UNIVERSITAT DE BARCELONA

Undergraduate Thesis

MAJOR IN MATHEMATICS and BUSINESS ADMINISTRATION

A vision of Two-Sided Matching Markets

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Barcelona, June 2021

Abstract

In this thesis we study an implementation of auctions in assignment markets. First we lay the basis for understanding auctions of a single object with private valuations, both at first price and at second price. As a main result, we find the revenue equivalence principle concerning certain types of auction. We focus on multi-object auctions and their classes.

Then, taking as a reference Shapley and Shubik (1971) [16], we explain the assignment market and its associated assignment game. We also present in detail the core and its lattice structure, as well as the buyers and the sellers optima.

Finally, we put the two issues together through the writing of Demange, Gale and Sotomayor (1986) [4] - the article *Multi-Item Auctions.* To find the buyers-optimal core allocation we present two mechanisms. We only need the information about the valuations and the minimum price at which the seller is willing to give up the item.

Resumen

En esta tesis estudiamos una implementación de subastas en mercados de asignación. En primer lugar, sentamos las bases para entender las subastas de un solo objeto con valoraciones privadas, tanto a primer precio como a segundo. Como resultado principal, encontramos el principio de equivalencia de ingresos relativo a ciertos tipos de subasta. Nos centramos en las subastas multiobjeto y en sus clases.

A continuación, tomando como referencia Shapley y Shubik (1971) [16], explicamos el mercado de asignación y su juego de asignación asociado. También presentamos en detalle el núcleo y su estructura reticular, así como los óptimos de los compradores y los vendedores.

Por último, unimos los dos temas a través del escrito de Demange, Gale y Sotomayor (1986) [4] - el artículo *Multi-Item Auctions.* Para encontrar la asignación óptima para los compradores en el núcleo presentamos dos mecanismos. Sólo necesitamos la información sobre las valoraciones y el precio mínimo al que el vendedor está dispuesto a ceder el artículo.

2010 Mathematics Subject Classification. 91A12, 91A25, 91B26

Acknowledgements

With this work ends, as my dear grandfather used to say, the Maria Antònia of Barcelona. I return to my land, Mallorca, and with it a new stage begins.

Throughout the trip I have been accompanied by very important people who have always been there and they have encouraged me to follow when I saw everything black. I could not present this work today without the presence of some people for which I want to thank them.

First, I would like to express my most sincere gratitude to my tutors, Javier Martínez de Albéniz and Xavier Jarque, for all their dedication to this project and their patience during these months. Thank you for sharing your knowledge with me, without your help this work would not have been possible.

To the roommates, people with whom I shared learning moments, both in kindergarten and at Raval library nights. With them everything is more enjoyable. They have always been and will be. Next I would like to mention 'Equipo A', the family that I met in the race. Although it is not easy for me to express my feelings, you know that you have been a fundamental part of this stage of my life and have come to stay.

Thank the double degree, which has taught me more about myself than any other field. To fight for what you want and to improve yourself, whatever the cost. As Cisco García says, a wheelchair tennis player that I admire, grit your teeth and move onward.

Finally and to whom I dedicate this work, my family. Those people who could not receive a decent education and have offered me what they could never have. Thank you for your incalculable effort that has allowed my sister and I to be who we are.

"You don't know how strong you are until being strong is the only option. You think you're not going to be able and you clench your teeth and you can if you have the right attitude."

Cisco García

Contents

Introduction

In economics, the concept of *market* is very important. It is defined as an entity that relates the individual who seeks a good, product or service with the individual who offers it. Similarly, the market is the physical or virtual place where sellers and buyers come together to make transactions, following the principles of supply and demand. Markets are present even without use of money, whenever it is necessary to assign students to schools, and agents have preferences over the other side of the market. See Roth (2002) [13] in *The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics.*

For a better understanding of the market, two important concepts must be understood: supply is the quantity of goods and services that sellers are willing to offer at a given price, while demand is the expressed formulation of a desire that is conditioned by the available resources of the individual or entity seeking a good or service.

There are three classic classifications of market types according to the sector of the economy in which they are located:

- Stock market - financial.
  In the financial market, financial assets, products and instruments are traded and a public price for the assets is set by the interaction of supply and demand.

- Labour market.
  These are the relationships established between a group of job seekers and a group of employers who demand certain professional profiles for their companies or projects.

  Depending on the country, the labour market is delimited by laws that establish some relevant aspects such as minimum wage, agreements and benefits for workers, number of working hours allowed, etc.

- Services and goods market.
  The goods and services market is where all kinds of goods - such as clothes, food, household appliances, etc. and services such as health, education, aesthetic services, etc. - are bought and sold.

This project deals with an application of auctions to "solve" an assignment market, giving a natural way to obtain a competitive equilibrium of the assignment market. To this end, we review some aspects of auctions and the game-theoretic tools to analyze them. Then we study assignment markets and the cooperative game version of it, where we define the competitive equilibrium, a matter of microeconomics. And lastly a way to find the competitive prices which are best for one side of the market, using a mechanism related to auctions. In this sense much of this work is on the interface between economics and operations research.

Auctions

Selling objects to the highest bidder or procuring valuable services from the lowest bidder have existed since time immemorial. Even so, there is no doubt that auctions are certainly of far greater importance today than at any time in the past.

We live in a globalized world in which a large number of agreements and economic transactions take place daily, not only between countries, but also between companies and individuals. Much of these operations are produced through auctions, so its study seems essential to understand a little better the functioning of the environment in which we live. Commodities such as fish and fresh flowers are sold in auctions, as they have been for centuries. Governments also rely on auctions for selling rights to timber, minerals or petroleum.

In recent years, online auctions have become increasingly popular. Platforms such as eBay rely on auctions to facilitate business-to-business, business-to-consumer, and consumer-to-consumer transactions; search engines like Google and Yahoo! employ auctions to sell keyword positions and advertisements. These type of auction break down and remove the physical limitations of traditional auctions such as geography or presence.

Traditionally, auctions are known as mechanisms for selling art and collectibles; a method of allocation where the price increases until a single buyer remains. However, the auction concept goes much further, and encompasses a broad set of mechanisms where rules and strategies are complicated to arrive at complex, but potentially very efficient market designs for pricing associated with an allocation of goods.

The design of auctions is intrinsically linked to a fundamental field of mathematics: game theory, which uses models to analyse strategies and behaviour in a wide range of problems in various application areas, among which auctions stand out.

The analysis of auctions as games originates in the work of William Vickrey (1961) [19]. He makes a systematic and formal study of auctions, which can only be understood from Bayesian games. A good reference for this topic where we will follow the notation and the basic model is the book by V. Krishna, (2002) [9].

The 1996 Nobel Prize in Economics awarded to the canadian economist William Vickrey. Then, the 2007 Prize in Economic Sciences awarded to Roger B. Myerson, for contributions to mechanism design. The 2014 Prize was also awarded to Jean Tirole, for contributions to the theory of regulation and competition policy. Finally, the last award given, the 2020 Nobel Prize in Economic Sciences, was awarded to Paul R. Milgrom and Robert B. Wilson. All of them won thanks to contributions to auction theory and the invention of new forms of auctions. Their works have enlightened others to investigate this area.

Assignment market

Decisions are clearly present in our interactions. Negotiations are made every day by different agents. The actions and choices of all the agents affect the outcome of each other. Therefore Game Theory analyses the decision-making process of several agents in mutually dependent situations.

The key pioneers of Game Theory were mathematician (and many other specializations) John von Neumann and economist Oskar Morgenstern in the 1940s. They introduced for the first time this term in *Theory of Games and Economic Behavior* (1944) [20].

The focus of game theory is the game, which serves as a model of an interactive situation among rational players. The key to game theory is that one player's payoff is contingent on the strategy implemented by the other(s) player(s). The game identifies the players' identities, preferences, and available strategies and how these strategies affect the outcome. Depending on the model, various other requirements or assumptions may be necessary.

Mathematician John Nash is regarded by many as providing the first significant extension of the von Neumann and Morgenstern work. Thanks to his contributions to game theory and bargaining processes, he won the Nobel Prize in Economics in 1994, and also the Abel Prize in 2015.

The assignment game is a model for a two-sided market in which a product that comes in large, indivisible units (e.g., houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit. In this setting, there are two disjoint sets that consist of m buyers and n sellers respectively. The units need not be alike, and the same unit may have different values to different participants.

This model of cooperative game was introduced by Shapley and Shubik (1971) [16]. It was subsequently studied by Roth and Sotomayor (1990) [14]. It is a well-established model of a market and a lot of works deals with the analysis of the game.

It is shown here that the outcomes in the core of such a game, those that cannot be improved upon by any subset of players, are the solutions of a certain linear programming problem dual to the optimal assignment problem. In this regard, Gale (1960) [5] defines competitive equilibrium and competitive equilibrium prices for more general markets and shows that they exist for any allocation problem.

These outcomes correspond exactly to the price-lists that competitively balance supply and demand. The geometric structure of the core is then described and interpreted in economic terms.

Demange (1982) [2] and Leonard (1983) [10] prove that in the buyers-optimal core allocation each buyer attains his marginal contribution and likewise for the sellers-optimal core allocation. The survey by Izquierdo, Núñez and Rafels (2012) [7], specialised in this area and professors of the Universitat de Barcelona, gives the main results on the bilateral assignment game. Also the survey in Núñez and Rafels (2015) [12] is relevant to explain the utility of the model.

Demange (1982) [2], Leonard (1983) [10], and Demange and Gale (1985) [3] have considered an allocation mechanism that turns out to be a generalization of the well-known second-price auction first described by Vickrey (1961) [19].

This project has been carried out with the aim of finding a non-cooperative basis for the issue of markets. The work of Demange, Gale and Sotomayor (1986) [4] shows that the buyers' optimal solution can be reached by a direct mechanism. The purpose of their paper is to show that there is another familiar property of the single-item auction that generalizes to the multi-item case.

Namely, instead of a one-shot sealed bid auction it is possible to achieve the minimum equilibrium price allocation by dynamic auctions. These are natural generalizations of the familiar auctions that occur in practice, in which the auctioneer systematically raises the price of an item until all but one of the bidders has dropped out.

In these auctions the sale price will then be approximately the second highest bid since, presumably, the highest bidder will try to outbid the competition by as small as amount as possible.

Two different dynamic auction mechanisms for the multi-item case will be presented. The first one explained here will be the most structured of the two and will lead us to produce in a finite number of steps the exact minimum price equilibrium. The second mechanism, the approximate, will simulate the competition in real auctions, where bidders increase the price they bid. Eventually we will reach a point as close as desired to the minimum price equilibrium.

These mechanisms only need to know the reserve price and the valuation of each buyer with respect to each object for sale. Therefore, these methods are more feasible and simpler, since it requires less information than trying to allocate with the well-known second-price auction.

About this work

The main part of this project is devoted to implement auctions in assignment markets.

In Chapter 1 we lay the basis for understanding auctions. We focus on multi-object auctions and their classes.

Chapter 2 introduces assignment market and its associated assignment game, taking as a reference Shapley and Shubik (1971) [16], which it has a key role in our study. We also present in detail the core and its lattice structure, as well as the buyers and the sellers optimum.

Finally, in Chapter 3, we present two mechanisms to find the buyers-optimal core allocation which minimise the procedure that would be carried out using the auction mechanism.

Chapter 1

Auctions

In this first chapter we will introduce Bayesian games, or incomplete information games, in order to explain auctions, which is an application of Bayesian games.

Although auctions are familiar to most people, we will go into the different types of auctions, explaining in detail the first-price and second-price auctions and the dominant strategies of each of them, and realising that the expected revenue in a first-price auction is the same as the expected revenue in a second-price auction.

We will then focus on multi-object auctions, which is a lesser-known type of auction, where more than one object is allocated, as the name suggests. Finally, we will present each type of multi-object auction with an example to understand it better.

This will create a basis for a better understanding of the next chapters. For this we will follow Krishna (2002) [9], as it defines the basic concepts of auctions. Moreover, the final project of Berta Serra (2019) [15] has been also an example of single-item auctions because she also used the work I have just mencioned.

1.1 Bayesian games

A standard assumption in *game theory* is that the structure of the game - number of players, their strategy sets, their preferences - are shared knowledge, what is called common knowledge. Consequently, all players know this information, all know that the other players know this information, and so on. One may wonder how more realistic situations can be studied where players have less information about the structure of the game.

Nash (1951)[11] defines the difference in between cooperative and non-cooperative games. There are many examples of situations that can be modeled as a game, but for a more realistic setting we can think that not all information is available to all players.

For example, the final payoff may depend on an unknown state of nature, about which the players have private information. Or a player may not know exactly some relevant characteristic of other players, their preferences or their beliefs about some important information in the game. All these examples have one factor in common, there is an information asymmetry in some relevant aspect of the game; the players do not have complete information about the structure of the game, and so in this situation we talk about *games with incomplete information.*

In a game with incomplete information, players possess private information about preferences and skills when choosing their strategies. The fact that they choose their strategies in the presence of information asymmetries is the distinguishing feature of these games. This property contrasts with the situation in games with imperfect information, in which the asymmetry is generated after the strategies have been selected. The seminal book of game theory is Von Neumann and Morgenstern (1944) [20].

Harsanyi (1967) [6] provides the necessary tools to understand these situations by introducing the concept of Bayesian games and considering these as the appropriate model for games with incomplete information. There are several issues about which players may have different information: they may differ about the number of players, about the strategy spaces available to each player, about how decisions influence the outcomes of the game, and about players' preferences over these outcomes. Harsanyi [6] shows that all these uncertainties can be transformed into ignorance about utility functions.

This section is just an introduction to Bayesian games. In the first place we include a brief introduction to game theory. To this end we define what a game is.

Definition 1.1. A *game* is a triple $G = (N, S, (\pi_i)_{i \in N})$, where

1. $N = \{1, 2, \ldots, n\}$ is a finite set, the players' set.

2. $S = \Pi_{i \in N} S_i$ is the set of strategies, with a set S_i for $i = 1, 2, \ldots, n$ the strategy set of each player.

3. $\pi_i(s_1, s_2, \ldots, s_n) \in \mathbb{R}$ for $i \in \{1, 2, \ldots, n\}$ is the payoff function of each player $i \in N$.

   The function π_i represents the payoff that the player i obtains if the strategy combination $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ occurs.

Any rational person wants to make as much profit as possible in any field. Focusing on games, each player expects to obtain certain payoffs and wants to maximise them. This requires the player to choose a strategy that, if he knew how others would behave, would give him the maximum expected payoff. This strategy is called a *best response,* to the strategy profile of the other players. In an environment where one knows the strategies that others are going to take, players respond better and the behaviour of the players is consistent. So, the player's strategies are mutual best responses. The idea of mutual best response is one of the many contributions of Nobel laureate John Nash to game theory. Nash [11] used the term *equilibrium* to refer to this term, but now we call *Nash equilibrium.*

On an informal basis, a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ and its corresponding payoffs represent a Nash equilibrium if no player can increase his payoff by changing his strategy, provided that the other players do not intend to change their chosen strategies. Lets define it now in a more formal way.

Definition 1.2. Given a game $G = (N, S, \pi_i)$, a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S_1 \times \ldots \times S_n$ is a *Nash equilibrium* if and only if

$$\pi_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq \pi_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)$$

for every $s_i \in S_i$ and each player $i \in N$.

Let us formalize the definition of an incomplete information game, also called Bayesian game.

Definition 1.3. *A Bayesian game with set of players* $N = \{1, 2, \ldots, n\}$ is a 5-tuple $BG = (N, (S_i)_{i\in N}, (T_i)_{i\in N}, \phi, (\pi_i)_{i\in N})$, where:

- The strategy set of each player, S_i for $i \in N$. We denote a player's i action as $s_i \in S_i$.
- The possible types set T_i for each player $i \in N$. We denote as $t_i \in T_i$ the type of the player i. Given a types' vector, one for each player, $(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)$ we write t_{-i} to represent the vector of types of players different from i.
- The conjecture that each player $i \in N$ has about the other players' types, i.e. the probability distribution, that maybe could be conditioned by his type, $\phi_i(t_{-i}|t_i)$. Initially all players have a common-knowledge distribution of types, and whenever the player knows his own type, the player updates the beliefs on the other players using Bayes' formula.
- The payoff functions $\pi_i(s_1, s_2, \ldots, s_n; t_1, \ldots, t_n) \in \mathbb{R}$ for $i \in N$.

Using the concepts of equilibrium and incomplete information games defined above, let us define Bayesian Nash equilibrium.

Definition 1.4. In a Bayesian game BG, where

$$BG = (N; S_1, \ldots, S_n; T_1, \ldots, T_n; \phi_1, \ldots, \phi_n; \pi_1, \ldots, \pi_n),$$

the strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a *Bayesian Nash equilibrium* if for each $i \in N$ and each type, $t_i \in T_i$, the strategy $s_i^*(t_i)$ is a solution of

$$\max_{s_i\in S_i} \sum_{t_{-i}\in T_{-i}} \phi_i(t_{-i}|t_i)\pi_i(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), s_i, s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n); t_1, \ldots, t_n).$$

Everything explained so far will be used for auctions where the type will be the valuations of the participants, since auctions are Bayesian games because the valuations of the other bidders are not available. This is self-evident since in an auction no one knows what value the object being sold brings to the other bidders. This is why an auction is an incomplete information game. Now we are going to study the auctions. Here we will observe that a Bayesian game appears in which the types of players follow some established distributions which we will analyze.

1.2 Auctions

Normally objects on the market have a fixed price, which buyers must accept if they want to obtain the object. In this case, the objects are sold through a competitive game. The basis of this way of defining the price is that the buyer who likes the product more will be able to buy it. It links the idea of paying more with the idea of liking it more. Whoever likes the product more will be willing to pay more for it. This competitive game is called an auction. The participants who want to buy the product are called bidders and the amount of money they offer is a bid.

Definition 1.5. An *auction* is a sales transaction in which interested parties compete with each other for the good or service to be sold at auction. During the auction, which is regulated by a set of rules, participants bid one or several amounts of money and the highest bid is the winner. The objective of this sales procedure is to maximise the profit from the sale and determine its equilibrium price.

Auctions are one of the first applications of static incomplete information games since the payoff functions are unknown to the bidders. Participants only know their own payoff function. The characteristic feature of auctions is the uncertainty of the values that bidders assign to the object. This creates the confrontation between sellers and buyers, since if the seller were aware of this, he would offer the object for the value he is willing to pay.

Martin Shubik [17] was an American economist who said that the existence of organised auctions dates back to Babylonian times, 500 BC. During this period, each village held an annual wives' market. At this time, women of marriageable age were gathered in the square, where men stood in a circle, and assigned by auction. The sale was carried out in succession, so that the most beautiful maiden received the highest number of bids and therefore the highest price.

Other examples cited by Shubik [17] are those of ancient Greece, where the auction method was used for the concession of mines, or in Rome, where the sale of slaves by auction was very common. However, with the fall of the Roman Empire, auctions lost their interest until well into the 18th century, when they once again became important and new methods began to emerge. One of these, with which we are very familiar, is the increasingly widespread use of the hammer to award the goods or the incorporation of time limits for submitting bids.

It was the French who introduced the fixing of time by establishing that an auction was not closed until three candles were consumed, which were lit immediately after a bid was made. If a new bid was made during this time, two more candles were lit, and so on until they were consumed.

It was not until the 20th century that the importance of auction transactions became significant. The advent of the internet and the possibilities opened up by online auctions has extended this system as a means of sale to a large number of buyers and sellers (see Bradley (2013) [1]).

1.2.1 Single Object Auctions

One particular type of auction is the sealed-bid auction. It means that each bidder bids an amount that no one else knows. When all participants have written their bids, they are made public and the highest bid wins.

We assume that there are n people interested in the product. The set of bidders is denoted by $N = \{1, 2, \dots n\}$. Each of these people values the object differently, which will be the maximum you are willing to pay. That is why we call X_i the maximum amount that player i is willing to pay, where i ranges from 1 to n. Each X_i is a random variable independently and identically distributed on some interval $[0, v], v \in \mathbb{R}_+$. Let us suppose that X_i follows a distribution function F and admits a continuous density function f.

A *private valuation auction* is one in which each bidder knows the value that the auctioned object has for him and, moreover, no bidder knows with certainty the valuation of the other bidders on the same object. That is why if we value an object in x currency units is indifferent to owning the object than having this money (x currency units). $V_i(x_1, \dots, x_n) = V_i(x_i)$ where V_i denotes the value that the bidder i assigns to the object. Knowledge of the valuations of other bidders does not affect the valuation itself. The actual valuation of the object is related to the offer we will make in an auction. That will be the strategy.

When bids are submitted, each player knows his valuation and the valuations of the other bidders are distributed independently, i.e. the valuations are private. The function, which assigns to any possible valuation the bid of each bidder i, is:

$$\beta_i : [0, v] \to \mathbb{R}_+$$

$$x_i \longmapsto \beta_i(x_i)$$

This function is known as *strategy*, which gives an offer for each private valuation x_i of the player $i \in N$.

Risk-neutral individuals are people who show indifference to the usefulness of a secure income and an uncertain one with the same expected value. Bidders are risk-neutral, it means that their usefulness is measured in the benefit they obtain.

Four main basic types of auctions have been considered. The first two are open auctions - the English and the Dutch - while the last two are sealed auctions - the first and second price formats. Another feature is that the first two have to be face-to-face, i.e. all bidders and sellers have to be gathered in the same room, as it is done at a certain time. On the other hand, bidders in the last two types of auctions can send their bids via email. Below is a brief description of each of these types.

Ascending or English auction.

This is the most commonly used type of auction. Its defining characteristic is the fact that the price is successively increased until there is only one buyer, who is awarded the good at the final price. The systems by which prices are increased can be different. Perhaps the best known is where the bidders themselves "call out" their bids (either orally or by entering them into an electronic mechanism). Buyers can submit as many bids as they wish as long as they satisfy the condition of outbidding the highest bid in place. Normally, when auction theorists discuss this type of auction, implicitly or explicitly, they are referring to a variant of the ascending auction in which the price is continuously raised (either by the seller or in an automated manner) and buyers successively withdraw when the price reaches levels they are unwilling to pay (once someone withdraws they are not allowed to rejoin). When a buyer withdraws, the remaining candidates observe the price at which they have left, and the process continues until only one buyer remains active, who is awarded the good at the price at which the last candidate left.

A well-known example is the auction of paintings. Firstly, when a painting is auctioned, the seller starts with an entrance price so that people who can not afford the painting do not enter to the auction. Then, the price of the painting starts rising until only one buyer remains. This one wins the object and pays the seller an amount equal to the price at which the second-last bidder dropped out.

Descending or Dutch auction.

This is the reverse of the above mechanism. In this case, the auctioneer starts with a very high price, which is successively lowered until a buyer accepts it. In this case, the best known example is the fish auction. The product starts with a price high enough so that presumably no bidder is interested in buying the object at that price. Then, the price is gradually going under until some bidder indicates his interest.

Second-price sealed-bid auction

These auctions are closed auctions, i.e. only one bid is made at the same time as the other buyers and they do not know how much the other buyers have bid. The successful bidder pays the second highest bid. So the profit for a participant if he has won is the value for him of the object bid minus the highest bid excluding his own bid. If there is a higher bid than his, he does not win the auction and his profit is 0. In particular, if there are two bids of the same amount and they correspond to the highest bid, they win with equal probability.

To sum up, the payoff function is:

$$\pi_i(b_1,\dots,b_n) = \begin{cases} x_i - \max_{j\neq i} b_j, & \text{if } b_i > \max_{j\neq i} b_j, \\ 0, & \text{if } b_i < \max_{j\neq i} b_j, \\ \frac{1}{\#\{j\in N | b_j = b_i\}}(x_i - b_i) & \text{if } b_i = \max_{j\neq i} b_j. \end{cases}$$

where b_i is the bid of i.

The strategy in second-price auctions is easier than in first-price auctions, as we will see later on, since in this case the weak dominance solution is that each player bids the value that the auctioned object has for him.

Proposition 1.1. *In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta(x) = x$.*

Proof. We want to see that if we bid $\beta(x_i) \neq x_i$, we will be worse off. Let's suppose that $b_k = \max_{j\neq i} b_j$ and $\beta(x_i) = x_i \pm \epsilon$, where $\epsilon \in \mathbb{R}_+$. We call b_i the bid that the bidder i offers and we will see that it is worse than offering the value that the auctioned object has for him, x_i.

Assume that $b_i > x_i$. Then, there are three possibilities:

- $b_k \geq b_i$, the bidder i does not win the auction because there is a higher bid.

  If he had offered his valuation, he would have been awarded the item.

- $b_i > b_k \geq x_i$, the bidder i wins the auction but obtains losses of $x_i - b_k$.

  If he had offered his valuation, he would not have made a loss as he would have lost the auction.

- $b_i > x_i > b_k$, the bidder i wins the auction and makes a profit of $x_i - b_k$.

  If he had offered his valuation, he would have obtained the same benefit.

We now assume the opposite, $b_i < x_i$. There are also three possibilities:

- $b_k \geq x_i > b_i$, the bidder i does not win the auction.

  If he had offered his valuation, he would have lost anyway.

- $x_i > b_k \geq b_i$, the bidder i does not win the auction.

  If he had offered his valuation, he would have made a profit of $x_i - b_k$.

- $x_i > b_i > b_k$, the bidder i wins the auction and makes a profit of $x_i - b_k$.

We have just seen that not betting the value that the object has for him worsens his chances of winning, and he can even make a loss. Therefore, the best strategy is that $b_i = x_i$. □

Hence, as we have seen in Definition 1.2, since it is not in the interest of any bidder to change strategy, bidding one's own valuation is a Nash equilibrium.

First-price sealed-bid auction

This type of auction is the same as the previous one, the winner is the highest bidder, but unlike the second-price auction, the winner does not pay the second highest bid, he pays the amount bid. So the profit is the difference between the maximum you are willing to pay for the item minus what you actually pay for it, if your bid is the highest. On the other hand, if it is not the highest, your profit is 0. A rare, but possible, case is when two participants bid the same amount and it is the highest. In this situation, as in the second-price auctions, they win with equal probability. In other words, the winner is decided by rolling a die. For example, if N people bid the maximum bid amount, the probability of one of them winning would be 1 in N.

To sum up, the payoff function is:

$$\pi_i(b_1, \ldots, b_n) = \begin{cases} x_i - b_i, & \text{if } b_i > \max_{j \neq i} b_j, \\ 0, & \text{if } b_i < \max_{j \neq i} b_j, \\ \frac{1}{\#\{j \in N | b_j = b_i\}}(x_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j. \end{cases}$$

where b_i is the bid of i.

Equilibrium behaviour is complicated as there is a confrontation of strategies. No one would bid an amount equal to its value as their profit would be 0. If they bid an amount higher than its value, there is a greater chance of winning and the lower the bid is relative to its value, the harder it is to win. Therefore, you always have to take into account the benefits you can get, that is, you have to maximise the difference between the maximum value you are willing to bid and your bid, in case you are the winner. In this context, if x is the value of the object from my point of view, then the strategy is to bid a value strictly lower than x. What I want to bid is exactly the highest bid of the other players, because if I bid more than that, I still win the auction but pay more.

Before continuing, it is necessary to introduce some concepts of probability and statistics. For this section we will consider absolutely continuous aleatory variables. As mentioned above, $X_1, X_2, \ldots, X_n$ independent and identically distributed aleatory variables associated with the distribution function F and admits a continuous density function f. Now let's rearrange them from highest to lowest, calling them $Y_1, Y_2, \ldots, Y_n$. That is, $Y_1 \geq Y_2 \geq \ldots \geq Y_n$. Consequently, for a $k \in N$, we denote by F_k the distribution function of Y_k. These aleatory variables are called *order statistics*.

By definition, Y_1 is the maximum statistical order and its distribution is obtained as follows. We see that, as $Y_1 \geq X_k \geq y$, for all $k \in N$. As the variables X_k are independent and identically distributed and have the same distribution function, we obtain

$$F_1(y) \equiv F(y)$$

and the corresponding density function is

$$f_1(y) = n \cdot F(y)^{(n-1)} \cdot f(y),$$

where $F(y)^{(n-1)}$ means that it corresponds to $n-1$ variables.

We consider the aleatory variable Y_1 that corresponds to the maximum valuation of the all bidders. Now, Y_1 will mean the same but for $N \setminus \{i\}$ bidders and denote as G its distribution function and as g its density function.

Proposition 1.2. *Symmetric equilibrium strategies in a first-price auction are given by*

$$\beta(x) = E[Y_1 | Y_1 < x]$$

where Y_1 is the highest of $N-1$ independently drawn values.

Proof. Let's assume that the strategy β is adopted by $n-1$ bidders. The demonstration will be based on seeing that the best the missing bidder can do is also follow the strategy β.

We take it for granted that β is an increasing and continuous function; and that in equilibrium and following this strategy, the auction is awarded to the one who values the object more, since he will have bid more than the rest.

It is not optimal for the missing bidder, called i, to bid $b_i > \beta(v)$, where v is the maximum value of bidder i, as it is overpriced. Therefore, the optimal offer will be $b_i \leq \beta(v)$. So the bidder i wins the auction when $b_i \geq \max_{j \neq i} b_j$ which is equivalent to $b_i \geq \beta(Y_1)$. So, the expected benefits of the bidder i offering b are

$$\text{Benefits}(b) = G(\beta^{-1}(b))(x-b).$$

Maximizing respect to the variable b

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x-b) - G(\beta^{-1}(b)) = 0.$$

We can rewrite it as

$$G(x) \cdot \beta'(x) + g(x) \cdot \beta(x) = x \cdot g(x)$$

because $b = \beta(x)$ and

$$G(x) \cdot \beta'(x) + g(x) \cdot \beta(x) = \frac{d}{dx}(G(x)\beta(x)),$$

so

$$\frac{d}{dx}(G(x)\beta(x)) = x \cdot g(x).$$

Since we know that $\beta(0) = 0$, because if the object has no value for the bidder he will not bid, then the best strategy is

$$\beta(x) = \frac{1}{G(x)} \int_0^x t g(t) dt = E[Y_1 | Y_1 < x].$$

If bidder i with value x bids $b_i = \beta(z)$, that is a different amount from $\beta(x)$, then the payoff function is

$$\begin{aligned}\pi(b,x) = \pi(\beta(z),x) =& G(z)[x-b] = G(z)[x-\beta(z)] = \\ =& G(z)\cdot x - G(z)E[Y_1|Y_1<z] = G(z)\cdot x - \int_0^z y\cdot g(y)dy = \\ =& G(z)\cdot x - [y\cdot G(y)]_0^z - \int_0^z G(y)dy = G(z)(x-z) + \int_0^z G(y)dy.\end{aligned}$$

On the other hand, if the bidder i bids $\beta(x)$ the payoff function is

$$\pi(\beta(x),x) = (x-x)G(x) + \int_0^x G(y)dy = \int_0^x G(y)dy.$$

Now that we have calculated the two profits to be made by bidding the two different amounts, we see that the difference between whether the bidder bids x or z is

$$\begin{aligned}\pi(\beta(x),x) - \pi(\beta(z),x) &= G(z)(z-x) + \int_0^z G(y)dy - \int_0^x G(y)dy = \\ &= G(z)(z-x) + \int_x^z G(y)dy \geq 0.\end{aligned}$$

In short, regardless of whether a higher or lower offer is made than the valuation, the benefits will be less. Therefore, we have just demonstrated that a bidder will earn his maximum profit if and only if his bid is $\beta(x)$.

□

Hence, as we have seen in Definition 1.2, since it is not in the interest of any bidder to change strategy, bidding one's own valuation is a Nash equilibrium. The first-price strategy is to bid what I expect the second winning player to have bid, which is the highest value of the other players' bids.

Revenue Comparison

In the study of sealed-bid auctions we have discussed so far, one has different possible rules for allocating the object, different assumptions about the nature and knowledge of the valuations, and also the attitude to risk of the bidders. For each situation, there is a special type of symmetric equilibrium and, consequently, a predictable behaviour of the bidders that strongly depends on the characteristics of each situation.

Once we have characterised the equilibrium strategies of the auction formats, a question arises: which of the auctions yields the highest revenue for the seller? The following theorem states that, from the seller's point of view, within a wide range of auction types, the characteristics of the auction do not affect the expected revenue.

Proposition 1.3. *With independently and identically distributed private values, the expected revenue in a first-price auction is the same as the expected revenue in a second-price auction.*

It is surprising that the expected sale revenues coincide in both first-price and second-price auctions since the sale price varies according to the auction. What is meant by the above proposition is that on average, the revenues coincide regardless of the distribution function. In addition, revenues from second-price auctions are more variable than first-price auctions since second-price auctions range from 0 to v and first-price auctions range from 0 to $E[Y_1]$ and the second-price auction carries more risk for the seller.

All the sealed-bid auctions we have studied in this chapter, which allocate the good to a random bidder, are called *standard* auctions because they have the property of awarding the good to the highest bidder. Moreover, we say that they are *symmetric* bidders if they follow the same valuation distribution function.

Theorem 1.1. ***(Principle of revenue equivalence)*** *We consider all standard sealed-bid auctions of a good, with a number n of risk-neutral bidders, whose valuations are independent and identical random variables over $[0, \bar{v}]$, and whose acceptable bids are also in $[0, \bar{v}]$.*

Then, in any symmetric Bayesian equilibrium in strictly increasing strategies in which any player with zero valuation makes a payment with zero expected value, the seller of the good obtains the same expected revenue.

Proof. To demonstrate this it is necessary first of all to see what the expected payout is for a bidder. In a second-price auction for bidder i with valuation x is:

$$\begin{aligned} m(x) &= Prob[\text{win}] \cdot E[\text{ the second highest offer } |x \text{ is the highest offer}] \\ &= Prob[x \text{ is the highest offer }] \cdot E[\text{ the second highest offer } |x \text{ is the highest value}] \\ &= G(x) \cdot E[Y_1|Y_1 < x]. \end{aligned}$$

In a first-price auction for bidder i with valuation x is:

$$m(x) = Prob[\text{win}] \cdot \text{quantity offered} = G(x) \cdot E[Y_1|Y_1 < x].$$

We note that the expected revenues for the first and second price auctions coincide.

The demonstration can now begin. Let's consider a standard auction. Let's set a symmetrical and increasing equilibrium β and $m(x)$ is the expected equilibrium auction payment for a bidder with a valuation x. Notice that $m(0) = 0$.

Assume that all other bidders, except $i \in N$, follow the β-equilibrium strategy. Suppose that the bidder i bids an amount different from $\beta(x)$. Let's call $\beta(z)$ the new bid. Then, bidder i wins the auction if $\beta(z) > \beta(Y_1)$. So, bidder i only wins if $z > Y_1$.

Accordingly, the expected payouts are

$$\pi(z, x) = G(z) \cdot x - m(z)$$

where $m(z)$ does not depends on x.

Now let's find the maximum:

$$\frac{d}{dz}\pi(z, x) = g(z) \cdot x - \frac{d}{dz}m(z) = 0.$$

Using $z = x$ we obtain the optimal. So, for all y,

$$g(y) \cdot y = \frac{d}{dy}m(y).$$

As $m(0) = 0$,

$$m(x) = m(0) + \int_0^x y \cdot g(y)dy = \int_0^x y \cdot g(y)dy = G(x) \cdot E\left[Y_1 \mid Y_1 < x\right].$$

We note thus that the payment does not depend on the type of auction. $\square$

Now that we have a general understanding of single-item auctions, let's focus on auctions with more than one object for sale.

1.2.2 Multiple Object Auctions

By the beginning of the 1990s, the main research focus shifted from single-object auctions to multi-object auctions. This shift was largely due to a desire to use markets for trading a wide range of objects - like spectrum - frequency bands, electricity, and batches of "troubled debt" - that had previously been allocated in other ways. In this section we will look at multi-object auctions.

The objects to be auctioned can be either identical or physically different. An illustration of this is an auction of many boxes of the same wine or an auction of different paintings by a painter. The wine is the same and each box is identical to the other. The paintings, on the other hand, are different from each other. The wine would correspond to auctions of the same products and the paintings to auctions of physically different products.

In this chapter we will look in detail at auctions of identical objects. In Chapter 3, on the other hand, we deal with the case of heterogeneous products. We have therefore studied both cases. If they are identical objects, they can be:

- Perfectly divisible object. In this case each participant is interested in a fraction of the whole object. For example, electricity or shares. It is the most common case.
- Discrete number of objects. In this case each participant is interested in one or more units. For example, 3G licenses.

Therefore, if the seller has several objects that he wants to sell, he can decide whether to make a single auction selling all of them or to sell them separately, making an auction for each object, by doing single-object auctions. If you make a single auction, it does not mean that a single bidder buys all the items. This is the case that we are going to study in more detail now, where they are identical objects.

The marginal value of the first item is higher than the second item, as utility is decreasing. Each bidder is asked to bid on each number of units to see how much they are willing to pay. That is, we want to see his decreasing demand function with the increasing number of units. Let's give an example.

Suppose we are doing a multi-item auction of identical objects. Assume bidder i is willing to pay 20 currency units for the first unit, 13 currency units for the second unit and 10 currency units for the third unit, then he is willing to pay 20 currency units for one unit, 33 currency units for two units and 43 currency units for three units. We refer to this sequence of numbers as *the bid vector*. To clarify, there are K bids per buyer and there are n buyers. So, there are $n \cdot K$ bids. Among of all these bids, the items are awarded to the highest bids. In other words, if they are K units for sale then the way to express this vector is

$$b^i = (b^i_1, b^i_2, \ldots, b^i_K).$$

Accordingly with the case of our example, the vector is $b^i = (20, 13, 10)$.

The *demand function* of player i for a particular price p is:

$$d^i(p) = \max\{k : p \leq b^i_k\}$$

which means the maximum number of units will be wanted for the price set p.

Example 1.1. Suppose there are six identical objects to sell, so $K = 6$, and there are three players interested in acquiring it, $n = 3$. So, there are 18 bids. Assume that the bid vectors corresponding to each buyer are

$$b^1 = (25, 23, 20, 16, 7, 2),$$
$$b^2 = (21, 14, 10, 6, 3, 1),$$
$$b^3 = (22, 17, 12, 7, 4, 3).$$

Of all these 18 bids, the six highest bids are $(25, 23, 22, 21, 20, 17)$. Marking them in red on the vectors, to make it easier to understand, it looks like this:

$$b^1 = (25, 23, 20, 16, 7, 2),$$
$$b^2 = (21, 14, 10, 6, 3, 1),$$
$$b^3 = (22, 17, 12, 7, 4, 3).$$

It is clear that bidder 1 is awarded three units, bidder 2 gets one unit and bidder 3 gets two units.

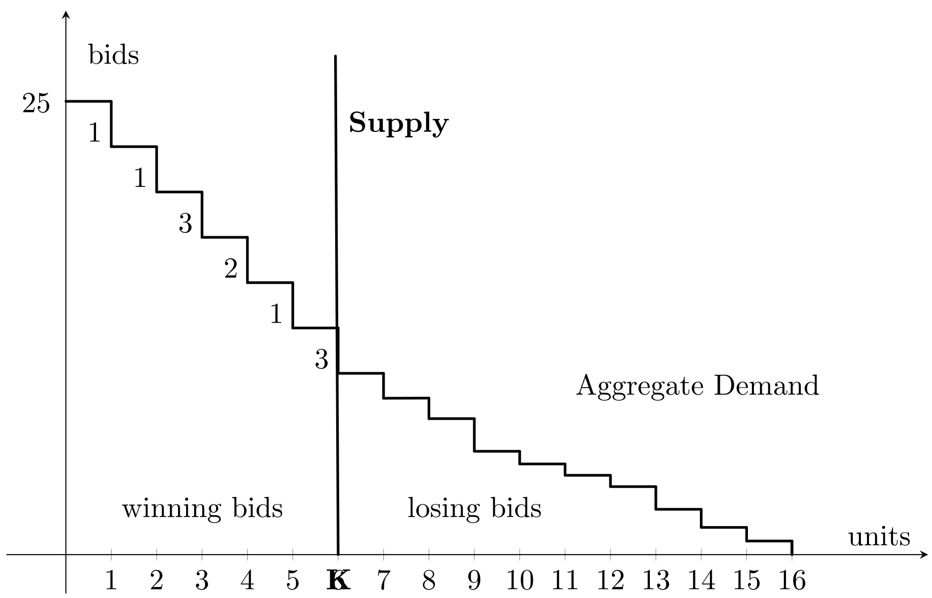

Figure 1.1: Aggregate Demand and Supply

First, market demand is derived by "horizontally" adding up the n individual demands. Then, all bids to the left of the intersection between market demand and supply functions, the K highest bids, are deemed winning bids. In Figure 1.1 we take the winning and losing bids of our previous example. It shows that the six objects have been awarded to the different bidders. The ordinate axis, or y-axis, corresponds to the bids, where the highest bid is 25 currency units. The abscissa axis, or x-axis, corresponds to the number of bids placed. In accordance with the theory, the graph is decreasing as the utility is reducing. Finally, in the part of the winning bids, the numbers related to each level correspond to who has been awarded that item.

As in the context auctions, we designate *standard multi-object auctions* to any mechanism which assign the objects to the K highest bids.

We will now explain the three multi-item auction formats, where the difference between them lies in the price to be paid by the winners.

Discriminatory Auctions

Let n be the number of players or bidders and let K be the number of objects to be auctioned. The discriminatory auction is the multiple-item version of the single-object first-price auction. That is, each winning buyer pays exactly the amount he bid for those goods. It can also be expressed in terms of *residual supply*, denoted as $s^{-i}(p)$, for a bidder i and a price p.

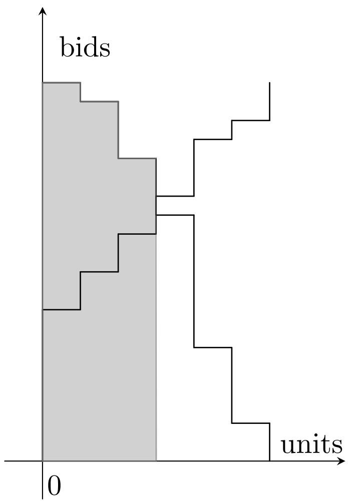

Figure 1.2: Payments and Pricing Rules of the Discriminatory Auction

The residual supply is:

$$s^{-i}(p) = \max\{K - \sum_{j \neq i} d^j(p), 0\}$$

which is the number of items for sale in the auction minus the sum of what is demanded by the other buyers. In Example 1.1,

$$b^1 = (25, 23, 20, 16, 7, 2)$$

where the winning bids are marked in red, bidder 1 wins three items, so he will have to pay the sum of the items he has been awarded. Accordingly, bidder 1 will pay 25+23+20=

68 currency units and the same goes for the other two participants. Figure 1.2 shows the case of bidder 1 in the example above. It represents the residual supply function where the grey area is what bidder 1 has to pay in total in this type of multi-object auction, i.e. the 68 currency units.

This type is an extension of the first price sealed auction as can be seen if the number of items for sale is one, $K = 1$.

Uniform-Price Auctions

The uniform-price auction is the multiple-item version of the single-object second-price auction. In this case the K units are sold at a *market-clearing price*. In other words, the total quantity demanded is equal to the total quantity delivered.

This market price is variable, as it can be any price between the highest losing bid, i.e. the highest bid that has been left without being awarded any good, and the lowest winning bid. For us, the market-clearing price will be the highest losing bid.

If we order the bids from highest to lowest offered amount competitors have bid for, all bidders' bids $j \neq i$, and take the first K positions, we get a K-vector of *competing bids* facing bidder i named c^{-i}. So in the first position there will be the highest bid, in the second position the second highest bid and so on.

In short, bidder i wins exactly $k^i > 0$ units if and only if

$$b^i_{k^i} > c^{-i}_{K-k^i+1} \quad \text{and} \quad b^i_{k^i+1} > c^{-i}_{K-k^i}.$$

This can be more easily understood by reproducing it in the proposed example.

The market-clearing price is

$$p = \max\{b^i_{k^i+1}, c^{-i}_{K-k^i+1}\}.$$

If a bidder is awarded with x items, where $x \in \{0, 1, \ldots, N\}$, then $p \times x$ must be paid, as each object has the same price.

In the Example 1.1, we look at the amount bidder 1 has to pay. First we will order all bids from highest to lowest.

Putting all the bids of the three participants together results in the following:

$$(25, 23, 22, 21, 20, 17, 16, 14, 12, 10, 7 \text{ (twice)}, 6, 4, 3 \text{ (twice)}, 2, 1).$$

Now we remove those of bidder 1:

$$(22, 21, 17, 14, 12, 10, 7, 6, 4, 3 \text{ (twice)}, 1),$$

and we take the first K, in this case $K = 6$, so the vector c^{-1} is

$$c^{-1} = (22, 21, 17, 14, 12, 10).$$

As $b^1 = (25, 23, 20, 16, 7, 2)$ and

$$b^1_3 > c^{-1}_4 \text{ but } b^1_4 < c^{-1}_3,$$

bidder 1 wins three units, as we have seen before. Applying the formula, the market-clearing price is

$$p = \max\{b^1_{3+1}, c^{-1}_{6-3+1}\} = \max\{b^1_4, c^{-1}_4\} = \max\{16, 14\} = 16.$$

For each item won, 16 currency units are paid. As bidder 1 is awarded three products, he pays a total of $16 \cdot 3 = 48$ currency units. Figure 1.3 shows the uniform-price auction case, where the darker area represents the amount to be paid by bidder 1.

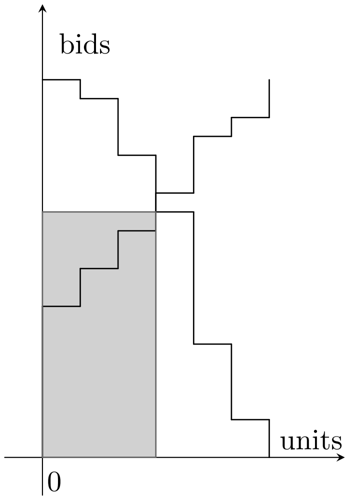

Figure 1.3: Payments and Pricing Rules of the Uniform-Price Auction

This type is an extension of the second price sealed auction as it can be seen if the number of items for sale is one, $K = 1$.

Vickrey Auctions

In Vickrey auctions, whoever wins k^i units pays the highest k^i bids of the competitors who have lost, not including their own. Using the vector c^{-i} defined above, it stands to reason that for one to win a unit, it must be higher than any component of the vector c^{-i}. That is $b_1^i > c_K^{-i}$. To win the kth unit, i's kth highest bid must defeat the kth lowest competing bid. If bidder i wins k^i units, then the amount to be paid is

$$\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i}$$

Using Example 1.1,

$$b^1 = (25, 23, 20, 16, 7, 2)$$
$$c^{-1} = (22, 21, 17, 14, 12, 10)$$

As bidder 1 wins three objects out of the six objects for sale, $K = 6$ and $k^i = 3$, then

$$\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i} = c_{6-3+1}^{-1} + c_{6-3+2}^{-1} + c_{6-3+3}^{-1} = c_4^{-1} + c_5^{-1} + c_6^{-1} = 14 + 12 + 10 = 36.$$

In a nutshell, each bidder is required to pay an amount equal to the externality he exerts on the others competing bidders. The shaded area of Figure 1.4 represents the area lying under the residual supply function facing bidder 1.

This type is an extension of the first price sealed auction as can be seen if the number of items for sale is one, $K = 1$.

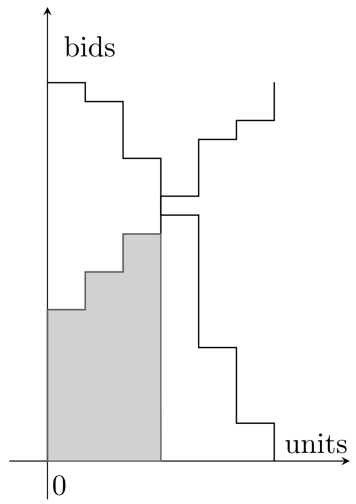

Figure 1.4: Payments and Pricing Rules of Vickrey Auction

In this chapter we have considered three basic auction formats for the sale of multiple identical units. There are auctions of identical units of the same good, such as auctions of public debt, and auctions of heterogeneous goods or services, such as auctions of services in different parts of a country.

In Chapter 3 we discuss cases of auctions of various heterogeneous objects. The demand functions, in this case, are not decreasing and people value each object differently due to interests. Furthermore, it is considered that all bidders are interested in buying only one unit.

Each bidder is assumed to place a monetary value on each of the items, and, given a price vector, he will demand that item or those items that maximize his surplus, the difference between his valuation and the price of the item, assuming that this surplus is positive

Chapter 2

Assignment markets and assignment games

In this chapter we describe assignment markets, where both buyers and sellers interact to exchange indivisible goods and make a profit. In a market, participants, let's say buyers and sellers, interact and exchange items by a price.

Then we will focus on assignment games, which are each associated with an assignment market. Shapley and Shubik (1971) [16] define and study assignment games and their interesting features. These assignment games are cooperative games that reflect the nature of the market. We see that prices are in a natural way reflected in allocations of the game, and see the meaning of competitive prices. We will see that some matchings between buyers and sellers are optimal, leading to the concept of the core.

Finally, we study the structure of the core, which is a lattice in which there is a best point or allocation for all the sellers at the same time, the sellers-optimal core allocation, and also the buyers-optimal core allocation.

In addition, Roth and Sotomayor (1990) [14] presents this topic from the point of view of matching procedures. A useful survey about assignment games is Izquierdo, Núñez and Rafels (2012) [7], from where I have obtained information for this chapter, or Núñez and Rafels (2015) [12]. It is worth noting that the project by Rubén Ureña (2017) [18] who also drew on this article to write his work, has been used for this work.

2.1 Assignment markets

Assignment problems were originated from the logistic efforts related to the Second World war, and Koopmans and Beckmann (1957) [8] study it. Gale (1960) [5] introduces the idea of linear markets and competitive equilibrium prices. Following Gale's ideas, Shapley and Shubik (1971) [16] defines the assignment market, from the point of view of cooperative games. Let's start by defining each term we use in the following definitions.

Purchase is the action by which an agent (the buyer), acquires a good from another agent (the seller), in exchange for a monetary or in-kind consideration. The buying process is always shown in contrast to the selling process. In this sense, the two agents that must be present for the process to take place are:

- Buyer: The person who wants the good or service. This is the person who pays the consideration in exchange for the good or service. This may be an individual or a legal entity.
- Seller: The person who possesses the good or service that the other person wants. He establishes the consideration he wishes to receive for the good or service. In this case, it can also be a natural or legal person.

Thus, when both parties reach an agreement, a process known as buying and selling takes place.

An *assignment market* consists of a finite set M of m buyers, who each want to buy a single good, and a finite set M' of m' sellers, who each want to sell exactly one good. So we can say that an agreement will only be reached between the two parties, sellers and buyers, when the price offered by the person interested in obtaining the object is higher than the minimum price at which the seller is willing to sell.

Let $h_i^j \in \mathbb{R}_+$ be the value that the buyer $i \in M$ has of seller $j \in M'$'s object and let $c_j \in \mathbb{R}_+$ be the minimum value at which the seller $j \in M'$ is willing to sell the product. In other words, seller $j \in M'$ has a reservation price $c_j \in \mathbb{R}_+$ below which he/she will not sell his good. So there will be an agreement when $h_i^j \geq c_j$. The agreed price will be a value between the value of the good to the buyer and the minimum price the seller is willing to accept.

Thus, the matrix that captures the profit of each possible pairing is matrix A with

$$a_{ij} = \max\{h_i^j - c_j, 0\} \quad \text{for all } (i,j) \in M \times M', \tag{2.1}$$

where the profit is 0 if there no possible agreement between the parties. Notice that the price of the transaction is in between h_{ij} and c_j. Therefore if the transaction takes place, a part of a_{ij} goes to the buyer and other part goes to the seller.

Formally, we denote this market by $\gamma = (M, M'; A)$.

Shapley and Shubik (1971) [16] associate to each assignment market (M, M', A) a cooperative game that they call the *assignment game*.[1] It is defined by a set of players, which are the buyers and sellers, $M \cup M'$, and the characteristic function w_A that associates to each coalition of agents the maximum profit they can achieve by assigning buyers and sellers.

This means that we want to match a buyer with a seller in such a way as to achieve the best benefit for different buyers and sellers, where we recall that each seller only sells one object and each buyer is only interested in one object. In addition, they are free to buy or not to buy, as they are not obliged to do so.

Another feature is that the objects to be allocated are indivisible. An example is that you want to allocate different flats to potential buyers. Each assigned buyer is given the whole flat, not just the kitchen.

Definition 2.1. A *matching* μ between M and M' is a subset of the Cartesian product, $M \times M'$, such that each agent belongs to at most one pair.

[1] A *cooperative game* is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players and a function $v : \mathcal{P}(N) \longrightarrow \mathbb{R}$ that assigns to any coalition $S \subseteq N$ a real number $v(S)$ with $v(\emptyset) = 0$.

Usually and to avoid inconsistencies, we assume that the cardinality of the matching is that of the smaller set, that is $|\mu| = \min\{|M|, |M'|\}$.

The set of all possible matchings is denoted as $\Gamma(M, M')$. Now we see when the matching is optimal, that is, it gives a maximum value, for a matrix A of possible profits.

Definition 2.2. The matching $\mu^* \in \Gamma(M, M')$ is *optimal* for the market $\gamma = (M, M'; A)$ if

$$\sum_{(i,j)\in\mu^*} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}, \qquad \text{for all } \mu' \in \Gamma(M, M').$$

We denote $\Gamma_A(M, M')$ the set of all optimal matchings for the market $\gamma = (M, M'; A)$. An optimal matching μ can be found by solving the linear assignment problem, which we describe now. The optimal matching is denoted as μ^*.

For the assignment market $\gamma = (M, M'; A)$ define the following linear assignment problem:

$$\begin{aligned} \max \qquad & z = \sum_{i\in M}\sum_{j\in M'} a_{ij}\mu_{ij} \\ \text{where} \qquad & \sum_{i\in M} \mu_{ij} \leq 1, \qquad \text{for all } j \in M', \\ & \sum_{j\in M'} \mu_{ij} \leq 1, \qquad \text{for all } i \in M, \\ & \mu_{ij} \in \{0,1\}, \qquad \text{for all } (i,j) \in M \times M'. \end{aligned} \tag{2.2}$$

Notice that, because of its definition, this linear program is an integer linear program, and if we think in terms of the matrix $(\mu_{ij})_{i\in M, j\in M'}$, it has at most one non-zero entry for each row and column.

Thus, when $\mu_{ij} = 1$, it means that buyer i buys seller j's house, and we shall write $(i,j) \in \mu$, or equivalently $\mu(i) = j$ or $\mu^{-1}(j) = i$.

So, if $\mu = (\mu_{ij})_{(i,j)\in M\times M'}$ is a workable allocation of buyers to sellers such that

$$\mu_{ij} = \begin{cases} 1, & \text{means that buyer } i \text{ is assigned to seller } j, \\ \\ 0, & \text{means that buyer } i \text{ and seller } j \text{ are not assigned.} \end{cases}$$

The optimal value of program (2.2) is the value of the total coalition, $w_A(M \cup M')$.

We now consider the continuous case of this integer linear program. We state as our next linear program (2.3). It is worth mentioning that the matrices $(\mu_{ij})_{(i,j)\in M\times M'}$, which are solutions of program (2.2), are also solutions of the one described below.

$$\begin{aligned} \max \qquad & z = \sum_{i\in M}\sum_{j\in M'} a_{ij}\mu_{ij} \\ \text{where} \qquad & \sum_{i\in M} \mu_{ij} \leq 1, \qquad \text{for all } j \in M', \\ & \sum_{j\in M'} \mu_{ij} \leq 1, \qquad \text{for all } i \in M, \\ & \mu_{ij} \geq 0, \qquad \text{for all } (i,j) \in M \times M'. \end{aligned} \tag{2.3}$$

The solution for (2.3) is attained when $\mu_{ij} \in \{0,1\}$, for all $(i,j) \in M \times M'$. Hence this implies a solution to the assignment problem (2.2).

As the solution of the assignment problem deals with a linear program, one can consider the following dual program, that is dual to the program (2.3):

$$\begin{aligned} \min \quad & z = \sum_{i \in M} u_i + \sum_{j \in M'} v_j \\ \text{where} \quad & u_i + v_j \geq a_{ij} \qquad \text{for all } (i,j) \in M \times M', \\ & u_i \geq 0, \qquad \text{for all } i \in M, \\ & u_j \geq 0, \qquad \text{for all } j \in M'. \end{aligned} \tag{2.4}$$

Correspondingly, because of the fundamental duality theorem for linear programming, we know that the solution of the dual program (2.4) coincides with the solution of the linear program (2.2) if they exist.

The characteristic function w_A has already been mentioned, but let's define it in more detail. It associates to each coalition of agents the maximum profit they can achieve by allocating buyers and sellers, mentioned above. An analogous linear program is used to evaluate the worth of any coalition. Therefore, let S be a coalition of $N = M \cup M'$,

$$w_A(S) = \max_{\mu \in \Gamma(S \cap M, S \cap M')} \sum_{(i,j) \in \mu} a_{ij} \qquad \text{for all } S \subseteq N.$$

If $n = m + m'$ is the number of players, buyers and sellers, one linear program must be solved for each coalition. This means solving $2^n - 1$ linear programs. This type of game is called a *combinatorial optimization game*.

2.2 The core

The previous section defines the assignment game $(M \cup M', w_A)$ where, as we have seen above, M is the set of buyers, M' the set of sellers, $M \cup M'$ the total of them and w_A is the characteristic function of the matrix A. Now we look at the main solution of cooperative games, the core.

An *imputation* is a vector $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ of payments such that all coordinates are more that the individual coalitions worth, in this case 0. Hence, $u \in \mathbb{R}^M_+$ and $v \in \mathbb{R}^{M'}_+$. Moreover it has to be efficient, that is the total worth allocated to the players must be equal to the grand coalition's worth.

The first component of the vector is the profit that buyer i makes on acquiring the object, compared to the value he places on it, and the second component of the vector is the profit that seller j makes on selling the object.

As a result,

$$\sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_A(M \cup M'). \tag{2.5}$$

The set of all imputations is denoted by $I(w_A)$.

Notice that an imputation is an allocation of the total worth such that all the individual coalitions get at least its worth. When we ask that an efficient allocation will give to any coalition at least its worth, we obtain the concept of core. We can now define the core.

Definition 2.3. The *core of an assignment game* $(M \cup M', w_A)$ is the set of those imputations such that every coalition receives, at least, its worth according to the characteristic function:

$$C(w_A) = \left\{ (u, v) \in I(w_A) \mid \sum_{i \in S \cap M} u_i + \sum_{j \in S \cap M'} v_j \geq w_A(S) \text{ for all } S \subseteq M \cup M' \right\}.$$

Shapley and Shubik (1971) [16] shows that any assignment game has a non-empty core. Moreover, the core coincides with the set of dual solutions to the linear assignment problem.

Clearly, if $\mu \in \Gamma_A(M, M')$ is an optimal matching, the optimally assigned pairs get exactly their matrix entry. For any other pair, the sum of their allocations must be at least the matrix entry. Notice also that only mixed-pair coalitions matter to determine the core of the game. Then $(u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ belongs to $C(w_A)$ whenever

$$\begin{cases} u_i + v_j \geq a_{ij}, & \text{if } \ i \in M \ \text{ and } \ j \in M', \\ u_i + v_j = a_{ij}, & \text{if } \ (i, j) \in \mu, \\ u_i = 0, & \text{if } \ i \in M \ \text{ is unmatched by } \ \mu, \\ v_j = 0, & \text{if } \ j \in M' \ \text{ is unmatched by } \mu. \end{cases}$$

Theorem 2.1. *Let $\gamma = (M, M'; A)$ be an assignment market. Then, its corresponding assignment game (N, w_A), where $N = M \cup M'$, has a non-empty core. Additionally, the core coincides with the set of dual solutions to the linear assignment problem.*

Proof. An optimal matching μ can be found by solving this linear assignment problem:

$$\begin{aligned} \max \quad & z = \sum_{i \in M} \sum_{j \in M'} a_{ij} \mu_{ij} && (2.6) \\ \text{where} \quad & \sum_{i \in M} \mu_{ij} \leq 1, \quad && \text{for all } j \in M', \\ & \sum_{j \in M'} \mu_{ij} \leq 1, && \text{for all } i \in M, \\ & \mu_{ij} \in \{0, 1\}, && \text{for all } (i, j) \in M \times M'. \end{aligned}$$

As mentioned earlier, the solution of the above integer linear program coincides with its linear program relaxation, which is the related continuous linear program with $\mu_{ij} \geq 0$ for all $(i, j) \in M \times M'$. The fundamental duality theorem states that every linear program can be transposed into a dual form. Moreover, if the original program has a solution, then the optimal values of both programs coincide. Subsequently, the dual of the linear program relaxation of the primal program is:

$$\begin{aligned} \min \quad & z = \sum_{i \in M} u_i + \sum_{j \in M'} v_j && (2.7) \\ \text{where} \quad & u_i + v_j \geq a_{ij} \quad && \text{for all } (i, j) \in M \times M', \\ & u_i \geq 0, && \text{for all } i \in M, \\ & u_j \geq 0, && \text{for all } j \in M'. \end{aligned}$$

Consequently, the fundamental duality theorem tells that this linear program has a solution and taking into account the restrictions of the set results in

$$\min \sum_{i\in M} u_i + \sum_{j\in M'} v_j = \max \sum_{i\in M}\sum_{j\in M'} a_{ij}x_{ij} = w_A(M\cup M').$$

In short, a payoff vector (u, v) is an element of the core of (N, w_A) if and only if it is a solution of the dual program (2.7).

□

Example 2.1. Consider an assignment game with the set of buyers $M = \{1, 2\}$ and the set of sellers $M' = \{1', 2'\}$, where the matrix A of mixed-pair coalitions is given by Table 2.1. The optimal matching is shown in boldface.

Buyers/Sellers	1’	2’
1	**5**	1
2	2	**3**

Table 2.1: Matrix A of Example 2.1.

Now we describe the core and depict it.

$$C(w_A) = \left\{(u_1, u_2; v_1, v_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \mid u_1 + v_1 = 5;\ u_2 + v_2 = 3;\ u_1 + v_2 \geq 1;\ u_2 + v_1 \geq 2\right\}.$$

The equalities and inequalities that define the core non-negative coordinates are:

$$u_1 + v_1 = 5 \tag{2.8}$$
$$u_2 + v_2 = 3 \tag{2.9}$$
$$u_1 + v_2 \geq 1 \tag{2.10}$$
$$u_2 + v_1 \geq 2. \tag{2.11}$$

Let’s start with the equation (2.8). Knowing that the unknowns are not negative, and that the sum of u_1 and v_1 has to be 5, u_1 can go from 0 to 5. The same is applicable to v_1.

For now we have that $0 \leq u_1 \leq 5$ and $0 \leq v_1 \leq 5$.

The same for equation (2.9), so $0 \leq u_2 \leq 3$ and $0 \leq v_2 \leq 3$.

In equation (2.10) we use equation (2.9) to obtain:

$$\begin{cases} v_2 = 3 - u_2, \\ u_1 + v_2 \geq 1. \end{cases}$$

So, $u_1 + 3 - u_2 \geq 1$. Hence we have $u_1 - u_2 \geq -2$.

In equation (2.11), let’s substitute the first clear equation:

$$\begin{cases} v_1 = 5 - u_1, \\ u_2 + v_1 \geq 2. \end{cases}$$

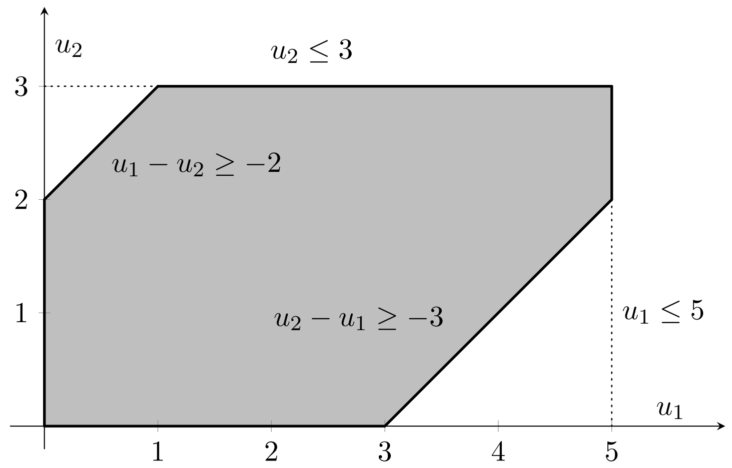

Figure 2.1: The core of Example 2.1.

So, $u_2 + 5 - u_1 \geq 2$. Hence we have $u_2 - u_1 \geq -3$.

In short, having in mind the equalities (2.8) and (2.9),

$$\begin{aligned}
&u_1 \in [0,5],\\
&u_2 \in [0,3],\\
&u_1 - u_2 \geq -2,\\
&u_2 - u_1 \geq -3.
\end{aligned}$$

All points that meet these conditions are points of the core. Graphically, in the projection over the first two coordinates the core is drawn in Figure 2.1.

2.3 Lattice structure of the core

As can be inferred from the example, the core of the assignment game has a very special shape. Let's see now that the core has the structure of a complete lattice, with a partial order properly defined. Then the join and meet are also appropriately defined.

The partial order is defined taking into account only the payoffs to one side of the market. That is, for $(u,v),(u',v') \in C(w_A)$ we define

$$(u,v) \leq_M (u',v') \quad \text{if and only if} \quad u_i \leq u'_i \ \text{ for all } \ i \in M.$$

Then we can state the following result. Its proof is straightforward.

Theorem 2.2. *Let $\gamma = (M, M'; A)$ be an assignment market and $(M \cup M', w_A)$ its associated assignment game. If (u,v) and (u',v') are two elements of $C(w_A)$, the core, then:*

- *The join belongs to the core.*

$$(u,v) \vee (u',v') = ((\max\{u_i, u'_i\})_{i\in M}, (\min\{v_j, v'_j\})_{j\in M'})$$

- *The meet belongs to the core.*

$$(u,v) \wedge (u',v') = ((\min\{u_i, u'_i\})_{i\in M}), (\max\{v_j, v'_j\})_{j\in M'})$$

The core is a complete lattice and a compact set. This implies that there are extreme core points (related to the order), and they will also be in the core. These extreme points are those all agents from the same side, buyers or sellers, reach their maximum payoff within the core. To see it just consider $\underline{u}_i = \min_{(u,v)\in C(w_A)}\{u_i\}$ for all $i \in M$ and $\overline{u}_i = \max_{(u,v)\in C(w_A)}\{u_i\}$ for all $i \in M$. In the same way we define $\underline{v}_j$ and $\overline{v}_j$. Since we are in a compact set, this is attained at some point in the core. Use now the join and meet to obtain the desired extreme core points. Let us denote these extreme core points. In the core we have:

· The buyers-optimal core element is denoted by $(\overline{u}^A, \underline{v}^A)$. In it all buyers get the maximum possible payoff in the core and all sellers get the minimum payoff, and at the same time.

· The sellers-optimal core element is denoted by $(\underline{u}^A, \overline{v}^A)$. In it all sellers get the maximum possible payoff in the core and all buyers get the minimum payoff, and at the same time.

What is somehow surprising is that, even the buyers look for the maximum payoff, they compete in fact not with agents of the same side, but on the other side.

Theorem 2.3. *Let $\gamma = (M, M'; A)$ be an assignment market and $(M \cup M', w_A)$ its associated assignment game. The core of the assignment game, $C(w_A)$, always contains a buyers optimum $(\overline{u}^A, \underline{v}^A)$ and a sellers optimum $(\underline{u}^A, \overline{v}^A)$.*

The proof of this theorem has been explained above.

A formula to find the maximum payoff of any agent was studied, independently, by Demange (1982) [2] and Leonard (1983) [10]. The following proposition gives the precise formula.

Proposition 2.1. *Let $\gamma = (M, M'; A)$ be an assignment market and $(M \cup M', w_A)$ its associated assignment game. The maximum core payoff of an agent is his/her marginal contribution to the grand coalition, that is,*

$$\begin{aligned}
\overline{u}_i^A &= w_A(M \cup M') - w_A((M\backslash\{i\}) \cup M'), && \text{for all } i \in M, && (2.12)\\
\overline{v}_j^A &= w_A(M \cup M') - w_A((M \cup (M'\backslash\{j\})), && \text{for all } j \in M', && (2.13)\\
\underline{u}_i^A &= a_{i\mu(i)} - \overline{v}_{\mu(i)} && \text{if } i \in M \text{ is assigned by } \mu, && (2.14)\\
\underline{v}_j^A &= a_{\mu^{-1}(j)j} - \overline{u}_{\mu^{-1}(j)} && \text{if } j \in M' \text{ is assigned by } \mu. && (2.15)
\end{aligned}$$

These formulas are due to Demange (1982) [2] and Leonard (1983) [10] who show us that the maximum payment of an agent in the core corresponds to its marginal contribution.

Example 2.2. Consider an assignment game with the set of buyers $M = \{1, 2, 3\}$ and the set of sellers $M' = \{1', 2', 3', 4', 5'\}$, where the values of the objects for the buyers are shown in Table 2.2. In this case, Table 2.2 corresponds to the $h_i^j, i \in M$ and $j \in M'$ of the formula (2.1). Each buyer has to value each object, where we assume that a seller only sells one item. Hence, there is one object per seller.

In addition, there is a price for which the seller is not willing to give up the object. This minimum price is called *the reserve price* and if the offer does not exceed or equal this price, the seller is not willing to part with the item. This vector, denoted c_j in the formula (2.1), is given in Table 2.3.

	Sellers (objects)				
Buyers	1’	2’	3’	4’	5’
1	8	30	42	14	27
2	50	17	9	41	23
3	12	38	20	4	18

Table 2.2: Valuations of items of Example 2.2.

Reserve price of Sellers	1’	2’	3’	4’	5’
Reserve price	10	20	45	15	3

Table 2.3: Reservation prices of Example 2.2.

Therefore, using (2.1) and the information in Tables 2.2 and 2.3 we can obtain the matrix A of mixed-pair coalitions. In short, we will deduct the reserve price from the valuations and if it is negative, we will assign a 0 to that cell of the matrix.

This means that the matrix A is

$$A = \begin{pmatrix} 0 & 10 & 0 & 0 & 24 \\ 40 & 0 & 0 & 26 & 20 \\ 2 & 18 & 0 & 0 & 15 \end{pmatrix}$$

where the rows represent the 3 buyers and the columns represent the 5 sellers or objects for sale.

Buyer 3, third row of A, earns either 2 currency units by buying item 1 or 18 currency units by buying item 2 or 15 currency units by buying item 5. Therefore, he will always choose item 2.

The optimal matching is given in boldface. So,

$$A = \begin{pmatrix} 0 & 10 & 0 & 0 & \mathbf{24} \\ \mathbf{40} & 0 & 0 & 26 & 20 \\ 2 & \mathbf{18} & 0 & 0 & 15 \end{pmatrix}.$$

Notice that sellers 3 and 4 are not assigned to any buyer.

So the restrictions of the core are

$$\begin{aligned} &u_2 + v_1 = 40, && u_3 + v_2 = 18, && u_1 + v_5 = 24, && u_3 + v_1 \geq 2, \\ &u_1 + v_2 \geq 10, && u_2 + v_4 \geq 26, && u_2 + v_5 \geq 20, && u_3 + v_5 \geq 15. \end{aligned}$$

Now we are going to find out what is best for buyers, using (2.12) and (2.5),

$$\overline{u}_1^A = w_A(M \cup M') - w_A((M \backslash \{1\}) \cup M') = (40 + 18 + 24) - (40 + 18) = 82 - 58 = 24.$$

Accordingly, we also obtain that $\overline{u}_2^A = 40$ and that $\overline{u}_3^A = 18$.

So, the best for the buyers is

$$(\overline{u}^A, \underline{v}^A) = (24, 40, 18; 0, 0, 0, 0, 0).$$

Now we are going to find out what is best for sellers, using (2.13) and (2.5).

The best for the sellers is

$$(\underline{u}^A, \overline{v}^A) = (0, 26, 0; 14, 18, 0, 0, 24).$$

2.4 The core and competitive equilibrium

The assignment market and also other markets have been analyzed from the point of view of microeconomics by Gale (1960) [5]. Gale considers the market $\gamma = (M, M'; A)$ and think of buyers and sellers of a house. A competitive equilibrium is a matching of buyers and sellers, that is those willing to buy and those willing to sell, and prices for the houses such that support the matching, i.e. no pair of agents is willing to change their partners at the present prices.

As in Roth and Sotomayor (1990) [14], we assume that M' contains as many copies as necessary of a null object $O \in \mathcal{O}$ such that $a_{io} = 0$ for all $i \in M$. That is, no benefit is derived from it and therefore as many 0's as required can be added to the matrix since it makes no change. Then, for any matching μ, all buyers are assigned an object and this object can be real or an imaginary object O that is attributed a 0 to the matrix A. Now we define the demand set of a buyer.

Definition 2.4. Let $\gamma = (M, M'; A)$ be an assignment market. Given a vector of non-negative prices $p \in \mathbb{R}^{M'}$, with $p_o = 0$, the *demand set* of buyer $i \in M$ at prices p is

$$D_i(p) = \left\{ j \in M' \quad | \quad a_{ij} - p_j = \max_{k \in M'} \{a_{ik} - p_k\} \right\}.$$

The buyer i requests those objects that give him the maximum profit, given by the difference of valuation and price. We can also write

$$D_i(p) = \arg \max_{k \in M'} \{a_{ik} - p_k\}.$$

A pair (p, μ) is a *competitive equilibrium* if $\mu(i) \in D_i(p)$ for all $i \in M$ and $p_j = 0$ whenever $j \in M'$ is unassigned by μ. Under these circumstances, p is said to be a *competitive equilibrium price vector*. In addition, the payoff vector (u, v) is a *competitive equilibrium payoff vector* if (p, μ) is a competitive equilibrium, $u_i = a_{i\mu(i)} - p_{\mu(i)}$ for all $i \in M$ and $v_j = p_j$ for all $j \in M'$.

The following theorem relates the competitive equilibrium with the payoffs in the core.

Theorem 2.4. *For any assignment game, the set of solutions of the dual program of (2.7) coincides with the set of competitive equilibrium payoff vectors.*

Proof. Let (u, v) be a solution of the dual program, and define $p = v \in \mathbb{R}_+^{M'}$. If μ^* is an optimal matching, then

$$\sum_{(i,j) \in \mu} a_{ij} = \sum_{i \in M} u_i + \sum_{j \in M'} v_j \quad \text{and} \quad u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in \mu^*,$$

shows that $p_j = v_j = 0$ for all unassigned object $j \in M'$ and $u_i + v_j = a_{ij}$ if $(i, j) \in \mu^*$. Furthermore, for all $i \in M$,

$$a_{i\mu(i)} - p_{\mu(i)} = u_i \geq a_{ij} - p_j \quad \text{for all } j \in M',$$

This is why p is a competitive price vector. Then there exists $\mu \in \Gamma(M, M')$ such that $p_j = 0$ if j is unassigned by μ and

$$\mu(i) \in D_i(p) \quad \text{for all} \quad i \in M.$$

Defining $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ by

$$v_j = p_j \text{ for all } j \in M' \quad \text{ and } \quad u_i = a_{i\mu(i)} - p_{\mu(i)} \text{ for all } i \in M.$$

If $i \in M$ is assigned to a null object, then $u_i = 0$. Also, $v_j = 0$ if $j \notin \mu(M)$. Now let's verify that (u, v) is a solution of the dual problem.

If (p, μ) is a competitive equilibrium, then μ is an optimal matching. Take another matching $\mu' \in \Gamma(M, M')$ and since $a_{i\mu(i)} - p_{\mu(i)} \geq a_{i\mu'(i)} - p_{\mu'(i)}$ for all $i \in M$, we have

$$\begin{aligned}
\sum_{(i,j)\in\mu} a_{ij} = \sum_{i\in M} a_{i\mu(i)} &\geq \sum_{i\in M} (a_{i\mu'(i)} - p_{\mu'(i)}) + \sum_{i\in M} p_{\mu(i)} \\
&= \sum_{i\in M} a_{i\mu'(i)} - \sum_{j\in\mu'(M)} p_j + \sum_{j\in\mu(M)} p_j \\
&= \sum_{i\in M} a_{i\mu'(i)} - \sum_{j\in\mu'(M)\setminus\mu(M)} p_j + \sum_{j\in\mu(M)\setminus\mu'(M)} p_j \\
&\geq \sum_{i\in M} a_{i\mu'(i)}
\end{aligned}$$

where the last inequality follows from the fact that (p, μ) is a competitive equilibrium and hence $p_j = 0$ for all $j \notin \mu(M)$. Since μ^* is an optimal matching, then

$$w_A(M \cup M') = \sum_{i\in M} a_{i\mu(i)} = \sum_{i\in M} u_i + v_{\mu(i)} = \sum_{i\in M} u_i + \sum_{j\in M'} v_j,$$

which means (u, v) is efficient. Finally, for all $i \in M$ and for all $j \in M'$,

$$\begin{aligned}
u_i + v_j = u_i + p_j &= a_{i\mu(i)} - p_{\mu(i)} + p_j \\
&\geq a_{ij} - p_j + p_j = a_{ij},
\end{aligned}$$

which concludes the proof that (u, v) is a solution of the dual program. □

From the previous result we have the following equivalence:

Proposition 2.2. *For any assignment market $\gamma = (M, M'; A)$ with the associated assignment game $(M \cup M', w_A)$, the following four sets coincide:*

- *The core, $C(w_A)$.*
- *The set of dual solutions to the assignment problem, see (2.7).*
- *The set of competitive equilibrium payoff vectors of the market.*
- *The set of pairwise-stable payoff vectors.*

In the next chapter we analyze a mechanism based on auction theory and prices that gives the buyers-optimal core element.

Chapter 3

Auctions in the allocation market to find the buyers-optimal core allocation

In this chapter we study the use of auctions to find competitive prices in the assignment market. We discusses the difficulties that information asymmetries between economic agents imply for the relationships between these agents and presents the mechanisms developed to solve those obstacles. Demange et al. (1986) [4] use what they call Multi-Item Auctions to find the seller-optimal core allocation, that is the best prices for the sellers.

This article investigates the assignment market by means of multi-item auctions. We first briefly introduce what a mechanism is. Then, it describes two dynamic auction mechanisms: one achieves this equilibrium and the other approximates it to any desired degree of accuracy.

3.1 Mechanisms

Mechanisms are an important tool in microeconomics. They have been shown to have many applications in modeling and designing problem solutions with asymmetric information. These applications can be found in the fields of engineering, computer science, e-commerce and economics. In Algorithmic Mechanism Design the goal is to construct efficient mechanisms that will handle the selfish behavior of the players. In particular, we are interested in designed truthful mechanisms, that is, mechanisms in which the dominant strategy of each player is to simply reveal his or her true valuation. Notice that this is a way of generalizing the auctions we know.

Let $\mathcal{X}$ be the set of valuations of the agents, that is $\mathcal{X} = \Pi_{i \in N} \mathcal{X}_i$, and these are our primitives. Next, we define in general terms what it is a selling mechanism, using Krishna (2002)[9].

Definition 3.1. A *selling mechanism* is a triplet $(\mathcal{B}, \pi, \mu)$, where

- $\mathcal{B} = \Pi_{i \in N} \mathcal{B}_i$, where $\mathcal{B}_i$ is the set of possible "messages" or offers for each player $i \in N$. An element of $\mathcal{B}$ will be like $b = (b_1, b_2, \ldots, b_n)$.

- $\pi : \mathcal{B} \longrightarrow \Delta$ is an assignment rule where the simplex $\Delta = \{(p_1, p_2, \ldots, p_n) \in [0,1]^n \mid \sum_{i \in N} p_i = 1\}$ is the set of probability distributions over the set of buyers N. An allocation rule determines, depending on the bids, the probability for each bidder $i \in N$, of getting the object, $\pi_i(b)$.
- $\mu : \mathcal{B} \longrightarrow \mathbb{R}^n$ is a payment rule. A payment rule determines, depending on the bids, the expected payment for each bidder $i \in N$. We denote it as $\mu_i(b)$.

This definition shows that both first and second price auctions are mechanisms. The message space can be regarded as the bids in an auction, and the assignment rule allows for results giving some ties. The methods can be quite complicated, but in this case, we are going to use what are called *direct mechanisms*, which are a simpler class of methods as each bidder is asked to report the value of the item.

Formally, a *direct mechanism* $(\mathcal{Q}, \mathcal{M})$ consists of a pair of functions:

$$\begin{aligned} \mathcal{Q} : &\quad \mathcal{X} \longrightarrow \Delta \\ \mathcal{M} : &\quad \mathcal{X} \longrightarrow \mathbb{R}^N \end{aligned}$$

where $\mathcal{Q}_i(x)$ is the probability that i will get the object and $\mathcal{M}_i(x)$ is the expected payment by i.

Since a mechanism is a triplet $(\mathcal{B}, \pi, \mu)$, a direct mechanism corresponds to the fact that $\mathcal{B} = \mathcal{X}, \pi = \mathcal{Q}$ and $\mu = \mathcal{M}$. Thus we denote direct mechanisms as $(\mathcal{Q}, \mathcal{M})$.

This mechanism has a truthful equilibrium if it is an equilibrium for each buyer to reveal his true value, i.e, $\beta(x) = x$. That is, a equilibrium of a direct mechanism is an assignment of the bidders to win the object and a payment expected by all of them, which means that there is no option to improve the situation for any bidder. The mechanism will be supported by sincerity.

Now we show that the outcomes resulting from any equilibrium is a truthful equilibrium. The principle of revelation is fundamental in the theory of design mechanisms. It demostrates that results obtained as a consequence of using a mechanism in an equilibrium can be reproduced by a sincere equilibrium using direct mechanisms. This means that there is no loss of generality when we focus on direct mechanisms. We state the result without proof (see Krishna (2002) [9]).

Proposition 3.1. ***(Revelation Principle)*** *Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which*

- *It is an equilibrium for each buyer to report his value truthfully,*
- *The outcomes are the same as in the given equilibrium of the original mechanism.*

The idea behind this proposition is simple. Assume we have a mechanism and an equilibrium of this mechanism. Now we can avoid knowing bids and apply rules to know who gets the object and who pays for it. One can simply ask bidders to state their valuations x_i.

A direct mechanism is said to be *incentive compatible* if it maximizes the utility for the agent whenever he/she reports his/her true valuation. All incentive compatible mechanisms are alike, and the payoff to the agents is the same up to a constant. It depends only on $\mathcal{Q}$, and not on the payment rule. It can be stated as the revenue equivalence principle in this context.

3.2 The Progressive Auction Mechanisms

In this chapter we see that the allocation of the minimum equilibrium price can be achieved by means of dynamic auctions, also known as progressive auctions. Two different types of dynamic auctions will be presented.

In contrast with the case of multi-object auction of identical objects, in our case the auction is also multi-object but with heterogeneous items. Think with different houses, that may have nothing to do with each other. Moreover, the buyer only wants to acquire one house, a fact that also contrasts with the auctions of homogeneous items. Just as the electric public auctions would be an example of an auctioned object in the first chapter, in our case, an example would be the auction of houses. Each house is different, either in terms of location or interior design. Due to interests, people value each house differently. In addition, the interested parties only want to buy one of them. Notice also that we want to apply auctions to the assignment market.

Taking as a starting point that there are different bidders, in terms of the assignment market, these would be the set of m buyers, M. From the basis of an assignment market, valuations of the buyers and reservation prices of the sellers, we can generate the matrix A. Notice that we will generate an auction where each buyer is a bidder. In addition, there are different objects to be distributed among the bidders, but each bidder will acquire at most one good, being able to bid for more than one, but when he obtains one, he will withdraw.

Therefore, as we have seen above, each bidder have to submit a sealed bid listing his valuations of all the items. We call $v^i = (v_1^i, v_2^i, \dots, v_k^i)$ where there are k units for sale. We use k to denote the number of sellers, that is of different objects. Therefore we know $|M'| = k$. Although each bidder bids a monetary amount for each object that is auctioned, there is only one single vector with equilibrium prices that is optimal. These prices are those prices at which each unit is attributed, i.e. the highest prices, winners of the auction. An *equilibrium price vector* is one in which each bidder is assigned an object from its demand vector v^i but in which no two bidders are both assigned the same object.

For each product α, the minimum price the seller will accept is $s(\alpha)$. But using the minimum equilibrium price, there will be products that are not sold because, if $p(\alpha)$ is the minimum equilibrium price, $s(\alpha) > p(\alpha)$. In other words, the seller is not willing to concede it for the price $p(\alpha)$. In this case, if the item α is not sold, the price of this object in equilibrium will correspond to the reserve price, and nobody asks for it. In short, if the item is not sold, $p(\alpha) = s(\alpha)$.

The multi-item model treated here is unsymmetrical in that each seller specifies only one number, his reservation price, while buyers specify $|I|$ numbers, their valuations for each of the items. The two mechanisms start out the same. A vector p_0 of prices is announced. This indicates the minimum price that the seller is willing to accept for each α, which we have called $s(\alpha)$ before. Thus, if there are five items to be auctioned, the vector will have five positions. The way to express this initial price vector is

$$p_0 = (p_0^{\alpha_1}, p_0^{\alpha_2}, \dots, p_0^{\alpha_k}),$$

where there are k units for sale.

A main point is that "incentive compatibility" of the single item auction carries over to the multi-item case, meaning that submitting true valuation is a weak dominant strategy for the bidders. Therefore, by jointly falsifying valuations, no subset of bidders can improve the outcome for all its members.

As mentioned above, we will set out two progressive auction mechanisms.

3.2.1 The exact auction mechanism

Let's assume that the prices and valuations displayed from now on are integers. We will also assume that the currency units used to value the quantities shown are dollars or hundreds of dollars, in order to reduce the number if the product being auctioned is highly valuable. Now it is time for the buyers. Each of them has to choose which one of the k items they want to purchase at the starting price. We want to reach equilibrium, i.e. each object is assigned to a different bidder.

Let us propose a mechanism to calculate it. It is necessary to assume that at all times buyers demand all those products that maximise their profits at the specific price of that particular moment. We must assume that all bidders behave honestly in valuing each item at all stages.

- If, at the time of the buyers' choice, each item is allocated to a bidder who is interested in acquiring it, then equilibrium has been reached and the allocation mechanism ends here.

- If the previous case does not occur and equilibrium is not reached in the first allocation, different actions are carried out as explained subsequently.

  We say that a subset of items is $over-demanded$, if more buyers that the cardinality of the subset want this subset. It is a concept used in microeconomics and economic theory to define the situation in which the quantity demanded is higher than the quantity offered at the current price. The subset that a buyer asks is just its demand set at those prices. Therefore, it requires the absence of over-demand for a possible equilibrium in the allocation between buyers and items. Furthermore, this condition is sufficient. However, if we find over-demand, we can locate a set that is over-demanded but none of its proper subsets is over-demanded. Hence, we can locate a minimum over-demanded set. The auctioneer can now locate this set and increase the price of each item by one unit.

  After raising the price, the bidders' demands are asked again. We now return to the same point as before. It may be that at these prices equilibrium has been reached, or it may be that the minimum over-demanded set has to be located again. We enter into a loop where if we find an over-demanded set, we have to raise the price and ask for the price vector again. But this loop is finite because if the price of a good is increased too much, it will not be demanded. For example, if the price of the product is higher than all the bidders' valuations, this item will not belong to any over-demanded set.

So by this method, either with one set of prices or several sets of prices in the case of a high demand, we will arrive at an equilibrium assignment. There exists, consequently, an equilibrium. It is now necessary to prove that the prices obtained correspond to the minimum equilibrium prices.

Convergence of the Exact Auction Mechanism

We are naming the sets that we have been referring to in the previous sections, the sets of buyers with the set that must be associated, the set of items. So we call B, of bidders, the set of buyers and I, of items, the set of goods for sale. Recall that for each item, of the set we have just designated as I, has a specific selling price. Let α be an item, then the minimum selling price is denoted by $s(\alpha)$. The value of the item α for bidder b will be denoted as v_α^b and this value will never be negative. The price vector p is a function defined as:

$$\begin{aligned} p: \quad & I \longrightarrow \mathbb{R}^+ \\ & \alpha \longmapsto p(\alpha). \end{aligned}$$

The positions of this vector p will increase until each object is assigned, obtaining the selling price $s(\alpha)$ for each item α. Recall (see Definition 2.4) that for bidder b, the demand set at price p is defined by

$$D_b(p) = \left\{ \alpha \in I \mid v_\alpha^b - p(\alpha) = \max_{\beta \in I} \left[v_\beta^b - p(\beta) \right] \right\}. \tag{3.1}$$

The price p is called *competitive price* if there is an assignment

$$\begin{aligned} \mu: \quad & B \longrightarrow I \\ & b \longmapsto \mu(b) \in D_b(p), \end{aligned}$$

and if $b' \neq b$ and $\mu(b) = \mu(b')$ then, $\mu(b) = \alpha_0$. In other words, competitive pricing consists of setting the price at the same level as one's competitors. Therefore, the pair (p, μ) is an equilibrium if

$$\begin{cases} p \text{ is competitive}, \\ \text{if } \alpha \notin \mu(B), \text{ then } p(\alpha) = s(\alpha). \end{cases}$$

Let's see that the Exact Auction Mechanism converges to the competitive minimum price p. This is the next theorem.

Theorem 3.1. *Let p be the price vector obtained from the exact auction mechanism and let q be any other competitive price. Then $p \leq q$.*

Proof. This theorem consists of proving that the price at which is assigned is the lowest possible price, i.e. the best price for the bidders. This demonstration will be proved by contradiction. Therefore we assume that for some q we have $p \nleq q$. This is the same as saying that for some $\alpha \in I, p(\alpha) > q(\alpha)$. In summary, we assume that the price vector obtained is not the vector with the lowest possible components.

We can suppose, without loss of generality, that the reservation prices are set to zero. If we place us at the beginning of the auction, that is at time 0, $t = 0$, then we have $p_0 = 0$. So, $0 = p_0 \leq q$.

Now, if we place us at the last stage of the auction where $p_t \leq q$, so the next stage will already correspond to $p_{t+1} > q$. Then we denote the set of objects such that the price surpasses by $S_1 = \{\alpha \mid p_{t+1}(\alpha) > q(\alpha)\}$. If S is the over-demanded set whose prices are raised at stage $t+1$, then $S = \{\alpha \mid p_{t+1}(\alpha) > p_t(\alpha)\}$. So $S_1 \subset S$.

We are going to show that $S \setminus S_1$ is nonempty and over-demanded; hence S is not a minimal over-demanded set, contrary to the rules of the auction.

Define now the following sets of bidders:

$$\begin{cases} T = \{b \mid \quad D_b(p_t) \subset S\}\,, \\ T_1 = \{b \mid b \in T \text{ and } D_b(p_t) \cap S_1 \neq \emptyset\}\,. \end{cases}$$

As S is over-demanded, which means that there are more interested bidders than objects, it thus results in the following

$$|T| > |S|. \tag{3.2}$$

We claim that $D_b(q) \subset S_1$ for all b in T_1. Indeed, choose α in $S_1 \cap D_b(p_t)$. If $\beta \notin S$, then b prefers α to β at price p_t because $b \in T$, but $p_t(\beta) \leq q(\beta)$ and $p_t(\alpha) = q(\beta)$, so b prefers α to β at price q.
If $\beta \in S \setminus S_1$, then b likes α at least as well as β at price p_t. But $p_t(\beta) < p_{t+1}(\beta) \leq q(\beta)$. So $p_t(\alpha) = q(\alpha)$. So b prefers α to β at price q.

Now there are no over-demanded sets at price q, since q is competitive. So,

$$|T_1| \leq |S_1|. \tag{3.3}$$

Notice that $T \setminus T_1 = \{b \mid b \in T \text{ and } D_b(p_t) \in S \setminus S_1\}$.

From expressions (3.2) and (3.3), we obtain

$$|T \setminus T_1| > |S \setminus S_1|.$$

This means that $S \setminus S_1$ is over-demanded, so we have reached a contradiction. □

We see now that we can choose a specific assignment μ to ensure that the pair (p, μ) is an equilibrium.

Theorem 3.2. *There is an assignment μ^* such that (p, μ^*) is a competitive equilibrium if p is the minimum competitive price.*

Proof. Let μ be an assignment corresponding to p. An item α is overpriced if it is not assigned and $p(\alpha) > s(\alpha)$, which means that the price of this product is higher than the reserve price assigned by its seller.

If (p, μ) is not an equilibrium, this means that there is at least one overpriced item, otherwise the above-mentioned loop would have ended. To eliminate overpriced items, we will give a procedure for altering μ. For this purpose we construct a directed graph whose vertices are $B \cup I$. There are two types of arcs:

- If $\mu(b) = \alpha$, there is an arc from b to α. Means α is already assigned.
- If $\alpha \in D_b(p)$, there is an arc from α to b, where α is one of the objects that bidder b demands at price p.

Now let α_1 be an overpriced item. Then $\alpha_1 \in D_b(p)$ for some $b \in B$, for if not one could decrease $p(\alpha)$ and still have competitive prices. Let $\widehat{B} \cup \widehat{I}$ be all vertices that can be reached by a directed path starting from α_1.

- Case 1: Set $\widehat{B}$ contains an unassigned bidder b.

  Let $(\alpha_1, b_1, \alpha_2, b_2, \ldots, \alpha_k, b)$ be the path from α_1 to b. Then we may change μ by assigning b_1 to α_1, b_2 to $\alpha_2, \ldots, b$ to α_k.

  The assignment is still competitive and α_1 is no longer overpriced, so the number of overpriced items has been reduced.

- Case 2: All b in $\widehat{B}$ are assigned.

  Then we claim that there must be some α in I such that $p(\alpha) = s(\alpha)$. In other words, there is an object that is never over-demanded. It is therefore assigned at the minimum price at which the seller is willing to sell it. Suppose not. By definition of $\widehat{B} \cup \widehat{I}$ we know that if $b \notin \widehat{B}$ then b does not demand any item in $\widehat{I}$. Therefore, we can decrease the price of each item in I by some positive t and still have competitiveness, contradicting the minimality of p, since p is the minimum competitive price.

  So choose $\alpha \in \widehat{I}$ such that $p(\alpha) = s(\alpha)$ and let $(\alpha_1, b_1, \alpha_2, b_2, \ldots, b_k, \alpha)$ be the path from α_1 to α. Again change μ by assigning b_i to α_i for all i, leaving α unassigned. Again the number of overpriced items has been reduced.

We have reached a contradiction since in both cases the number of overvalued products has been reduced, which contradicts the definition of equilibrium. Therefore, an assignment at a certain price vector can only be a competitive equilibrium if the price vector is the minimum competitive price, otherwise the number of over-demanded products could be reduced. □

Example 3.1. We follow Example 2.2 of the previous chapter. With this example we see a case of what the Exact Auction Mechanism would look like.

First of all we could set all prices to zero. In this way all buyers compute which is the item(s) that gives them the maximum profit, and this is their demand sets. Even if they will be interested (meaning they have some profit) in acquiring all items, they show interest in their demand set. But sellers have a minimum price, or reserve price, which they will not sell for lower quantities than this. So the mechanism begins at this stage.

As we have seen in the example of the previous section, the reserve prices are the ones shown in the vector $(10, 20, 45, 15, 3)$.

Notice that the idea is to increase the prices by one unit, in case there are over-demanded sets, but we can omit this part at this stage, since until we reach the reserve prices, there will not be any transaction. So we take as the original price the vector of reserve prices. That is,

$$p_0 = (10, 20, 45, 15, 3).$$

Recall that matrix A gives the difference between the valuation and the reservation price. we have seen that:

Buyer 1 is interested in object 2 and 5. With product 2, as it is valued at 30 currency units and the reserve price is 20, he only earns 10 currency units, and with product 5 he earns 24 monetary units since he values it at 27 and the reserve price is 3. Therefore,

$$\begin{aligned} h_1^2 - c_2 =& 30 - 20 = 10, \\ h_1^5 - c_5 =& 27 - 3 = 24. \end{aligned}$$

Using the formula of the demand set (3.1),

$$D_1(p_0) = \arg\max_{\beta \in I}\{v^1_\beta - p(\beta)\} = \{5\}.$$

The demand sets of buyer 2 and buyer 3 are computed in the same way and

$$D_2(p_0) = \arg\max_{\beta \in I}\{v^2_\beta - p(\beta)\} = \{1\},$$
$$D_3(p_0) = \arg\max_{\beta \in I}\{v^3_\beta - p(\beta)\} = \{2\}.$$

Since all demand sets are not over-demanded, we have the assignment ready. For the current time, the price vector looks like this

$$p_0 = (\mathbf{10}, \mathbf{20}, 45, 15, \mathbf{3}).$$

and prices in boldface mean that the corresponding sellers are already assigned.

In this example we see how each buyer is interested in acquiring a different object. Therefore, there is no over-demand. This causes the assignment to occur in the first iteration without the need to resolve any subset demanded by two or more bidders.

To reflect the case where more than one buyer is interested in a product and we have to break the tie by raising the price by one unit, we present a new example with two buyers and three objects to be assigned, where two buyers have the same preference on a particular product.

Example 3.2. Consider a set of 2 agents $\{1, 2\}$ and a set of 3 objects $\{1', 2', 3'\}$. We assume that buyers' valuations are reflected in Table 3.1.

	Sellers (objects)		
Buyers	1'	2'	3'
1	12	9	8
2	10	9	4

Table 3.1: Valuations of items of Example 3.2.

Sellers have reserve prices, for which they will not sell for less than that amount. This vectoris given in Table 3.2.

Reserve price of Sellers	1'	2'	3'
Reserve price	6	6	5

Table 3.2: Reservation prices of Example 3.2.

This means that the matrix A is

$$A = \begin{pmatrix} 6 & 3 & 3 \\ 4 & 3 & 0 \end{pmatrix}$$

where the rows represent the two buyers and the columns represent the three sellers or objects for sale.

The first price is $p_0 = (6, 6, 5)$.

Now at this price, the buyers compute their demand set:

$$D_1(p_0) = \arg\max_{\beta \in I}\{v^1_\beta - p_0(\beta)\} = \{1'\},$$
$$D_2(p_0) = \arg\max_{\beta \in I}\{v^2_\beta - p_0(\beta)\} = \{1'\}.$$

We see that both buyers are interested in the same product with these prices. We have just reached an over-demanded subset as more than one bidder wants product 1'. It is time to break the tie in the manner described above.

So we increase the price of this item by one unit and we get $p_1 = (7, 6, 5)$. So, with these prices, the demand set are:

$$D_1(p_1) = \arg\max_{\beta \in I}\{v^1_\beta - p_1(\beta)\} = \{1'\},$$
$$D_2(p_1) = \arg\max_{\beta \in I}\{v^2_\beta - p_1(\beta)\} = \{1', 2'\}.$$

We can see that by raising the price of the first product by one currency unit, bidder 1 is still interested in acquiring that product, as we can see from the demand set that it is the one that provides him with the highest margin.

On the other hand, when the price for the first product is 7 currency units, the demand set of the second bidder changes. At this price, both the first and the second product bring the same profit.

Therefore, by assigning the second bidder the second product we solve the over-demand that existed. In summary,

- Product 1' is assigned to buyer 1 at the price of 7, obtaining a margin of 5 currency units.
- Product 2' is assigned to buyer 2 at the price of 6, obtaining a margin of 3 currency units.
- The price vector is $p = (\mathbf{7}, \mathbf{6}, 5)$.

3.2.2 Approximate Auction Mechanism

As mentioned above, the two mechanisms start out the same. The auctioneer announces the initial sale price for each item. We refer to the so-called vector p_0.

These starting prices are so low that any buyer is willing to bid on any of the k items for sale. In addition, the person bidding is said to be committed to that item, as he or she agrees to pay the fixed price for that auctioned object.

The idea is to start at the point where all bidders are interested in all products and to gradually narrow these sets as the price goes up. Thus each decreasing subset will be interested in a smaller and smaller subset of items. Generally speaking, we could say that some subset of bidders will be committed to some subset of items at some set of prices.

In the scenario where we have bidders who are not engaged to an item, or committed to it, there are three options:

- Can bid on a product that is not yet assigned. In this case, he or she would be bound to that product at the initial price. That is, commitment to pay that amount if it is allocated to you.
- Can bid for a good that has already been allocated. In this case, he will be committed to that product at a higher price than the one assigned to it, let's call the increase in this price δ. As a result, the bidder who was previously engaged will no longer be committed to the item.
- Can drop out of the bidding. In this case, no item will be associated with him.

The auction ends when there are no more bidders without an assigned item, i.e. there are no uncommitted bidders. At the end of the transaction, the bidder with an assigned item must buy the product at the current price.

We must assume that bidders calculate the difference between the value of the product to them and the initial price of the unassigned product or, in the other case, that they choose the item for which they will bid their price plus δ, and only bid if the value to them is higher than the price to be paid. So, we suppose that the bidders behave in accordance with the linear surplus utility functions.

We may think that, as bids can be modified during the course of the auction, it is a very attractive rate for bidders. But we can see that this variation is limited.

Let's imagine that all bidders change product, i.e. they choose the second section of the options they have when they are uncommitted. Then the final prices would vary from the minimum equilibrium price as maximum $k \cdot \delta$ units, since k is the number of products and δ is the increased price when moving from one bidder to another.

$$\text{Variation of final price} = p \cdot k \cdot \delta - p \cdot k = k \cdot \delta$$

where p are the prices, k the number of items auctioned and δ the margin overpaid for taking the product from another bidder.

From now on, we suppose that $k = \min(|I|, |B|)$. So, if δ is sufficiently small, the deviation is reduced and we can conclude that the result is arbitrarily close to the minimum equilibrium price.

Convergence of the Approximate Auction Mechanism

In this section, we argue that, using this method, the final price differs from the price obtained by the exact method, at most in $k \cdot \delta$.

Theorem 3.3. *If the price of an item α at time t is*

$$p_t(\alpha) \geq p(\alpha) + k \cdot \delta,$$

no buyer is going to bid for it.

Proof. Basically we want to see that, using the approximate auction mechanism, the final price of an item tends to the minimum equilibrium price for a sufficiently small δ. To prove this theorem we need to cite and prove two lemmas. To do so, let us define the nomenclature. Let us call an item α expensive at time t if $p_t(\alpha) > p(\alpha)$ and let μ be any assignment corresponding to p.

Lemma 3.1. *If b bids for an expensive item α, then he is assigned by μ.*

Proof. If b were unassigned means that no one is demanding it and therefore, $v^b_\alpha - p(\alpha) \leq 0$. In the lemma we have assumed that α is expensive, and thereby, b would not bid for α at time t because $v^b_\alpha - p_t(\alpha) \leq 0$.

□

Lemma 3.2. *If $\mu(b) = \alpha$ and b bids for β at time t, then $p_t(\alpha) - p(\alpha) \geq p_t(\beta) - p(\beta)$.*

Proof. This indicates that the bidder's opinion has changed in terms of demand for the products. This leads us to assume that the price has changed for one or both of the two products causing preferences to change. The price of the products has changed and he now prefers item β where before he preferred the item α. We have $v^b_\beta - p_t(\beta) \geq v^b_\alpha - p_t(\alpha)$, since b bids for β, but also $v^b_\alpha - p(\alpha) \geq v^b_\beta - p(\beta)$ since $\mu(b) = \alpha$ in the first stage. Adding these inequalities gives the asserted result. □

Now that we have seen the two lemmas, let's suppose that b_1 bids for α when $p_t(\alpha) \geq p(\alpha) + k \cdot \delta$. If α is expensive, from Lemma 3.1, b_1 is assigned under μ to some item α_1. By using Lemma 3.2 and supposing that $p_t(\alpha) \geq p(\alpha) + k \cdot \delta$, we see that

$$p_t(\alpha_1) - p(\alpha_1) \geq p_t(\alpha) - p(\alpha) \geq k \cdot \delta.$$

So,

$$p_t(\alpha_1) \geq p(\alpha_1) + k \cdot \delta > p(\alpha_1) \geq k \cdot \delta.$$

In this way, at time t, bidder b_2 is assigned to α_1. Therefore there must be more than one object to allocate. In short, $k > 1$. Assume that there is only a single object, then $\alpha = \alpha_1$ and α_1 would be expensive resulting in b_2 having to be assigned to an item by μ, which is a contradiction since $b_1 \neq b_2$.

Accordingly bidder b_2 must have bid for α_1 at price $p_t(\alpha_1) - \delta \geq p(\alpha_1) + (k-1) \cdot \delta > p(\alpha_1)$, so α_1 was expensive. By Lemma 3.1, b_2 is assigned under μ to some α_2. Since b_1 and b_2 are both assigned, we have $k \geq 2$ as for each person assigned there must be the item that is awarded and there may be more items that have not yet been assigned to any bidder.

By Lemma 3.2 and supposing that $p_t(\alpha) \geq p(\alpha) + k \cdot \delta$, we see that

$$p_t(\alpha_2) - p(\alpha_2) \geq p_t(\alpha_1) - \delta - p(\alpha_1) \geq (k-1) \cdot \delta.$$

So

$$p_t(\alpha_2) \geq p(\alpha_2) + (k-1) \cdot \delta > p(\alpha_2) \geq s(\alpha_2),$$

recalling that $s(\alpha_2)$ is the reserve price of object α_2, the minimum price that sellers are willing to sell it for.

So b_3 is committed to α_2 at price $p_t(\alpha_2)$. This means that $k > 2$, for if not $\alpha \in \{\alpha_1, \alpha_2\}$ and α_2 would be expensive, and so b_3 should be matched by μ, which is a contradiction. So b_3 must have bid for α_2 at price $p_t(\alpha_2) - \delta \geq p(\alpha_2) + (k-2) \cdot \delta > p(\alpha_2)$, so α_2 was expensive. By Lemma 3.1, b_3 is assigned under μ to some α_3. Since b_1, b_2 and b_3 are assigned, there must be three or more objects, as three bidders have already been assigned.

At this point it is noted that this process can never terminates, thus k is unbounded and there can be innumerable objects to allocate. This is impossible because there are only k items and we can not allocate more objects than those that currently exist. So we have reached a contradiction. □

Now let $\widehat{p}$ be the final price for an approximate auction. We must show that no price will be very much lower than the minimum equilibrium price p.

Theorem 3.4. *For any item α,*

$$\widehat{p}(\alpha) \geq p(\alpha) - k \cdot \delta.$$

Proof. We are going to show that if there is some α_1 such that $\widehat{p}(\alpha_1) < p(\alpha_1) - k \cdot \delta$ then there must be more than k items α such that $p(\alpha) > s(\alpha)$, which is impossible. It is proved in the same way as Lemma 3.2, since one sees that this would contradict equilibrium since at most k items can be assigned, so it will have been shown that $\widehat{p}(\alpha) \geq p(\alpha) - k \cdot \delta$.

We assume that $p(\alpha_1) \geq \widehat{p}(\alpha_1) + (k+1) \cdot \delta > s(\alpha_1)$. So α_1 is assigned under μ, e.g. to the bidder b_1. There must be some other bidder b_1' who demands α_1 at price p for if not one could decrease $p(\alpha_1)$, until $s(\alpha_1)$, and still have equilibrium.

It follows that b_1', the other α_1 claimant, is committed at $\widehat{p}$ to some item α_2 and $\widehat{p}(\alpha) \leq p(\alpha_2) - k \cdot \delta$; namely,

$$v_{\alpha_2}^{b_1'} - \widehat{p}(\alpha_2) \geq v_{\alpha_1}^{b_1'} - \widehat{p}(\alpha_1) - \delta,$$

otherwise he would not have stopped demanding the item α_1 and

$$v_{\alpha_1}^{b_1'} - p(\alpha_1) \geq v_{\alpha_2}^{b_1'} - p(\alpha_2),$$

because before changing the price he preferred item α_1 rather than item α_2. Adding the inequalities and some manipulation gives

$$p(\alpha_2) - \widehat{p}(\alpha_2) \geq p(\alpha_1) - \widehat{p}(\alpha_1) - \delta \geq k \cdot \delta, \tag{3.4}$$

since the profit margin he makes when the price of the objects changes is higher for item α_2. It follows, using (3.4), that $p(\alpha_2) \geq \widehat{p}(\alpha_2) + k \cdot \delta > s(\alpha_2)$. So α_2 is assigned under μ, e.g. to the bidder b_2. It is matched at equilibrium.

There must be some third bidder b_2' who demands either α_1 or α_2 for if not $p(\alpha_1)$ and $p(\alpha_2)$ could be decreased until $s(\alpha_1)$ and $s(\alpha_2)$, and still have equilibrium. Bidder b_2' is committed at $\widehat{p}(\alpha_3)$ to some item α_3 and $p(\alpha_3) - \widehat{p}(\alpha_3) \geq (k-1) \cdot \delta$.

α_3 is assigned at price p to some b_3 because $p(\alpha_3) \geq s(\alpha_3) + (k-1) \cdot \delta$.

At this point it is noted that this process can never terminates, thus k is unbounded and there can be innumerable objects to allocate. This is impossible, because we can not assign $k+1$ items as there are only k. So we have reached a contradiction. □

To find the buyers-optimal core allocation, we have seen that the two mechanisms explained above converge. Therefore, it is feasible to use them to assign several items to certain bidders. To this aim, it is necessary to emphasise the veracity of the buyers' valuations of each product, since trying to cheat the mechanism can lead to worse margins.

This way, it facilitates the process because without the need for sealed auctions, you arrive at the same allocation. In other words, we obtain the same equilibrium in both cases.

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