight line" on an abstract Riemann manifold:

Definition 4.2.9. *A* geodesic *on a Riemann manifold (M, g) is a smooth path $\gamma : (a, b) \to M$ satisfying*

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0, \tag{4.1}$$

where ∇ is the Levi-Civita connection.

Proposition 4.2.10. *Let (M, g) be a Riemann manifold of dimension m. For any compact subset $K \subset TM$ there exists $\epsilon > 0$ such that for any $(x, X) \in K$ there exists a unique geodesic $\gamma = \gamma_{(x,X)} : (-\epsilon, \epsilon) \to M$ such that*

(i) $\gamma(0) = x$

(ii) $\dot{\gamma}(0) = X$

Proof. Using local coordinates $\mathbf{x} = (x^1, \ldots, x^m)$ we can rewrite Equation 4.1 as a second order, nonlinear system of ordinary differential equations (for details, see ([17], subsection 13.3)):

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \ \forall k = 1, \ldots, m \tag{4.2}$$

Since the Chrystoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ij}(p)$ depend smoothly upon $p \in M$, the proof boils down to an application of the classical theorem of existence for initial-value problems (see [3], section 31).

□

Definition 4.2.11. *One can think of a geodesic as defining a path in the tangent bundle $t \mapsto (\gamma(t), \dot{\gamma}(t))$. Proposition 4.2.10 shows that the geodesics define a local flow Φ on TM by $\Phi^t(x, X) = (\gamma(t), \dot{\gamma}(t))$, $\gamma = \gamma_{(x,X)}$.*
This is called the geodesic flow *of the Riemann manifold (M, g).*

4.2.2 The exponential map and normal coordinates

There are severe deviations of Riemannian geometry from the classical Euclidean geometry. For instance, two distinc geodesics on a Riemann manifold may intersect at more than one point (see ([41], 4.1.2)). However, one can define some special collections of local coordinates on (M, g) in which the expression of the metric closely resembles the Euclidean situation $g_0 = \delta_{ij} dy^i dy^j$.

More precisely, let $\mathbf{x} = (x^1, \ldots, x^m)$ be local coordinates on the open set $U \subset M$. Let $q \in U$ be the point with coordinates $(0, \ldots, 0)$. Via a linear change in coordinates we may assume without loss of generality that $g_{ij}(q) = \delta_{ij}$. We say in this case that (g_{ij}) is *Euclidean up to order zero*.

We would like to extend the above equality to an entire neighbourhood of q. To achieve this we try to find local coordinates $\mathbf{y} = (y^j)$ near q such that in these new coordinates the metric is *Euclidean up to order one*, i.e.,

$$g_{ij}(q) = \delta_{ij} \text{ and } \frac{\partial g_{ij}}{\partial y^k}(q) = \frac{\partial \delta_{ij}}{\partial y^k}(q) = 0, \ \forall i, j, k = 1, \ldots, m$$

It is the purpose of this subsection is to explain how to construct, based on the above-presented geodesic flow, these useful sets of coordinates.

Definition 4.2.12. *(a) Denote by $\gamma_{(q,X)}(t)$ the geodesic from q with initial direction $X \in T_qM$. Note the following easy-to-check fact:*

$$\forall s > 0,\ \gamma_{(q,sX)}(t) = \gamma_{(q,X)}(st)$$

Hence, there exists a small neighbourhood V of $0 \in T_qM$ such that, for any $X \in V$, the geodesic $\gamma_{(q,X)}(t)$ is defined for all $|t| \leq 1$.

We define the exponential map at q *by* $\exp_q : V \subset T_qM \to M$

$$X \mapsto \gamma_{(q,X)}(1)$$

(b) The tangent space T_qM is a Euclidean space, and we can define $\mathbf{D}_q(r) \subset T_qM$, the open "disk" of radius r centered at the origin.

Proposition 4.2.13. *Let (M, g) and $q \in M$ as above. Then, there exists $r > 0$ such that the exponential map* $\exp_q : \mathbf{D}_q(r) \subset T_qM \to M$ *is a diffeomorphism.*

The supremum of all radii r with this property is denoted by $\rho_M(q)$ and it is called the injectivity radius of M at q.

The infimum $\rho_M := \inf_q \rho_M(q)$ is called the injectivity radius of M.

Proof. See ([41], p.148). □

Definition 4.2.14. *Choose an orthonormal frame $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of T_qM and denote by $(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ the resulting Cartesian coordinates in T_qM. Notice that for $0 < r < \rho_M(q)$, any point $p \in \exp_q(\mathbf{D}_q(r))$ can be uniquely written as*

$$p = \exp_q(\boldsymbol{x}^i \boldsymbol{e}_i),$$

so that the collection $(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ induces a coordinatisation of the open set $\exp_q(\mathbf{D}_q(r)) \subset M$. The coordinates thus obtained are called normal coordinates at q.

The open set $\exp_q(\mathbf{D}_q(r))$ is called a normal neighbourhood *and will be denoted by $\mathbf{B}_r(q)$.*

Lemma 4.2.15. *In normal coordinates $(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ at q, the Christoffel symbols Γ^k_{ij} vanish at q.*

Proof. See ([41], p. 149). □

With this lemma, we can finally complete our programme:

Proposition 4.2.16. *Let $(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ be normal coordinates at $q \in M$ and denote by $\boldsymbol{g}_{ij}$ the expression of the metric tensor in these coordinates. Then, we have:*

$$\boldsymbol{g}_{ij}(q) = \delta_{ij} \text{ and } \frac{\partial \boldsymbol{g}_{ij}}{\partial \boldsymbol{x}^k}(q) = 0,\ \forall i, j, k = 1, \dots, m$$

Proof. See ([41], p. 149). □

4.3 Riemannian curvature

With the introduction of the *Riemann curvature*, we are already approaching some deeper understanding of our main result. Indeed, the original Gauss-Bonnet theorem states that this *local object* encodes *global* information about the surface. In Chapter 7 we will see how this extends to arbitrary dimension via the so-called *Euler class* of a vector bundle $E \xrightarrow{\pi} M$., defined using the curvature of a connection on E.

Definition 4.3.1. *Let (M, g) be a Riemann manifold, and denote by ∇ the Levi-Civita connection.*

The Riemann curvature tensor *is the tensor field $R = R(g) := F(\nabla)$, where $F(\nabla)$ is the curvature of this connection.*

4.3.1 Theorema Egregium

Definition 4.3.2. *Let (M, g) be a Riemannian manifold and $(S, \iota^* g) \subset (M, g)$ a Riemann submanifold of M. Define, for a given $p \in S$, a vector $v_p \in T_p M$ to be* normal to S *whenever $g(v_p, w_p)$ for all $w_p \in T_p S$ (i.e., v_p is orthogonal to $T_p S$). The set $N_p S$ of all such v_p is called the* normal space to S at p.

Analogously to the definition of the tangent bundle (see 2.1.12), the total space of the normal bundle NS to S *is defined to be*

$$NS := \bigsqcup_{p \in S} N_p S$$

Local trivialisations are then constructed in the obvious way, by composition of local trivialisations of the tangent bundle with the linear orthogonal projection.

Definition 4.3.3. *Let (M, g) be a Riemannian manifold and $(S, \iota^* g) \subset (M, g)$ a Riemann submanifold of M. Let ∇^M be the Levi-Civita connection on TM.*

(a) The first fundamental form of $S \hookrightarrow M$ *is the induced metric $g_S = \iota^* g$.*

(b) The second fundamental form of $S \hookrightarrow M$ *is the map*

$$\mathcal{N} : \mathrm{Vect}(S) \times \mathrm{Vect}(S) \to C^\infty(NS)$$

$$(U, V) \mapsto (\nabla^M_V U)^\nu,$$

where ν is the orthogonal projection with respect to g_S.

The second fundamental form can be used to establish a relationship between the curvatures of M and S. In particular:

Theorem 4.3.4. (Gauss' Theorema Egregium.) *Let (M, g) be a Riemannian manifold and $(S, \iota^* g) \subset (M, g)$ a Riemann submanifold of M.*

Let R^M denote the Riemann curvature of (M, g) and, respectively, let R^S denote that of $(S, \iota^* g)$. Then, for any $X, Y, Z, T \in$ Vect(S), we have:

$$g(R^M(X,Y)Z, T) = g_S(R^S(X,Y)Z, T) + g_S(\mathcal{N}(X,Z), \mathcal{N}(Y,T)) - g_S(\mathcal{N}(X,T), \mathcal{N}(Y,Z))$$

Proof. See ([41], p.175). □

Chapter 5

De Rham cohomology

Cohomology is a basic technique in algebraic topology, which may be defined as the field that studies procedures of associating to topological spaces algebraic objects that encode information about their topological properties. Cohomology is the dual of *homology*, whose intuitive original motivation can be described as acquiring a technique for "counting holes" on a space. See ([2], Lecture 1) for more details.

In particular, we will be presenting the De Rham cohomology. Intuitively, this technique can be used to study the global solvability of equations of the form $du = \alpha$, where u, α are forms on a certain smooth manifold M. Notice that if such an equation has at least one solution u, then $0 = d^2u = d\alpha$, so this is a necessary condition for solvability. The Poincaré lemma states that this also is sufficient *locally* (see ([41], subsection 7.1.1) for details).

We mainly follow ([41], Ch. 7) throughout.

5.1 De Rham cohomology

Definition 5.1.1. *Let M be an n-dimensional smooth manifold.*

(a) We define the exact k-forms *on M to be*

$$B^k(M) \coloneqq \{d\omega \in \Omega^k(M) : \omega \in \Omega^{k-1}(M)\}$$

and the closed k-forms *to be*

$$Z^k(M) \coloneqq \{\eta \in \Omega^k(M) : d\eta = 0\}$$

(b) Clearly $B^k \subset Z^k$. We construct the quotient: $H^k(M) \coloneqq Z^k(M)/B^k(M)$.

This vector space is called the k-th De Rham cohomology group *of M. It is clearly diffeomorphism invariant.*

Intuitively, it consists of the closed k-forms ω for which the equation $du = \omega$ has no global solution $u \in \Omega^{k-1}(M)$, as hinted at above.

(c) Thus, to any smooth manifold M we can now associate the $\mathbb{Z}$-graded vector space $H^\bullet(M) \coloneqq \bigoplus_{k\geq 0} H^k(M)$.

Definition 5.1.2. *Let M be an n-dimensional smooth manifold.*

(a) The k-th Betti number of M *is $b_k(M) \coloneqq \dim H^k(M)$*

(b) The Euler characteristic of M *is the alternating sum $\chi(M) \coloneqq \sum_k (-1)^k b_k(M)$.*

These are special instances of the general framework of (co)homology theory, so that it is useful to introduce a little terminology from homological algebra. We assume all rings to be unitary and commutative.

Definition 5.1.3. *(a) Let R be a ring and let $C^\bullet = \bigoplus_{n\in\mathbb{Z}} C^n$, $D^\bullet = \bigoplus_{n\in\mathbb{Z}} D^n$ be two $\mathbb{Z}$-graded left R-modules.*

A degree k-morphism *$\phi : C^\bullet \to D^\bullet$ is an R-module morphism such that $\phi(C^n) \subset D^{n+k}$, $\forall n \in \mathbb{Z}$.*

(b) Let $C^\bullet$ be a $\mathbb{Z}$-graded R-module as above. A boundary *(respectively* coboundary*) operator is a degree -1 (respectively degree 1) endomorphism $d : C^\bullet \to C^\bullet$ such that $d^2 = 0$.*

A chain *(respectively* cochain*)* complex *over R is a pair $(C^\bullet, d)$, where $C^\bullet$ is a $\mathbb{Z}$-graded R-module, and d is a boudary (respectivley a coboundary) operator.*

Definition 5.1.4. *Let*
$$\dots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \to \dots$$
be a cochain complex of R-modules. Set
$$Z^n(C) := \ker(d_n),\ B^n(C) := \operatorname{Im}(d_{n-1})$$
The elements of $Z^n(C)$ *are called* cocyles *and the elements of* $B^n(C)$ *are called* coboundaries.

Two cocycles $c, c' \in Z^n(C)$ *are said to be* cohomologous *if* $c - c' \in B^n(C)$. *The quotient module of equivalence classes of cohomologous cycles*
$$H^n(C) := Z^n(C)/B^n(C)$$
is called the n-th cohomology group *of* C.

Example 5.1.5. (The de Rham complex). Let M be an m-dimensional smooth manifold. Then the sequence
$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m(M) \to 0,$$
where d is the exterior derivative, is a cochain complex of real vector spaces called the *De Rham complex*. Its cohomology groups are the *De Rham cohomology groups* of the manifold.

Definition 5.1.6. *Let* $(A^\bullet, d)$ *and* $(B^\bullet, \delta)$ *be two cochain complexes of R-modules.*

(a) A cochain morphism *is a degree 0 morphism* $\phi : A^\bullet \to B^\bullet$ *such that* $\phi \circ d = \delta \circ \phi$, *i.e., the diagram below is commutative for any n:*

$$\begin{array}{ccc} A^n & \xrightarrow{d_n} & A^{n+1} \\ \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n+1}} \\ B^n & \xrightarrow{\delta_n} & B^{n+1} \end{array}$$

(b) Two cochain morphisms $\phi, \psi : A^\bullet \to B^\bullet$ *are said to be* cochain homotopic, *and we write* $\phi \simeq \psi$, *if there exists a degree -1 morphism* $\chi : A^\bullet \to B^\bullet$ *such that* $\phi(a) - \psi(a) = \pm(\delta \circ \chi)(a) \pm (\chi \circ d)(a)$.

(c) Two cochain complexes $(A^\bullet, d)$ *and* $(B^\bullet, \delta)$ *are said to be* homotopic *if there exist cochain morphisms* $\phi : A^\bullet \to B^\bullet$, $\psi : B^\bullet \to A^\bullet$ *such that* $\psi \circ \phi \simeq \mathbb{1}_A$ *and* $\phi \circ \psi \simeq \mathbb{1}_B$.

Proposition 5.1.7. *(a) Any cochain morphism* $\phi : (A^\bullet, d) \to (B^\bullet, \delta)$ *induces a degree-zero morphism in cohomology,* $\phi_* : H^\bullet(A) \to H^\bullet(B)$.

(b) If the cochain maps $\phi, \psi : A^\bullet \to B^\bullet$ *are cochain homotopic, then they induce identical morphisms in cohomology,* $\phi_* = \psi_*$.

(c) $(\mathbb{1}_A)_* = \mathbb{1}_{H^\bullet(A)}$, *and if* $(A_0^\bullet, d^0) \xrightarrow{\phi} (A_1^\bullet, d^1) \xrightarrow{\psi} (A_2^\bullet, d^2)$ *are cochain morphisms, then* $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Proof. See ([41], p. 231). □

Corollary 5.1.8. *If two cochain complexes* $(A^\bullet, d)$ *and* $(B^\bullet, \delta)$ *are cochain homotopic, then their cohomology groups are isomorphic.*

Remark 5.1.9. Let $\phi : M \to N$ be a smooth map between smooth manifolds. The pullback $\phi^* : \Omega^\bullet() \to \Omega^\bullet(M)$ is a cochain morphism (i.e., $\phi^* d_N = d_M \phi^*$). Thus, ϕ^* induces a morphism in cohomology which we use the same notation to denote, $\phi^* : H^\bullet(N) \to H^\bullet(M)$.

Definition 5.1.10. *(a) Two smooth maps* $\phi_0, \phi_1 : M \to N$ *are said to be* (smoothly) homotopic, *and we write this* $\phi_0 \simeq_{sh} \phi_1$ *if there exists a smooth map* $\Phi : I \times M \to N$, $(t, m) \mapsto \Phi_t(m)$ *such that* $\Phi_i = \phi_i$, *for* $i = 0, 1$.

(b) A smooth map $\phi : M \to N$ *is said to be a* (smooth) homotopy equivalence *if there exists a smooth map* $\psi : N \to M$ *such that* $\phi \circ \psi \simeq_{sh} \mathbb{1}_N$ *and* $\psi \circ \phi \simeq_{sh} \mathbb{1}_M$.

(c) Two smooth manifolds M and N are said to be homotopy equivalent *if there exists a homotopy equivalence $\phi : M \to N$.*

Proposition 5.1.11. *Let $\phi_0, \phi_1 : M \to N$ be two homotopic smooth maps. Then they induce identical maps in cohomology $\phi_0^* = \phi_1^* : H^\bullet(N) \to H^\bullet(M$.*

Proof. According to Proposition 5.1.7, it suffices to show that the pullbacks are cochain homotopic. See ([41], p. 236).

□

The following result is a fundamental tool for performing computations.

Theorem 5.1.12. (Mayer-Vietoris). *Let $M = U \cup V$ be an open cover of the smooth manifold M. Denote by $\iota_U : U \hookrightarrow M$ and $\iota_V : V \hookrightarrow M$ the inclusions. These induce the restriction maps $\iota_U^* : \Omega^\bullet(M) \to \Omega^\bullet(U)$, $\omega \mapsto \omega|_U$ and $\iota_V^* : \Omega^\bullet(M) \to \Omega^\bullet(V)$, $\omega \mapsto \omega|_V$.*

We get a cochain morphism $r : \Omega^\bullet(M) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V)$, $\omega \mapsto (\iota_U^*\omega, \iota_V^*\omega)$.

We have another cochain morphism given by $\delta : \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V)$, $(\omega, \eta) \mapsto -\omega|_U + \eta|_{U\cap V}$.

Then, there exists a long exact sequence

$$\cdots \to H^k(M) \xrightarrow{r} H^k(U) \oplus H^k(V) \xrightarrow{\delta} H^k(U \cap V) \xrightarrow{\partial} H^{k+1}(M) \to \ldots,$$

where ∂ are called the connection morphisms*.*

It is called the long Mayer-Vietoris sequence*.*

Proof. See ([41], p. 237).

□

The De Rham cohomology vector spaces introduced in 5.1.1 may, *a priori*, be infinite-dimensional. We now give a sufficient condition for finite-dimensionality.

Definition 5.1.13. *A smooth m-manifold M is said to be of* finite type *if it can be covered by finitely many open sets $U_1, \ldots, U_n$ such that any non-empty intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ $(k \geq 1)$ is diffeomophic to $\mathbb{R}^m$. Such a cover is called a* good cover*.*

In particular, any compact manifold is of finite type.

Proposition 5.1.14. *Any finite type manifold has finite Betti numbers (see Definition 5.1.2).*

Proof. See ([41], p. 244).

□

We close this section stating a seminal result in cohomology theory, proved by Georges de Rham himself in 1931. In particular, it holds that the De Rham cohomology of a smooth manifold M (encoding information about the solutions of certain differential equations on M) is isomorphic to its singular cohomology with coefficients in $\mathbb{R}$ (encoding information of the number of "holes" in each dimension). Thus, it shows a connection between the analytical and the topological properties of a smooth manifold. A proof can be given relying on Stokes' theorem, which establishes a duality between de Rham cohomology and singular homology. It can be shown that this is in fact an isomorphism:

Theorem 5.1.15. (De Rham). *Let M be an n-dimensional manifold. Then, for any $k \geq 0$, its de Rham cohomology group $H^k_{dR}(M)$ is isomorphic to its singular cohomology group with real coefficients $H^k(M; \mathbb{R})$.*

Proof. A complete proof using the programme we have outlined above can be found in [42]. The necessary background in singular (co)homology is remarkably well covered in ([23], Ch. 2), and a very good expanded treatment of de Rham theory is to be found in the classic work ([11], Ch.1).

□

5.2 The cohomological Poincaré duality

In this section, we present the (de Rham) *cohomological* Poincaré duality, which relies on integration of differential forms (cfr. Subsection 3.2.1) and establishes a pairing between the de Rham cohomology groups and the de Rham cohomology groups with compact supports of a certain type of manifolds. For the (singular) Poincaré duality, which relies on the so-called cap product and provides a relationship between singular homology and singular cohomology, see ([23], section 3.3).

5.2.1 Cohomology with compact support

Definition 5.2.1. *Let M be a smooth m-dimensional manifold. Denote by $\Omega^k_{cpt}(M)$ the space of smooth compactly-supported k-forms. Then, $0 \to \Omega^0_{cpt}(M) \xrightarrow{d} \Omega^1_{cpt}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m_{cpt}(M) \to 0$ is a cochain complex.*

Its cohomology is denoted by $H^\bullet_{cpt}(M)$ and it is called the de Rham cohomology with compact supports*. Note that when M is compact this coincides with the usual de Rham cohomology.*

Remark 5.2.2. The are important differences between this and the usual de Rham cohomology, the most obvious of which beign the fact that the pullback of a smooth form by an arbitrary smooth map may not have compact support, so smooth maps no longer induce, in general, cohomology morphisms.

However, if $\phi : M \to N$ is an embedding between two smooth manifolds of the same dimension, we can identify M with an open subset of N, and then any $\eta \in \Omega^\bullet_{cpt}(M)$ can be extended by 0 outside $M \subset N$. This extension by zero defines a *push-forward map* $\phi_* : \Omega^\bullet_{cpt}(M) \to \Omega^\bullet_{cpt}(N)$. It is easy to check that it is a cochain map, so that it induces a morphism in cohomology.

An analog to the Mayer-Vietoris sequence from Theorem 5.1.12 exists also for compact supports. From it one can deduce:

Proposition 5.2.3. *Any finite type manifold has finite Betti numbers with compact supports.*

5.2.2 The Poincaré duality

Definition 5.2.4. *Let M be an n-dimensional, finite-type,* oriented *smooth manifold.*

There is a natural pairing $\langle \cdot, \cdot \rangle_\kappa : \Omega^k(M) \times \Omega^{n-k}_{cpt} \to \mathbb{R}$

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta,$$

called the Kronecker pairing*. We can extend this pairing to any $(\omega, \eta) \in \Omega^\bullet(M) \times \Omega^\bullet_{cpt}$ as*

$$\langle \omega, \eta \rangle_\kappa = \begin{cases} 0 & \text{if } \deg \omega + \deg \eta \neq n \\ \int_M \omega \wedge \eta & \text{if } \deg \omega + \deg \eta = n \end{cases}$$

Remark 5.2.5. (a) The Kronecker pairing induces maps $\mathcal{D} = \mathcal{D}^k : \Omega^k(M) \to (\Omega^{n-k}_{cpt}(M))^*$ defined by $\langle \mathcal{D}(\omega), \eta \rangle = \langle \omega, \eta \rangle_\kappa$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between a vector space and its dual (cfr. Example A.3.2).

(b) Stokes' theorem implies that $\mathcal{D}(\omega)$ vanishes on the space of exact, compactly-supported $(n-k)$-forms and that if ω is furthermore exact, $\mathcal{D}(\omega)$ is identically zero (for details, see ([41], p.250)). Hence $\mathcal{D}$ descends to a map in cohomology $\mathcal{D} : H^k(M) \to (H^{n-k}_{cpt}(M))^*$.

Equivalently, this means that the Kronecker pairing descends to a pairing in cohomology $\langle \cdot, \cdot \rangle_\kappa : H^k(M) \times H^{n-k}_{cpt}(M) \to \mathbb{R}$.

Theorem 5.2.6. (Poincaré duality). *The Kronecker pairing in cohomology is a duality for all oriented, smooth manifolds of finite type.*

Proof. See ([41], p. 251). ∎

Corollary 5.2.7. *If M is an oriented, smooth n-dimensional manifold of finite type, then $H^{\bullet}_{cpt}(M) \cong \left(H^{n-\bullet}(M)\right)^*$.*

Proof. As H^k_{cpt} is finite-dimensional for every k, the transpose $\mathcal{D}^{\dagger}_M : \left(H^{n-k}_{cpt}(M)\right)^{**} \to \left(H^k(M)\right)^*$ is an isomorphism. From here it only remains to recall that for any finite-dimensional vector space V, there exists a natural isomorphism $V^{**} \cong V$.

□

Corollary 5.2.8. *Let M be a compact, connected, oriented, n-dimensional manifold. Then, the pairing*

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is a duality.

In particular, $b_k(M) = b_{n-k}(M)$, $\forall k$. □

5.3 Intersection theory

5.3.1 Cycles and their duals

Definition 5.3.1. *Let M be a smooth manifold. A k-*dimensional cycle in M *is a pair (S, ϕ), where S is a compact, oriented k-manifold and $\phi : S \to M$ is a smooth map. We denote by $\mathcal{C}_k(M)$ the set of k-dimensional cycles in M.*

Definition 5.3.2. *(a) Two cycles $(S_0, \phi_0), (S_1, \phi_1) \in \mathcal{C}_k(M)$ are said to be* cobordant, *and we write this $(S_0, \phi_0) \sim_c (S_1, \phi_1)$, if there exists a compact, oriented manifold with boundary Σ (see Definition B.2.9) and a smooth map $\Phi : \Sigma \to M$ such that:*

(i) $\partial\Sigma = (-S_0) \sqcup S_1$, where $-S_0$ denotes S_0 with the reversed orientation.

(ii) $\Phi|_{S_i} = \phi_i$, $i = 0, 1$.

(b) A cycle $(S, \phi) \in \mathcal{C}_k(M)$ is called trivial *if there exists a $(k+1)$-dimensional, oriented manifold Σ with (oriented) boundary S, and a smooth map $\Phi : \Sigma \to M$ such that $\Phi|_{\partial\Sigma} = \phi$. We denote by $\mathcal{T}_k(M)$ the set of trivial cycles.*

Remark 5.3.3. For an expanded presentation of the notion of (co)bordism and its relationship to (co)homology, see [5]. The basic background can be gathered from ([53], section 21).

Example 5.3.4. Consider M a *compact*, smooth, oriented manifold of finite type. In $M \times M$ we can consider the *diagonal cycle* $\Delta = \Delta_M : M \to M \times M$, $x \mapsto (x, x)$.

Definition 5.3.5. *Suppose M is a smooth, oriented manifold of finite type. Any k-cycle (S, ϕ) defines a linear map $H^k(M) \to \mathbb{R}$ given by $\omega \mapsto \int_S \phi^*\omega$. Stokes' theorem guarantees that this map is well defined, i.e., that is is independent of the closed form representing a cohomology class on account of its vanishing on exact forms.*

In other words, each cycle defines an element in $(H^k(M))^*$. Via the Poincaré duality we identify this space with $H^{n-k}_{cpt}(M)$. Thus, there exists $\delta_S \in H^{n-k}_{cpt}(M)$ such that

$$\int_M \omega \wedge \delta_S = \int_S \phi^*\omega, \; \forall\omega \in H^k(M)$$

The compactly-supported cohomology class δ_S is called the Poincaré dual of (S, ϕ).

Proposition 5.3.6. *Let M be a smooth, oriented manifold of finite type. Suppose that $(S_i, \phi_i) \in \mathcal{C}_k(M)$ $(i = 0, 1)$ are two k-cycles in M.*

(a) If $(S_0, \phi_0) \sim_c (S_1, \phi_1)$, then $\delta_{S_0} = \delta_{S_1}$ in $H^{n-k}_{cpt}(M)$.

(b) If (S_0, ϕ_0) is trivial, then $\delta_{S_0} = 0$ in $H^{n-k}_{cpt}(M)$.

Proof. Consider a compact manifold Σ with boundary $\partial\Sigma = -S_0 \sqcup S_1$ and a smooth map $\Phi : \Sigma \to M$ such that $\Phi|_{\partial\Sigma} = \phi_0 \sqcup \phi_1$. For any closed k-form $\omega \in \Omega^k(M)$, we have

$$0 = \int_\Sigma \Phi^*(d\omega) = \int_\Sigma d\Phi^*\omega \overset{Stokes}{=} \int_{\partial\Sigma} \omega = \int_{S_1} \phi_1^*\omega - \int_{S_0} \phi_0^*\omega$$

□

Definition 5.3.7. *Let M be a smooth, oriented manifold of finite type and S a k-dimensional, compact, oriented submanifold of M. We denote the inclusion by $\iota : S \hookrightarrow M$ so that (S, ι) is a k-cycle.*

(a) A smooth map $\phi : T \to M$ from a $(n-k)$-dimensional, oriented manifold T is said to be transversal to S, *and we write $S \pitchfork \phi$, if the following hold:*

   (i) $\phi^{-1}(S)$ is a finite subset of T.

   (ii) For every $x \in \phi^{-1}(S)$, we have $\phi_*(T_xT) \oplus T_{\phi(x)}S = T_{\phi(x)}M$.

(b) If $S \pitchfork \phi$, then for each $x \in \phi^{-1}(S)$ we define the local intersection number at x *to be*

$$i_x(S,T) = \begin{cases} 1 & \text{if } \boldsymbol{or}(T_{\phi(x)}S) \wedge \boldsymbol{or}(\phi_* T_xT) = \boldsymbol{or}(T_{\phi(x)}M) \\ -1 & \text{if } \boldsymbol{or}(T_{\phi(x)}S) \wedge \boldsymbol{or}(\phi_* T_xT) = -\boldsymbol{or}(T_{\phi(x)}M) \end{cases}$$

(c) We define the intersection number of S with T *to be $S \cdot T \coloneqq \sum_{x\in\phi^{-1}(S)} i_x(S,T)$.*

Proposition 5.3.8. *Let M be a a smooth, oriented manifold of finite type. Consider a compact, oriented, k-dimensional submanifold $S \hookrightarrow M$ and $(T,\phi) \in \mathcal{C}_{n-k}(M)$ a $(n-k)$-dimensional cycle intersection S transversally, i.e., $S \pitchfork \phi$.*

Then, $S \cdot T = \int_M \delta_S \wedge \delta_T$.

Proof. See ([41], p.259).

□

5.3.2 The Thom isomorphism

Definition 5.3.9. *Let $p : E \to B$ be an orientable fibre bundle (cfr. Definition 2.3.17) with standard fibre F and compact, oriented basis B of dimensions r and m, respectively. Then, the total space E is a compact, orientable manifold which we equip with the natural orientation (cfr. Remark 2.3.18).*

We know that the integration along fibres $p_* = \int_{E/B} : \Omega^\bullet_{cpt}(E) \to \Omega^{\bullet-r}_{cpt}(B)$ satisfies $p_* d_E = (-1)^r d_B p_*$ (see Proposition 3.2.15), so that it induces a map in cohomology $p_* : H^\bullet_{cpt}(E) \to H^{\bullet-r}_{cpt}(B)$ called the Gysin map.

Remark 5.3.10. Any smooth section $\sigma : B \to E$ of a bundle as above defines an embedded cycle in E of dimension $m = \dim B$. Denote by δ_σ its Poincaré dual in $H^r_{cpt}(E)$. Notice that:

(1) Using the properties of the integration along fibres we deduce that, for any $\omega \in \Omega^m(B)$, we have

$$\int_E \delta_\sigma \wedge p^*\omega = \int_B \left(\int_{E/B} \delta_\sigma \right) \omega$$

(2) On the other hand, by Poincaré duality we get

$$\int_E \delta_\sigma \wedge p^*\omega = (-1)^{rm} \int_E p^*\omega \wedge \delta_\sigma$$

$$= (-1)^{rm} \int_B \sigma^* p^* \omega = (-1)^{rm} \int_B (p\sigma)^*\omega = (-1)^{rm} \int_B \omega$$

Thus, combining the two results we infer $p_*\delta_\sigma = \int_{E/B} \delta_\sigma = (-1)^{rm} \in \Omega^0(B)$.

Proposition 5.3.11. *Let $p : E \to B$ be a bundle as above. If it admits at least one section σ, then the Gysin map $p_* : H^\bullet_{cpt}(E) \to H^{\bullet - r}(B)$ is surjective.*

Proof. It suffices to show that p_* has a right inverse. Denote by τ_σ the map $\tau_\sigma : H^\bullet(B) \to H^{\bullet + r}_{cpt}(E)$, $\omega \mapsto (-1)^{rm}\delta_\sigma \wedge p^*\omega = p^*\omega \wedge \delta_\sigma$.

Then, we have:

$$\omega = (-1)^{rm} p_*\delta_\sigma \wedge \omega = (-1)^{rm} p_*(\delta_\sigma \wedge p^*\omega) = p_*(\tau_\sigma \omega),$$

as we wanted.

□

Remark 5.3.12. The map p_* is not injective in general. For example, if (S, ϕ) is a k-cycle in F, then it defines a cycle in any fibre $\pi^{-1}(b)$, and consequently in E. Denote by δ_S its Poincaré dual in $H^{m+r-k}_{cpt}(E)$. Then, for any $\omega \in \Omega^{m-k}(B)$, we have

$$\int_B (p_*\delta_S) \wedge \omega = \int_E \sigma_S \wedge p^*\omega = \pm \int_S \phi^* p^* \omega = \int_S (p \circ \phi)^* \omega = 0,$$

since $p \circ \phi$ is constant. Hence $p_*\delta_S = 0$, and we conclude that if there are non-trivial cycles in F then $\ker p_*$ may not be trivial.

We will now see that if the fibre is a vector space, then p_* is an isomorphism, an important result due to Thom.

Definition 5.3.13. *Let $p : E \to B$ be an orientable rank-r vector bundle over the compact oriented manifold B of dimension m. The* Thom class *of E, denoted by $\tau_E \in H^r_{cpt}(E)$, is the Poincaré dual of the cycle defined by the zero section $\zeta_0 : B \to E$, $b \mapsto 0 \in E_b$.*

We state now a preliminary technical result:

Lemma 5.3.14. *Let $p : E \to B$ be an orientable rank-r real vector bundle. Denote by p_* the integration-along-fibre map. Then, there exists a smooth bilinear map*

$$\mathfrak{m} : \Omega^i_{cpt}(E) \times \Omega^j_{cpt}(E) \to \Omega^{i+j-r-1}_{cpt}(E)$$

such that

$$p^*p_*\alpha \wedge \beta - \alpha \wedge p^*p_*\beta = (-1)^r d(\mathfrak{m}(\alpha, \beta)) - \mathfrak{m}(d\alpha, \beta) + (-1)^{\deg(\alpha)}\mathfrak{m}(\alpha, d\beta)$$

Proof. See ([9], p.52).

□

Theorem 5.3.15. (Thom isomorphism.) *Let $p : E \to B$ be an orientable rank-r vector bundle over the compact oriented manifold B of dimension m. Then, the map*

$$\tau : H^\bullet(B) \to H^{\bullet + r}(E)$$

$$\omega \mapsto \tau_E \wedge p^*\omega$$

is an isomorphism called the Thom isomorphism. *Its inverse is the map $(-1)^{rm}p_* : H^\bullet_{cpt}(E) \to H^{\bullet - r}(B)$.*

Proof. We have already established in Proposition 5.3.11 that $p_*\tau = (-1)^{rm}$. To prove the reverse equality, we deduce from Lemma 5.3.14 that, for any $\beta \in \Omega^\bullet_{cpt}(E)$, we have

$$(p^*p_*\tau_E) \wedge \beta - \tau_E \wedge (p^*p_*\beta) = (-1)^r d(\mathfrak{m}(\tau_E, \beta)),$$

where $\mathfrak{m}(\tau_E, \beta) \in \Omega^\bullet_{cpt}(E)$. Note that the two additional addends in Lemma 5.3.14 vanish, since cohomology classes are represented by closed forms and $\mathfrak{m}$ is bilinear.

Now, since $p^*p_*\tau_E = (-1)^{rm}$, we deduce that

$$(-1)^{rm}\beta = \tau_E \wedge p^*(p_*\beta) + \text{ exact form } \Rightarrow (-1)^{rm}\beta = \tau_E \circ p_*(\beta) \text{ in } H^\bullet_{cpt}(E)$$

□

Remark 5.3.16. Note that this is equivalent to saying that the Thom class of E is the unique element in $H^\bullet(E)$ the integration of which along every fibre is identically 1. In fact, this is a usual formulation of the above theorem (see, for example, ([8], p.4)).

5.3.3 The Euler class

In this subsection, we will establish an important result for our main theorem: namely, we will compute the integral of the *topological Euler form* of a certain manifold M and we will see it coincides with the Euler characteristic of E.

Definition 5.3.17. *Let $E \to M$ be a real orientable vector bundle over the compact, oriented, n-dimensional, smooth manifold M. Denote by $\tau_E \in H^n_{cpt}(E)$ the Thom class of E. The* Euler class of E *is defined by*

$$\boldsymbol{e}(E) \coloneqq \zeta_0^* \tau_E \in H^n(M),$$

where ζ_0 denotes the zero section.

The Euler class of M, *denoted by $\boldsymbol{e}(M)$, is $\boldsymbol{e}(TM)$.*

Proposition 5.3.18. *Let $\sigma_0, \sigma_1 : M \to TM$ be two sections of TM. They determine cycles in TM of complementary dimension, which means their intersection number is a well-defined integer independent of the two sections. Any two such cycles are homotopic: it suffices to pick an affine homotopy along the fibres of TM.*

Then, $\int_M \boldsymbol{e}(M) = \sigma_0 \cdot \sigma_1$. In particular, if $\dim M$ is odd, then $\int_M \boldsymbol{e}(M) = 0$.

Proof. We have that the sections σ_0, σ_1 are cobordant, and their Poincaré dual in $H^n_{cpt}(TM)$ is the Thom class τ_M. Hence, by proposition 5.3.8:

$$\begin{aligned} \sigma_0 \cdot \sigma_1 &= \int_{TM} \delta_{\sigma_0} \wedge \delta_{\sigma_1} = \int_{TM} \tau_M \wedge \tau_M \\ &= \int_{TM} \tau_M \wedge \sigma_{\zeta_0} = \int_M \zeta_0^* \tau_M = \int_M \boldsymbol{e}(M) \end{aligned}$$

If M is odd-dimensional, then

$$\int_M \boldsymbol{e}(M) = \sigma_0 \cdot \sigma_1 = -\sigma_1 \cdot \sigma_0 = -\int_M \boldsymbol{e}(M)$$

□

To prove the general result, we present a set of preliminary lemmas. Let M be a compact, oriented, n-dimensional, smooth manifold.

Lemma 5.3.19. *Denote by Δ the diagonal cycle in $M \times M$ (see Example 5.3.4). Then, $\chi(M) = \Delta \cdot \Delta$.*

Proof. See ([41], p. 270).

□

Lemma 5.3.20. *Denote by $\exp$ the exponential map of a Riemann metric g on $M \times M$. Regard Δ as a submanifold in N_Δ via the embedding given by the zero section. Then, there exists an open neighbourhood U of $\Delta \subset N_\Delta \subset T(M \times M)$ such that*

$$exp|_U : U \to M \times M$$

is an embedding. (N_Δ is defined in the proof of Theorem 5.3.21).

Proof. See ([41], p. 271).

□

Theorem 5.3.21. *Let M be a compact, oriented, n-dimensional manifold and denote by $\boldsymbol{e}(M) \in H^n(M)$ its Euler class. Then,*

$$\int_M \boldsymbol{e}(M) = \chi(M)$$

In particular, $\chi(M) = 0$ if M is odd-dimensional (see Proposition 5.3.18).

Proof. We use the description of $\chi(M)$ given by Lemma 5.3.19. Note that the tangent bundle of $M \times M$ restricts to the diagonal Δ as a rank-$2n$ vector bundle. If we choose a Riemann metric on $M \times M$ (Proposition 4.2.5 ensure that this can be done), then we get an orthogonal splitting

$$T(M \times M)|_\Delta = N_\Delta \oplus T\Delta$$

The diagonal map $M \to M \times M$ identifies M with Δ so that $T\Delta \cong TM$.

We claim, furthermore, that $N_\Delta \cong TM$. Indeed, it suffices to use the isomorphisms $T(M \times M)|_\Delta \cong T\Delta \oplus N_\Delta \cong TM \oplus N_\Delta$ and the fact that $T(M \times M)|_\Delta = TM \oplus TM$. From this we immediately deduce the equality of Thom classes

$$\tau_{N_\Delta} = \tau_M \tag{5.1}$$

Now, let U be a neighbourhood of $\Delta \subset N_\Delta$ as in Lemma 5.3.20. Set $\mathcal{N} := \exp(U)$. Denote by δ_Δ^U the Poincaré dual of Δ in U, $\delta_\Delta^U \in H^n_{cpt}(U)$ and by $\delta_\Delta^{\mathcal{N}} \in H^n_{cpt}(\mathcal{N})$ the Poincaré dual of Δ in $\mathcal{N}$. Then,

$$\int_U \delta_\Delta^U \wedge \delta_\Delta^U = \int_{\mathcal{N}} \delta_\Delta^{\mathcal{N}} \wedge \delta_\Delta^{\mathcal{N}} = \int_{M\times M} \delta_M \times \delta_M = \chi(M),$$

where δ_M is the Poincaré dual of Δ (cfr. Example 5.3.4).

The cohomology class δ_Δ^U is the Thom class of the bundle $N_\Delta \to \Delta$, which on account of Equation (5.1) means that $\delta_\Delta^U = \tau_{N_\Delta} = \tau_M$. We therefore obtain

$$\int_U \delta_\Delta^U \times \delta_\Delta^U = \int_\Delta \delta_\Delta^U = \int_\Delta \zeta_0^* \tau_{N_\Delta} = \int_M \zeta_0^* \tau_M = \int_M \boldsymbol{e}(M)$$

Thus,

$$\int_M \boldsymbol{e}(M) = \chi(M),$$

as we wanted to show.

□

5.3.4 The Poincaré-Hopf theorem

We end this chapter with a brief subsection where we touch upon the celebrated Poincaré-Hopf theorem. Although we will not make use of it in our proof of the Gauss-Bonnet-Chern theorem, it was a key ingredient in the original proof by Chern, which motivates its inclusion in the present work. We refer to the classic [36] for a very good approximation to this result via Sard's theorem (see, in particular, Ch. 6).

Definition 5.3.22. *Let M be a compact, oriented, n-dimensional, smooth manifold. Let X be a smooth vector field on M*

(a) *The* graph of X in TM *is the n-dimensional submanifold of TM $\Gamma_X = \{(x, X(x)) \in T_xM : x \in M\}$.*

(b) *A point $x_0 \in M$ is said to be a* non-degenerate zero of X *if $X(x_0) = 0$ and*

$$det\left(\frac{\partial X_i}{\partial x^j}\right)|_{x=x_0} \neq 0$$

for some (or, equivalently, any) local coordinates $\mathbf{x} = (x^i)$ near x_0 such that the orientation of $T^*_{x_0}M$ is given by $dx^1 \wedge \cdots \wedge dx^n$.

From Definiton 5.3.7 it is easy to deduce the following result:

Proposition 5.3.23. *If X and x_0 are as above, then the local intersection number of Γ_X with M at x_0 is given by*

$$i_{x_0}(\Gamma_X, M) = \text{sign}\det\left(\frac{\partial X_i}{\partial x^j}\right)|_{x=x_0}$$

This is called the local index of X at x_0 *and it is denoted by $i(X, x_0)$.*

This proposition ultimately implies:

Theorem 5.3.24. (Poincaré-Hopf). *If X is a vector field along a compact, oriented manifold M, with only non-degenerate zeros $x_1, \ldots, x_k$, then*

$$\chi(M) = \sum_{j=1}^{k} i(X, x_j)$$

□

Chapter 6

Characteristic classes

The Gauss-Bonnet-Chern theorem is a classic result in *differential topology*, which studies the interactions between the topology and the smooth structure of a manifold. John Milnor, a towering figure in the field, points out:

> The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, where-by one passes from the manifold M to its tangent bundle, and thence to a cohomology class in M which depends on this bundle ([35], p.1).

We will present here the Chern-Weil theory, a method for producing so-called characteristic classes of arbitrary vector bundles, exposing the necessary ingredients for it. We will also review some of the most important examples. One of them, the Euler class of an oriented, even-rank vector bundle over a smooth manifold, will play a fundamental role in the statement of our main theorem.

We mainly follow ([41], Ch. 8) throughout.

6.1 Frame bundles

One of the elements necessary for the Chern-Weil construction is a principal bundle. In particular, we will be working with frame bundles of vector bundles. It is important to bear the following remarks in mind:

Remark 6.1.1. Let $\pi : E \to M$ be a vector bundle over a smooth manifold M and (ρ, P) a G-structure on E (cfr. Definition 2.3.13). If the collection $\{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$ defines a connection on the principal bundle P, then the collection $\{\rho_*(A_\alpha)\}_{\alpha\in\Lambda}$, where $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ denotes the differential of ρ at $\mathbb{1} \in G$, defines a connection on the vector bundle E. A connection on E obtained in this manner is said to be *compatible with the G-structure*. Note that if $\{F_\alpha\}_{\alpha\in\Lambda}$ is the curvature of the connection on P, then the collection $\{\rho_*\big(F(A_\alpha)\big)\}_{\alpha\in\Lambda}$ coincides with the curvature $\{F\big(\rho_*(A_\alpha)\big)\}_{\alpha\in\Lambda}$ of the connection $\{\rho_*(A_\alpha)\}_{\alpha\in\Lambda}$.

In particular, a connection compatible with some metric on a vector bundle is compatible with the orthogonal/unitary structure of that bundle (more details on this will be given below). The curvature of such a connection is skew-symmetric.

Remark 6.1.2. Let $\pi : E \to M$ be a complex (respectively real) vector bundle over a smooth manifold M.

The existence of Hermitian (respectively Riemann) metrics is guaranteed under weak assumptions on the base: it suffices for it to be paracompact. Indeed, if this holds, there exists a locally-finite cover $\mathcal{U} = \{U_i\}$ over which the bundle is trivial. Then, we can choose any Hermitian (respectively Riemann) metric h_i on the trivialisations $U_i \times \mathbb{C}^n$ (respectively $U_i \times \mathbb{R}^n$) and use a partition of unity $\{f_i\}$ subordinate to $\mathcal{U}$ to get a global metric $h = \sum_i f_i h_i$. This procedure is completely analogous to that of Proposition 4.2.5.

Now, any locally compact, second countable topological space is paracompact (see ([17], p.5)). In particular, any smooth manifold is. Therefore, it is not with loss of generality that we can assume metrics to exist on the arbitrary vector bundles appearing in this section.

6.1.1 The orthogonal frame bundle of a real vector bundle

Let $E \xrightarrow{\pi} M$ be a rank k real vector bundle over a smooth manifold M equipped with a Riemannian metric g. The latter allows us to consider *orthogonal* frames of each fibre E_x. We denote by F_x the set of all such frames at a point $x \in M$, and set $O(E) := \bigsqcup_{x\in M} F_x$.

$O(E)$ can be given the structure of a principal $O(k)$-fibre bundle over M. There is a natural action of $O(k)$ on each fibre F_x induced by the restriction of the natural action of $GL(k,\mathbb{R})$ on F_x. As it is known that, given a basis of a vector space, there exists a unique invertible linear map that sends it to another basis, this action is free.

Now, the elements of $O(E)$ can be thought of as pairs $(x, \mathbf{b})$, where $x \in M$ and $\mathbf{b}$ is a basis of E_x. There is then a natural projection $\widetilde{\pi} : O(E) \to M$, $(x, \mathbf{b}) \mapsto x$.

Finally, we will describe the bundle structure of $O(E)$. Pick a local trivialisation (U_i, φ_i) of E. This determines, for each $x \in U_i$, a linear isomorphism $\varphi_{i|x} : E_x \to \mathbb{R}^k$. Now, we can produce local trivialisations of the orthogonal frame bundle by defining

$$\psi_i : \widetilde{\pi}(U_i) \to U_i \times GL(k,\mathbb{R})$$

$$\big(x, \mathbf{b} = (b_1, ..., b_k)\big) \mapsto \big(x, [\varphi_{i|x}(b_1), ..., \varphi_{i|x}(b_k)]\big)$$

and then applying the Gram-Schmidt orthonormalisation technique.

In short, a Riemann metric on a rank-r vector bundle E induces an $O(r)$-structure on it. $\rho : O(r) \hookrightarrow GL(r,\mathbb{R})$ is the *fundamental representation* given by the natural group inclusion.

Remark 6.1.3. Given two different metrics g, h, on the vector bundle E, one easily deduces from Proposition A.5.3 that the induced orthogonal frame bundles $O_g(E)$, $O_h(E)$ are isomorphic as principal bundles.

In other words, *the* orthogonal frame bundle of a vector bundle is independent of the metric chosen to construct it.

Remark 6.1.4. If the vector bundle E is additionally *oriented*, then we can speak of *positively oriented orthonormal* frames at a point $x \in M$. All the above constructions and results are valid in this case, so that we can consider $SO(E)$, ***the** orthonormal frame bundle of an oriented vector bundle E*. This is a $SO(r)$-structure on E.

6.1.2 The unitary frame bundle of a complex vector bundle

Definition 6.1.5. *(a) A complex square matrix U is said to be* unitary *if its conjugate transpose is also its inverse.*

(b) The unitary group of degree n, *$U(n)$ is the group of $n \times n$ unitary matrices. It is a Lie group of dimension n^2.*

Definition 6.1.6. *Let $\pi : E \to M$ be a complex vector bundle of rank r over the smooth manifold M endowed with a Hermitian metric h. A* unitary frame of E *is a complex linear frame of E that is orthonormal with respect to h.*

In a manner completely analogous to that of the previous subsection, we can now construct the *unitary frame bundle of E* with respect to h, and see that it is in fact independent of the Hermitian metric used to construct it. This is a $U(r)$-structure on E. $\rho : U(r) \hookrightarrow GL(r,\mathbb{C})$ is the *fundamental representation* given by the natural group inclusion.

6.2 Invariant polynomials

Remark 6.2.1. Let V a vector space over $\mathbb{K} = \mathbb{C}, \mathbb{R}$ and consider the symmetric power $S^k(V^*) \subset (V^*)^{\otimes k}$, which consists of symmetric, multilinear maps $\phi : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{K}$.

(a) Note now that any $\phi \in S^k(V^*)$ is completely determined by its *polynomial form* $P_\phi(v) \coloneqq \phi(v, \ldots, v)$, which follows immediately from the well-known *polarisation formula*

$$\phi(v_1, \ldots, v_k) = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \ldots \partial t_k} P_\phi(t_1 v_1 + \cdots + t_k v_k)$$

Thus, if dim$V = n$ and we fix a basis of V, we can identify $S^k(V^*)$ with the space of degree-k homogeneous polynomials in n variables (for details on this identification and a proof of the above formula, see ([43], section 3.2)).

(b) Now, let $\mathcal{A}$ be a unitary algebra over $\mathbb{K}$. Given $\phi \in S^k(V^*))$ we can obtain a $\mathbb{K}$-multilinear map $\phi_{\mathcal{A}} : \overbrace{(\mathcal{A}\otimes V) \times ... \times (\mathcal{A}\otimes V)}^{k \text{ times}} \to \mathcal{A}$, uniquely determined by

$$\phi_{\mathcal{A}}(a_1 \otimes v_1, ..., a_k \otimes v_k) = \phi(v_1, ..., v_k)a_1a_2...a_k \in \mathcal{A}$$

The above form is symmetric if $\mathcal{A}$ is commutative.

Definition 6.2.2. *Consider now a matrix Lie group G. The adjoint action of G on its Lie algebra $\mathfrak{g}$ (see Definition 3.4.1) induces an action on $S^k(\mathfrak{g}^*)$ still denoted by* Ad. *We denote by $I^k(G)$ the subset of those elements of the symmetric power $S^k(\mathfrak{g}^*)$ that are invariant under the adjoint action of G on its Lie algebra $\mathfrak{g}$: i.e., $\phi \in S^k(\mathfrak{g}^*)$ belongs to $I^k(G)$ if, and only if:*

$$\phi(gX_1g^{-1}, \dots, gX_kg^{-1}) = \phi(X_1, \dots, X_k),\ \forall X_1, \dots, X_k \in \mathfrak{g},\ \textit{and}\ \forall g \in G$$

Set $I^\bullet(G) := \bigoplus_{k\geq 0} I^k(G)$. The elements of $I^\bullet(G)$ are called invariant polynomials.

Proposition 6.2.3. *Let $\phi \in I^k(G)$. Then, for any $X, X_1, \dots, X_k \in \mathfrak{g}$ we have*

$$\phi([X, X_1], X_2, \dots, X_k) + \dots + \phi(X_1, X_2, \dots, [X, X_k]) = 0$$

Proof. It follows immediately from the equality

$$\frac{d}{dt}|_{t=0}\phi(e^{tX}X_1e^{-tX}, \dots, e^{tX}X_ke^{-tX}) = 0$$

□

Remark 6.2.4. Let $P \in I^k(\mathfrak{g})$, U be an open subset of $\mathbb{R}^n$ and

$$F_i = \omega_i \otimes X_i \in \Omega^{d_i}(U) \otimes \mathfrak{g},\ A = \omega \otimes X \in \Omega^d(U) \otimes \mathfrak{g}$$

Then,

$$P(F_1, \dots, F_{i-1}, [A, F_i], F_{i+1}, \dots, F_k) = (-1)^{d(d_1+\dots+d_{i-1})}\omega\omega_1 \dots \omega_k P(X_1, \dots, [X, X_i], \dots, X_k)$$

In particular, if $F_1, \dots, F_{k-1}$ have even degree, we deduce that for every $i = 1, \dots, k$ we have

$$P(F_1, \dots, F_{i-1}, [A, F_i], F_{i+1}, \dots, F_k) = \omega\omega_1 \dots \omega_k P(X_1, \dots, [X, X_i], \dots, X_k)$$

Summing over i, and using the Ad-invariance of the polynomial P, we deduce

$$\sum_{i=1}^{k} P(F_1, \dots, F_{i-1}, [A, F_i], F_{i+1}, \dots, F_k) = 0 \tag{6.1}$$

$\forall F_1, \dots, F_{k-1} \in \Omega^{\text{even}}(U) \otimes \mathfrak{g}; F_k, A \in \Omega^\bullet(U) \otimes \mathfrak{g}$.

6.3 The Chern-Weil theory

Definition 6.3.1. *The* Chern-Weil construction *is a tool that enables one to produce invariants of a principal fibre bundle over a smooth manifold. In particular, we obtain a class in the de Rham cohomology of the manifold which is independent of the connection on the principal fibre bundle chosen to construct it.*

It requires three ingredients:

(i) A principal G-bundle $P \xrightarrow{\pi} M$ over a smooth manifold M, defined by an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$, and gluing cocycles $g_{\alpha\beta} : U_{\alpha\beta} \to G$, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$ (see Subsection 2.3.1).

(ii) A connection A on P, defined by the collection $\{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$, satisfying, on $U_{\alpha\beta}$, the transition rules $A_\beta = g_{\alpha\beta}^{-1}dg_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta}$.

Its curvature is then defined by the collection $\{F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] \in \Omega^2(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$, which satisfies, on $U_{\alpha\beta}$, the transition rules $F_\beta = g_{\alpha\beta}F_\alpha g_{\alpha\beta}^{-1}$ (see Section 3.4).

(iii) An invariant polynomial $\phi \in I^k(G)$.

Let us now consider the case $\mathcal{A} = \Omega^{even}(U_\alpha)$, $V = \mathfrak{g}$. We can define as in Remark 6.2.1 (b), given $\phi \in I^k(G)$:

$$P_\phi(F_\alpha) \coloneqq \phi(\overbrace{F_\alpha, ..., F_\alpha}^{k\ times}) \in \Omega^{2k}(U_\alpha).$$

From the transition rules for the curvature along with the Ad-invariance of ϕ we deduce that $P_\phi(F_\alpha) = P_\phi(F_\beta)$ on $U_{\alpha\beta}$. This means that the locally defined forms $P_\phi(F_\alpha)$ are compatible so that they patch up to a global $2k$-form on M denoted by $P_\phi(F_A)$.

Theorem 6.3.2. (Chern-Weil).

(a) The form $P_\phi(F_A)$ is closed for any connection A on P.

(b) If A^0, A^1 are two connection on P, then the forms $P_\phi(F_{A^0})$ and $P_\phi(F_{A^1})$ are cohomologous.
In particular, this means that the closed form $P_\phi(F_A)$ defines a cohomology class in $H^{2k}(M)$ which is independent of the connection A.

Proof. (a) We use the Bianchi identity $dF_\alpha = -[A_\alpha, F_\alpha]$ (see Proposition 3.4.5). The Leibniz rule yields

$$\begin{aligned} d\phi(F_\alpha, \dots, F_\alpha) &= \phi(dF_\alpha, F_\alpha, \dots, F_\alpha) + \dots + \phi(F_\alpha, \dots, F_\alpha, dF_\alpha) \\ &= -\phi([A_\alpha, F_\alpha], F_\alpha, \dots, F_\alpha) - \dots - \phi(F_\alpha, \dots, F_\alpha, [A_\alpha, F_\alpha]), \end{aligned}$$

which vanishes by Proposition 6.2.3.

(b) Let the connections A_i on P, $(i = 0, 1)$, be defined by the collections $\{A^i_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}_{\alpha\in\Lambda}$. Set $C_\alpha \coloneqq A^1_\alpha - A^0_\alpha$. For $0 \le t \le 1$, we define $A^t_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ by $A^t_\alpha \coloneqq A^0_\alpha + tC_\alpha$.
The collection $\{A^t_\alpha\}_{\alpha\in\Lambda}$ defines a connection A^t on P, and $t \mapsto A^t$ is a path connecting A^0 to A^1. Note that $C \coloneqq \{C_\alpha\}_{\alpha\in\Lambda} = \frac{dA^t}{dt}$. Denote by $F^t = \{F^t_\alpha\}_{\alpha\in\Lambda}$ the curvature of A^t. It is easy to compute that

$$F^t_\alpha = F^0_\alpha + t(dC_\alpha + [A^0_\alpha, C_\alpha]) + \frac{t^2}{2}[C_\alpha, C_\alpha], \tag{6.2}$$

from which one deduces

$$\frac{dF^t_\alpha}{dt} = dC_\alpha + [A^0_\alpha, C_\alpha] + t[C_\alpha, C_\alpha] = dC_\alpha + [A^t_\alpha, C_\alpha]$$

Consequently,

$$\begin{aligned} \phi(F^1_\alpha) - \phi(F^0_\alpha) &= \int_0^1 \Big[\phi(\frac{dF^t_\alpha}{dt}, F^t_\alpha, \dots, F^t_\alpha) + \dots + \phi(F^t_\alpha, \dots, F^t_\alpha, \frac{dF^t_\alpha}{dt})\Big] dt \\ &= \int_0^1 \Big[\phi(dC_\alpha, F^t_\alpha, \dots, F^t_\alpha) + \dots + \phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha)\Big] dt \\ &+ \int_0^1 \Big[\phi([A^t_\alpha, C_\alpha], F^t_\alpha, \dots, F^t_\alpha) + \dots + \phi(F^t_\alpha, \dots, F^t_\alpha, [A^t_\alpha, C_\alpha])\Big] dt \end{aligned}$$

Now, as the algebra $\Omega^{\text{even}}(U_\alpha)$ is commutative, we deduce that $\phi(\omega_{\sigma(1)}, \dots, \omega_{\sigma(k)}) = \phi(\omega_1, \dots, \omega_k)$ for all $\in \mathcal{S}_k$ and any $\omega_1, \dots, \omega_k \in \Omega^{\text{even}}(U_\alpha) \otimes \mathfrak{g}$. Hence, by reordering to the right:

$$\phi(F^1_\alpha) - \phi(F^0_\alpha) = k \int_0^1 \phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]) dt$$

We claim that $\phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]) = d\phi(F^t_\alpha, \dots, F^t_\alpha, C_\alpha)$.
By using again the Bianchi identity, we get

$$\begin{aligned} d\phi(F^t_\alpha, \dots, F^t_\alpha, C_\alpha) &= \phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha) + \phi(dF^t_\alpha, \dots, F^t_\alpha, C_\alpha) + \dots + \phi(F^t_\alpha, \dots, dF^t_\alpha, C_\alpha) \\ &= \phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha) - \phi(C_\alpha, [A^t_\alpha, F^t_\alpha], F^t_\alpha, \dots, F^t_\alpha) - \dots - \phi(C_\alpha, F^t_\alpha, \dots, F^t_\alpha, [A^t_\alpha, F^t_\alpha]) \end{aligned}$$

$$= \phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]) - \phi(F^t_\alpha, \dots, F^t_\alpha, [A^t_\alpha, C_\alpha])$$
$$-\phi([A^t_\alpha, F^t_\alpha], F^t_\alpha, \dots, F^t_\alpha, C_\alpha) - \dots - \phi(F^t_\alpha, \dots, F^t_\alpha, [A^t_\alpha, F^t_\alpha], C_\alpha)$$
$$\stackrel{(6.1)}{=} \phi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]) = \phi(dC_\alpha + [A^t_\alpha, C_\alpha], F^t_\alpha, \dots, F^t_\alpha),$$

which implies

$$\phi(F^1_\alpha) - \phi(F^0_\alpha) = d\int_0^1 k\phi(\frac{dA^t_\alpha}{dt}, F^t_\alpha, \dots, F^t_\alpha)dt \tag{6.3}$$

We set

$$T_\phi(A^1_\alpha, A^0_\alpha) := \int_0^1 k\phi(\frac{dA^t_\alpha}{dt}, F^t_\alpha, \dots, F^t_\alpha)dt$$

As $\frac{dA^t_\beta}{dt} = C_\beta = g_{\alpha\beta}C_\alpha g^{-1}_{\alpha\beta}$ and $F_\beta = g_{\alpha\beta}F_\alpha g^{-1}_{\alpha\beta}$ on $U_{\alpha\beta}$, we conclude from the Ad-invariance of ϕ that the collection $\{T_\phi(A^1_\alpha, A^0_\alpha)\}_{\alpha\in\Lambda}$ defines a global $(2k-1)$-form on M which we denote by $T_\phi(A^1, A^0)$ and we name it the *ϕ-transgression from A^0 to A^1*. We have thus established the *transgression formula*

$$\phi(F_{A^1}) - \phi(F_{A^0}) = dT_\phi(A^1, A^0), \tag{6.4}$$

that guarantees that the difference above is exact, thus vanishing in cohomology, as we wanted to see.

□

Definition 6.3.3. *We have just proved that the cohomology class $P_\phi(F_A) \in H^{2k}(M)$ is independent of the connection A on P used to define its representative. In other words, it is only dependent on the principal bundle P. We capture this by referring to it as $\phi(P)$.*

Thus, the principal bundle P defines a map, called the Chern-Weil correspondence*:*

$$\mathfrak{cw}_P : I^\bullet(G) \to H^\bullet(M)$$
$$\phi \mapsto \phi(P)$$

The following results are straightforward from the naturality of cohomology (cfr. ([41], subsection 7.1.4)) and the definition of pulled-back principal bundles (completely analogous to Definition 2.2.11):

Proposition 6.3.4. *(a) If P is a trivial G-bundle over the smooth manifold M, then $\phi(P) = 0 \in H^\bullet(M)$ for any $\phi \in I^\bullet(G)$.*

(b) Let $f : M \to N$ ne a smooth map between smooth manifolds. Then, for every principal G-bundle over N and any $\phi \in I^\bullet(G)$ we have $\phi(f^*(P)) = f^*(\phi(P))$.

Remark 6.3.5. (a) We have thus solved one direction of the characterisation of *trivial* G-principal bundles over a smooth manifold M (see the discussion in Remark 2.2.16).

(b) There are no other characteristic classes (defined with the requirements stated in ([41], p. 323) generalising the above laid-out properties) valued in the De Rham cohomology other than those obtainable via the Chern-Weil construction. We have not provided the necessary background in this work to supply a proof for that, yet we refer the interested reader to the reference work [37] (see, in particular, Appendix C).

(c) There exist characteristic classes valued in cohomology groups other than the De Rham ones, for instance, the so-called *Čech* cohomology groups. Cfr. the above-referenced work or the classical [49] (see, in particular, part III).

6.4 Important examples

Definition 6.4.1. *(a) A* torus *in a compact Lie group G is a compact, connected, Abelian Lie subgroup of G of dimension n (which is then isomorphic to the standard n-dimensional torus $T^n = S^1 \times \dots \times S^1 = U(1) \times \dots \times U(1)$).*

(b) A torus T in a compact Lie group G is maximal *if for any torus T' such that $T \subset T'$, it holds that $T = T'$.*

6.4.1 Chern classes

Recall that in Subsection 6.1.2 we saw that any Hermitian metric on a rank-r complex vector bundle E over a smooth manifold M induces a metric-independent $U(r)$-structure in E, so that E can be identified with the principal $U(r)$-bundle of unitary frames of E. This will be the principal bundle that will be used in the Chern-Weil construction of the characteristic classes of E.

Furthermore, recall from Remark 6.1.1 that a connection on this $U(r)$-bundle is equivalent to a connection ∇ on E compatible with a Hermitian metric (a notion completely analogous to that given in Definition 4.2.6). To elucidate the characteristic classes of E, then, it only remains to compute the ring of invariants $I^\bullet(U(r))$, i.e., the symmetric, r-linear maps $\mathfrak{u}(r) \times \cdots \times \mathfrak{u}(r) \to \mathbb{C}$ invariant with respect to the adjoint action $\mathfrak{u}(r) \ni X \mapsto TXT^{-1} \in \mathfrak{u}(r)$, for $T \in U(r)$, where $\mathfrak{u}(r)$ denotes the Lie algebra of $r \times r$ complex skew-Hermitian matrices. We know a skew-Hermitian matrix is diagonalisable and that its eigenvalues are all purely imaginary (and possibly zero) (see [26], section 2.5). In other words, for any $X \in \mathfrak{u}(r)$, there exists $T \in U(r)$ such that $TXT^{-1} = \boldsymbol{i}\ \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$, for some $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

Definition 6.4.2. *(a) The set of diagonal matrices in $\mathfrak{u}(r)$ is called the* Cartan algebra of $\mathfrak{u}(r)$ *and we denote it by $\mathcal{C}_{\mathfrak{u}(r)}$. It is a maximal Abelian Lie subalgebra of $\mathfrak{u}(r)$.*

(b) Consider the stabiliser

$$\mathcal{S}_{\mathfrak{u}(r)} := \{T \in U(r) : TXT^{-1} = X,\ \forall X \in \mathcal{C}_{\mathfrak{u}(r)}\}$$

and the normaliser

$$\mathcal{N}_{\mathfrak{u}(r)} := \{T \in U(r) : T\mathcal{C}_{\mathfrak{u}(r)}T^{-1} \subset \mathcal{C}_{\mathfrak{u}(r)}\}$$

(c) Since $\mathcal{S}_{\mathfrak{u}(r)} \trianglelefteq \mathcal{N}_{\mathfrak{u}(r)}$, we can define the Weyl group of $U(r)$ *as the quotient $\mathcal{W}_{\mathfrak{u}(r)} := \mathcal{N}_{\mathfrak{u}(r)}/\mathcal{S}_{\mathfrak{u}(r)}$.*

From this definitions and the diagonal form of a skew-Hermitian matrix, it is clear that $\phi \in I^\bullet(U(r))$ is Ad-invariant if, and only if, its restriction to the Cartan algebra is invariant under the action of the Weyl group.

Remark 6.4.3. Simple computations show that $\mathcal{S}_{\mathfrak{u}(r)} = \mathcal{C}_{\mathfrak{u}(r)}$ and that $\mathcal{W}_{\mathfrak{u}(r)}$ is the set of *permutation matrices*, from which we deduce that $\mathcal{W}_{\mathfrak{u}(r)} \cong \mathcal{S}_r$.

Now, the Cartan algebra is the Lie algebra of the maximal torus T^r consisting of diagonal unitary matrices. We introduce, then, angular coordinates on T^r: $0 \le \theta^i \le 2\pi$, $1 \le i \le r$ and set $x_i := -\frac{1}{2\pi\boldsymbol{i}} d\theta^i$. We can identify the restriction of $\phi \in I^\bullet(U(r))$ to the Cartan algebra with a polynomial in the variables $x_1, \ldots, x_r$ (cfr. Remark 6.2.1 (a)). Therefore, as the Weyl group $\mathcal{S}_r$ simply permutes these variables, $\phi \in I^\bullet(U(r))$ is Ad-invariant if, and only if, it is a symmetric polynomial in its variables.

According to the *fundamental theorem of symmetric polynomials*, the ring of these polynomials is generated as an $\mathbb{R}$-algebra by the *elementary* ones (see ([4], p. 479)):

$$\begin{aligned} c_1 &= \textstyle\sum_i x_i \\ c_2 &= \textstyle\sum_{i<j} x_i x_j \\ \vdots \ &\ \vdots\ \vdots \\ c_r &= \textstyle\prod_i x_i \end{aligned}$$

Thus, $I^\bullet(U(r)) \cong \mathbb{R}[c_1, \ldots, c_r]$.

Definition 6.4.4. *For a matrix $X \in \mathfrak{u}(r)$, the following equality holds:*

$$\sum_k c_k(X) t^k = \det\left(\mathbb{1} - \frac{t}{2\pi\boldsymbol{i}} X\right) \in I^\bullet(U(r))[t]$$

This is immediate if only one, on the one hand, realises that the functions c_j are symmetric with respect to the eigenvalues of X and that the variables x_i are precisely these, normalised, and, on the other hand, bears in mind Proposition A.2.9.

The above polynomial is known as the universal rank-r Chern polynomial, *and its coefficients are called the* universal, rank-r Chern classes.

Returning to our rank-r vector bundle E:

Definition 6.4.5. *The* Chern classes of E *are the coefficients* $c_k(E) \coloneqq c_k(F(\nabla)) \in H^{2k}(M)$ *of the* Chern polynomial of E, $c_t(E) \coloneqq \det\left(\mathbb{1} - \frac{t}{2\pi i}F(\nabla)\right) \in H^\bullet(M)[t]$, *where* $F(\nabla)$ *denotes the curvature of a connection compatible with a Hermitian metric on* E.

6.4.2 Pontryagin classes

Again, recall that in Subsection 6.1.1 we saw that any metric on a rank-r real vector bundle E over a smooth manifold M induces a metric-independent $O(r)$-structure in E, so that E can be identified with the principal $O(r)$-bundle of orthogonal frames of E. This will be the principal bundle that will be used in the Chern-Weil construction of the characteristic classes of E. Again, a connection on this principal bundle can be viewed as a metric-compatible connection in the associated vector bundle, and to describe this set of characteristic classe we will need to understand the ring of invariant $I^\bullet(O(r))$. As above, we will identify the elements of $I^k(O(r))$ with the degree-k, Ad-invariant polynomials on the Lie algebra $\mathfrak{o}(r)$ consisting of skew-symmetric $r \times r$ real matrices.

Definition 6.4.6. *Set* $m = [r/2]$ *(where* $[\cdot]$ *denotes the integer part) and denote by* J *the* 2×2 *matrix* $J \coloneqq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Consider the Cartan algebra of $\mathfrak{o}(r)$

$$\mathcal{C}_{\mathfrak{o}(r)} = \begin{cases} \{\lambda_1 J \oplus \cdots \oplus \lambda_m J \in \mathfrak{o}(r) : \lambda_j \in \mathbb{R}\} & r = 2m \\ \\ \{\lambda_1 J \oplus \cdots \oplus \lambda_m J \oplus 0 \in \mathfrak{o}(r) : \lambda_j \in \mathbb{R}\} & r = 2m+1 \end{cases},$$

which is the Lie algebra of the maximal torus

$$T^m = \begin{cases} R_{\theta_1} \oplus \cdots \oplus R_{\theta_m} \in O(r) : \lambda_j \in \mathbb{R}\} & r = 2m \\ \\ R_{\theta_1} \oplus \cdots \oplus R_{\theta_m} \oplus \mathbb{1}_{\mathbb{R}} \in O(r) & r = 2m+1 \end{cases},$$

where for each $\theta \in [0, 2\pi]$ *we denoted by* R_θ *the* 2×2 *rotation* $R_\theta \coloneqq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

As above, we introduce the variables $x_i \coloneqq -\frac{1}{2\pi} d\theta^i$, $i = 1, \ldots, m$. *By Proposition A.4.7, it holds that for every* $X \in \mathfrak{o}(r)$, *there exists* $T \in O(r)$ *such that* $TXT^{-1} \in \mathcal{C}_{\mathfrak{o}(r)}$. *Thus, any Ad-invariant polynomial is uniquely defined by its restriction to the Cartan algebra.*

Definition 6.4.7. *In a manner completely analogous to that of the previous subsection, we can define the Weyl group of* $O(r)$ *as the quotient* $\mathcal{W}_{\mathfrak{o}(r)} \coloneqq \mathcal{N}_{\mathfrak{o}(r)}/\mathcal{S}_{\mathfrak{o}(r)}$.

It is easy to see that:

Proposition 6.4.8. $\mathcal{W}_{\mathfrak{o}(r)}$ *is the subgroup of* $\mathrm{GL}(m, \mathbb{R})$ *generated by the involutions*

$$\sigma_{ij} : (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_m) \mapsto (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_m)$$

$$\epsilon_j : (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_m) \mapsto (\lambda_1, \ldots, -\lambda_j, \ldots, \lambda_m)$$

projected to the quotient by $\mathcal{S}_{\mathfrak{o}(r)}$.

Again, the restriction of a certain $\phi \in I^k(O(r))$ to $\mathcal{C}_{\mathfrak{o}(r)}$ can be identified with a degree-k homogeneous polynomial in the variables $x_1, \ldots, x_m$ invariant under the action of the Weyl group. Using the result above, we infer that ϕ must be a symmetric polynomial separately even in each variable. If we invoke, as in the previous subsection, the fundamental theorem of symmetric polynomials, we conclude that ϕ must be a polynomial in the elementary symmetric ones:

$$\begin{aligned} p_1 &= \textstyle\sum_i x_i^2 \\ p_2 &= \textstyle\sum_{i<j} x_i^2 x_j^2 \\ \vdots &\;\;\vdots\;\vdots \\ p_m &= \textstyle\prod_i x_i^2 \end{aligned}$$

Thus, $I^\bullet(O(r)) \cong \mathbb{R}[p_1, \ldots, p_m]$.

Definition 6.4.9. *For a matrix $X \in \mathfrak{u}(r)$, the following equality holds:*

$$\sum_k p_k(X)t^{2k} = \det\left(\mathbb{1} - \frac{t}{2\pi}X\right) \in I^\bullet(O(r))[t]$$

by the same arguments as in the previous subsection.

The above polynomial is known as the universal rank-r Pontryagin polynomial*, and its coefficients are called the* universal, rank-r Pontryagin classes*.*

Returning to our rank-r vector bundle E:

Definition 6.4.10. *The* Pontryagin classes of E *are the coefficients $p_k(E) \coloneqq p_k(F(\nabla)) \in H^{4k}(M)$ of the* Pontryagin polynomial of E, $p_t(E) \coloneqq \det\left(\mathbb{1} - \frac{t}{2\pi}F(\nabla)\right) \in H^\bullet(M)[t]$*, where $F(\nabla)$ denotes the curvature of a connection compatible with a metric on E.*

6.4.3 The Euler class

Remark 6.4.11. An object from linear algebra, the *Pfaffian*, which operates on skew-symmetric linear endomorphisms of a vector space, is an important ingredient in the construction of the Euler class of an oriented vector bundle. If the reader is not well acquainted with it, we suggest they gather now the necessary background material from Subsection A.4.1.

In the previous subsection, we pointed out that any metric on a rank-r real vector bundle E over a smooth manifold M induces a metric-independent $O(r)$-structure in E, so that E can be identified with the principal $O(r)$-bundle of orthogonal frames of E. If E is furthermore oriented, we can analogously consider the principal $SO(r)$-bundle of positively oriented orthonormal frames (see Subsection 6.1.1).

Now, we know that the groups $O(r)$ and $SO(r)$ share the same Lie algebra $\mathfrak{o}(r) = \mathfrak{so}(r)$, and the inclusion $\iota : SO(r) \hookrightarrow O(r)$ induces a morphism of $\mathbb{R}$-algebras $\iota^* : I^\bullet(O(r)) \to I^\bullet(SO(r))$ (we can think of it as simply introducing in each component the differential of ι at $\mathbb{1} \in SO(r)$, $\iota_* : \mathfrak{so}(r) \to \mathfrak{o}(r)$). Its injectivity is immediately deduced from the equality $\mathfrak{o}(r) = \mathfrak{so}(r)$, and it is easy that ι^* is an isomorphism in the odd-rank case.

Definition 6.4.12. *To describe the ring of invariants $I^\bullet(SO(2m))$ when r is even, $r = 2m$, we need again to study the* Cartan algebra $\mathcal{C}_{\mathfrak{o}(2m)} = \{\lambda_1 J \oplus \cdots \oplus \lambda_m J \in \mathfrak{o}(2m) : \lambda_j \in \mathbb{R}\}$ *and the corresponding* Weyl group *action. The latter is defined as usual as the quotient $\mathcal{W}_{\mathfrak{o}(2m)} \coloneqq \mathcal{N}_{\mathfrak{o}(2m)}/\mathcal{S}_{\mathfrak{o}(2m)}$.*

It is easy to see that:

Proposition 6.4.13. *$\mathcal{W}_{\mathfrak{o}(2m)}$ is the subgroup of $\mathrm{GL}(m,\mathbb{R})$ generated by the involutions*

$$\sigma_{ij} : (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_m) \mapsto (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_m)$$

$$\epsilon : (\lambda_1, \ldots, \lambda_m) \mapsto (\epsilon_1\lambda_1, \ldots, \epsilon_m\lambda_m),$$

where $\epsilon_1, \ldots, \epsilon_m = \pm 1$ such that $\epsilon_1 \ldots \epsilon_m = 1$, projected to the quotient by $\mathcal{S}_{\mathfrak{o}(2m)}$.

We introduce again the variables $x_i \coloneqq -\frac{1}{2\pi}d\theta^i$, $i = 1, \ldots, m$ as in Definition 6.4.6. The Pontryagin $O(2m)$ invariants

$$p_j(x_1, \ldots, x_m) = \sum_{1 \le i_1 \le \cdots \le i_j \le m} (x_{i_1} \ldots x_{i_j})^2$$

continue to be invariant under the action of $\mathcal{W}_{\mathfrak{o}(2m)}$.

However, in the even-rank case a new invariant appears: $\Delta(x_1, \ldots, x_m) \coloneqq \prod_i x_i$. In terms of a certain $X \in \mathcal{C}_{\mathfrak{o}(2m)}$, we can write

$$\Delta(X) = \left(\frac{-1}{2m}\right)^m \boldsymbol{Pf}(X),$$

where $\boldsymbol{Pf}(X)$ denotes the Pfaffian of the skew-symmetric matrix X viewed as a linear map $\mathbb{R}^{2m} \to \mathbb{R}^{2m}$, where $\mathbb{R}^{2m}$ is endowed with the canonical orientation. Indeed, the above equality is immediate: one needs only note that X is of the form described in Proposition A.4.7 (b) and then apply the arguments used the prove Proposition A.4.8, bearing in mind that the x_i are again the normalised eigenvalues of X. Note that $p_m = \Delta^2$.

With a little more work, one can show that:

Proposition 6.4.14.

$$I^\bullet(SO(2m)) \cong \mathbb{R}[Z_1, Z_2, \ldots, Z_m; Y]/(Y^2 - Z_m) \ (Z_j = p_j, Y = \Delta),$$

where $(Y^2 - Z_m)$ *denotes the ideal generated by the polynomial* $Y^2 - Z_m$.

Proof. See ([41], p.330).

□

Remark 6.4.15. Note that the above result contains an abstract-algebraic formulation of Proposition A.4.8.

Remark 6.4.16. (a) One can easily see that $\boldsymbol{Pf}(\cdot) \in I^\bullet(SO(2m))$ without resorting to computing the whole ring of invariants. Indeed, the adjoint action of $SO(2m)$ is simply conjugation by an orthonormal matrix, i.e., a change of basis. As our first definition of the Pfaffian was *intrinsic*, it is clear that it is invariant under this action (see Definition A.4.5). It only remains to heed the identification from Remark 6.2.1 (a), and recall that Proposition A.4.9 tells us that $\boldsymbol{Pf}(\cdot)$ is a homogeneous polynomial of degree $2m$ in $2m \times 2m$ variables (the entries of the matrix it operates on).

(b) Let E be a real vector bundle over a smooth manifold M and choose a connection ∇ compatible with some metric on E. Let $F(\nabla)$ denote is curvature. Remark 3.4.6 combined with the final note of Remark 6.1.1 shows that the expression $\boldsymbol{Pf}(F(\nabla))$ makes sense.

At last, we have:

Definition 6.4.17. *(a) The* universal Euler class *is defined to be*

$$\boldsymbol{e}(X) := \frac{1}{(2\pi)^m}\boldsymbol{Pf}(-X) \in I^m(SO(2m))$$

(b) The Euler class of E *is the cohomology class* $\boldsymbol{e}(E) \in H^{2m}(M)$ *represented by the* Euler form

$$\boldsymbol{e}(\nabla) := \frac{1}{(2\pi)^m}\boldsymbol{Pf}(-F(\nabla)) \in \Omega^{2m}(M)$$

Remark 6.4.18. As stated by the Chern-Weil theorem 6.3.2, the above cohomology class is independent of the connection. As we pointed out above, the bundle $SO(E)$ is metric-independent, so that the cohomology class $\boldsymbol{e}(E)$ is both metric and connection independent, thus making it an invariant of the vector bundle E. This is captured by our notation.

Chapter 7

The generalised Gauss-Bonnet-Chern theorem

We have arrived at the final stage of our route. We will now prove a link between the topological and the geometric properties of an arbitrary vector bundle over a smooth vector bundle, which reduces to a generalisation of the classical Gauss-Bonnet theorem when applied to the tangent bundle.

If $E \to M$ is a real oriented vector bundle over a smooth, compact, oriented manifold M, two apparently distinct notions of Euler classes naturally associated to E can be defined:

(1) The *topological Euler class*:

$$\boldsymbol{e}_{top}(E) = \zeta_0^* \tau_E,$$

where $\zeta_0 : M \to E$ is the zero section and τ_E is the Thom class of E.

(2) The *geometrical Euler class* exposed in the previous chapter, defined as

$$\boldsymbol{e}_{geo}(E) = \boldsymbol{e}(\nabla) \coloneqq \frac{1}{(2\pi)^k} \boldsymbol{Pf}(-F(\nabla)) \in H^{2k}(M)$$

for vector bundles of even rank $2k$. We extend this definition to arbitrary vector bundles by setting it to be identically zero in the odd-rank case.

We set out to prove that these two notions in fact coincide.

Remark 7.0.1. Recall that the unit sphere bundle of E is $S(E) \coloneqq \{u \in E; |u|_g = 1\}$. The fibres are then copies of S^{2k-1}.

We denote by π_0 the natural projection $S(E) \to M$, and by $\pi_0^*(E)$ the pullback of E to $S(E)$ via the map π_0.

We need at this point a technical preliminary lemma:

Lemma 7.0.2. *There exists* $\Psi = \Psi(\nabla) \in \Omega^{2k-1}(S(E))$ *such that*

(i) $d\Psi(\nabla) = \boldsymbol{e}(\pi_0^*(E))$

(ii) $\int_{S(E)/M} \Psi(\nabla) = -1$

The form $\Psi(\nabla)$ *is sometimes referred to as the* global angular form of the pair (E, ∇) *(see ([11], subsection I.6.6)).*

Proof. See Appendix II, section E.

□

Theorem 7.0.3. (Generalised Gauss-Bonnet-Chern). *Let* $E \xrightarrow{\pi} M$ *be a real, oriented vector bundle over a smooth, compact, oriented manifold* M*.*

Then

$$\boldsymbol{e}_{top}(E) = \boldsymbol{e}_{geo}(E)$$

Proof. We will distinguish two cases, according to the parity of the rank.

(1) Let the rank of E be odd. We consider the automorphism of E

$$\mathfrak{i} : E \to E$$

$$u \mapsto -u$$

Since the fibres of E are odd dimensional, we deduce that $\mathfrak{i}$ is orientation-reversing on the fibres. If we denote by π_* the integration-along-fibres operator, this implies

$$\pi_* \mathfrak{i}^* \tau_E = -\pi_* \tau_E = \pi_*(-\tau_E)$$

The Thom isomorphism theorem implies that π_* is an isomorphism, so that we deduce

$$\mathfrak{i}^* \tau_E = (-\tau_E). \tag{7.1}$$

On the other hand, note that

$$\zeta_0^* \mathfrak{i}^* = (\mathfrak{i}\zeta_0)^* = (-\zeta_0)^* = (\zeta_0)^* \tag{7.2}$$

Hence, we finally obtain:

$$\boldsymbol{e}_{top}(E) = \zeta_0^* \tau_E \overset{(7.1)}{=} -\zeta_0^* \mathfrak{i}^* \tau_E \overset{(7.2)}{=} -\zeta_0^* \tau_E$$

so that $\boldsymbol{e}_{top}(E) = 0$ and the equality is proven.

(2) Let the rank of E be $2k$.

Let ∇ be a connection on E that is compatible with some metric g. First of all, we outline the strategy of the proof.

Our goal is to explicitly construct a closed form $\omega \in \Omega^{2k}_{cpt}(E)$ such that the following hold:

(i) $\pi_* \omega = 1 \in \Omega^0(M)$

(ii) $\zeta_0^* \omega = \boldsymbol{e}(\nabla) = \frac{1}{(2\pi)^k} \boldsymbol{Pf}(-F(\nabla))$

Condition *(i)* coupled with the Thom isomorphism theorem 5.3.15 implies that ω represents the Thom class in $H^{2k}_{cpt}(E)$, so that then condition *(ii)* simply expresses the desired equality.

We start by recalling that the above-defined vector bundle $\pi_0^*(E)$ is endowed with a $SO(2k)$-structure (see Remark 6.1.4), and that, furthermore, it admits a tautological, nowhere-vanishing section:

$$\sigma : S(E_x) \ni e \mapsto e \in E_x \cong (\pi_0^*(E)_x)_e,$$

for some $x \in M$.

If we denote by $\boldsymbol{e}(\pi_0^*(E))$ the element of $H^{2k}(S(E))$ represented by the differential form

$$\boldsymbol{e}(\pi_0^* \nabla) = \frac{1}{(2\pi)^k} \boldsymbol{Pf}(-F(\pi_0^* \nabla)) \in \Omega^{2k}(S(E)),$$

Proposition D.0.4 tells us that the existence of such a section implies that $\boldsymbol{e}(\pi_0^*(E))$ vanishes.

Thus, there must exist some $\Psi \in \Omega^{2k-1}(S(E))$ such that $d\Psi = \boldsymbol{e}(\pi_0^*(E))$. This is indeed the case, as ensured by Lemma 7.0.2.

Now, let us denote by $r : E \to \mathbb{R}_+$ the radial function $E \ni e \mapsto |e|_g$. This is of course invariant along spheres. If we set $E^0 = E \setminus \{\text{zero section}\}$ (that is, we remove the 0 element from each fibre), then the following identification holds:

$$E^0 \cong (0, \infty) \times S(E)$$

$$e \mapsto (|e|, \frac{1}{|e|} e)$$

If we consider a smooth cutoff function (cfr. Subsection B.2.2) $\rho = \rho(r) : [0, \infty) \mapsto \mathbb{R}$, such that $\rho(r) = -1$ for $r \in [0, \frac{1}{4}]$, and $\rho(r) = 0$ for $r \geq \frac{3}{4}$, we can define:

$$\omega \coloneqq \omega(\nabla) = -\rho'(r)dr \wedge \Psi(\nabla) - \rho(r)\pi^*(\boldsymbol{e}(\nabla))$$

It holds that ω satisfies condition *(ii)*, since $\zeta_0^*\omega = -\rho(0)\zeta_0^*\pi^*\boldsymbol{e}(\nabla) = \boldsymbol{e}(\nabla)$, as sections are right inverses of the projection and $\rho'(r) \equiv 0$ near the zero section.

We proceed to show that it also verifies condition *(i)*. We have that, by definition of r:

$$\int_{E/M} \rho(r)\pi^*\boldsymbol{e}(\nabla) = \int_{S(E)/M} \left(\int_0^\infty \rho(r)\right)\pi_0^*\boldsymbol{e}(\nabla) = 0,$$

as the interior integral is just a scalar that we can take out, and the exterior is an along-fibers integral of a form pulled back to the base, which vanishes by definition (cfr. Proposition 3.2.14).

Then:

$$\int_{E/M} \omega = -\int_{E/M} \rho'(r)dr \wedge \Psi(\nabla) = -\int_0^\infty \rho'(r)dr \cdot \int_{S(E)/M} \Psi(\nabla) =$$

$$= -(\rho(1) - \rho(0)) \int_{S(E)/M} \Psi(\nabla) \overset{7.0.2\,(ii)}{=} 1$$

It only remains to show, to complete our programme, that ω is closed. Indeed:

$$d\omega = \rho'(r)dr \wedge d\Psi(\nabla) - \rho'(r) \wedge \pi^*(\boldsymbol{e}(\nabla)) \overset{7.0.2\,(i)}{=} \rho'(r)dr \wedge [\pi_0^*\boldsymbol{e}(\nabla) - \pi^*\boldsymbol{e}(\nabla)]$$

The above form is identically zero, since $\pi_0^*\boldsymbol{e}(\nabla) = \pi^*\boldsymbol{e}(\nabla)$ (in particular, this equality holds on the support of ρ').

Hence ω is closed and the theorem is proved.

□

We have finally derived the generalisation we endeavoured to proof. We need only bear in mind Theorem 5.3.21 and apply the above result to the tangent bundle and the Levi-Civita connection:

Corollary 7.0.4. (Gauss-Bonnet-Chern). *Let (M, g) be a compact, oriented, Riemann manifold of dimension $2n$. If R denotes the Riemann curvature, then*

$$\chi(M) = \frac{1}{(2\pi)^n} \int_M \boldsymbol{Pf}(-R)$$

□

Chapter 8

Conclusions

We have finally acquired the generalisation we were chasing. To reach it, it was necessary to develop a wide range of techniques and definitions in modern differential geometry. We have tried to intertwine several remarks and references to different problems with the main unifying thread, so as to give the reader a feel for the breadth and vitality of the area. Any of the pointed-at paths are worth following.

However, we would perhaps like to elaborate on one of them: *index theory*. It is indeed common to introduce the seminal *Atiyah-Singer index theorem*, a landmark of XXth-century mathematics, as a generalisation of the Gauss-Bonnet-Chern theorem. A more precise statement would be to say that the former result, relating the *topological* and the *analytical indexes* of certain *elliptic partial differential operators* on manifolds, contains as a particular case the latter result, via a certain of said operators, $d + d^*$, where d is the exterior derivative and d^* is its adjoint, constructed using the *Hodge star operator*. Acquiring the necessary background to state and prove in detail this remarkable theorem would be the desired, and natural, continuation of the present work. Very good companions on that journey might be: ([41], Chapters 10 and 11), and [10]. Furthermore, the proof of the Gauss-Bonnet-Chern via the construction of the Heat Kernel (see [32], Ch. 5) finds a natural coupling with one of the approaches to the Atiyah-Singer index theorem (see [45]).

Somewhere *in between* our present position and a proof of the Atiyah-Singer index theorem lies *Hodge theory*, whose goal is to study the cohomology groups of a smooth manifold using partial differential equations. A good starting point, building on the notions we have introduced so far, might be [22].

Lastly, another natural prolongation of our work could be to understand more deeply the theory of characteristic classes (for which we have already given excellent references in the corresponding chapters), in particular, to understand how Chern classes fail to capture *torsion* (see the influential paper [29]).

Appendix I

Basic preliminaries

Section A

Linear-algebra preliminaries

We present here some elementary notions of linear algebra, which will be carried on to the field of geometry via tensor calculus.

In this section, we have mainly selected material from ([41], Ch. 2). We have occasionally given alternative definitions and proofs where we have deemed it appropriate, especially in the section concerning the Pfaffian.

We assume all the introduced vector spaces to be finite dimensional.

A.1 Tensor products

Definition A.1.1. Let E and F be two vector spaces over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Consider the direct sum

$$\mathcal{T}(E, F) \coloneqq \bigoplus_{(e,f)\in E\times F} \mathbb{K}$$

That is, the $\mathbb{K}$-free vector space over $E \times F$, each element of which is the formal linear combination of finitely many elements of the Cartesian product of E and F with coefficients in $\mathbb{K}$.

Inside $\mathcal{T}(E, F)$ sits the linear subspace $\mathcal{R}(E, F)$ spanned by subsets of the form

$$\{\lambda(e, f) - (\lambda e, f), \lambda(e, f) - (e, \lambda f), (e + e', f) - (e, f) - (e', f), (e, f + f') - (e, f) - (e, f')\},$$

where $e, e' \in E$, $f, f' \in F$ and $\lambda \in \mathbb{K}$.

We now define the *tensor product of the vector spaces E and F over $\mathbb{K}$* to be the quotient $E \otimes_{\mathbb{K}} F \coloneqq \mathcal{T}(E, F)/\mathcal{R}(E, F)$

The field of scalars may be dropped in the notation when it is clear from context. If we denote by π the canonical projection $\pi : \mathcal{T}(E, F) \to E \otimes F$, we set $e \otimes f \coloneqq \pi((e, f))$

That is, $e \otimes f$ denotes the class of (e, f) within $E \otimes F$.

Remark A.1.2. We get a natural map $\iota : E \times F \to E \otimes F$, $(e, f) \mapsto e \otimes f$. The identifications in Definition A.1.1 make it obvious that ι is bilinear.

Note that if $\{e_i\}$, $\{f_j\}$ are bases of E and F, respectively, then $\{e_i \otimes f_j\}$ is a basis of $E \otimes F$, which implies that $\dim_{\mathbb{K}}(E \otimes_{\mathbb{K}} F) = \dim_{\mathbb{K}}(E) \cdot \dim_{\mathbb{K}}(F)$.

One fundamental property of the above construction is the so-called *universality property*, which ensures that $\mathbb{K}$-bilinear maps $E \times F \to H$ are in one-to-one correspondence with $\mathbb{K}$-bilinear maps $E \otimes F \to H$:

Proposition A.1.3. *For any linear map $\Phi : E \times F \to H$ there exists a* unique *linear map $\tilde{\Phi} : E \otimes F \to H$ such that the diagram below is commutative:*

$$\begin{tikzcd} E \times F \arrow[r, hook, "\iota"] \arrow[dr, "\Phi"'] & E \otimes F \arrow[d, dashed, "\tilde{\Phi}"] \\ & H \end{tikzcd}$$

Remark A.1.4. The previous construction can be iterated, so that, apparently, with a triple of vector spaces we can construct two different tensor products.

Using Definition A.1.1 and the universality property from Proposition A.1.3, it is easy to see that, given vector spaces E, F, H over a field $\mathbb{K}$, the following canonical isomorphisms exist:

(a) $E \otimes F \cong F \otimes E$
$e \otimes f \mapsto f \otimes e$

(b) $(E \otimes F) \otimes H \cong E \otimes (F \otimes H)$
$(e \otimes f) \otimes h \mapsto e \otimes (f \otimes h)$

(c) $\mathbb{K} \otimes_{\mathbb{K}} E \cong E$
$k \cdot e \mapsto e$

Isomorphism (b), in particular, implies that there exists an up-to-isomorphism *unique* triple tensor product, that we choose to denote by $E \otimes F \otimes H$. It is consequently clear that we may now consider, without ambiguity, arbitrary finite tensor products $E_1 \otimes \cdots \otimes E_n$.

Definition A.1.5. *(a) Given arbitrary vector spaces E, F over a field $\mathbb{K}$, we denote by $Hom_{\mathbb{K}}(E, F)$ the space of $\mathbb{K}$-linear maps $E \to F$. Again, the field may be dropped in the notation so long as it is clear from context.*

(b) The dual of a $\mathbb{K}$-vector space E *is the vector space $E^* := Hom_{\mathbb{K}}(E, \mathbb{K})$ of $\mathbb{K}$-linear maps $E \to \mathbb{K}$. For any $e^* \in E^*$ and $e \in E$, we set $\langle e^*, e\rangle := e^*(e)$.*

Using this duality construction and the definition of tensor product, the result below can be easily proven:

Proposition A.1.6. *(a) There exists a natural isomorphism $E^* \otimes F^* \cong (E \otimes F)^*$, uniquely defined by: $e^* \otimes f^* \mapsto L_{e^* \otimes f^*}$, where $L_{e^* \otimes f^*}$ is determined by*

$$\langle L_{e^* \otimes f^*}, x \otimes y\rangle = \langle e^*, x\rangle \cdot \langle f^*, y\rangle \in \mathbb{K}, \forall x \in E, y \in F$$

Note that, combined with the universality property from Proposition A.1.3, this implies that $E^* \otimes F^*$ can be naturally identified with the space of bilinear maps $E \times K \to \mathbb{K}$.

(b) The adjunction morphism *$E^* \otimes F \to Hom(E, F)$, given by $e^* \otimes f \mapsto T_{e^* \otimes f}$, where $T_{e^* \otimes f}$ is determined by*

$$\langle T_{e^* \otimes f}, x\rangle = \langle e^*, x\rangle \cdot f, \forall x \in E,$$

is an isomorphism.

Definition A.1.7. *(a) Let V be a $\mathbb{K}$-vector space. For $r, s \geq 0$, set $\mathcal{T}^r_s(V) := V^{\otimes r} \otimes (V^*)^{\otimes s}$, where by definition $V^{\otimes 0} = (V^*)^{\otimes 0} = \mathbb{K}$. An element of $\mathcal{T}^r_s(V)$ is called a* tensor of type (r,s).

A tensor of type $(r, 0)$ is called contravariant*, while a tensor of type $(0, s)$ is called* covariant.

(b) The tensor algebra *is defined to be $\mathcal{T}(V) := \bigoplus_{r,s \geq 0} \mathcal{T}^r_s(V)$.*

The use of the term algebra *is justified by the fact that the tensor product induces bilinear maps*

$$\otimes : \mathcal{T}^r_s(V) \times \mathcal{T}^{r'}_{s'}(V) \to \mathcal{T}^{r+r'}_{s+s'}(V)$$

Indeed, it is easy to see that $(\mathcal{T}(V), +, \otimes)$ is an associative algebra.

Remark A.1.8. Proposition A.1.6 tells us, in particular, that a tensor of type $(1, 1)$ can be identified with a linear endomorphism of V, i.e., $\mathcal{T}^1_1(V) \cong \mathrm{End}(V)$, while a tensor of type $(0, k)$ can be identified with a k-linear map $\underbrace{V \times \cdots \times V}_{k} \to \mathbb{K}$.

It is often useful to represent tensors in coordinate systems. We end this subsection by providing such a description of these objects and reformulating in this way one of the above-introduced identifications:

Remark A.1.9. (a) Let $\{e_i\}$ be a basis of a vector space V and let $\{e^i\}$ denote the dual basis of V^*, uniquely defined using, as usual, the Kronecker delta, by $\langle e^i, e_j\rangle := \delta^i_j$.

We then obtain a basis of $\mathcal{T}^r_s(V)$ by

$$\{e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes e^{j_1}\cdots\otimes e^{j_s}/1\leq i_\alpha, j_\beta\leq \dim V\},$$

so that any element $T\in\mathcal{T}^r_s(V)$ has thus a decomposition

$$T = T^{i_1\ldots i_r}_{j_1\ldots j_s} e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes e^{j_1}\cdots\otimes e^{j_s}$$

(b) As we know, using the adjunction morphism in Proposition A.1.6, we can identify the space $\mathcal{T}^1_1(V)$ with the space $\mathrm{End}(V)$ of linear isomorphisms. Using the bases introduced above, we can now explicitly describe the adjunction identification as the correspondence that associates to the tensor $T = T^i_j e_i\otimes e^j\in\mathcal{T}^1_1(V)$ the linear operator $L_T : V\to V$, $\lambda_j e_j\mapsto T^i_j\lambda_j e_i$.

A.2 Symmetric and skew-symmetric tensors. The exterior algebra

Remark A.2.1. Let V be a vector space over $\mathbb{K}=\mathbb{R},\mathbb{C}$. We set $\mathcal{T}^r(V) := \mathcal{T}^r_0(V)$, and we denote by $\mathcal{S}_r$ the group of permutations or r objects. When $r=0$, we set $S_0 := \{\mathbb{1}\}$.

Now, every permutation $\sigma\in\mathcal{S}_r$ uniquely determines a linear map $\mathcal{T}^r(V)\to\mathcal{T}^r(V)$ by the correspondences:

$$v_1\otimes\cdots\otimes v_r\mapsto v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(r)}, \forall v_1,\ldots,v_r\in V.$$

We denote this action of $\sigma\in\mathcal{S}_r$ on an arbitrary element $T\in\mathcal{T}^r(V)$ by σT. In this subsection, we will describe two subspaces invariant under this action, one of which is especially relevant for geometric purposes, as it is the set underlying the so-called exterior algebra.

Definition A.2.2. *(a) Define* $\boldsymbol{S}_r : \mathcal{T}^r(V)\to\mathcal{T}^r(V)$

$$T\mapsto \frac{1}{r!}\sum_{\sigma\in\mathcal{S}_r}\sigma T$$

and $\boldsymbol{A}_r : \mathcal{T}^r(V)\to\mathcal{T}^r(V)$

$$T\mapsto\begin{cases}\frac{1}{r!}\sum_{\sigma\in\mathcal{S}_r}\epsilon(\sigma)\sigma T & if \quad r\leq dimV\\ \\ 0 & if \quad r> dimV\end{cases}$$

Above, we denoted by $\epsilon(\sigma)$ *the signature of the permutation* σ*. Note that* $\boldsymbol{S}_0=\boldsymbol{A}_0=\mathbb{1}_{\mathbb{K}}$.

(b) A tensor $T\in\mathcal{T}^r(V)$ *is called* symmetric *(respectively* skew-symmetric*) if* $\boldsymbol{S}_r(T)=T$ *(respectively* $\boldsymbol{A}_r(T)=T$*).*

(c) The natural number r *is called the* degree *of one such tensor.*

The space of symmetric tensors (respectively skew-symmetric ones) of degree r *will be denoted by* $\boldsymbol{S}^r(V)=T$ *(respectively by* $\Lambda^r(V)$*).*

We set $\boldsymbol{S}^\bullet V := \bigoplus_{r\geq 0}\boldsymbol{S}^r(V)$ *and* $\Lambda^\bullet V := \bigoplus_{r\geq 0}\Lambda^r(V)$.

Definition A.2.3. *The* exterior product *is the bilinear map* $\wedge : \Lambda^r(V)\times\Lambda^s(V)\to\Lambda^{r+s}(V)$

$$(\omega^r,\eta^s)\mapsto\omega^r\wedge\eta^s := \frac{(r+s)!}{r!s!}\boldsymbol{A}_{r+s}(\omega^r\otimes\eta^s)$$

Proposition A.2.4. *The exterior product has the following properties:*

*(i) (*Associativity*)* $(\alpha\wedge\beta)\wedge\gamma=\alpha\wedge(\beta\wedge\gamma)$, $\forall\alpha,\beta,\gamma\in\Lambda^\bullet V$.

In particular,

$$v_1\wedge\cdots\wedge v_k = k!\boldsymbol{A}_k(v_1\otimes\cdots\otimes v_k)=\sum_{\sigma\in S_k}\epsilon(\sigma)(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}), \forall v_i\in V$$

.

*(ii) (*Super-commutativity*)* $\omega^r \wedge \eta^s = (-1)^{rs} n^s \wedge w^r$, $\forall \omega^r \in \Lambda^r V, \eta^s \in \Lambda^s V$.

Proof. These are standard results, whose rigorous verification is however somewhat lengthy. We refer the reader to ([41], p.48) for a complete proof.

□

Definition A.2.5. *The space* $(\Lambda^\bullet V, +, \wedge)$ *is called the* exterior algebra of V*. It is a* $\mathbb{Z}$*-graded algebra, i.e.,*

$$(\Lambda^r V) \wedge (\Lambda^s V) \subseteq (\Lambda^{r+s} V), \forall r, s \geq 0$$

Remark A.2.6. As before, it is convenient to represent skew-symmetric tensors in coordinates. If $\{e_1, \ldots, e_n\}$ is a basis of the vector space V, then, for any $1 \leq r \leq n$, the family $\{e_{i_1} \wedge \cdots \wedge e_{i_r} / 1 \leq i_1 < \cdots < i_r \leq n\}$ is a basis of $\Lambda^r V$, so that any degree-r skew-symmetric tensor T can be uniquely represented as

$$T = \sum_{1 \leq i_1 < \cdots < i_r \leq n} T^{i_1 \ldots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}$$

Note that this means, in particular, that $\dim(\Lambda^r V) = \binom{n}{r}$.

Definition A.2.7. *Let* V *be an* n*-dimensional* $\mathbb{K}$*-vector space. The one-dimensional vector space* $\Lambda^n V$ *is called the* determinant line of V*, and it is denoted by* $detV$*.*

Remark A.2.8. One of the byproducts of the exterior-algebra construction is that it provides an appropriate framework in which to formalise the usual notion of *determinant* in a coordinate-free manner. This is the justification for the above-introduced definition.

In particular, let $L : V \to V$ be a linear endomorphism of V. It induces an endomorphism $\Lambda^n L : \det V \to \det V$, $v_1 \wedge \cdots \wedge v_n \mapsto L(v_1) \wedge \cdots \wedge L(v_n)$. The one-dimensionality of $\det V$ implies that $\Lambda^n L$ is just the multiplication by some scalar, which we choose to denote by $\det L$. One can easily show that this coincides with the usual notion of determinant (which we know is independent of the choice of basis) of a matrix representation of L.

Proposition A.2.9. *Suppose* V *is a complex* n*-dimensional vector space, and* A *is an endomorphism of* V*. For any natural number* r*, we denote by* $\sigma_r(A)$ *the trace of the induced endomorphism* $\Lambda^r L : \Lambda^r V \to \Lambda^r V$*, defined in a manner completely analogous to that of Remark A.2.8.*

(a) If A *is diagonalizable and its eigenvalues are* $\lambda_1, \ldots, \lambda_n$*, then*

$$\sigma_r(A) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \ldots \lambda_{i_r}$$

(b)

$$det(\mathbb{1}_V + tA) = \sum_{r \geq 0} \sigma_r(A) t^r,$$

where t *is an arbitrary indeterminate.*

Proof. (a) We will prove that, if we denote by A the matrix of L with respect to a chosen basis $\mathfrak{B}_V$ of V (for instance, a basis of eigenvectors), then $\Lambda^r A$, the matrix of $\Lambda^r L$ with respect to the induced basis $\mathfrak{B}_{\Lambda^r V}$ is what is frequently called in the literature the *r-th compound matrix* of A. The desired result is then a trivial corollary; this fact will become clear as soon as we introduce this notion.

We borrow from [39] the following definition: Let M be a $n \times m$ real or complex-valued matrix, and let $m_{i_1 \ldots i_r}^{j_1 \ldots j_r}$ be the minor of M determined by the rows $(i_1, \ldots, i_r)$ and the columns $(j_1, \ldots, j_r)$. Then, the *r-th compound matrix of M*, $M^{(r)}$, is the $\binom{n}{r} \times \binom{m}{r}$ matrix whose entries, written in lexicographical order, are $m_{i_1 \ldots i_r}^{j_1 \ldots j_r}$.

Now, choose a basis $\mathfrak{B}_V = \{e_i, i \in \{1, .., n\}\}$ of V. We know that a basis of $\Lambda^r V$ is then given by the collection $\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_r}, I = \{i_1, \ldots, i_r\} \subseteq N := \{1, .., n\}\}$. Let $A = (a_{ij})$ denote the matrix of L with respect to $\mathfrak{B}_V$. Then, the action of $\Lambda^r L$ on basis elements, of the form e_I, can be described in terms of A in the following way:

$$\Lambda^r L(e_I) = Ae_{i_1} \wedge \cdots \wedge Ae_{i_r} = \sum_{J = \{j_1, \ldots, j_r\} \subseteq N} a_{i_1 j_1} \ldots a_{i_r j_r} e_{j_1} \wedge \cdots \wedge e_{j_r} = \sum_{J \subseteq N} a_{IJ} e_J,$$

where $a_{IJ} = |a_{i_k j_l}|$, $k, l = 1, \ldots, r$, as we wanted to see.

(b) It is an immediate result if we bear in mind the above-proven equality. Indeed, we have that

$$\det(\mathbb{1}_V + tA) = \prod_{i=1}^{n}(1 + t\lambda_i)$$

Every one of the addends in the expansion of this product corresponds to a product resulting of the choice, for every $i \in \{1, \ldots, n\}$, of either 1 or $(t\lambda_i)$ (with the obvious restriction that each combination must appear only once, and we must exhaust all possibilities).

Clearly, the arising combinations are exactly those given by $\sigma_r(A)t^r$, if we let $r \in \{1, \ldots, n = \dim V\}$ and take into account the expression for $\sigma_r(A)$ proven above, thereby obtaining the desired equality.

□

A.3 Duality

As the reader may be aware, duality is a profound mathematical concept arising in many branches of mathematics. We list here some definitions and results from linear algebra that are relevant to our purposes; in particular, for tensor calculus and duality in cohomology.

Definition A.3.1. *(a) A* pairing between two K-vector spaces *V and W is a bilinear map $B : V \times W \to \mathbb{K}$.*

(b) Note that any pairing $B : V \times W \to \mathbb{K}$ defines a linear map $\mathbb{I}_B : V \to W^*$

$$v \mapsto B(v, \cdot) \in W^*$$

This is called the adjunction morphism associated to the pairing*.*

Conversely, any linear map $L : V \to W^*$ defines a pairing $B_L : V \times W \to \mathbb{K}$

$$(v, w) \mapsto (Lv)(w)$$

(c) Observe that $\mathbb{I}_{B_L} = L$. A pairing B is called a duality *if the adjunction map $\mathbb{I}_B$ is an isomorphism.*

Example A.3.2. The natural paiting $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{K}$ is a duality. Indeed, it is easy to see that $\mathbb{I}_{\langle \cdot, \cdot \rangle} = \mathbb{1}_{V^*}$. This pairing is called the *natural duality* between a vector space and its dual. For more details, cfr. ([16], section 4).

One can prove that the notion of duality is compatible with the constructions introduced above:

Proposition A.3.3. *Let $B_i : V_i \times W_i \to \mathbb{K}$, $(i = 1, 2)$ be two pairs of $\mathbb{K}$-vector spaces in duality. Then, there exists a natural duality $B = B_1 \times B_2 : (V_1 \otimes V_2) \times (W_1 \otimes W_2) \to \mathbb{K}$, uniquely determined by*

$$\mathbb{I}_{B_1 \otimes B_2} = \mathbb{I}_{B_1} \otimes \mathbb{I}_{B_2} \Leftrightarrow B(v_1 \otimes v_2, w_1 \otimes w_2) = B_1(v_1, w_1) \cdot B_2(v_2, w_2)$$

Remark A.3.4. (a) Iterating the result in Proposition A.3.3, we may infer that given two spaces in duality $B : V \times W \to \mathbb{K}$ there is a naturally induced duality $B^{\otimes r} : V^{\otimes r} \times W^{\otimes r} \to \mathbb{K}$.

This defines by restriction a pairing $\Lambda^r B : \Lambda^r V \times \Lambda^r W \to \mathbb{K}$ uniquely determined by

$$\Lambda^r B(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) \coloneqq det(B(v_i, w_j))_{1 \leq i,j \leq r}$$

(b) In particular, the natural duality $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{K}$ induces a duality $\langle \cdot, \cdot \rangle : \Lambda^r V \times \Lambda^r V^* \to \mathbb{K}$, and thus defines a natural isomorphism $\Lambda^r V^* \cong (\Lambda^r V)^*$.

This shows that we can regard the elements of $\Lambda^r V^*$ as skew-symmetric r-linear forms $V^r \to \mathbb{K}$.

(c) A duality $B : V \times W \to \mathbb{K}$ naturally induces a duality $B^\dagger : V^* \times W^* \to \mathbb{K}$ by

$$B^\dagger(v^*, w^*) \coloneqq \langle v^*, \mathbb{I}_B^{-1}(w^*) \rangle,$$

where $\mathbb{I}_B : V \to W^*$ is the adjunction morphism associated to the duality B.

The remarks above, in particular, imply the propagation of an inner product to the exterior algebra:

Proposition A.3.5. *Let V be a real Euclidean vector space. Denote its inner product by $(\cdot,\cdot)$. Then, $(\cdot,\cdot)$ naturally induces an inner product on the exterior algebra $\Lambda^\bullet V$.*

Proof. One needs only notice that the standard self-duality defined by $(\cdot,\cdot)$ induces a self-duality $(\cdot,\cdot):\Lambda^r V\times\Lambda^r V\to\mathbb{R}$ determined by

$$(v_1\wedge\cdots\wedge v_r, w_1\wedge\cdots\wedge w_r) \coloneqq det((v_i,w_j))_{1\leq i,j\leq r}$$

The right-hand side of the above equality is a Gramm determinant, so that the bilinear form it defines is symmetric and positive definite.

□

A.4 Orientation

This an important notion, as we will be working with *oriented* manifolds and oriented vector bundles over them.

Definition A.4.1. *Let V be a real vector space. A* volume form *on V is a nontrivial linear form on the determiant line of V, $\mu: detV\to\mathbb{R}$.*

Equivalently, a volume form on V is a nontrivial element of $detV^*$ ($n = dimV$). Since $detV$ is one-dimensional, a choice of a volume form corresponds to a choice of a basis of $detV$.

Definition A.4.2. *(a) An* orientation *on a vector space V is a continuous, surjective map $\boldsymbol{or}: detV\setminus\{0\}\to\{\pm 1\}$.*

We denote by $\boldsymbol{Or}(V)$ the set of orientations of V. Observe that it consists of exactly two elements.

(b) A pair $(V,\boldsymbol{or})$ is called an oriented vector space.

(c) Suppose $\boldsymbol{or}\in\boldsymbol{Or}(V)$. A basis ω of $detV$ is said to be positively oriented *if $\boldsymbol{or}(\omega)>0$. Otherwise, the basis is said to be* negatively oriented.

(d) To any basis $\{e_1,\ldots,e_n\}$ of V one can associate a basis $\{e_1\wedge\cdots\wedge e_n\}$ of $detV$. Note that a permutation of the indices changes the associated basis of $detV$ by a factor equal to the signature of the permutation. Thus, to define an orientation on a vector space, it suffices to specify a total ordering of given basis of the space.

An ordered basis of an oriented vector space $(V,\boldsymbol{or})$ is said to be positively oriented *if so is the associated basis of $detV$.*

Remark A.4.3. It is useful to present an equivalent way of looking at orientations. To describe it, note that any nontrivial volume form μ on V uniquely specifies an orientation $\boldsymbol{or}_\mu$ given by

$$\boldsymbol{or}_\mu(\omega)\coloneqq \mathrm{sign}(\mu(\omega)), \forall\omega\in\det V\setminus\{0\}$$

We can now define an equivalence relation on the space of nontrivial volume forms by declaring

$$\mu_1\sim\mu_2\Leftrightarrow\mu_1(\omega)\mu_2(\omega)>0,\forall\omega\in\det V\setminus\{0\}$$

Then,

$$\mu_1\sim\mu_2\Leftrightarrow\boldsymbol{or}_{\mu_1}=\boldsymbol{or}_{\mu_2}$$

To every orientation $\boldsymbol{or}$ we can associate an equivalence class $[\mu]_{\boldsymbol{or}}$ of volume forms such that

$$\mu(\omega)\boldsymbol{or}(\omega)>0,\forall\omega\in\det V\setminus\{0\}$$

Thus, we can identify the set of orientations with the set of equivalence classes of nontrivial volume forms.

Equivalently, to specify an orientation on V it suffices to specify a basis ω of $\det V$. The associated orientation $\boldsymbol{or}_\omega$ is uniquely characterised by the condition

$$\boldsymbol{or}_\omega(\omega)=1$$

Proposition A.4.4. *An orientation* $\boldsymbol{or}$ *on an Euclidean vector space* (V, g) *canonically selects a volume form on* V*, that we henceforth choose to denote by* $\mathrm{Det}_g{=}\mathrm{Det}_g^{\boldsymbol{or}}$.

Proof. Let V an Euclidean space where the Euclidean inner product is denoted by $g(\cdot,\cdot)$. The vector space $\det V$ has an induced Euclidean structure, as we saw in Proposition A.3.5, and, in particular, there exist *exactly* two length-one vectors in $\det V$. If we fix one of them, call it ω, and we think of it as a basis of $\det V$, note that it determines a volume form μ_g defined by $\mu_g(\lambda\omega) = \lambda$, thereby determining an orientation on V.

Conversely, an orientation $\boldsymbol{or} \in \boldsymbol{Or}(V)$ uniquely selects a length-one vector $\omega = \omega_{\boldsymbol{or}}$ in $\det V$, which determines a volume form $\mu_g = \mu_g^{\boldsymbol{or}}$.

□

A.4.1 The Pfaffian

We present here an object, the Pfaffian, that plays an important role in the proof of our aimed-for theorem, along with some useful basic results.

Definition A.4.5. *(a) Let* E *be an even dimensional vector space (*$\dim E = 2n$*) equipped with a Euclidean metric* g*. Let* $f : E \to E$ *be a skew-symmetric endomorphism.*

Then, f *induces a form* $\omega_f \in \Lambda^2 E$ *given by* $\omega_f(w, v) = g(f(w), v)$*. This is equivalent to defining*

$$\omega_f = \sum_{i<j} a_{ij} e_i \wedge e_j = \frac{1}{2}\sum_{i,j} a_{ij} e_i \wedge e_j,$$

where (a_{ij}) *is the matrix of* f *with respect to a* positively oriented orthonormal *basis of* E.

Now, if we consider the power $\omega_f^n = \overbrace{\omega_f \wedge \cdots \wedge \omega_f}^{n}$*, we have that* $\frac{1}{n!}\omega_f^n \in \Lambda^{2n} E$*, which is one-dimensional, so that we must have:*

$$\frac{1}{n!}\omega_f^n = P(e_1 \wedge e_2 \wedge \cdots \wedge e_{2n})$$

for some scalar P*, which we call* $\boldsymbol{Pf}(\omega_f)$*, the* Pfaffian of ω_f.

(b) Given a skew-symmetric endomorphism $f : E \to E$ *of an Euclidean vector space* (E, g)*, we define the* Pfaffian of f *to be* $\boldsymbol{Pf}(f) := \boldsymbol{Pf}(\omega_f)$.

Remark A.4.6. (a) Note that $\boldsymbol{Pf}(\cdot)$ can be seen as an operator on skew-symmetric endomorphisms of an Euclidean space or, equivalently, on skew-symmetric real $2n \times 2n$ matrices.

(b) $\boldsymbol{Pf}(f)$ is independent of the matrix representation chosen for f, as ensures the first *intrinsic* definition of it that we gave.

A well-known relation exists between the Pfaffian and the determinant. In particular, the Pfaffian is a square root of the determinant. To prove it, we first present a pair of preliminary results.

Lemma A.4.7. *Let* A *be a real skew-symmetric* $2n \times 2n$ *matrix.*

(a) Then, the non-zero eigenvalues of A *are pure imaginary and come in conjugate pairs.*

(b) Let $\{\pm i\lambda_k\}_{k=1,\dots,n}$ *be the eigenvalues of* A.

Then, there exists a matrix $B \in O(2n)$ *such that*

$$B^{-1}AB = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & A_n \end{bmatrix}, \text{ where } A_k = \begin{bmatrix} 0 & -\lambda_k \\ \lambda_k & 0 \end{bmatrix}.$$

This is sometimes refered to in the literature as the normal form of a skew-symmetric matrix *(see, for example, [50]).*

Notice that the above matrix may have a zero block, provided that $0 \in spec(A)$.

Proof. (a) Let λ be a non-zero eigenvalue of A and let v an eigenvector of λ, so that

$$Av = \lambda v \tag{A.1}$$

If we know multiply both sides of Equation A.1 by the conjugate of v, we have:

$$\overline{v}^\intercal Av = \lambda \overline{v}^\intercal v = \lambda ||v||^2 \tag{A.2}$$

Notice that the left side can be interpreted as the canonical scalar product of $\overline{v}^\intercal$ and Av in $\mathbb{C}^{2n}$, which is commutative, therefore obtaining:

$$\overline{v}^\intercal Av = (Av)^\intercal \overline{v} = v^\intercal A^\intercal \overline{v} = -v^\intercal A\overline{v}, \tag{A.3}$$

where the last equality follows from the skew-symmetry of A.

If we now take conjugates in Equation A.1, we obtain $\overline{Av} = \overline{\lambda}\overline{v}$, the substitution of which in Equation A.3 results in $\overline{v}^\intercal Av = -v^\intercal \overline{\lambda}\overline{v} = -\overline{\lambda} v^\intercal \overline{v} = -\overline{\lambda}||v||^2$.

If we now return to A.2, we deduce: $\lambda||v||^2 = -\overline{\lambda}||v||^2$, which implies, as v is non-zero on account of being an eigenvector, that $\lambda = -\overline{\lambda}$.

Only pure imaginary numbers verify such equation. We have thus proven that the non-zero eigenvalues of A are pure imaginary.

Finally, if λ is such an eigenvalue of A, then its conjugate, $-\lambda$, is clearly an eigenvalue of A.

(b) As conjugation of A by an othogonal matrix simply amounts to expressing f_A, the endomorphism of $\mathbb{R}^{2n}$ defined by A, in an orthogonal basis, we reduce the problem to giving an orthogonal basis in which the matrix of f_A has the desired form.

We start off by choosing an eigenvector e_1 of non-zero eigenvalue $i\lambda$, and we consider its conjugate $\overline{e_1}$. The set $\{(e_1+\overline{e_1}), i(e_1-\overline{e_1})\}$ contains real vectors that generate an invariant $\mathbb{R}$-vector space of dimension 2, which we call E_1. The restriction of f_A to that subspace has the form $A_{|E_1} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}$.

We switch to the orthogonal space $E_1^\perp$ and repeat the above procedure, thereby defining E_2. Again, we repeat this procedure in $(E_1 \oplus E_2)^\perp$, progressively adding the newly-obtained invariant 2-dimensional subspaces to the already established ones.

We iterate this process until we reach the $\{0\}$ subspace by projection to the orthogonal, at the end of which the matrix of f_A with respect to the union of the bases of the E_i subspaces will have the desired form. Note that the process is well-defined, as we are working over even dimension and we are producing 2-dimensional subspaces.

□

Proposition A.4.8. *Let $f : V \to V$ be a skew-symmetric endomorphism of an oriented Euclidean space V. Then,*

$$\boldsymbol{Pf}(f)^2 = det(f)$$

Proof. The result is now straightforward. On the one hand, if we consider the complex diagonalisation provided by Lemma A.4.7, (a), we have that $\det A = \lambda_1^2 \dots \lambda_n^2$.

On the other hand, it is easy to see that $\boldsymbol{Pf}(\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}) = \lambda$, which implies, by Lemma A.4.7, (b), that $\boldsymbol{Pf}(A) = \lambda_1 \dots \lambda_n$

□

It is useful to give an alternative definition of the Pfaffian that evinces its character of a polynomial on the entries of a skew-symmetric matrix A:

Proposition A.4.9. *Let $(V, g, \boldsymbol{or})$ be an oriented Euclidean space of dimension $2n$, $f : V \to V$ a skew-symmetric endomorphism and $\{e_1, \dots, e_{2n}\}$ a positively oriented orthonormal basis.*

Let $A = (a_{ij})$ be the matrix of A with respect to the basis (e_i). Then,

$$\boldsymbol{Pf}(A) = \frac{(-1)^n}{2^n n!} \sum_{\sigma \in \mathcal{S}_{2n}} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} \dots a_{\sigma(2n-1)\sigma(2n)}$$

Proof. The result is direct. Indeed, one needs only notice that when considering the n-th power of the form ω_f from Definition A.4.5, this expansion requires one to choose products of combinations of n factors of the form $\frac{1}{2} a_{ij} e_i \wedge e_j$, with the restriction that the only non-vanishing ones will be those in which each e_i appears exactly once, by the properties of the exterior product.

These combinations are in one-to-one correspondence with the different ways of ordering the set $\{1, \dots, 2n\}$. There only remains to remark that the factor $\frac{1}{n!}$ comes from the definition of the Pfaffian and $\frac{1}{2^n}$ from that of ω_f, whereas $(-1)^n$ arises by the super-commutativity of the exterior product (see Proposition A.2.4).

□

A.5 Some complex linear algebra

To finish off, we supply the reader with some elementary definitions in complex linear algebra necessary for the treatment of *complex* vector bundles.

Definition A.5.1. *(a) Let V be a complex vector space. Its* conjugate, *$\overline{V}$, is the complex vector space which coincides with V as a* real *vector spaces, but in which the multiplication by a scalar $\lambda \in \mathbb{C}$ is defined by $\lambda \cdot v := \overline{\lambda} v$, $\forall v \in V$.*

(b) The vector space V has a complex dual *V_c^* that can be identified with the space of complex-linear maps $V \to \mathbb{C}$. If we disregard the complex structure, we obtain a* real *dual V_r^* consisting of all real-linear maps $V \to \mathbb{R}$.*

Definition A.5.2. *A* Hermitian metric *is a complex bilinear map $(\cdot, \cdot) : V \times \overline{V} \to \mathbb{C}$ satisfying the following properties:*

(i) The bilinear form $(\cdot, \cdot)$ is positive definite, i.e., $(u, v) > 0, \forall v \in V \setminus \{0\}$.

(ii) For any $u, v \in V$, we have $(u, v) = \overline{(v, u)}$

Proposition A.5.3. *Let V be a complex (respectively real) n-dimensional vector space endowed with two distinct Hermitian (respectively Euclidean) metrics $g(\cdot, \cdot)$, $h(\cdot, \cdot)$. Then, there exists an endomorphism $f : V \to V$ such that*

$$g(v, w) = h(f(v), f(w)), \ \forall v, w \in V$$

Proof. It suffices to pick orthonormal bases $\{e_1, \dots, e_n\}$ with respect to g and $\{v_1, \dots, ev_n\}$ with respect to h (these can be produced from arbitrary ones via Gram-Schmidt) and define $f(e_i) = v_i$, $\forall i = 1, \dots, n$.

□

Remark A.5.4. (a) A Hermitian metric defines a duality $V \times \overline{V} \to \mathbb{C}$, and hence it induces a *complex-linear* isomorphism $\mathcal{L} : \overline{V} \to V_c^*$, $v \mapsto (\cdot, v)$.

(b) If V and W are complex Hermitian vector spaces, then any complex linear map $A : V \to W$ induces a complex-linear map $A^* : \overline{W} \to V_c^*$

$$w \mapsto \Big(v \mapsto \langle Av, w\rangle\Big) \in V_c^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural duality between a vector space and its dual. We can rewrite the above fact as $\langle Av, w \rangle = \langle v, A^* w \rangle$

(c) A complex-linear map $\overline{W} \to V_c^*$ is the same as a complex-linear map $W \to \overline{V_c^*}$. The metric duality defines a complex-linear isomorphism $\overline{V_c^*} \cong V$ so we can view the *adjoint* A^* as a complex-linear map $W \to V$.

Remark A.5.5. Let V be a complex vector space, and $\{e_1, \dots, e_n\}$ be a basis of V over $\mathbb{C}$. This is not a *real* basis of V, since $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V$. However, we can complete this to a real basis; in particular, $\{e_1, \boldsymbol{i} e_1 \dots, e_n, \boldsymbol{i} e_n\}$ is a real basis of V.

Definition A.5.6. *The* canonical orientation *of a complex vector space, $\dim_{\mathbb{C}} V = n$, is the orientation defined by $e_1 \wedge ie_1 \wedge \dots \wedge e_n \wedge ie_n \in \Lambda_{\mathbb{R}}^{2n} V$, where $\{e_1, \dots, e_n\}$ is any complex basis of V.*

Section B

Smooth manifolds

We mainly follow ([41], Chap. 1), with some slight modifications of the material where needed.

B.1 Some preliminary analysis

In this section, we briefly review some classic analytical definitions and results. Indeed, the study of multivariable calculus serves as propaedeutics for the more abstract theory of manifolds. For an exposition of this subject with a deep geometric flavour, we refer the reader to the standard work [47].

Definition B.1.1. *(a) Let X and Y be two Banach spaces (cfr. ([7], p.1)). We denote by $L(X,Y)$ the space of bounded linear operators $X \to Y$.*

(b) Let $F : U \subset X \to Y$ be a continuous function on an open subset U of X. The map is said to be Fréchet differentiable at $u_0 \in U$ *if there exists $T \in L(X,Y)$ such that*

$$||F(u_0 + h) - F(u_0) - Th||_Y = o(||h||_X) \text{ as } h \to 0$$

(c) We will use the notation $T = D_{u_0}F$ and we will call T the Fréchet derivative of F at u_0.

(d) Assume that the map $F : U \to Y$ is differentiable at each point $u \in U$. Then F is said to be of class C^1 *if the map $u \mapsto D_u F \in L(X,Y)$ is continuous.*

F is said to be of class C^2 *if the map $u \mapsto D_u F \in L(X,Y)$ is of class C^1.*

*One can define inductively C^k and C^∞ (*smooth*) maps.*

Remark B.1.2. Informally speaking, a continuous function is differentiable at a given point if, near said point, it can be best approximated by a linear map.

When F is differentiable at $u_0 \in U$, the operator T in the above definition is uniquely determined by

$$Th = \frac{d}{dt}|_{t=0} F(u_0 + th) = \lim_{t \to 0} \frac{1}{t}(F(u_0 + th) - F(u_0))$$

Example B.1.3. Let us consider $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Using Cartesian coordinates $\mathbf{x} = (x^1, \dots, x^n)$ in $\mathbb{R}^n$ and $\mathbf{y} = (y^1, \dots, y^m)$ in $\mathbb{R}^m$, we can think of F as a collection of m functions on U

$$F^1 = y^1(x^1, \dots, x^n), \dots, F^m = y^m(x^1, \dots, x^n)$$

The map F is differentiable at a point $p = (p^1, \dots, p^n) \in U$ if, and only if, the functions F^i are differentiable at p in the usual 1-dimensional sense. The Fréchet derivative of F at p is the linear operator $D_p F : \mathbb{R}^n \to \mathbb{R}^m$ given by the *Jacobian matrix*:

$$D_p F = \frac{\partial(y^1, \dots, y^m)}{\partial(x^1, \dots, x^n)} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1}(p) & \frac{\partial y^1}{\partial x^2}(p) & \dots & \frac{\partial y^1}{\partial x^n}(p) \\ \frac{\partial y^2}{\partial x^1}(p) & \frac{\partial y^2}{\partial x^2}(p) & \dots & \frac{\partial y^2}{\partial x^n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y^m}{\partial x^1}(p) & \frac{\partial y^m}{\partial x^2}(p) & \dots & \frac{\partial y^m}{\partial x^n}(p) \end{bmatrix}$$

The map F is smooth if, and only if, the functions $F^i(\mathbf{x})$ are smooth.

We now state the basic result that ensures the local propagation of invertibility of the Fréchet derivative of a smooth function to that function:

Theorem B.1.4. (Inverse function theorem). *Let X, Y be two Banach spaces, $U \subset X$ open and $F : U \subset X \to Y$ a smooth map.*

If at a point $u_0 \in U$ the derivative $D_{u_0} \in L(X, Y)$ is invertible, then there exists an open neighbourhood U_1 of u_0 in U such that $F(U_1)$ is an open neighbourhood of $v_0 = F(u_0)$ in Y and $F : U_1 \to F(U_1)$ is bijective, with smooth inverse.

Proof. It is a standard result. For a proof, we refer to ([41], p.4). This proof relies on the well-known Banach fixed-point theorem (see ([12], p.83)).

□

A direct corollary of the above theorem is the following additional basic result:

Theorem B.1.5. (Implicit function theorem). *Let X, Y, Z be Banach spaces, $\mathcal{U} \subset X$, $\mathcal{V} \subset Y$ open sets and $F : \mathcal{U} \times \mathcal{V} \to Z$ a smooth map. Let $(x_0, y_0) \in \mathcal{U} \times \mathcal{V}$, and $z_0 := F(x_0, y_0)$.*

Set $F_2 : \mathcal{V} \to Z$, $y \mapsto F(x_0, y)$.

Assume that $D_{y_0}F_2 \in L(X, Y)$ is invertible. Then, there exist open neighbourhoods $U \subset \mathcal{U}$ of x_0 in X, $V \subset \mathcal{V}$ of y_0 in Y, and a smooth map $G : U \to V$ such that the set S of solutions (x, y) of the equation $F(x, y) = z_0$ that lie inside $U \times V$ can be identified with the graph of G, i.e.,

$$\{(x, y) \in U \times V : F(x, y) = z_0\} = \{(x, G(x)) \in U \times V : x \in U\}$$

Proof. See ([41], p.5). □

Remark B.1.6. Intuitively, this theorem enables one to approach a problem of the form $F(x, y) = z_0$ as above. An equation of this sort does not define, in general, a function, but the implicit function theorem ensures the existence of a locally-valid $y = f(x)$. However, it doesn't tell us anything about how to find it.

Example B.1.7. With $X, Y, Z = \mathbb{R}$, we can consider the simple $x^2 + y^2 = 1$.

We cannot extract from this equation a global function $y(x)$, as it would fail to be a map in neighbourhoods of the points $(1, 0)$ and $(-1, 0)$, but a locally-valid function exists by virtue of the above theorem.

B.2 Smooth manifolds

We finally come to introduce the basic objects of study of this work, *smooth manifolds*. Intuitively, differential geometry studies the geometric properties of objects that are *independent of coordinates* (that is to say, independent of the way we choose to represent the object for the observer).

The prefiguration of the notion of a manifold is already contained in the work of Gauss on surfaces (cfr. [19]). However, as proves his wonder at Theorema Egregium, Gauss never ceased to see surfaces as lying withing the space $\mathbb{R}^3$. Thus, the truly foundational step, that emancipated manifolds as objects in its own right, was made by Riemann in his doctoral dissertation ([44]). Riemann understood surfaces as a particular case of a more general class of objects whose key characterisation was precisely their independence of the system of coordinates chosen to represent them. In fact, this would prove to be a seminal idea in the history of scientific thought through its enormous relevance in the theory of relativity (see [46] for more details).

B.2.1 Basic definitions

Definition B.2.1. *A* smooth manifold of dimension *m is a locally compact, second countable Hausdorff space M together with a collection, called* atlas, *consisting of the following:*

(a) An open cover $\{U_i\}_{i \in I}$ of M.

(b) A collection of homeomorphisms $\{\Psi_i : U_i \to \Psi_i(U_i) \subset \mathbb{R}^m\}_{i \in I}$ (called charts *or* local coordinates*) such that:*

(i) $\Psi_i(U_i)$ is open in $\mathbb{R}^m$

(ii) If $U_i \cap U_j \neq \varnothing$, then the transition map $\Psi_j \circ \Psi_i^{-1} : \Psi_i(U_i \cap U_j) \subset \mathbb{R}^m \to \Psi_j(U_i \cap U_j) \subset \mathbb{R}^m$ *is smooth. We say that the charts are* smoothly compatible.

Remark B.2.2. (a) As Chern himself pointed out: "A manifold is a sophisticated concept" ([15], p.344), as a result of which the working-out of particular cases may be of great help for a deeper understanding of this notion. As we do not have here the room to provide the reader with them in the required detail, we refer them to the classic work [48] for an abundant collection of examples (see, in particular, the second section of the first chapter).

(b) It is also possible to define infinite-dimensional manifolds by modelling them on Hilbert or Banach spaces. As the main theorem of this work falls within the finite-dimensional setting, we will restrict ourselves to it. We refer the interested reader to [20].

Remark B.2.3. (a) As M is second countable, we can always find an atlas that is *at most countable.*

(b) Trivially, from the definition of chart we deduce that any open subset A of an m-dimensional manifold inherits this structure.

(c) Each chart $\Psi_i : U_i \to \mathbb{R}^m$ can be viewed as a collection of m functions $(x^1, \ldots, x^m)$ on U_i, i.e,

$$\Psi_i(p) = (x^1(p), \ldots, x^m(p))$$

Consequently, any other chart Ψ_j can be seen as another collection of functions $\Psi_j(p) = (y^1(p), \ldots, y^m(p))$. The transition map $\Psi_j \circ \Psi_i^{-1}$ can then be thought of as a collection of maps

$$(x^1, \ldots, x^m) \mapsto (y^1(x^1, \ldots, x^m), \ldots, y^m(x^1, \ldots, x^m))$$

Remark B.2.4. Note that $\mathbb{R}^n$ can be trivially endowed with a manifold structure $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$.

The "local equivalency" of an n-dimensional manifold with the space $\mathbb{R}^n$ can be used to deploy all the usual machinery of analysis on $\mathbb{R}^n$ on manifolds. The following definitions establish both the first step towards that end and the notion of "equivalence" in the category of smooth manifolds.

Definition B.2.5. *(a) Let M, N be two smooth manifolds of dimensions m and n respectively. A continuous map $f : M \to N$ is said to be* smooth *if, for every $p \in M$ and for any local charts $\phi : U \to V$ of M with $p \in U$ and $\psi : W \to Z$ of N with $f(p) \in W$, the composition $\psi \circ f \circ \phi^{-1} : \phi(f^{-1}(W) \cap U) \subset \mathbb{R}^n \to Z \subset \mathbb{R}^m$ is smooth.*

(b) A smooth map $f : M \to N$ is called a diffeomorphism *if it is invertible and its inverse is also a smooth map.*

(c) If M is a smooth m-dimensional manifold, we denote by $C^\infty(M)$ the linear space of all smooth functions $f : M \to \mathbb{R}$. (It is straightforward to check that it is indeed a vector space with pointwise sum and real scalar multiplication as operations).

The above-laid out implicit function theorem turns out to give us a general recipe for producing manifolds, that we expose below. A lower-dimensional case of this result is a staple ingredient in differential-curves-and-surfaces courses, under the name of the *regular value theorem* (see [18], p.59).

Proposition B.2.6. *Let M be a smooth manifold of dimension m and $f_1, \ldots, f_k \in C^\infty(M)$. Define*

$$\mathcal{Z} = \mathcal{Z}(f_1, \ldots, f_k) \coloneqq \{p \in M : f_1(p) = \cdots = f_k(p) = 0\} = \bigcap_{i=1}^{k} f_i^{-1}(0)$$

Assume that the functions $f_1, \ldots, f_k$ are functionally independent *along $\mathcal{Z}$, that is, for each $p \in \mathcal{Z}$, there exist local coordinates $\mathbf{x} = (x^1, \ldots, x^m)$ defined in a neighbourhood of p in M such that $x^i(p) = 0$ for $i \in \{1, \ldots, m\}$ and the $k \times m$ matrix Jacobian matrix*

$$\frac{\partial f}{\partial \mathbf{x}}\Big|_p \coloneqq \begin{bmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial x^2} & \cdots & \frac{\partial f_1}{\partial x^m} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial x^2} & \cdots & \frac{\partial f_2}{\partial x^m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial x^1} & \frac{\partial f_k}{\partial x^2} & \cdots & \frac{\partial f_k}{\partial x^m} \end{bmatrix}_{x^1=\cdots=x^m=0}$$

has rank k. Then, $\mathcal{Z}$ has a structure of smooth manifold of dimension mk.

Proof. See ([41], p.8). □

The above result serves as motivation for the following definition:

Definition B.2.7. *Let M be an m-dimensional manifold. A* codimension-k submanifold of M *is a subset $S \subset M$ locally defined as the common zero locus of k functionally independent functions $\{f_1, \dots, f_k\} \subset C^\infty(M)$.*

Naturally, Proposition B.2.6 guarantees that any submanifold $N \subset M$ has a natural smooth structure so that it becomes a manifold in its own right. Furthermore, the inclusion map $i : N \hookrightarrow M$ is smooth.

We introduce now an object which will appear abundantly in the following chapters:

Definition B.2.8. *A* Lie group *is a smooth manifold G together with a group structure on it such that the map $G \times G$, $(g, h) \mapsto g \cdot h^{-1}$ is smooth (in other words, the group operations are).*

To end this subsection, we consider the notion of a *manifold with boundary*, needed to accommodate objects that would be manifolds were not for the fact that they have an "edge". Here is a visualisation:

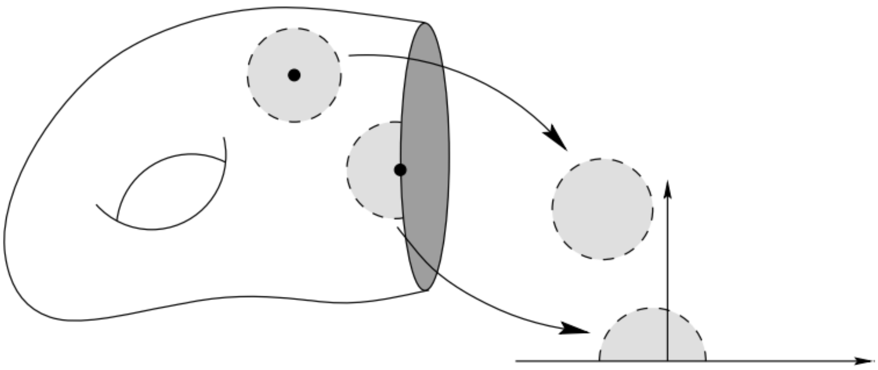

Figure B.1: Manifold with boundary. Extracted from ([31], p.25).

Definition B.2.9. *(a) For $n \geq 1$, we define $\mathbb{H}^n \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n : x^n \geq 0\}$. Naturally, $\mathbb{H}^n$ has the induced topology of $\mathbb{R}^n$.*

(b) A manifold with boundary *is defined in a manner completely analogous to that of Definition B.2.1, with the only modification that the definition of a chart is now extended to allow homeomorphisms of the form $\varphi : U \to \mathbb{H}^n$. These are called* boundary charts.

A point $p \in M$ is called a boundary point *if it lies in the domain of a boundary chart. A point $p \in M$ is called an* interior point *if it lies in the domain of a standard chart.*

The set of all boundary points of M is denoted by ∂M and the set of all interior points of M is denoted by Int*(M). It can be shown that these two sets are disjoint.*

An orientation on M induces an orientation on ∂M.

Remark B.2.10. If M is an n-manifold with boundary, then ∂M is an $(n-1)$-manifold (in the old sense) and $\mathrm{Int}(M)$ is an n-manifold (again in the old sense). In particular, $\partial\partial M = \varnothing$.

B.2.2 Cut-off functions

This is a brief technical subsection, where we state the existence of a certain type of differential functions that come in handy for different applications in geometry.

Definition B.2.11. *The* support *of a function $f \in C^\infty(M)$ is the closed set of M* $\mathrm{supp}(f) \coloneqq \overline{\{p \in M : f(p) \neq 0\}}$.

Proposition B.2.12. *Let U be an open set of M i $K \subset U$ compact.*

Then, there exists a differentiable function $f : M \to [0, 1]$ such that $f = 1$ in K, and $\mathrm{supp}(f)$ *is a compact subset of U.*

We call f a cut-off function associated to the pair (K, U).

Proof. See ([17], subsection 1.3.1) for a constructive proof. □

B.2.3 Partitions of unity

We state here a technical result that underpins some basic geometric facts. For instance, it is a standard requirement to prove the universal existence of Riemannian metrics on smooth manifolds and to define integration over them.

Definition B.2.13. *Let M be a smooth manifold and $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ an open cover of M. A (smooth)* partition of unity subordinated to this cover *is a family $\{f_\beta\}_{\beta\in\mathcal{B}} \subset C^\infty(M)$ satistying the following conditions:*

(i) $0 \leq f_\beta \leq 1$.

(ii) There exists some $\phi : \mathcal{B} \to \mathcal{A}$ such that supp$f_\beta \subset U_{\phi(\beta)}$, i.e., the support of every f_β is included in some open set of the cover.

(iii) The family $(\mathrm{supp} f_\beta)_{\beta\in\mathcal{B}}$ is locally finite, *i.e., any point $x \in M$ admits an open neighbourhood intersecting only finitely many of the supports $(\mathrm{supp} f_\beta)_{\beta\in\mathcal{B}}$.*

(iv) $\sum_{\beta\in\mathcal{B}} f_\beta(x) = 1$ *for all* $x \in M$.

Proposition B.2.14. *(a) For any open cover $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ of a smooth manifold M, there exists at least one smooth partition of unity $(f_\beta)_{\beta\in\mathcal{B}}$ subordinated to $\mathcal{U}$ such that* $\mathrm{supp}(f_\beta)$ *is compact for any $\beta \in \mathcal{B}$.*

(b) If we drop the requirement for compact supports, then we can find a partition of unity in which $\mathcal{A} = \mathcal{B}$ and $\phi = \mathbb{1}_{\mathcal{A}}$.

Proof. For a complete proof, we refer the reader to ([1], p.76).

□

Appendix II

Postponed proofs

Section C

Some computations in Riemannian geometry

We present here a pair of computations needed for the proof of Lemma 7.0.2:

Example C.0.1. (The second fundamental form of the sphere.) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a selfadjoint, invertible linear operator with at least one positive eigenvalue. Using Proposition B.2.6, one can show that the quadric

$$\mathcal{Q}_A = \{u \in \mathbb{R}^3 : g_0(Au, u) = 1\},$$

where g_0 is the usual Euclidean metric, is non-empty and smooth.

Now, let $u_0 \in \mathcal{Q}_A$ and consider the space $(Au_0)^\perp$. Set $\mathbf{n}(u) := \frac{Au}{|Au|}$. Consider an orthonormal frame (e_0, e_1, e_2) of $\mathbb{R}^3$ such that $e_0 = \mathbf{n}(u_0)$. Denote the Cartesian coordinates in $\mathbb{R}^3$ with respect to this frame by (x^0, x^1, x^2).

The second fundamental form of $\mathcal{Q}_A$ at u_0 is

$$\mathcal{N}_{\mathbf{n}}(\partial_{x^i}, \partial_{x^j}) = g_{0u_0}(\partial_{x^i}\mathbf{n}, \partial_{x^j})$$

Computing

$$\begin{aligned}\partial_{x^i}\mathbf{n} = \partial_{x^i}\left(\frac{Au}{|Au|}\right) &= \partial_{x^i}(g_0(Au,u)^{-\frac{1}{2}})Au + \frac{1}{|Au|}A\partial_{x^i}u \\ &= -\frac{g_0(\partial_{x^i}Au, Au)}{|Au|^{\frac{3}{2}}}Au + \frac{1}{|Au|}\partial_{x^i}Au\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{N}_{\mathbf{n}}(\partial_{x^i}, \partial_{x^j})|_{u_0} &= \frac{1}{|Au_0|}g_0(A\partial_{x^i}u, e_j)|_{u_0} \\ = \frac{1}{|Au_0|}g_0(\partial_{x^i}u, Aej)|_{u_0} &= \frac{1}{|Au_0|}g_0(e_i, Ae_j),\end{aligned}$$

which implies, in the particular case where $A = \frac{1}{r^2}\mathbb{1}$ so that $\mathcal{Q}_A$ is the sphere of radius r, that the first and the second fundamental forms of $\mathcal{Q}_A$ coincide.

Example C.0.2. (Area of the k-sphere.) Let $\mathbf{u} = (u^1, \dots, u^k)$ denote the coordinates on $S^k \hookrightarrow \mathbb{R}^{k+1}$ obtained via the stereographic projection from the south pole.

A straightforward (but tedious) calculation shows that the round metric g_0 on S^k (that is, the metric induced on S^k by the ordinary Euclidean metric) expressed in these coordinates is given by

$$g_0 = \frac{4}{(1+u^2)^2}\big((du^1)^2 + \cdots + (du^k)^2\big),$$

where $u := (u^1)^2 + \cdots + (u^k)^2$. (For details, see ([30], p.37)).

We want to compute the k-dimensional area of S^k, defined as

$$\sigma_k := \int_{S^k} dv_{g_o},$$

where dv_{g_o} is the volume form associated to the round metric.

Now, let $\mathbb{B}^n(r) \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq r^2\}$ be the ordinary *n-ball* in $\mathbb{R}^n$ and consider $\mathbb{S}^n(r) \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = r^2\}$.

Notice that to compute the n-dimensional area of $\mathbb{S}^n(r)$, defined analogously as the integral over $\mathbb{S}^n(r)$ of the volume form associated to the pull-backed Euclidean metric, it suffices to compute the volume of $\mathbb{B}^n(r)$ and derivate (with respect to r). It is well-know that this volume is

$$\mathrm{vol}(\mathbb{B}^n(r)) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{1}{2}n+1)} r^n,$$

where Γ is Euler's gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

.

(We do not have here the room to provide all the somewhat lengthy calculations, but a detailed and clear exposition is to be found in [21])). Therefore, the area of $\mathbb{S}^n(r)$ is given by

$$\frac{d\mathrm{vol}(\mathbb{B}^n(r))}{dr} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1}$$

If we reduce to the unitary-sphere case $S^k = \mathbb{S}^{k+1}(1)$, we finally obtain

$$\sigma_k = \frac{2\pi^{(k+1)/2}}{\Gamma(\frac{k+1}{2})} \tag{C.1}$$

Section D

Reductions of principal bundles

Definition D.0.1. *Let $\varphi : H \to G$ be a smooth morphism of Lie groups.*

(a) If P is a principal H-bundle over the smooth manifold M defined by the open cover (U_α), and gluing cocycle $h_{\alpha\beta} : U_{\alpha\beta} \to H$, then the principal G-bundle defined by the gluing cocycle $g_{\alpha\beta} \coloneqq \varphi \circ h_{\alpha\beta} : U_{\alpha\beta} \to G$ is said to be the φ-associate of P, *and it is denoted by $\varphi(P)$.*

(b) A principal G-bundle Q over M is said to be φ-reducible, *if there exists a principal H-bundle $P \to M$ such that $Q = \varphi(P)$.*

Remark D.0.2. Set $I^\bullet(G) \coloneqq \bigoplus_{k\geq 0} I^k(G)$ (see Subsection 6.2). The morphism $\varphi : H \to G$ in the above definition induces a morphism of $\mathbb{R}$-algebras $\varphi^* : I^\bullet(G) \to I^\bullet(H)$ (we can think of it as simply introducing in each component the differential of φ at $e \in H$, $\varphi_* : \mathfrak{h} \to \mathfrak{g}$).

The elements of $\ker \varphi^* \subset I^\bullet(G)$ are called universal identities.

Proposition D.0.3. *Let P be a principal G-bundle oer a smooth manifold M which can be reduced to a principal H-bundle Q.*

Then for every $\eta \in \ker \varphi^*$ we have

$$\eta(P) = 0 \in H^\bullet(M)$$

Proof. As above, if we denote by φ_* the differential of φ at $e \in H$, then $\varphi_* : \mathfrak{g} \to \mathfrak{h}$.

We pick a connection $\{A_\alpha\}$ on Q, and denote by $\{F_\alpha\}$ its curvature. Then the collection $\{\varphi_*(A_\alpha)\}$ defines a connection on P with curvature $\{\varphi_*(F_\alpha)\}$.

Finally,

$$\eta(\varphi_*(F_\alpha)) = (\varphi^*\eta)(F_\alpha) = 0$$

$\square$

Proposition D.0.4. *Let E be a rank $2k$ real, oriented vector bundle over the smooth manifold M.*

If E admits a nowhere-vanishing section σ, then the geometrical Euler class of E (see Subsection 6.4.3) verifies: $\boldsymbol{e}(E) = 0$.

Proof. To start off, choose an Euclidean metric on E (cfr. Remark 6.1.2) so that, E is now equipped with a $SO(2k)$-structure (see Remark 6.1.4).

Now, as σ is nonwhere-vanishing, the span $<\sigma(p)> \subset E_p$ is one dimensional for each $p \in M$. Thus, σ generates a real line subbundle of E. Therefore, E splits as an orthogonal sum $E = L \oplus L^\perp$.

Note that the orientation on E, and the orientation on L defined by σ induce an orientation on $L^\perp$, so that $L^\perp$ is endowed with a $SO(2k-1)$-structure.

That is to say that the $SO(2k)$-structure of E can be reduced to an $SO(1) \times SO(2k-1) \cong SO(2k-1)$-structure. Denote by i^* the inclusion-induced morphism $I^\bullet(SO(2k)) \to I^\bullet(SO(2k-1))$ and by $\boldsymbol{e}^k$ the universal Euler class in $I^\bullet(SO(2k))$ (see Definition 6.4.17).

Clearly, it is $i^*(\boldsymbol{e}^k) = 0$, so that $\boldsymbol{e}^k \in \ker i^*$. It is then derived from D.0.3 that $\boldsymbol{e}^k = 0$ and, therefore, $\boldsymbol{e}(E) = 0$, as we wanted to show.

$\square$

Section E

Proof of Lemma 7.0.2

Let $\pi : E \to M$ be a real oriented vector bundle over a smooth, compact, oriented manifold M. Denote by $S(E)$ the unitary sphere bundle of E (see Remark 7.0.1).

Lemma E.0.1. *There exists* $\Psi = \Psi(\nabla) \in \Omega^{2k-1}(S(E))$ *such that*

(i) $d\Psi(\nabla) = \boldsymbol{e}(\pi_0^*(E))$

(ii) $\int_{S(E)/M} \Psi(\nabla) = -1$

Proof. We denote by $\overline{\nabla}$ the pullback of ∇ to π_0^*E. The tautological section $\Upsilon S(E) \to \pi_0^*E$ can be used to produce an orthogonal splitting $\pi_0 E = L \oplus L^\perp$, where L is the real line bundle spanned by Υ (see the proof of Proposition D.0.4) and $L^\perp$ is its orthogonal complement in π_0^*E with respect to the pulled-back metric g. Denote by $P : \pi_0^*E \to \pi_0^*E$ the orthogonal projection onto $L^\perp$. Using P, we can produce a new metric-compatible connection $\tilde{\nabla}$ on π_0^*E by

$$\tilde{\nabla} = (\text{trivial connection on } L) \oplus P\overline{\nabla}P$$

We have an equality of differential forms $\pi_0^*\boldsymbol{e}(\nabla) = \boldsymbol{e}(\overline{\nabla}) = \frac{1}{(2\pi)^k}\boldsymbol{Pf}(-F(\overline{\nabla}))$.

Now, since the curvature of $\tilde{\nabla}$ splits as a direct sum $F(\tilde{\nabla}) = 0 \oplus F'(\tilde{\nabla})$, where $F'(\tilde{\nabla})$ denotes the curvature of $\tilde{\nabla}|_{L^\perp}$, we deduce that $\boldsymbol{Pf}(F'(\tilde{\nabla})) = 0$.

We denote by ∇^t the connection $\tilde{\nabla} + t(\overline{\nabla} - \tilde{\nabla})$ (in a manner analogous to Remark 3.3.4), so that $\nabla^0 = \tilde{\nabla}$, and $\nabla^1 = \overline{\nabla}$. If F^t is the curvature of ∇^t, we deduce from the transgression formula (6.4) that

$$\pi_0^*\boldsymbol{e}(\nabla) = \boldsymbol{e}(\overline{\nabla}) - \boldsymbol{e}(\tilde{\nabla}) = d\left[\left(\frac{-1}{2\pi}\right)^k k\int_0^1 \boldsymbol{Pf}(\overline{\nabla} - \tilde{\nabla}, F^t, \dots, F^t)dt\right]$$

We claim that the form

$$\Psi(\nabla) \coloneqq \left(\frac{-1}{2\pi}\right)^k k\int_0^1 \boldsymbol{Pf}(\overline{\nabla} - \tilde{\nabla}, F^t, \dots, F^t)dt$$

satisfies the conditions required.

By construction, $d\Psi(\nabla) = \pi_0^*\boldsymbol{e}(\nabla)$, so that all we need to prove is $\int_{S(E)/M} \Psi(\nabla) = -1 \in \Omega^0(M)$. It suffices to show that the integral of $\Psi(\nabla)$ along each fibre E_x of E is -1.

Along each fibre E_x, π_0^*E is naturally isomorphic with a trivial bundle:

$$\pi_0^*E|_{E_x} \cong (E_x \times E_x \to E_x)$$

Furthermore, the connection $\overline{\nabla}$ restricts to E_x as the trivial connection. $\pi_0^*E|_{E_x}$ can be identified, by the choice of an orthonormal basis of E_x, with the trivial bundle $\underline{\mathbb{R}}^{2k}$ over $\mathbb{R}^{2k}$. The unit sphere $S(E_x)$ is then identified with the unit sphere $S^{2k-1} \subset \mathbb{R}^{2k}$. The splitting $L \oplus L^\perp$ over $S(E)$ restricts over $S(E_x)$ as the splitting $\underline{\mathbb{R}}^{2k} = \nu \oplus TS^{2k-1}$, where ν denotes the normal bundle (see Definition 4.3.2) of $S^{2k-1} \hookrightarrow \mathbb{R}^{2k}$. The connection $\tilde{\nabla}$ is then the direct sum of the trivial connection on ν and the Levi-Civita connection on TS^{2k-1} (see Definition 4.2.7).

Fix a point $p \in S^{2k-1}$ and denote by $\mathbf{x} = (x^1, \dots, x^{2k-1})$ a collection of normal coordinates near p (see Definition 4.2.14), such that the basis $\{\partial_{x^i}|p\}_i$ is positively oriented. Set $\partial_i \coloneqq \partial_{x^i}$ for $i = 1, \dots, 2k-1$. Denote the unit outer normal vector field by ∂_0. For $\alpha = 0, 1, \dots, 2k-1$, set $f_\alpha = \partial_\alpha|p$. The vectors $\{f_\alpha\}$ form a positively-oriented orthonormal basis of $\mathbb{R}^{2k}$.

We will use Latin letters to denote indices running from 1 to $2k-1$, and Greek letters to denote indices running from 0 to $2k-1$. We can decompose

$$\overline{\nabla}_i \partial_\alpha = (\overline{\nabla}_i \partial_\alpha)^\nu + (\overline{\nabla}_i \partial_\alpha)^\tau,$$

where the superscript ν indicates the normal component, while the superscript τ indicates the tangential component. Since we are working with normal coordinates, it holds that, at p,

$$0 = \tilde{\nabla}_i \partial_j = (\overline{\nabla}_i \partial_j)^\tau,$$

from which we deduce that

$$\overline{\nabla}_i \partial_j = (\overline{\nabla} \partial_j)^\nu \text{ at } p$$

Hence, $\overline{\nabla}_i \partial_j = (\overline{\nabla}_i \partial_j, \partial_0)\partial_0 = -(\partial_j, \overline{\nabla}_i \partial_0)\partial_0$.

If we recall that $\overline{\nabla}$ is the trivial connection in $\mathbb{R}^{2k}$, we deduce that

$$\overline{\nabla}_i \partial_0|_p = \left(\frac{\partial}{\partial f_i}\partial_0\right)|_p = f_i = \partial_i|_p,$$

from which we infer that

$$\overline{\nabla}_i \partial_j = -\delta_{ji}\partial_0 \text{ at } p,$$

where δ_{ij} denotes the Kronecker symbol.

If we denote by $\{\theta^i\}$ the local frame of T^*S^{2k-1} dual to $\{\partial_i\}$, then we can reformulate the above equality as

$$\overline{\nabla} \partial_j = -(\theta^1 + \cdots + \theta^{2k-1}) \otimes \partial_0$$

On the other hand, $\overline{\nabla}_i \partial_0 = \partial_i$, i.e.,

$$\overline{\nabla} \partial_0 = \theta^1 \otimes \partial_1 + \cdots + \theta^{2k-1} \otimes \partial_{2k-1}$$

Since $\mathbf{x} = (x^1, \ldots, x^{2k-1})$ are normal coordinates with respect to the Levi-Civita connection $\tilde{\nabla}$, we deduce that $\tilde{\nabla}\partial_\alpha = 0$, $\forall \alpha$, so that

$$A := (\overline{\nabla} - \tilde{\nabla})|_p = \begin{bmatrix} 0 & -\theta^1 & \ldots & -\theta^{2k-1} \\ \theta^1 & 0 & \ldots & 0 \\ \theta^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \theta^{2k-1} & 0 & \ldots & 0 \end{bmatrix}$$

Denote now by F^0 the curvature of $\nabla^0 = \tilde{\nabla}$ at p. Then, $F^0 = 0 \oplus R$, where R denotes the Riemann curvature of $\tilde{\nabla}$ at p. The computations from Example C.0.1 show that the second fundamental form of the embedding $S^{2k-1} \hookrightarrow \mathbb{R}^{2k}$ coincides with the first fundamental form, the induced Riemann metric g_S. Using Theorema Egregium (cfr. 4.3.4) we get

$$g_S\big(R(\partial_i, \partial_j)\partial_k, \partial_l\big) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$$

In matrix format, we have $F^0 = 0 \oplus (\Omega_{ij})$, where $\Omega_{ij} = \theta^i \wedge \theta^j$. The curvature F^t at p of $\nabla^t = \tilde{\nabla} + tA$ can be computed using Equation (6.2) in the proof of the Chern-Weil theorem. We therefore get

$$F^t = F^0 + t^2 A \wedge A = 0 \oplus (1-t^2)F^0$$

We can now proceed to evaluate $\Psi(\nabla)$:

$$\Psi(\nabla)|_p = \left(\frac{-1}{2\pi}\right)^k k \int_0^1 \boldsymbol{Pf}(A, (1-t^2)F^0, \ldots, (1-t^2)F^0)dt$$

$$= \left(\frac{-1}{2\pi}\right)^k k \left(\int_0^1 (1-t^2)^{k-1} dt\right) \boldsymbol{Pf}(A, F^0, F^0, \ldots, F^0) \tag{E.1}$$

We set $F := F^0$ for simplicity. To evaluate the Pfaffian in the right-hand side of the above formula, we use the polarisation formula from Remark 6.2.1 (a) and Proposition A.4.9. We get:

$$\boldsymbol{Pf}(A, F, F, \ldots, F) = \frac{(-1)^k}{2^k k!} \sum_{\sigma \in \mathcal{S}_{2k}} \epsilon(\sigma) A_{\sigma(0)\sigma(1)} F_{\sigma(2)\sigma(3)} \cdots F_{\sigma(2k-2)} F_{\sigma(2k-1)}$$

For $i = 0, 1$, we define $\mathcal{S}^i = \{\sigma \in \mathcal{S}_{2k} : \sigma(i) = 0\}$. We deduce that:

$$2^k k! \boldsymbol{Pf}(A, F, F, \ldots, F) = (-1)^k \sum_{\sigma \in \mathcal{S}^0} \epsilon(\sigma)(-\theta^{\sigma(1)}) \wedge \theta^{\sigma(2)} \wedge \cdots \wedge \theta^{\sigma(2k-2)} \wedge \theta^{\sigma(2k-1)}$$

$$+(-1)^k \sum_{\sigma \in \mathcal{S}^1} \epsilon(\sigma)\theta^{\sigma(0)} \wedge \theta^{\sigma(1)} \wedge \cdots \wedge \theta^{\sigma(2k-2)} \wedge \theta^{\sigma(2k-1)}$$

For each $\sigma \in \mathcal{S}^0$ we get a permutation $\phi \coloneqq \phi_\sigma = \big(\sigma(1), \sigma(2), \ldots, \sigma(2k-1)\big) \in \mathcal{S}_{2k-1}$ such that $\epsilon(\sigma) = \epsilon(\phi)$ and, similarly, for each $\sigma \in \mathcal{S}^1$ we get a permutation $\phi \coloneqq \phi_\sigma = \big(\sigma(0), \sigma(2), \ldots, \sigma(2k-1)\big) \in \mathcal{S}_{2k-1}$ such that $\epsilon(\sigma) = \epsilon(\phi)$

Consequently,

$$2^k k! \boldsymbol{Pf}(A, F, F, \ldots, F) = 2(-1)^{k+1} \sum_{\phi \in \mathcal{S}_{2k-1}} \epsilon(\phi)\theta^{\phi(1)} \wedge \cdots \wedge \theta^{\phi(2k-1)}$$

$$= 2(-1)^{k+1}(2k-1)!\theta^1 \wedge \cdots \wedge \theta^{2k-1} = 2(-1)^{k+1} dV_{S^{2k-1}},$$

where $dV_{S^{2k-1}}$ denotes the Riemannian volume form on the unit sphere S^{k-1}. Using the second equality in Equation (E.1), we obtain

$$\Psi(\nabla)|_p = \left(\frac{-1}{2\pi}\right)^k k \left(\int_0^1 (1-t^2)^{k-1} dt\right) \cdot 2(-1)^{k+1}(2k-1)! dV_{S^{2k-1}}$$

$$= -\frac{(2k)!}{(4\pi)^k k!}\left(\int_0^1 (1-t^2)^{k-1} dt\right) dV_{S^{2k-1}}$$

Using the computations from Example C.0.2, we get that

$$\omega_{2k-1} \coloneqq \int_{S^{2k-1}} dV_{S^{2k-1}} = \frac{2\pi^k}{(k-1)!},$$

which implies that

$$\int_{S^{2k-1}} \Psi(\nabla) = -\omega_{2k-1} \frac{(2k)!}{(4\pi)^k k!}\left(\int_0^1 (1-t^2)^{k-1} dt\right)$$

$$= -\frac{(2k)!}{2^{2k-1} k!(k-1)!}\left(\int_0^1 (1-t^2)^{k-1} dt\right)$$

The above integral can be evaluated inductively using the substitution $t = \cos\varphi$. We have:

$$I_k \coloneqq \int_0^1 (1-t^2)^{k-1} dt = \int_0^{\pi/2} (\cos\varphi)^{2k-1} d\varphi$$

$$= \Big[(\cos\varphi)^{2k-2}\sin\varphi\Big]_0^{\pi/2} + (2k-2)\int_0^{\pi/2} (\cos\varphi)^{2k-3}(\sin\varphi)^2 d\varphi$$

$$= (2k-1)I_{k-1} - (2k-2)I_k,$$

so that

$$I_k = \frac{2k-3}{2k-2} I_{k-1}$$

It is then immediate to see that

$$\int_{S^{2k-1}} \Psi(\nabla) = -1,$$

as we wanted to show.

□

Bibliography

[1] Ilka Agricola and Thomas Friedrich, *Global Analysis: Differential Forms in Analysis, Geometry and Physics*, Springer, 2003.

[2] Pierre Albin, *Lectures on algebraic topology*, https://www.youtube.com/playlist?list=PLpRLWqLFLVTCL15U6N3o35g4uhMSBVA2b.

[3] Vladimir I. Arnold, *Ordinary Differential Equations*, Springer, 2006.

[4] Michael Artin, *Algebra*, Pearson, 2010.

[5] Michael Francis Atiyah, *Bordism and cobordism*, Pembroke College - Cambridge (1960).

[6] ______, *Vector Fields on Manifolds*, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, 1970, https://webspace.science.uu.nl/~caval101/homepage/Complex_Geometry_files/Atiyah-Vector%20Fields%20on%20Manifolds.pdf.

[7] Bernard Beauzamy, *Introduction to Banach spaces and their geometry*, Elsevier Science Ltd, 2016.

[8] Denis Bell, *The Gauss-Bonnet theorem for vector bundles*, Journal of Geometry **85** (2006), no. 1-2, 15–21, https://arxiv.org/abs/math/0702162v1.

[9] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat Kernels and Dirac Operators*, Springer, 2004.

[10] B. Booss and D.D. Bleecker, *Topology and Analysis The Atiyah-Singer Index Formula and Gauge-Theoretic Physics*, Springer, 1985.

[11] Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer, 1995.

[12] Haïm Brezis, *Analyse fonctionnelle - Théorie et applications*, Dunod, 2005.

[13] Élie Cartan, *Riemannian Geometry in an Orthogonal Frame*, World Scientific Publishing Co Pte Ltd, 2001.

[14] Shiing-Shen Chern, *A Simple Intrinsic Proof of the Gauss-Bonnet Formula for Closed Riemannian Manifolds*, The Annals of Mathematics **45** (1944), no. 4.

[15] ______, *From Triangles to Manifolds*, The American Mathematical Monthly **86** (1979), no. 5, 339–349.

[16] Keith Conrad, *Dual modules*, https://kconrad.math.uconn.edu/blurbs/linmultialg/dualmod.pdf.

[17] Carlos Currás Bosch, *Geometria diferencial: varietats diferenciables i varietats de Riemann*, Edicions Universitat de Barcelona, 2003.

[18] Manfredo P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[19] Carl Friedrich Gauss, *Disquisitiones generales circa superficies curvas*, Commentationes Societatis Regiae Scientiarum Gottingensis recentiores **6** (1828), 99–146, https://gdz.sub.uni-goettingen.de/id/PPN35283028X_0006_2NS.

[20] Robert Geroch, *Infinite-Dimensional Manifolds: 1975 Lecture Notes*, Minkowski Institute Press, 2013.

[21] Jake Gipple, *The Volume of n-balls*, Rose-Hulman Undergraduate Mathematics Journal **15** (2014), no. 238-248, https://scholar.rose-hulman.edu/rhumj/vol15/iss1/14/.

[22] Samuel I. Goldberg, *Curvature and Homology*, Dover Publications Inc., 2003.

[23] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2009, https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[24] ———, *Vector Bundles and K-Theory*, 2017, https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[25] Robert Hermann, *Yang-Mills, Kaluza-Klein, and the Einstein Program (Interdisciplinary Mathematics ; V. 19)*, Math Science Press, 1978.

[26] Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[27] Allyn Jackson, *Interview with Shiing Shen Chern*, Notices of the AMS **47** (1998), no. 7, 860–865, https://www.ams.org/notices/199807/chern.pdf.

[28] Sudin Kadam and Ravi Banavar, *Geometry of locomotion of the generalized Purcell's swimmer: Modelling, controllability and motion primitives*, IFAC Journal of Systems and Control **4** (2018).

[29] K. Kodaira and D. C. Spencer, *Groups of Complex Line Bundles Over Compact Kahler Varieties*, Proceedings of the National Academy of Sciences **39** (1953), no. 8.

[30] John M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Springer, 1997.

[31] ———, *Introduction to smooth manifolds*, Springer, 2012.

[32] Yin Li, *The Gauss-Bonnet-Chern Theorem on Riemannian Manifolds*, arXiv.org (2011), https://arxiv.org/abs/1111.4972.

[33] Daniel Litt, *The Poincaré Lemma and de Rham Cohomology*, The Harvard College Math Review **1** (2007), no. 2, http://math.columbia.edu/~dlitt/exposnotes/poincare_lemma.pdf.

[34] Jerrold E. Marsden, *Differential Forms and Stokes' Theorem*, http://www.cds.caltech.edu/~marsden/wiki/uploads/cds202-08/home/Differential_Forms.pdf.

[35] John Milnor, *Differential Topology: Lectures at Princeton University*, 1958, https://www.maths.ed.ac.uk/~v1ranick/papers/difftop.pdf.

[36] ———, *Topology from the Differentiable Viewpoint*, Princeton University Press, 1997.

[37] John Milnor and James D. Stasheff, *Characteristic Classes*, Princeton University Press, 1974.

[38] Klaus Mohnke, *Differential Geometry II: Connections of Vector Bundles*, 2020, https://www.math.hu-berlin.de/~mohnke/classes/diffgeo2_20/diffgeoII_9_final.pdf.

[39] James S. Muldowney, *Compound matrices and ordinary differential equations*, Rocky Mountain Journal of Mathematics **20** (1990), no. 4.

[40] Liviu I. Nicolaescu, *The many faces of the Gauss-Bonnet theorem*, 2003, https://www3.nd.edu/~lnicolae/GradStudSemFall2003.pdf.

[41] ———, *Lectures on the Geometry of Manifolds*, 2018, https://www3.nd.edu/~lnicolae/Lectures.pdf.

[42] Peter S. Park, *Proof of de Rham's theorem*, Department of Mathematics, Harvard University, https://scholar.harvard.edu/files/pspark/files/derham.pdf.

[43] Claudio Procesi, *Lie Groups: An Approach through Invariants and Representations*, Springer, 2007.

[44] Bernhard Riemann, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, Bernard Riemann's Gesammelte mathematische Werke und wissenschaftlicher Nachlass, 2014.

[45] Steven Rosenberg, *The Laplacian on a Riemannian Manifold: An Introduction to Analysis on Manifolds*, London Mathematical Society, 1997.

[46] R.K. Sachs and H. Wu, *General Relativity for Mathematicians*, Springer, 2012.

[47] Michael Spivak, *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*, CRC Press, 1971.

[48] ———, *A Comprehensive Introduction to Differential Geometry, Vol. 1*, Publish or Perish, 1999.

[49] Norman Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1999.

[50] G. Thompson, *Normal forms for skew-symmetric matrices and Hamiltonian systems with first integrals linear in momenta*, Proceedings of the American Mathematical Society **104** (1988), no. 3.

[51] Zuoqin Wang, *Peter-Weyl theorem and its applications*, Introduction to Lie groups, 2013, http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Notes/Lec%2023-24.pdf.

[52] Chris Wendl, *Lecture Notes on Bundles and Connections*, 2008, https://www.mathematik.hu-berlin.de/~wendl/connections.html.

[53] ______, *Topology I and II*, 2019, https://www.mathematik.hu-berlin.de/~wendl/Winter2018/Topologie2/lecturenotes.pdf.