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Fraïssé Limits

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Introduction

One of the basic premises of model theory is the formalization of objects and the properties they satisfy as independent entities, called structures and theories, respectively. It goes without saying that this dichotomy leaves room for a more thorough description and handling of the classes of such objects, as well as their theories.

Starting from a determinate set of properties from some field of mathematics, we may seek their translation into first-order logic and ask what kind of structures satisfy them. This will narrow down the possible characteristics of both our objects and their class as a whole. The question, as always, concerns what we can achieve from this approach.

We can ask ourselves how to relate these structures beyond them fulfilling the formulas from a same theory. Under certain conditions of their class, Fraïssé [Fra54] proved the existence of a countable structure which can embed any of the structures and satisfies some additional homogeneity criteria. In other words, we obtain new information about the maps between the elements of the class.

The constructions we will delve into nowadays constitute a paramount implement in some areas of model theory. Beyond this, their potential is manifested in their capability to attain various structures, which are pivotal in other fields of mathematics, solely through the application of a single method to different classes of lesser structures.

Our goal throughout this work is to describe the properties which support Fraïssé's framework, present the centrals results of his theory and provide other specific results for particular instances of structure classes. In this manner, we will be able to review and study some of the most celebrated examples of limit structures, while detailing a selection of their peculiarities.

As any other widespread result, Fraïssé's method is outlined in multiple reference texts on model theory, such as Hodges' [Hod93], Evans' [Eva94] or Tent and Ziegler's [TZ12]. We will follow the formalization of the latter to provide a background for the central theorem and point out some of its results. Some other properties (including the central result or the amalgamation of finite Boolean algebras) were discussed over unpublished materials from Casanovas [Cas09][Cas14][Cas22] and during some meetings with him.

Regarding the occurrences of the limit structure in algebraic and relational classes, we followed Evans' [Eva94] compendium for a reference on some examples and a few brief summaries of their methodologies. A selection of these cases was then extended with the help of other sources, aiming to provide a systematic study on several properties established beforehand.

The thesis is structured as follows:

- Chapter 1 presents a short collection of definitions and propositions which are basic to understand the subsequent results. An undergraduate level of mathematical knowledge is sufficient to comprehend its entirety.
- Following the previous framework, chapter 2 discerns the candidates for our classes of structures and develops the theoretical core of the work. After that, it introduces some supplementary results which further characterize Fraïssé constructions.
- Chapter 3 revolves around some first examples of classes, namely those of finite graphs, finite orders and vector spaces over a fixed finite field. For that purpose, it establishes a series of general properties so as to handle the different cases more efficiently.
- We finish the work with chapter 4, dealing with more complex structures which require the background of Fraïssé's theory, but introduce distinctive nuances from their respective fields.

I would like to acknowledge the unswerving commitment and insightful guidance of Enrique Casanovas, my advisor for this project. There is nothing but thankfulness for his time and dedication, which pass on his eagerness for the study of model theory.

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Abstract

Fraïssé limits are a fundamental construct in model theory. Their significance relies on the fact that they generate overarching structures for certain classes of non-logical objects. This work focuses on two main objectives: to lay the theoretical foundations for Fraïssé limits and develop several well-known instances of algebraic and relational structures. To do so, we introduce a series of intermediate results which will apply to finite relational languages or more general contexts. Lastly, we describe some properties for the ω-categoricity and quantifier elimination of theories, and verify which of our examples satisfy them.

Notation: We will employ terminology from first order-logic and provide a concise summary of the necessary concepts, following [TZ12] and [Eva94].

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Chapter 1

Preliminary notions

This preparatory chapter focuses on providing the essential concepts and tools to sufficiently develop the results which lead to the construction of Fraïssé limits, in line with the formalization of the model theory foundations in [TZ12]. In addition, some properties set the bases for the analysis and discussion of particular cases, which will be presented in chapters 3 and 4. The reader is presupposed to possess a level of knowledge equivalent or superior to an undergraduate introductory course on elementary logic: for reference on set theory and other fundamental topics, we adhere to [End77].

1.1 Structures and maps

Classical model theory revolves around the interaction of first-order formulas and the classes of objects which satisfy them. Characterizing the collections of structures by means of statements which hold in them, as another of the main subjects of study, illustrates the continuous perspective switch which pervades the distinctive methods of the field. Said formulas are constructed upon a concrete *language*, that is, a –possibly empty– set of constants, function symbols and predicates (also referred to as relation symbols). Accordingly, languages also underlie the structures which will interact with subsequent collections of formulas:

Definition 1.1. *Let L be a language. An L-*structure *is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$, where*

A	*is a non-empty set, the* universe *of $\mathfrak{A}$,*
$Z^{\mathfrak{A}} \in A$	*if Z is a constant,*
$Z^{\mathfrak{A}} : A^n \to A$	*if Z is an n-ary function symbol, and*
$Z^{\mathfrak{A}} \subseteq A^n$	*if Z is an n-ary relation symbol.*

Thinking of sets as universes of structures equipped with the same language allows us to define special maps which connect *interpretations* of a same element.

Definition 1.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. A map $h : A \to B$ is called a* homomorphism *if for all $a_1, \dots, a_n \in A$, and for all c constants, f function symbols and R predicates from L:*

$$
\begin{aligned}
h(c^{\mathfrak{A}}) &= c^{\mathfrak{B}} \\
h(f^{\mathfrak{A}}(a_1, \dots, a_n)) &= f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \\
R^{\mathfrak{A}}(a_1, \dots, a_n) &\Rightarrow R^{\mathfrak{B}}(h(a_1), \dots, h(a_n))
\end{aligned}
$$

Furthermore, h is called an embedding[1] *if it is injective and the converse of the last implication is also true; an* isomorphism *is a surjective embedding.*

Homomorphisms, embeddings and isomorphisms are denoted, respectively, by $h : \mathfrak{A} \to \mathfrak{B}$, $h : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$ and $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$. Hence, *automorphisms* of $\mathfrak{A}$ are defined as isomorphisms $\mathfrak{A} \xrightarrow{\sim} \mathfrak{A}$.

Definition 1.3. $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$ *is a* substructure *of* $\mathfrak{B} = (B, (Z^{\mathfrak{B}})_{Z \in L})$ *(equivalently, $\mathfrak{B}$ is an* extension *of $\mathfrak{A}$, or $\mathfrak{A} \subseteq \mathfrak{B}$) if $A \subseteq B$ and the inclusion map is an embedding $\mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$.*

Note that if $\emptyset \neq A \subseteq B$, where B is the universe of an L-structure, A is the universe of a uniquely determined substructure if and only if A contains all $c^{\mathfrak{B}}$ and it is closed under all functions $f^{\mathfrak{B}}$. This is particularly relevant for relational languages like graphs or lineal orders, where any non-empty subset of an L-structure will be the universe of an L-structure.

Similarly, we can find other structures called *reducts* (denoted by $\mathfrak{A} \upharpoonright L_0 = (A, (Z^{\mathfrak{A}})_{Z \in K}))$ and *expansions*, by considering sublanguages $L_0 \subset L$. As a related case, given $\mathfrak{A}$ and $B \subseteq A$, we name $\mathfrak{A}_B$ the structure with universe $L \cup B$ which interprets the elements of B as themselves (and maintains the interpretation of the remaining elements). The following definition presents the elements of a structure which determine the values of any homomorphism:

Definition 1.4. *Let $\mathfrak{B}$ be an L-structure and $\emptyset \neq S \subseteq B$. An L-structure $\mathfrak{A} \subseteq \mathfrak{B}$ is* generated *by S if every $\mathfrak{A}' \subseteq \mathfrak{B}$ containing S satisfies $A \subseteq A'$. We then write $\mathfrak{A} = \langle S \rangle^{\mathfrak{B}}$, and call $\mathfrak{A}$* finitely generated *if S is finite.*[2]

[1] Also called monomorphism in some literature, due to the general concept in category theory.
[2] Notice that we will often reserve $A, B, \dots$ for the associated universes of structures $\mathfrak{A}, \mathfrak{B}, \dots$

1.2 Theories

The previous definitions allow us to recursively present the notion of *L-term*, as any variable, constant from L, or sequence $ft_1 \dots t_n$, where $f \in L$ is an n-ary function symbol and t_i are L-terms. In order to construct formulas from L-terms, we must be able to interpret them by means of the elements of our L-structures. Let us *assign* elements $\vec{b} = (b_1, \dots, b_m)$ of an L-structure $\mathfrak{A}$ to a set of variables $v_1, \dots, v_m$. We then define the *interpretation* $t^{\mathfrak{A}}[\vec{b}]$ in $\mathfrak{A}$ of an L-term t as

$$v_i^{\mathfrak{A}}[\vec{b}] = b_i, \quad c^{\mathfrak{A}}[\vec{b}] = c^{\mathfrak{A}}, \quad ft_1 \dots t_n^{\mathfrak{A}}[\vec{b}] = f^{\mathfrak{A}}\left(t_1^{\mathfrak{A}}[\vec{b}], \dots, t_n^{\mathfrak{A}}[\vec{b}]\right),$$

for every $c \in L$ constant and $f \in L$ function symbol. We may also express $t^{\mathfrak{A}}[\vec{b}]$ as $t^{\mathfrak{A}}[b_1, \dots, b_m]$ and, given L-terms $t, t_1, \dots, t_n$, consider the L-term $t(t_1, \dots, t_n)$ obtained by substituting each v_i with t_i. From this, letting $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$ be an isomorphism, the following equalities are deduced:

$$t(t_1, \dots, t_n)^{\mathfrak{A}}[\vec{b}] = t^{\mathfrak{A}}\left[t_1^{\mathfrak{A}}[\vec{b}], \dots, t_n^{\mathfrak{A}}[\vec{b}]\right], \quad h\left(t^{\mathfrak{A}}[b_1, \dots, b_m]\right) = t^{\mathfrak{B}}[h(b_1), \dots, h(b_m)]$$

The objects which we have introduced are enough to define formulas from the elements of L (constants, function symbols and relation symbols), variables, parentheses, quantifiers ($\exists, \forall$) and other logical symbols ($\doteq, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$):

Definition 1.5. *Let L be a language. Given $t_1, \dots, t_n$ L-terms, $R \in L$ an n-ary relation symbol and x a variable, we denominate* formula *any sequence of symbols of the form*

$$t_1 \doteq t_2, \quad Rt_1 \dots t_n, \quad \neg\phi, \quad (\phi_1 \wedge \phi_2), \quad \exists x\phi,$$

where ϕ, ϕ_1, ϕ_2 are formulas. Formulas of the first two kinds are called atomic, *while they and their negations are sometimes referred to as* basic. *We use $\forall x\phi$, $\phi_1 \vee \phi_2$, $\phi_1 \rightarrow \phi_2$, $\phi_1 \leftrightarrow \phi_2$, as abbreviations for $\neg\exists x\neg(\phi)$, $\neg(\neg\phi_1 \wedge \neg\phi_2)$, $\neg\phi_1 \vee \phi_2$, $(\phi_1 \rightarrow \phi_2) \wedge (\phi_2 \rightarrow \phi_1)$, respectively.*

Having developed a syntax, now it is possible to provide a semantic relation which will allow for formulas to describe most of the attributes a structure may have. The subsequent is well-defined due to the fact that every formula has a unique decomposition, i.e., as long as two given formulas are of the same kind (from the definition), the terms or subformulas which define them will be equal.

Definition 1.6. *Let $\mathfrak{A}$ be an L-structure, φ an L-formula and $\vec{b}$ an assignment. We say that φ* holds *for $\vec{b}$ in $\mathfrak{A}$ (or that $\mathfrak{B}$* satisfies *φ) if the relation $\mathfrak{A} \models \varphi[\vec{b}]$, defined by the following statements, holds:*

$$\mathfrak{A} \models t_1 \doteq t_2[\vec{b}] \Leftrightarrow t_1^{\mathfrak{A}}[\vec{b}] = t_2^{\mathfrak{A}}[\vec{b}]$$

$$\mathfrak{A} \models Rt_1 \dots t_n[\vec{b}] \Leftrightarrow R^{\mathfrak{A}}\left(t_1^{\mathfrak{A}}[\vec{b}], \dots, t_n^{\mathfrak{A}}[\vec{b}]\right)$$
$$\mathfrak{A} \models \neg\phi[\vec{b}] \Leftrightarrow \mathfrak{A} \not\models \phi[\vec{b}]$$
$$\mathfrak{A} \models (\phi_1 \wedge \phi_2)[\vec{b}] \Leftrightarrow \mathfrak{A} \models \phi_1[\vec{b}] \text{ and } \mathfrak{A} \models \phi_2[\vec{b}]$$
$$\mathfrak{A} \models \exists x\phi[\vec{b}] \Leftrightarrow \text{there is } a \in A\text{: } \mathfrak{A} \models \phi\left[b_0, \dots, b_{i-1}, a, b_{i+1}, b_m\right], \text{ if } x = v_i.$$

Moreover, two formulas are equivalent *if they hold for the same assignments; and an element $a \in A$* realizes *a set of formulas $\Sigma(x)$ if a satisfies in $\mathfrak{A}$ all formulas from $\Sigma(x)$ ($\mathfrak{A} \models \Sigma(a)$).*

Retrieving one of the previous equalities, we can establish an equivalence sometimes known as the *substitution lemma*:

$$\mathfrak{A} \models \varphi(t_1, \dots, t_n)[\vec{b}] \Longleftrightarrow \mathfrak{A} \models \varphi\left[t_1^{\mathfrak{A}}[\vec{b}], \dots, t_1^{\mathfrak{A}}[\vec{b}]\right]$$

Formulas can be assigned several attributes regarding the occurrence of quantifiers and quantified variables. We denote formulas without any symbols $\exists, \forall$ as *quantifier-free* formulas. Every formula is equivalent to one in *negation normal form*, i.e., it is built from basic formulas by employing $\wedge, \vee, \exists, \forall$; these are also called *universal* (respectively, *existential*) if they do not contain existential (resp. universal) quantifiers.

On the other hand, we say that a variable x is *free* in a formula if does not appear in the range of some $\exists x$, and *bound* otherwise. In other words, x is free in $t_1 \doteq t_2$ or in $Rt_1 \dots t_n$ if it occurs in one of the t_i; it is free in $\neg\phi_0$ or in $(\phi_1 \wedge \phi_2)$ if it is free in some of the ϕ_i; and it is free in $\exists y\phi$ if it is free in ϕ and $x \neq y$. Then, we call a formula *sentence* if it does not contain free variables, and denote $\mathfrak{A} \models \varphi$ when it holds for some (equivalently, all) assignment. We also say that $\mathfrak{A}$ satisfies, is a *model* of or *models* φ.

A set of L-sentences is said to be *consistent* or *satisfiable*[3] if it has a model, that is, a structure which models all its sentences. Similarly, given a structure $\mathfrak{A}$ and $B \subseteq A$, a set of $L(B)$-formulas $\Sigma(x)$ is consistent or satisfiable in $\mathfrak{A}$ if it is realized by some $a \in \mathfrak{A}$. We extend the notion of satisfiability to *finite satisfiability* if these conditions are met for finite subsets of the set of sentences or formulas.

A sentence φ is said to *follow* from a set of sentences T ($T \vdash \varphi$) if it holds in all models of T. A set of L-sentences T is then called an *L-theory* if any φ which follows from T is in T. Thus, any *equivalent* theories S and T ($S \equiv T$), that is $S \vdash T$ and $T \vdash S$, are equal. A set of formulas Σ is said to be consistent with a theory T if $\Sigma \cup T$ is consistent. Lastly, a consistent L-theory T is said to be *complete* if, for all L-sentences φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$. holds.

[3]In first-order logic, the general notions of consistency and satisfiability are equivalent, so we may use the two terms interchangeably.

1.3 Models, elementarity and chains

Structures are inherently tied to theories and sets of formulas they are models of: for instance, the *atomic diagram* $\mathrm{Diag}(\mathfrak{A})$ is defined as the class of basic $L(A)$-sentences which hold in $\mathfrak{A}_A$, while *the theory* (of a structure) $\mathrm{Th}(\mathfrak{A}_A)$ is the class of $L(A)$-sentences which hold in $\mathfrak{A}_A$. As demonstrated in [TZ12], these constructions allow us to characterize collections of sentences and provide criteria for their properties:

Proposition 1.7. *Let T be a consistent L-theory. The following are equivalent:*

a) T is complete.
b) All models of T are elementarily equivalent, *i.e. have the same L-theory.*
c) There exists an L-structure $\mathfrak{A}$ with $T = \mathrm{Th}(\mathfrak{A})$.

There is a stronger distinction for maps between L-structures: whenever they preserve the validity of arbitrary formulas (that is, $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ just in case $\mathfrak{B} \models \varphi[h(a_1), \ldots, h(a_n)]$), they are called *elementary embeddings* and denoted by $h : \mathfrak{A} \xrightarrow{\preccurlyeq} \mathfrak{B}$, and also $\mathfrak{A} \preccurlyeq \mathfrak{B}$ if $A \subseteq B$. It follows that the models of $\mathrm{Diag}(\mathfrak{A})$ and $\mathrm{Th}(\mathfrak{A}_A)$ are the structures $(\mathfrak{B}, h(a))_{a \in A}$ for, respectively, embeddings $h : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$ and elementary embeddings $h : \mathfrak{A} \xrightarrow{\preccurlyeq} \mathfrak{B}$.

Theorem 1.8 (Tarski's Test)**.** *Let $\mathfrak{B}$ be an L-structure. $A \subseteq B$ is the universe of an elementary substructure if and only if every $L(A)$- formula $\varphi(x)$ satisfiable in $\mathfrak{B}$ is also satisfied by an element of A.*

We introduce a last bundle of results which ensure some properties carry over to higher structures: given a linear order $(I, \leq)$, a chain (resp. elementary chain) $(\mathfrak{A}_i)_{i \in I}$ of L-structures is such that, if $i \leq j$, then $\mathfrak{A}_i \subseteq \mathfrak{A}_j$ (resp. $\mathfrak{A}_i \preccurlyeq \mathfrak{A}_j$).

Lemma 1.9. *Let $(\mathfrak{A}_i)_{i \in I}$ be a chain of L-structures. Then, $A = \bigcup_{i \in I} A_i$ is the universe of a uniquely determined L-structure $\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i$ which is:*

i) an extension of all $\mathfrak{A}_i$.
ii) (Tarski's Chain Lemma) *an elementary extension of all $\mathfrak{A}_i$, if $(\mathfrak{A}_i)_{i \in I}$ is elementary.*

Chapter 2

Theoretical construction

From a naive perspective, the interest of Fraïssé constructions is two-fold: they constitute an overarching structure which both extends a collection of smaller models and induces more specific theories with stronger properties. In order to elaborate on these matters, this chapter is laid out in the following fashion: the first section presents some necessary notions regarding theories and satisfiability of formulas and sentences; section 2.2 is the core of the theoretical half of the paper; and, lastly, a series of results are developed as implications of Fraïssé limits.[1]

2.1 ω-categoricity

Let us begin by stating one of the classical results in first-order logic, which provides an essential shift in the methods of proving certain properties for sets of formulas and theories in particular, as shown in [Hod93]:

Theorem 2.1 (Compactness Theorem)**.** *A set of formulas is satisfiable if and only if it is finitely satisfiable.*

As a consequence, a sentence follows from a theory T if and only if it follows from some finite subset of T; furthermore, a theory is consistent with a set of formulas Σ if and only if it is consistent with every finite subset of Σ. Along with the following definition and an immediate result, the consequences of the compactness theorem motivate the concept of type, which is pivotal for this work.

First note that a finite subset of a finitely satisfiable Σ in $L(A)$ is realized in $\mathfrak{A}$ if and only if it is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. Then, as seen in section 1.3, the latter occurs if and only if there exists an elementary extension of $\mathfrak{A}$ where Σ is realized: this now becomes an equivalent condition for $\Sigma(x)$ being finitely satisfiable in $\mathfrak{A}$.

[1] Let $\omega = |\mathbb{N}|$. We will make use the notation $n < \omega$ to express n is a natural number.

Definition 2.2. *Let $\mathfrak{A}$ be an L-structure and B a subset of A. A set $p(x_1, \ldots, x_n)$ of $L(B)$-formulas is called an n-*type *over B (referred to as its* domain*) if it is maximal finitely satisfiable in $\mathfrak{A}$. We denote the set of n-types over B by $\mathrm{S}_n(B) = \mathrm{S}_n^{\mathfrak{A}}(B)$.*

In particular, we may refer to 1-types simply as *types* and denote their set by $\mathrm{S}(B) = \mathrm{S}^{\mathfrak{A}}(B)$. We will usually follow the convention that n-types are *complete n-types*, i.e., so-called *partial n-types* (sets of formulas defined accordingly, but without the maximality requirement) will often be introduced with that distinction.

Moreover, n-types being maximal with respect to inclusion implies the following equivalence: an n-tuple $\bar{a}$ from $\mathfrak{A}$ realizes the n-type $p \in \mathrm{S}_n(B)$ if and only if $p = \mathrm{tp}(\bar{a}/B)$, which is defined as the set of $L(B)$-formulas satisfied by $\bar{a}$ in $\mathfrak{A}$. We deduce from this fact that both n-types and the set of n-types are preserved over elementary extensions of $\mathfrak{A}$.

Types motivate the definition of a kind of structures which are fundamental to Ryll-Nardzewski's Theorem and to determine uniqueness up to isomorphism for models of cardinality ω: we say an L-structure is *ω-saturated* if all types over finite subsets of A are realized in $\mathfrak{A}$.[2]

Precisely, as powerful as the Compactness Theorem may be, it also presents us with some hindrances when it comes to describing the models which satisfy a theory. Following the properties about types gathered from [TZ12], the subsequent well-known results show that finding an infinite model automatically implies the existence of many more structures which satisfy the same theory:

Theorem 2.3 (Löwenheim-Skolem)**.** *Let $\mathfrak{B}$ be an L-structure, S a subset of B and κ an infinite cardinal. Then,*

1. *$\mathfrak{B}$ has an elementary substructure of cardinality κ containing S if*

$$\max(|S|, |L|) \leq \kappa \leq |\mathfrak{B}|$$

2. *$\mathfrak{B}$ has an elementary extension of cardinality κ if $\mathfrak{B}$ is infinite and*

$$\max(|\mathfrak{B}|, |L|) \leq \kappa$$

Corollary 2.4. *An L-theory satisfied by an infinite model has a model in every cardinality $\kappa \geq \max(|L|, \omega)$.*

Thus, in order to classify theories according to the specificity of their models, we must limit ourselves to considering only individual cardinalities.

[2]It can be seen that the restriction to 1-types can be suppressed without consequences.

Definition 2.5. *Let κ be a cardinal. An L-theory T is κ-*categorical *if it has exactly one model of cardinality κ up to isomorphism.*[3]

The upcoming section provides tools to determine whether classes of finitely generated structures which respond to specific theories can be extended to a model of size ω, thus deeming the theory ω-categorical.

2.2 Fraïssé limits

In order to prove some of the results in this chapter, we need an auxiliary method to connect the aforementioned notions with a practical way of checking the ω-categoricity of structures. [Ber15] describes the following result, which allows us to construct isomorphisms between structures, and relates its implications:

Theorem 2.6 (Back-and-forth)**.** *Let L be a language and $\mathfrak{M}$, $\mathfrak{N}$ be L-structures of size ω. Consider $I : \mathfrak{M} \cong_p \mathfrak{N}$, the nonempty set of isomorphisms between finitely generated substructures of $\mathfrak{M}$ and $\mathfrak{N}$ with the properties:*

- **Back***: for every $f \in I$ and every $n \in N$, there exists some $f' \in I$ which extends f and contains n in its image.*
- **Forth***: for every $f \in I$ and every $m \in M$, there exists some $f' \in I$ which extends f and contains m in the domain.*

Then, there exists an isomorphism $h : \mathfrak{M} \xrightarrow{\sim} \mathfrak{N}$.

We will now present a weak version of the Fraïssé theorem, which does not ensure the uniqueness of the constructed structure. Departing from a fixed theory and a limited class of its models, our goal is to reach a countable[4] structure which also satisfies the theory and behaves properly with the aforementioned class.

Definition 2.7. *The* age[5] *$\mathcal{K}$ of an L-structure $\mathfrak{M}$,* Age($\mathfrak{M}$)*, is the class of all finitely generated L-structures which are isomorphic to a substructure of $\mathfrak{M}$.*

In other words, a finitely generated L-structure belongs to Age($\mathfrak{M}$) exactly when it is embeddable into $\mathfrak{M}$. For the sake of simplicity, let us also handle the elements of classes in terms of the equivalence relation $\cong$, of isomorphism between structures. Throughout the following two definitions, consider a class $\mathcal{K}$ of finitely generated structures closed under isomorphism, such that there are countably many isomorphism types of said structures (that is, $|\mathcal{K}/\cong| \leq \omega$).

[3] We can also attribute the property of κ-categoricity to structures inasmuch their theories may be κ-categorical.

[4] In this work, countable refers to a cardinality equal or less than ω.

[5] Also referred to as *skeleton* by some sources.

Definition 2.8. *The class $\mathcal{K}$ has the* Hereditary Property *(***HP***) if, for any $\mathfrak{A}$ in $\mathcal{K}$,* $\mathrm{Age}(\mathfrak{A})$ *is a subset of $\mathcal{K}$.*

Definition 2.9. *The class $\mathcal{K}$ has the* Joint Embedding Property *(***JEP***) if, for any $\mathfrak{B}_0, \mathfrak{B}_1$ in $\mathcal{K}$, there are some $\mathfrak{D} \in \mathcal{K}$ and embeddings $g_i : \mathfrak{B}_i \overset{\subseteq}{\to} \mathfrak{D}$ ($i \in \{0,1\}$).*

These properties are sufficient to construct a "weak" *Fraïssé limit* of the class $\mathcal{K}$, which is not uniquely determined with respect to isomorphism:

Theorem 2.10. *Let L be a countable language and $\mathcal{K}$ a class of finitely generated L-structures. Suppose $\mathcal{K}$ is closed under isomorphism and $|K/\cong| \leq \omega$. Then, there exists a countable L-structure $\mathfrak{M}$ such that $\mathcal{K} = \mathrm{Age}(\mathfrak{M})$ if and only if $\mathcal{K}$ satisfies* **HP** *and* **JEP***.*

Proof. $\mathrm{Age}(\mathfrak{M})$ has the **HP** since any $\mathfrak{A} \in \mathrm{Age}(\mathfrak{M})$ embeds into $\mathfrak{M}$, and every $\mathfrak{A}_0 \in \mathrm{Age}(\mathfrak{A})$ is finitely generated and embeds into $\mathfrak{A}$. For the **JEP**, consider some $\mathfrak{B}_0, \mathfrak{B}_1 \in \mathrm{Age}(\mathfrak{M})$ and find $\langle\{b_0^1, \ldots, b_0^{n_0}\}\rangle^{\mathfrak{M}} \cong \mathfrak{B}_0, \langle\{b_1^0, \ldots, b_1^{n_1}\}\rangle^{\mathfrak{M}} \cong \mathfrak{B}_1$. The structure $\mathfrak{D} = \langle\{b_0^0, \ldots, b_0^{n_0}, b_1^0, \ldots, b_1^{n_1}\}\rangle^{\mathfrak{M}} \in \mathrm{Age}(\mathfrak{M})$, embeds both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ as substructures.

Conversely, assume $\mathcal{K}$ satisfies **HP** and **JEP**, and let $(\mathfrak{B}_i : i < \omega)$ be an enumeration of the isomorphism representatives of the elements of $\mathcal{K}$. We will construct the limit $\mathfrak{M}$ as the union $\bigcup_{i<\omega} \mathfrak{A}_i$ of a chain $(\mathfrak{A}_i : i < \omega)$ of structures of $\mathcal{K}$, in a way that every $\mathfrak{B}_i$ embeds into $\mathfrak{A}_i$. We begin by setting $\mathfrak{A}_0 = \mathfrak{B}_0$: for each $i < \omega$, there exists some $\mathfrak{A}'_{i+1} \in \mathcal{K}$ and embeddings $g_i^0 : \mathfrak{A}_i \overset{\subseteq}{\to} \mathfrak{A}'_{i+1}$ and $g_i^1 : \mathfrak{B}_{i+1} \overset{\subseteq}{\to} \mathfrak{A}'_{i+1}$, due to **JEP**. Additionally, we may consider a structure $\mathfrak{A}_{i+1} \supseteq \mathfrak{A}_i$ in $\mathcal{K}$ such that $\mathfrak{A}_{i+1} \cong^h \mathfrak{A}'_{i+1}$. Then, $g_i^0 = h \circ \mathrm{Id}_{A_i}$, and we can embed $\mathfrak{B}_{i+1}$ into $\mathfrak{A}_{i+1}$ via $h^{-1} \circ g_i^1$. Thus, we have constructed a proper chain of $(\mathfrak{A}_j)$ which assimilates the representatives $\mathfrak{B}_i$ of isomorphism type in every step.

It remains to be checked that $\mathcal{K} = \mathrm{Age}(\mathfrak{M})$: by construction, every $\mathfrak{B}_i$ is embeddable into $\mathfrak{M}$, so $\mathcal{K} \subseteq \mathrm{Age}(\mathfrak{M})$; conversely, there exists an embedding from every finitely generated structure in $\mathfrak{M}$ into some $\mathfrak{A}_i$, which completes the proof. □

Example 2.11. While this method works for countable structures in general (in the sense their cardinality is not greater than ω), only one of the implications is preserved when switching to the finite case. Specifically, considering $|\mathcal{K}/\cong| < \omega$ does not necessarily yield a finite model $\mathfrak{M}$: for instance, a class based on the set of integers with the successor and predecessor function symbols, $\mathcal{K} = (\mathbb{Z}, S, P)$, only produces the trivial isomorphism type $\langle 0 \rangle$, whereas any model which results as the Fraïssé limit of $\mathcal{K}$ will have an infinite model into which $\mathbb{Z}$ embeds.

However, we may require additional properties in order to obtain a unique Fraïssé limit. For the remainder of the section, let us suppose L is a countable language and $\mathfrak{M}$ is a countable L-structure. We temporarily divert from Fraïssé limits so that we can adhere to the homogeneity approach in [Cas09], and define:

Definition 2.12. *$\mathfrak{M}$ is* ultrahomogeneous *if any isomorphism between its finitely generated substructures can be extended to an automorphism of $\mathfrak{M}$. $\mathfrak{M}$ is* algebraically ω-homogeneous *if, for any every isomorphic finitely generated substructures of $\mathfrak{M}$, $\mathfrak{A} \cong^h \mathfrak{B}$, and for each $a \in M$, there exists $b \in M$ such that $h \cup \{(a,b)\}$ can be extended to an isomorphism between $\langle Aa \rangle^{\mathfrak{M}}$ and $\langle Bb \rangle^{\mathfrak{M}}$. More weakly, $\mathfrak{M}$ is* strongly ω-homogeneous *if every finite elementary map in $\mathfrak{M}$ can be extended to an automorphism of $\mathfrak{M}$. Finally, $\mathfrak{M}$ is* ω-homogeneous *if, for any $a \in M$, every finite elementary map h in $\mathfrak{M}$ can be extended to an elementary map $h \cup \{(a,b)\}$, for some $b \in M$.*

Let us write $\bar{a} \equiv^{qfr} \bar{b}$ whenever two finite tuples $\bar{a}, \bar{b} \in M$ satisfy the same quantifier-free formulas. This is equivalent to the condition that the map $\bar{a} \mapsto \bar{b}$ can be extended to a (unique) isomorphism $\langle \bar{a} \rangle^{\mathfrak{M}} \xrightarrow{\sim} \langle \bar{b} \rangle^{\mathfrak{M}}$.[6] Given this fact, every isomorphism between finitely generated substructures of $\mathfrak{M}$ is elementary if and only if for any $\bar{a}, \bar{b} \in M$, $\bar{a} \equiv^{qfr} \bar{b}$ implies $\bar{a} \equiv \bar{b}$. Hence:

Proposition 2.13. *The following conditions are equivalent:*

1. *$\mathfrak{M}$ is algebraically ω-homogeneous.*
2. *For any finite tuples $\bar{a} \equiv^{qrf} \bar{b}$ of $\mathfrak{M}$ and for each $a \in M$, there exists some $b \in M$ such that $\bar{a}a \equiv^{qfr} \bar{b}b$.*
3. *For all finitely generated $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$, every embedding $f : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{M}$ can be extended to an embedding $g : \mathfrak{B} \xrightarrow{\subseteq} \mathfrak{M}$.*

The second condition implies that every isomorphism between structures is elementary, so $\mathfrak{M}$ is algebraically ω-homogeneous if and only if it is ω-homogeneous and every isomorphism between substructures is elementary. In general, a structure is ultrahomogeneous if and only if it is strongly ω-homogeneous and every isomorphism between its substructures is elementary. Therefore, if $\mathfrak{M}$ is countably generated, $\mathfrak{M}$ being ultrahomogeneous is equivalent to it being algebraically ω-homogeneous. From definition 2.12, we derive:

Lemma 2.14. *Let $\mathcal{K}$ be the age of and L-structure $\mathfrak{M}$. Then, the following are equivalent:*

1. *$\mathfrak{M}$ is algebraically ω-homogeneous.*

[6] A further characterization is provided in section 2.3.

2. *For all* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ *and embeddings* $f_0 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{M}$, $f_1 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$, *there is some embedding* $g : \mathfrak{B} \xrightarrow{\subseteq} \mathfrak{M}$ *such that* $g \circ f_1 = f_0$.

3. *For all* $\mathfrak{A} \subseteq \mathfrak{M}, \mathfrak{B}$ *in* $\mathcal{K}$ *and* $f_1 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$, *there is some embedding* $g : \mathfrak{B} \xrightarrow{\subseteq} \mathfrak{M}$ *such that* $g \circ f_1 = \mathrm{Id}_A$.

Definition 2.15. *The L-structure* $\mathfrak{M}$ *is said to be* rich[7] with respect to $\mathcal{K}$ *if its age is* $\mathcal{K}$ *and for all* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ *and embeddings* $f_0 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{M}$, $f_1 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$, *there is some embedding* $g : \mathfrak{B} \xrightarrow{\subseteq} \mathfrak{M}$ *such that* $g \circ f_1 = f_0$.

With this definition, we may proceed using interchangeable equivalences of richness depending on the context. We can now return to Fraïssé theory to prove a result and introduce the last concept before showing the central theorem:

Theorem 2.16. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be countably generated L-structures rich with respect to a same* $\mathcal{K}$. *Then,* $\mathfrak{M} \cong \mathfrak{N}$.

Proof. Let $(m_i : i < \omega)$ and $(n_i : i < \omega)$ be the generators of $\mathfrak{M}$ and $\mathfrak{N}$, respectively. We will apply a back-and-forth approach to construct an isomorphism f between $\mathfrak{M}$ and $\mathfrak{N}$, as the union of a chain $(f_i : i < \omega)$ of isomorphisms between finitely generated structures $\mathfrak{M}_i \subseteq \mathfrak{M}$ and $\mathfrak{N}_i \subseteq \mathfrak{N}$, such that $m_i \in \mathfrak{M}_{i+1}$ and $n_i \in \mathfrak{N}_{i+1}$ for every $i < \omega$. Since $\mathfrak{M}$ and $\mathfrak{N}$ have age $\mathcal{K}$, we can choose any substructure $\mathfrak{M}_0$ and find an isomorphism f_0 into some $\mathfrak{N}_0 \subseteq \mathfrak{N}$. Now, consider $f_i : \mathfrak{M}_i \xrightarrow{\sim} \mathfrak{N}_i$ and the inverse over an extended codomain $f_i^{-1} : \mathfrak{N}_i \xrightarrow{\subseteq} \langle \mathfrak{M}_i m_i \rangle^{\mathfrak{M}}$. Using lemma 2.14 we obtain an embedding $g : \langle \mathfrak{M}_i m_i \rangle^{\mathfrak{M}} \xrightarrow{\subseteq} \mathfrak{N}'_i \subseteq \mathfrak{N}$ such that $g \circ f_i^{-1} = \mathrm{Id}_{N_i}$, i.e. an embedding $g \supseteq f_i$. Applying the same reasoning to $g : \langle \mathfrak{M}_i m_i \rangle^{\mathfrak{M}} \xrightarrow{\subseteq} \langle \mathfrak{N}'_i n_i \rangle^{\mathfrak{N}}$ we find an extension f_{i+1} of g from $\mathfrak{M}_{i+1} \ni m_i$ to $\mathfrak{N}_{i+1} \ni n_i$. The union $\bigcup_{i<\omega} f_i$ is an isomorphism $\mathfrak{M} \xrightarrow{\sim} \mathfrak{N}$ since it regards every m_i, n_i. □

Definition 2.17. *The class* $\mathcal{K}$ *has the* Amalgamation Property *(***AP***) if, for any* $\mathfrak{A} \neq \emptyset, \mathfrak{B}_0, \mathfrak{B}_1$ *in* $\mathcal{K}$, *and for all embeddings* $f_0 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}_0$, $f_1 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}_1$, *there are some* $\mathfrak{D} \in \mathcal{K}$ *and embeddings* $g_i : \mathfrak{B}_i \xrightarrow{\subseteq} \mathfrak{D}$ $(i \in \{0,1\})$ *such that* $g_0 \circ f_0 = g_1 \circ f_1$.

We restrict $\mathfrak{A}$ to non-empty sets in order to avoid **AP** implying **JEP**. Proving this third Property may be somewhat troublesome when focusing in particular cases, but it will provide the following central theorem:

[7] [TZ12] reach this definition by other means and choose the name $\mathcal{K}$-saturated.

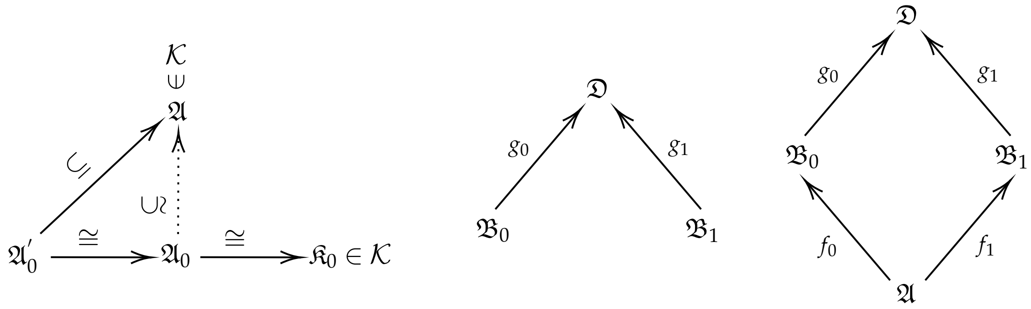

Figure 2.1: Diagrams for **HP**, **JEP** and **AP**, respectively

Theorem 2.18 (Fraïssé)**.** *Let L be a countable language and $\mathcal{K}$ a class of finitely generated L-structures. Suppose $\mathcal{K}$ is closed under isomorphism and $|\mathcal{K}/\cong| \leq \omega$. Then, there exists a unique (up to isomorphism) countable L-structure $\mathfrak{M}$ rich with respect to $\mathcal{K}$ if and only if $\mathcal{K}$ satisfies* **HP**, **JEP** *and* **AP**. *In this case, $\mathfrak{M}$ is called the* Fraïssé limit *of $\mathcal{K}$.*

Proof. First, if $\mathcal{K}$ is the age of such $\mathfrak{M}$, any structure from the age of any of its elements will be again isomorphic to some substructure of $\mathfrak{M}$; and any two elements of $\mathcal{K}$ will be finitely generated and naturally embedded into a larger, finitely generated structure of $\mathcal{K}$. For the **AP**, let $\mathfrak{A}, \mathfrak{B}_0, \mathfrak{B}_1, f_0, f_1$ be as required. There exists an embedding $h : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{M}$, due to the fact that $\mathfrak{A} \in \mathcal{K}$. Take the restriction $f_0 : \mathfrak{A} \xrightarrow{\sim} f_0(\mathfrak{A}) \subseteq \mathfrak{B}_0$ of f_0 and consider the composition $h \circ f_0^{-1} : f_0(\mathfrak{A}) \xrightarrow{\subseteq} \mathfrak{M}$, which can be extended to some $h_0 : \mathfrak{B}_0 \xrightarrow{\subseteq} \mathfrak{M}$ thanks to the richness of $\mathfrak{M}$. Now, considering $f_1 \circ f_0^{-1} : f_0(\mathfrak{A}) \xrightarrow{\subseteq} \mathfrak{B}_1$, we can apply that $\mathfrak{M}$ is rich with respect to $\mathcal{K}$ to find an embedding $h_1 : \mathfrak{B}_1 \xrightarrow{\subseteq} \mathfrak{M}$ such that $h_0 = h_1 \circ (f_1 \circ f_0^{-1})$. If we define g_0, g_1 as the restrictions of g_0 to $h_0(\langle \mathfrak{B}_0, \mathfrak{B}_1 \rangle^{\mathfrak{M}})$ and g_1 to $h_1(\langle \mathfrak{B}_0, \mathfrak{B}_1 \rangle^{\mathfrak{M}})$, respectively, we can check that $g_0 \circ f_0 = g_1 \circ f_1$ holds for any element $a \in A$:

$$(g_0 \circ f_0)(a) = (g_1 \circ (f_1 \circ f_0) - 1) \circ f_0)(a) = (g_1 \circ f_1 \circ (f_0^{-1} \circ f_0))(a) = (g_1 \circ f_1)(a)$$

Conversely, assume $\mathcal{K}$ has **HP**, **JEP** and **AP**. Guided by the reasoning in [Cas09], $\mathfrak{M}$ will be constructed as the union $\bigcup_{i<\omega} \mathfrak{A}_i$ of a chain of finitely generated L-structures $\mathfrak{A}_i \in \mathcal{K}$, hence it will be countable. Consider a representative list $(\mathfrak{B}_i : i < \omega)$ of the isomorphism types in $\mathcal{K}$, and observe that, for any $\mathfrak{B}_i$, there is a countable amount of embeddings from all finitely generated substructures of any given $\mathfrak{A} \in \mathcal{K}$ into $\mathfrak{B}_i$: these can be enumerated as $(f_r^{\mathfrak{A},i} : r < \omega)$.

Taking $\mathfrak{A}_0 = \mathfrak{B}_0$, we can follow a reasoning similar to theorem 2.10 and apply the **JEP** whenever $\mathfrak{A}_{2n}$ is known, to find some $\mathfrak{A}_{2n+1}$ such that $\mathfrak{A}_{2n} \subseteq \mathfrak{A}_{2n+1}$ and $\mathfrak{B}_n \subsetneqq \mathfrak{A}_{2n+1}$. In the case $\mathfrak{A}_i = \mathfrak{A}_{2n+1}$ is already constructed, we will apply **AP**

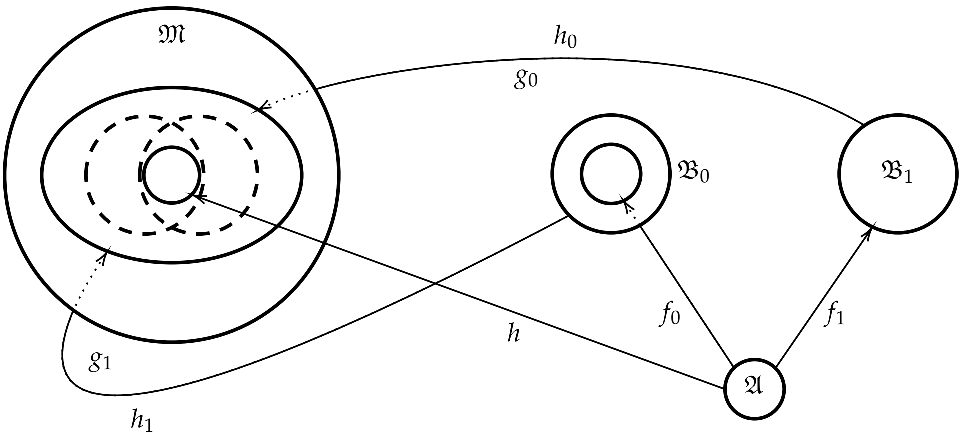

Figure 2.2: Diagram for the proof of the **AP**

repeatedly (a finite amount of times) to ensure that any embedding needed for the condition of algebraic ω-homogeneity exists. Following figure 2.3, we will construct a finite chain $\mathfrak{A}_i \subseteq \mathfrak{A}_i^1 \subseteq \ldots \subseteq \mathfrak{A}_i^m =: \mathfrak{A}_{i+1}$ by using the amalgamation property on every $\mathfrak{A}_j \subseteq \mathfrak{A}_i$, our new structures $\mathfrak{A}_i^s$ and every $\mathfrak{B}_k$, $k \leq i$, taken as embeddings the inclusion and every correspondent $f_r^{\mathfrak{A}_j,k}$ up to $r \leq i$ (thus, we invoke **AP** at most i^3 times): this allows us to find new structures of the chain and an embedding $g : \mathfrak{B}_k \xrightarrow{\subseteq} \mathfrak{A}_i^s \subseteq \mathfrak{A}_{i+1}$ such that $g \circ f_r^{\mathfrak{A}_j,k} = \mathrm{Id}_{A_j}$. Actually, we obtain some structure $\mathfrak{A}_i^{s\prime}$ and a morphism $g' : \mathfrak{A}_i \xrightarrow{\subseteq} \mathfrak{A}_i^{s\prime}$, but g' can be assumed to be the inclusion if we consider $\mathfrak{A}_i^s \cong^h \mathfrak{A}_i^{s\prime}$:

$$\mathrm{Id}_{A_j} = \mathrm{Id}_{A_i} \circ \mathrm{Id}_{A_j} = (h^{-1} \circ g') \circ \mathrm{Id}_{A_j} = h^{-1} \circ (g' \circ \mathrm{Id}_{A_j}) = h^{-1} \circ g'' \circ f_r^{\mathfrak{A}_j,k}$$

We define $\mathfrak{A}_{i+1}$ to be the last of the elements in the chain, which is finite and is expanded at each new $\mathfrak{A}_{2n'+1}$ in a way that every embedding between substructures is eventually covered. This works because the $f_r^{\mathfrak{A}_j,k}$ are involved gradually as n grows larger.

To prove the resultant $\mathfrak{M}$ is rich with respect to $\mathcal{K}$, we proceed analogously to theorem 2.10 for the computation of $\mathrm{Age}(\mathfrak{M})$. By lemma 2.14, there only remains to check that, for every finitely generated $\mathfrak{A} \subseteq \mathfrak{M}$ and for every $k < \omega$, any embedding $f : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}_k$ induces some $g : \mathfrak{B}_k \xrightarrow{\subseteq} \mathfrak{M}$ such that $g \circ f$ is the identity. But $f = f_r^{\mathfrak{A}_j,k} \restriction A$ for some $r < \omega$, as $\mathfrak{A} \subseteq \mathfrak{A}_j$ is finitely generated: by construction, this yields the requested embedding $\mathfrak{B}_k \xrightarrow{\subseteq} \mathfrak{A}_{i+1} \subseteq \mathfrak{M}$ when selecting any odd $i \geq j,k,r$. □

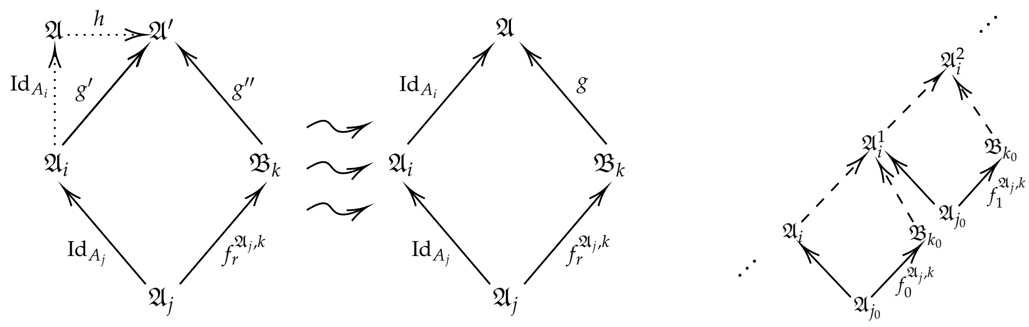

Figure 2.3: During the second part of theorem 2.18, we can assume the left side to be the inclusion (left); amalgamation process (right)

In practical terms, during chapter 4, we will consider fixed, restricted languages (which will yield more powerful results, as in the case of finitely relational languages) or certain structures which will require additional properties to ensure some of the hypotheses are met (such as universally local finitude, needed for finite generatedness).

2.3 Subsequent properties

By requesting some further conditions, we can obtain stronger properties for several of the newly-created structures. Let us introduce:

Theorem 2.19. *Let L be a countable language. Any two elementarily equivalent, countable, ω-saturated L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

Proof. Following a back-and-forth argument, we will construct an isomorphism $f : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$ as the union of a chain of elementary maps $\mathfrak{A}_i \to \mathfrak{B}_i$ between finite substructures of $\mathfrak{A}$ and $\mathfrak{B}$. Let us begin by setting enumerations for $A = \{a_i \mid i < \omega\}$ and $B = \{b_i \mid i < \omega\}$, and f_0 the empty map, which is elementary as $\mathfrak{A} \equiv \mathfrak{B}$.

If f_i, $i = 2n$, is constructed, we consider the 1-type $\mathrm{tp}(a_n / A_i)$ of $L(A_i)$-formulas satisfied by a_n in $\mathfrak{A}$, in order to extend f_i to $A_{i+1} = A_i \cup \{a_n\}$. f_i is elementary due to its construction, thus $f_i(p)(x)$ is a type over B_i in $\mathfrak{B}$. $\mathfrak{B}$ being ω-saturated implies the existence of an element $b' \in B$ which realizes $f_i(p)(x)$. Then, given any tuple $\overline{a} \subseteq A$ and some $L(A_i)$-formula φ, $\mathfrak{A} \models \varphi(a_n, \overline{a})$ implies $\mathfrak{B} \models \varphi(b', f_i(\overline{a}))$: this proves that the map $f_{i+1} : A_{n+1} \to B_{n+1} := B_n \cup b'$, with $f_{i+1}(a_n) = b'$, is an elementary extension of f_i.

Finally, A_j and B_j are defined for every $j < \omega$ and have as a union A and B, respectively. Therefore, the isomorphism $f : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$ emerges as the union of a well-defined chain $\{f_i \mid i < \omega\}$ of elementary maps between the A_j and the B_j. □

This result shows that ω-saturated models are ω-homogeneous. It is applied by [TZ12] to prove the Ryll-Nardzewski theorem and infer some of the remaining propositions in this section, whose demonstrations we will develop.

Theorem 2.20 (Ryll-Nardzewski)**.** *Let L be a countable language and T a complete L-theory. T is ω-categorical if and only if, for every n, there are only finitely many formulas $\varphi(x_1, \dots, x_n)$ up to equivalence relative to T.*

Ryll-Nardzewski's theorem is well-spread and may be expressed under a third equivalent condition, which is every type over T being *isolated*: this means there exists an L-formula $\varphi(x)$ consistent with T such that every formula $\sigma(x)$ in the type satisfies $T \vdash \forall x(\varphi(x) \to \sigma(x))$. It also provides a criterion for ω-categoricity which requires a preparatory notion (and its direct consequence):

Definition 2.21. *An L-theory T is said to have* quantifier elimination *if every L-formula*[8] *$\varphi(x_1, \dots, x_n)$ in the theory is equivalent, modulo T, to some quantifier-free formula $\rho(x_1, \dots, x_n)$.*

Proposition 2.22. *An L-theory T has quantifier elimination if and only if any models $\mathfrak{M}, \mathfrak{N}$ of T with a common substructure $\mathfrak{A}$ are elementary equivalent for $L(A)$-sentences (i.e., $\mathfrak{M}_A \equiv \mathfrak{N}_A$).*

A celebrated example of a quantifier-free theory is the theory of rational dense lineal orders without endpoints, which will be introduced in 3. Precisely, finite relational languages (which contain no constants or function symbols, thus every finitely generated structure is finite) need few conditions to imply other properties:

Lemma 2.23. *Let L be a finite relational language. A complete L-theory T with quantifier elimination is ω-categorical.*

Proof. Given some $n < \omega$, we can show in two steps that there is a finite amount of non-equivalent quantifier-free formulas $\rho(x_1, \dots, x_n)$. First, there are finitely many atomic L-formulas, since they can either be equalities between variables or be derived from the finite predicates of the language. Then, every quantifier-free formula is equivalent modulo T to a formula in conjunctive normal form, which is expressed as $\bigwedge_{i<m} \bigvee_{j<m_i} \pi_{ij}$, where π_{ij} is an atomic L-formula or its negation. As in Boolean algebra, given that our finite many atomic L-formulas can or cannot be satisfied, we can check there is a finite amount of non-equivalent formulas in conjunctive normal form, which stem from their combination. Therefore, quantifier elimination implies all formulas $\varphi(x_1, \dots, x_n)$ are equivalent to some $\rho(x_1, \dots, x_n)$ modulo T. By theorem 2.20, T is ω-categorical. □

[8] In the strict sense, excluding sentences and having exactly the required number of free variables.

We finish this section by proving one last equivalence theorem and presenting a corollary which sums up the results for a particular case. Let us call any $\varphi = \exists y \rho$, with a quantifier-free ρ, a simple existential formula.

Theorem 2.24. *Let L be a finite relational language and T a complete L-theory. Given a countably infinite model $\mathfrak{M}$ of T, the following are equivalent:*

1. *T has quantifier elimination.*
2. *Any isomorphism between finite substructures of $\mathfrak{M}$ is elementary.*
3. *The domain of any isomorphism between finite substructures of $\mathfrak{M}$ can be extended to any further element.*

Proof. (1) $\Rightarrow$ (2): We first prove that any isomorphism $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$ between finite substructures of $\mathfrak{M}$ preserves the validity of quantifier-free formulas. Formulas of the form $f(t_1, \ldots, t_n) \doteq f'(t'_1, \ldots, t'_n)$ hold their truth value: for every assignment $\vec{b}$, every $z \doteq z'$ constants or variables satisfy $h(z^{\mathfrak{A}}[\vec{b}]) = z^{\mathfrak{B}}[h(\vec{b})] \doteq (z')^{\mathfrak{B}}[h(\vec{b})] = h((z')^{\mathfrak{A}}[\vec{b}])$. By induction, assume that, for every assignment $\vec{b}$, any terms $t \doteq t'$ with as many function symbols as t_i, t'_i satisfy $h(t^{\mathfrak{A}}[\vec{b}]) = h((t')^{\mathfrak{A}}[\vec{b}])$. Then,

$$h\left(f^{\mathfrak{A}}\left(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}}\right)[\vec{b}]\right) = f^{\mathfrak{B}}\left(h\left(t_1^{\mathfrak{A}}[\vec{b}]\right), \ldots, h\left(t_n^{\mathfrak{A}}[\vec{b}]\right)\right) \doteq$$
$$(f')^{\mathfrak{B}}\left(h\left((t'_1)^{\mathfrak{A}}[\vec{b}]\right), \ldots, \left((t'_n)^{\mathfrak{A}}[\vec{b}]\right)\right) = h\left((f')^{\mathfrak{A}}\left((t'_1)^{\mathfrak{A}}, \ldots, (t'_n)^{\mathfrak{A}}\right)[\vec{b}]\right).$$

Formulas of the form $R(t_1, \ldots, t_n)$, where R is a predicate follow analogously due to the third condition of embeddings, $R^{\mathfrak{A}}(a_1, \ldots, a_n) \Leftrightarrow R^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$. Finally, it is easy to see that h preserves formulas of the form $\neg\rho(\overline{x})$ or $\rho_1(\overline{x^1}) \vee \rho_2(\overline{x^2})$. The implication follows from every quantifier-free formula being equivalent to one in disjunctive normal form.

(2) $\Rightarrow$ (1): Let us prove, in the first place, that all n-tuples $\overline{a}$ which satisfy in $\mathfrak{M}$ the same quantifier-free n-type $\mathrm{tp}_{\mathrm{qf}}(\overline{a}) = \{\rho(\overline{x}) \mid \mathfrak{M} \models \rho(\overline{a}),\ \rho(\overline{x}) \text{ quantifier-free}\}$ satisfy a same collection of simple existential formulas. That is, given $\overline{a}, \overline{b} \in M$, if $\mathrm{tp}_{\mathrm{qf}}(\overline{a}) = \mathrm{tp}_{\mathrm{qf}}(\overline{b})$, then $\mathrm{tp}_{\mathrm{sp}}(\overline{a}) = \mathrm{tp}_{\mathrm{sp}}(\overline{b})$, with $\mathrm{tp}_{\mathrm{sp}}(\overline{a}) = \{\exists y \rho(\overline{x}, y) \mid \mathfrak{M} \models \exists y \rho(\overline{a}, y),\ \rho(\overline{x}, y) \in \mathrm{tp}_{\mathrm{qf}}(\overline{a})\}$. The equality between quantifier-free n-types allows us to construct some well-defined isomorphism $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$, $h(\overline{a}) = \overline{b}$, which is elementary due to (2). Hence, $\mathrm{tp}_{\mathrm{sp}}(\overline{a}) = \mathrm{tp}_{\mathrm{sp}}(\overline{b})$.

Now, to show the implication, we need to prove that every simple existential formula[9] $\varphi(x_1, \ldots, x_n) = \exists y \rho(x_1, \ldots, x_n, y)$ is equivalent to a quantifier-free formula

[9] A supplementary result justifies that every existential formula can be expressed equivalently as a simple existential one.

modulo T. Consider the set $\mathrm{TP}(\varphi)$ of quantifier-free n-types $\{\mathrm{tp}_{\mathrm{qf}}(\overline{a^i}) \mid i \leq m\}$ of the n-tuples $\overline{a^i}$ which satisfy $\varphi(\overline{x})$ in $\mathfrak{M}$: since L is finite relational, $\mathrm{TP}(\varphi)$ is finite and we can denote the conjunction of the formulas in each of the $\mathrm{tp}_{\mathrm{qf}}(\overline{a^i})$ by $\rho_i(\overline{x})$. We need to prove that

$$T \vdash \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \bigvee_{i \leq m} \rho_i(\overline{x})).$$

For the right implication, any $\overline{a}$ satisfying φ in $\mathfrak{M}$ will necessarily satisfy $\mathrm{tp}_{\mathrm{qf}}(\overline{a^i})$ for some $i \leq m$, due to our construction, and thus $\mathfrak{M} \models \rho_i(\overline{a})$. Conversely, every $\overline{a}$ which satisfies $\bigvee_{i \leq m} \rho_i(\overline{x})$ in $\mathfrak{M}$ will also satisfy, in particular, some $\rho_i(\overline{x})$ and will realize $\mathrm{tp}_{\mathrm{qf}}(\overline{a^i})$. Then, by the preliminary property, $\overline{a}$ will satisfy the same simple existential formulas as $\overline{a^i}$ ($\mathrm{tp}_{\mathrm{sp}}(\overline{a}) = \mathrm{tp}_{\mathrm{sp}}(\overline{a^i})$), including $\mathfrak{M} \models \varphi(\overline{a})$.

(1) $\Rightarrow$ (3): Lemma 2.23 implies T is ω-categorical. As it is also complete, all types over finite subsets of A are realized in its countable model $\mathfrak{M}$, thus $\mathfrak{M}$ is ω-saturated: the proof of 2.19 additionally ensures they are ω-homogeneous. By the equivalence of (1) and (2), any isomorphism between finite substructures is elementary, so it can be extended to any element due to ω-homogeneity.

(3) $\Rightarrow$ (2): Again, we can use an inductive argument to show the elementarity of finite isomorphisms: given the class $I : \mathfrak{M} \cong_p \mathfrak{M}$ of isomorphisms between finite substructures of $\mathfrak{M}$, we show any $f \in I$ is elementary. The empty map is elementary, because we are considering satisfiability with regards to a same structure $\mathfrak{M}$. Isomorphisms preserve the validity of quantifier-free formulas, so –for any f– the only case left to check is $\varphi(x_1, \ldots, x_n) = \exists y \rho(a_1, \ldots, a_n, y)$, with $a_1, \ldots, a_n \in \mathrm{dom}(f)$. (3) allows us to find a value b which extends the isomorphism along with the corresponding $y = a$ (i.e., $f(a) = b$), so we can establish:

$$\begin{aligned} \mathfrak{M} \models \exists y \rho(a_1, \ldots, a_n, y) \Leftrightarrow \exists a \in M : \mathfrak{M} \models \rho(a_1, \ldots, a_n, a) \Leftrightarrow \\ \Leftrightarrow \exists b \in N : \mathfrak{N} \models \rho(f(a_1), \ldots, f(a_n), b) \Leftrightarrow \mathfrak{N} \models \exists y \rho(f(a_1), \ldots, f(a_n), y) \end{aligned}$$

The second implication holds since $\rho(x_1, \ldots, x_n, y)$ is a quantifier-free formula. □

Corollary 2.25. *Let L be a finite relational language and $\mathcal{K}$ a class of finite L-structures. If the Fraïssé limit of $\mathcal{K}$ exists, its theory is ω-categorical and has quantifier elimination.*

Proof. The (countably infinite) Fraïssé limit of $\mathcal{K}$, $\mathfrak{M}$, is unique up to isomorphism thanks to theorem 2.18: therefore, its theory is ω-categorical. $\mathfrak{M}$ is also algebraically ω-homogeneous due to lemma 2.14, so –in particular– the domain of any isomorphism between finite substructures of $\mathfrak{M}$ can be extended to any further element. Thus, by theorem 2.24, the theory of $\mathfrak{M}$ has quantifier elimination. □

Chapter 3

Basic examples

The latter half of this work is devoted to the study of widespread instances of classes which the Fraïssé theorem can be applied to, specifically from [Eva94] and [KT17]. For this purpose, most of the times we will need to introduce necessary lemmata before we can resort to the previous results. Afterwards, we will be able to introduce the corresponding languages and theories which represent the aforementioned classes of structures, which will comply with **HP**, **JEP** and **AP**.

In most of the cases, we will provide and justify a concrete definition for each of the Fraïssé limits with the help of bibliographic references from other areas of mathematics. However, our methods to demonstrate the existence of this kind of structures do not produce an explicit axiomatization.

This chapter is mostly centered on finite relational languages. As we saw in section 2.3, they induce useful properties to structures and theories based on them. We will start with a brief, trivial archetype of a theory based on finite relational language, sets without structure:

Definition 3.1. *Let* $L_\emptyset = \emptyset$ *be the empty language.*[1] *The theory* InfSet *of infinite sets consists of –for every natural* $n \geq 1$*– the axioms*[2]

$$\exists x_0 \dots x_{n-1} \bigwedge_{i<j<n} \neg x_i \doteq x_j$$

This theory defines indeed an infinite (in particular, possibly countable) set, as we can select an arbitrary natural number n and eventually find n different elements. In terms of Fraïssé, this countable infinite set is relevant as the unique limit of the class of finite sets without structure:

[1] Notice that the equality symbol $\doteq$ is implicitly featured in the construction of formulas due to our definition in 1, and not as a part of any language.

[2] Also note that, usually, natural numbers designate indexes for variables or constants in our formulas, so they are not part of the language either.

Proposition 3.2. *The structure $\mathfrak{N} = (\mathbb{N})$ models the $L_\emptyset$-theory* InfSet *and is the unique (up to isomorphism) Fraïssé limit of the class $\mathbb{S}$ of finite sets without structure.*

Proof. $\mathbb{S}$ is closed under isomorphism and its elements are finite structures. An isomorphism between any two of them can be established whenever they have the same cardinality, so the quotient $|\mathbb{S}/\cong|$ is countable. For the **HP**, the age of any set is a collection of finite sets, which is clearly a subset of $\mathbb{S}$. To prove the **JEP**, it suffices to consider the set-theoretical union of the considered structures and their embedding through inclusion. The **AP** is shown by considering any pair of embeddings defined to be the inverse of our existing embeddings (over the respective images of the initial set).

The set $\mathbb{N}$ has countably infinitely many different elements and is therefore a model of InfSet. To see that it is the Fraïssé limit of $\mathbb{S}$, we show that it is rich with respect to $\mathbb{S}$: the $L_\emptyset$-structures which are embeddable into $\mathfrak{N}$ are exactly the finite sets. And, secondly, any embedding $f_1 : \mathfrak{A} \overset{\subseteq}{\to} \mathfrak{B}$ between structures in $\mathbb{S}$ induces some $g : \mathfrak{B} \overset{\subseteq}{\to} \mathfrak{N}$ such that $g \circ f_1 = \mathrm{Id}_A$: it can be defined defined as the trivial extension of a set-theoretic inverse of f_1 over the image of f_1, i.e., $g \restriction f_1(A) = f_1^{-1}$, $g \restriction (B \setminus f_1(A)) = \mathrm{Id}_{B \setminus f_1(A)}$. □

Having proven this proposition, we can state that InfSet is ω-categorical and has quantifier elimination.

3.1 Finite graphs

In order to show the existence of a Fraïssé limit, we need to prove that the class we are addressing is countable up to isomorphism, as well as the **HP**, **JEP** and **AP**. In the next pages, we will present a series of intermediate results (preceding those in [Hod93]) which will guarantee these properties or will serve as a significant shortcut. Recall that, without constants or function symbols in L, finitely generated L-structures are in particular finite.

Proposition 3.3. *Let L be a finite relational language, consisting of k_i-ary relation symbols $\{R_i\}_{i \leq m}$, and $\mathcal{K}$ a class of finite L-structures. Then, $|\mathcal{K}/\cong| \leq \omega$.*

Proof. We shall prove that, for every $n < \omega$, there is a finite amount of isomorphism types in the subclass $\mathcal{K}_n \subseteq \mathcal{K}$ of L-structures of cardinality n. This will allow us to introduce an enumeration for $\mathcal{K} = \bigcup_{n<\omega} \mathcal{K}_n$.

We may assume the universe of our structures is $\{0, \dots, n-1\} = \mathbb{N}$: essentially, the only conditions for a map h to be an isomorphism between $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}_n$ are it being a bijection and satisfying, for any $i \leq m$, for any $a_1, \dots, a_{k_i} \in A^{k_i}$,

$$R_i^{\mathfrak{A}}(a_1, \dots, a_{k_i}) \Leftrightarrow R_i^{\mathfrak{B}}(h(a_1), \dots, h(a_{k_i}))$$

So the isomorphism types will be exclusively determined by the elements which are related in the different relations. For any $i \leq m$, any of the n^{k_i} k_i-tuples of elements can be related by R_i, which adds up to $2^{n^{k_i}}$ possible combinations for each R_i. Overall, taking into account the combinations for the m relations R_i, we still obtain a finite number $\prod_{i\leq m} 2^{n^{k_i}}$ of possibilities. □

Given the same conditions, we can also infer **HP** from an argument based on the finitude of substructures and their behavior with respect to embeddings. Notice that the classes which are object of our study are defined by the finite models which satisfy a set of axioms. Then,

Proposition 3.4. *Let L be a finite relational language and $\mathcal{K}$ a class of finite L-structures. Then, $\mathcal{K}$ has the Hereditary Property, i.e., for any $\mathfrak{A}$ in $\mathcal{K}$, $\mathrm{Age}(\mathfrak{A})$ is a subset of $\mathcal{K}$.*

In general, it is more effective to provide case-by-case arguments when proving the **JEP**. Nevertheless, most of them are based on creating a separated copy of one of the structures and embedding through inclusion: the particularities of each example rely on whether the resultant structure needs to be "connected" any further by the relation. There is not a general method to prove the **AP** either, but we may assume the initial elements are arranged in a particular, simplified manner. Let us first prove a required lemma, which was excerpted from a communication based in [Cas09] with the author:

Lemma 3.5. *Let L be a language, $\mathfrak{A}$ and $\mathfrak{B}$ L-structures and $f : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}$. For every set X, there exists an L-structure $\mathfrak{C} \supseteq \mathfrak{A}$ and an isomorphism $g : \mathfrak{C} \xrightarrow{\sim} \mathfrak{B}$ such that $(C \setminus A) \cap X = \emptyset$ and $g \restriction A = f$.*

Proof. We choose some set D such that $D \cap X = \emptyset$ and $|D| = B \setminus f(A)$. This is sufficient to build a set-theoretical bijection $h : D \to B \setminus f(A)$. We can now define C as $D \cup A$ and $g = f \cup h$, which is again a bijection whose inverse naturally defines an L-structure $\mathfrak{C}$ on the universe C. Then, $g : \mathfrak{C} \xrightarrow{\sim} \mathfrak{B}$ is a proper isomorphism between the structures. □

The lemma essentially ensures that we can extend any embedding between structures to an isomorphism in a way that the universe of the new structure can avoid certain sets. We will apply the result to prove:

Theorem 3.6 (Free amalgamation)**.** *Let L be a language and $\mathcal{K}$ a class of L-structures closed under isomorphism. Suppose that, for any L-structures $\mathfrak{M}, \mathfrak{N}_0 \supseteq \mathfrak{M}, \mathfrak{N}_1 \supseteq \mathfrak{M}$ in $\mathcal{K}$ with $N_0 \cap N_1 = M$, there exist some $\mathfrak{D} \in \mathcal{K}$ and embeddings $g'_0 : \mathfrak{N}_0 \xrightarrow{\subseteq} \mathfrak{D}$ and $g'_1 : \mathfrak{N}_1 \xrightarrow{\subseteq} \mathfrak{D}$ such that $g'_0 \restriction M = g'_1 \restriction M$. Then, $\mathcal{K}$ has the Amalgamation Property.*

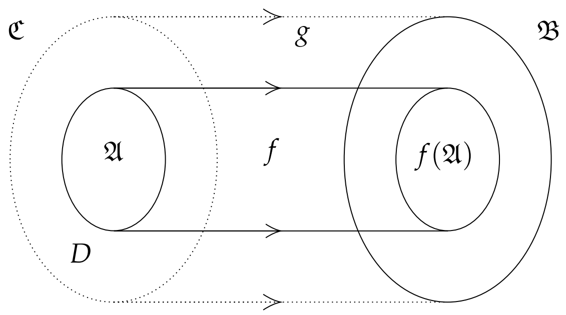

Figure 3.1: The embedding f in lemma 3.5 naturally extends to $\mathfrak{C}$.

Proof. Let us consider the initial setting to prove the **AP**: let $\mathfrak{A} \neq \emptyset, \mathfrak{B}_0, \mathfrak{B}_1$ be in $\mathcal{K}$ and $f_0 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}_0$, $f_1 : \mathfrak{A} \xrightarrow{\subseteq} \mathfrak{B}_1$. We want to find some $\mathfrak{D} \in \mathcal{K}$ and embeddings $g_i : \mathfrak{B}_i \xrightarrow{\subseteq} \mathfrak{D}$ ($i \in \{0,1\}$) such that $g_0 \circ f_0 = g_1 \circ f_1$. Using lemma 3.5, we can find:

- An L-structure $\mathfrak{N}_0 \supseteq \mathfrak{A}$ in $\mathcal{K}$ and an isomorphism $f_0' : \mathfrak{N}_0 \xrightarrow{\sim} \mathfrak{B}_0$ which extends f_0 ($f_0 \subseteq f_0'$).
- An L-structure $\mathfrak{N}_1 \supseteq \mathfrak{A}$ in $\mathcal{K}$, with $(N_1 \setminus A) \cap N_0 = \emptyset$, and an isomorphism $f_1' : \mathfrak{N}_1 \xrightarrow{\sim} \mathfrak{B}_1$ which extends f_1 ($f_1 \subseteq f_1'$).

Notice that $N_0 \cap N_1 = A$. This places us in the situation of the theorem hypothesis: $\mathfrak{A}$ is a substructure of both $\mathfrak{N}_0$ and $\mathfrak{N}_1$, and its universe is exactly the intersection of the M_i. Thus, there are some $\mathfrak{D} \in \mathcal{K}$ and embeddings $g_0' : \mathfrak{N}_0 \xrightarrow{\subseteq} \mathfrak{D}$ and $g_1' : \mathfrak{N}_1 \xrightarrow{\subseteq} \mathfrak{D}$ such that $g_0' \restriction A = g_1' \restriction A$. If we take $\mathfrak{D}$ as our overarching structure and, for $i \in \{0,1\}$, set $g_i : \mathfrak{B}_i \xrightarrow{\subseteq} \mathfrak{D}$ with $g_i = g_i' \circ (f_i')^{-1}$, it is easy to see that, for every $a \in A$,

$$(g_0 \circ f_0)(a) = (g_0' \circ (f_0')^{-1} \circ f_0)(a) = g_0'(a) = \\ = g_1'(a) = (g_1' \circ (f_1')^{-1} \circ f_1)(a) = (g_1 \circ f_1)(a)$$

□

Additionally, the following instances of $\mathfrak{D}, g_0', g_1'$ might suffice sometimes: $D = N_0 \cup N_1$, $Z^{\mathfrak{D}} = Z^{\mathfrak{N}_0} \cup Z^{\mathfrak{N}_1}$ (for any relation or function symbol $Z \in L$), $c^{\mathfrak{D}} = c^{\mathfrak{N}_0} = c^{\mathfrak{N}_1}$ (for any constant $c \in L$), $g_0' = \mathrm{Id}_{N_0}$, $g_1' = \mathrm{Id}_{N_1}$. We then call $\mathfrak{D}$ the *free amalgam* of $\mathfrak{M}, \mathfrak{N}_0, \mathfrak{N}_1$.

Having displayed these supplementary results, we can introduce the first example of structures build upon finite relational languages. Let us start with the class of finite (undirected, simple) graphs.

Any graph G is defined as a pair (V, E), where V is a set of elements called *vertices* and E is a set of pairs of vertices (known as *edges*). The notion of graph

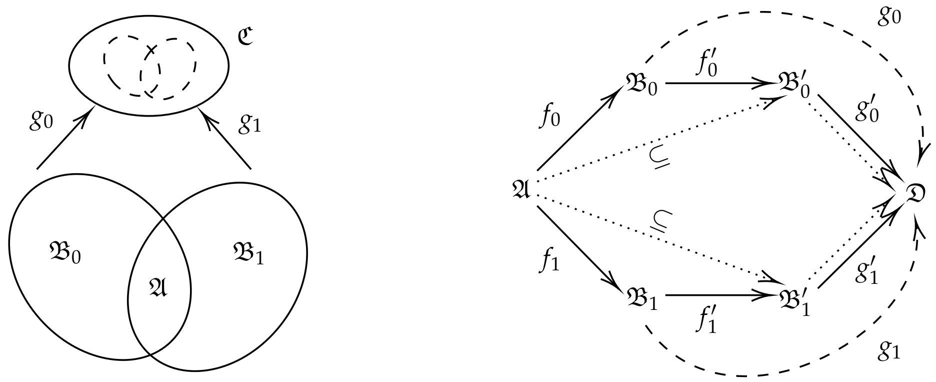

Figure 3.2: Initial setting of free amalgamation (left) and diagram for the **AP** maps in theorem 3.6

easily translates into model theory, as a structure with a universe of vertices which interprets a binary relation symbol as the set of edges. In order to completely axiomatize our selected graphs, we need to impose that the relation is irreflexive and symmetric: this way, there will be no loops (edges starting and ending at the same vertex) and edges will not regard any orientation, so the graph will be *simple*[3] and *undirected*.

Definition 3.7. *Let E be a binary predicate and $L_{\mathsf{Graph}} = \{E\}$ the language of graphs.* Graph, *the theory of graphs,*[4] *consists of the following axioms:*

- $\forall x(\neg xEx)$ *(irreflexivity)*
- $\forall x \forall y(xEy \to yEx)$ *(symmetry)*

Given a finite set V^G, a structure $G = (V^G, E^G)$ is called a finite graph *if $G \models$* Graph.

For the remainder of the section, let us assume $L := L_{\mathsf{Graph}}$.[5] We can now consider the class $\mathcal{K}$ of all finite graphs. $\mathcal{K}$ is closed under isomorphism, since any additional graph which is isomorphic to some that we already considered will be also finite. Propositions 3.3 and 3.4 guarantee, respectively, that $\mathcal{K}$ has a countable

[3]Actually, simpleness requires that no more than one edge exists between two vertices, but that is redundant due to the way we have defined the relation (it would need to be a multiset or some other kind of specific collection).

[4]Unless stated otherwise, graphs are considered to be simple and undirected.

[5]We shall proceed this way in every other example.

amount of isomorphism types and satisfies the Hereditary Property. To prove it has a Fraïssé limit, we will prove it also satisfies the **JEP** and the **AP**.

Proposition 3.8. *The class $\mathcal{K}$ of all finite graphs satisfies the Joint Embedding Property.*

Proof. Let $G_0 = (V_0, R_0), G_1 = (V_1, R_1) \in \mathcal{K}$.[6] We need to find an L-structure $H \in \mathcal{K}$, a finite graph, into which G_0 and G_1 can be embedded. Consider an isomorphic copy $G_1' \cong^h G_1$ over an arbitrary set of vertices $V_1' \cong^h V_1$ such that $V_0 \cap V_1' = \emptyset$, hence E_1 is preserved in E_1' with regard to V_1'. The finite graph $H := (V_0 \cup V_1', E_0 \cup E_1')$ works as a joint embedding, as the maps $g_0 : G_0 \to D$ ($g_0 = \mathrm{Id}_{V_0}$) and $g_1 : G_1 \to D$ ($g_1 = \mathrm{Id}_{V_1'} \circ h^{-1}$) both are injective and preserve the edge structure, i.e., they are embeddings. □

Although it may look as an insignificant nuance, we generate a copy of one of the structures in order to avoid contradictory configurations of edges: if we consider two graphs G_0, G_1 based on the same two vertices, G_0 connected and G_1 disconnected, we can no longer use the inclusion to find some embeddings in a trivial way. Note that the core of the issue resides in the naming of the vertices in the final graph: there is no binding property which forces G_0 and G_1 to preserve the structure of common graphs, since they are separate structures despite being built from the same set.

In general, this duplication method fails if we try to apply some variation to the process of proving the **AP**, since now there is a common substructure $\mathfrak{A}$ which ties the interpretations over the images $f_0(\mathfrak{A}) \subseteq \mathfrak{B}_0$ and $f_1(\mathfrak{A}) \subseteq \mathfrak{B}_1$. However, the free amalgam of any graph $H = G_0 \cap G_1$ is again a finite graph which includes G_0 and G_1, without the need of establishing further connections. Therefore, as seen in theorem 3.6, we have established the following:

Proposition 3.9. *The theory $\mathcal{K}$ of all finite graphs satisfies the Amalgamation Property and has a unique Fraïssé limit.*

Definition 3.10. *The* random graph *is the countable L-structure $G = (V, E)$, unique up to isomorphism, which satisfies the L-theory* RG, *made up of:*

- *The axioms of* Graph.
- *For every $m, n < \omega$, the axiom $\forall x_0 \ldots \forall x_{m-1} \forall y_0 \ldots \forall y_{n-1}$*

$$\bigwedge_{i,j} \neg x_i \doteq y_j \to \exists z \left(\left(\bigwedge_{i<m} zEx_i \right) \wedge \left(\bigwedge_{i<n} \neg zEy_j \wedge \neg z \doteq y_j \right) \right)$$

[6] From now on, for any example (whether it is graph-related), we may use notations like $V_i := V^{G_i}$, $E_i := E^{G_i}$.

That is, the random graph is countably infinite and, for every disjoint pair of finite sets $V_1, V_2 \subseteq V$, it contains a vertex which is connected to every element of V_1 by an edge, but not to any vertex in V_2. Also known as Rado's graph, it receives its name due to the fact that it can be constructed (up to isomorphism) by choosing with a fixed probability $p \in (0,1)$ whether to connect each pair of vertices. Both axiomatizations and their relationship are detailed in [Cam13], which we used as a reference for the class of finite graphs. To see that its Fraïssé limit is the random graph, we need to check it is rich with respect to the class:

Proposition 3.11. *The random graph $G = (V, E) \models \mathsf{RG}$ is the Fraïssé limit of the class $\mathcal{K}$ of finite graphs.*

Proof. In the first place, $\mathrm{Age}(G) = \mathcal{K}$ must hold: any $G_0 \in \mathrm{Age}(G)$ is embedded into G by some map $f : G_0 \xrightarrow{\subseteq} G$, so it is a finite structure which inherits the interpretation E of the relation. Thus, E^{G_0} is irreflexive and symmetric, and G_0 a finite graph. Conversely, let us show any $H = (V^H, E^H) \in \mathcal{K}$ is isomorphic to some finite substructure of G: given an enumeration $\{v_i \in V^H \mid 1 \leq i \leq |H|\}$ of the elements of H, we will prove inductively that, for every $r \leq |H|$, there exists an isomorphism h_r between $H_r = (V^{H_r} = \{v_i \in V^H \mid i \leq r\}, E^H \restriction V^{H_r} \times V^{H_r})$ and some finite subgraph $G_r \subseteq G$, and define $h : H = H_{|H|} \xrightarrow{\sim} G_{|H|} \subseteq G$.

Start with $\{v_1\}$: the structures generated by $v_1 \in H$ and any $v'_1 \in G$ are trivially isomorphic. Now suppose H_r, G_r and $h_r : H_r \xrightarrow{\sim} G_r$ are known for some $r < |H|$, and consider the partition $V^{H_r} = V_1 \cup V_2$ such that $\{(v, v_{r+1}) \mid v \in V_1\} \subseteq E^H$ and $\{(v, v_{r+1}) \mid v \in V_2\} \cap E^H = \emptyset$. Choosing $m = |V_1|, n = |V_2|$, the corresponding axiom in RG provides the existence of a vertex $v'_{r+1} \in V^G \setminus V^{\{G_r\}}$, in a way that any pair $(v'_{r+1}, h_r(v))$ with $v \in V_1$ represents an edge of G and any pair $(v'_{r+1}, h_r(v))$ with $v \in V_2$ does not. This extends h_r to some $h_{r+1} : H_{r+1} \xrightarrow{\sim} G_{r+1}$, so the union $h_{|H|} = \bigcup_{r \leq |H|} h_r$ is an isomorphism between H and some finite $G_{|H|} \subseteq G$.

To prove that G is rich with respect to $\mathcal{K}$, consider any finite graphs H_0, H_1 and the embeddings $f_0 : H_0 \xrightarrow{\subseteq} G$, $f_1 : H_0 \xrightarrow{\subseteq} H_1$. Restricting f_0 to $f'_0 : H_0 \xrightarrow{\sim} f_0(H_0)$, we can apply the **AP** to obtain a graph $H \in \mathcal{K}$ and two embeddings $g_0 : f_0(H_0) \xrightarrow{\subseteq} H$, $g_1 : H_1 \xrightarrow{\subseteq} H$ such that $g_0 \circ f'_0 = g_1 \circ f_1$. We can apply lemma 3.5 to extend g_0 to some $g'_0 : H'_0 \xrightarrow{\sim} H$, where H'_0 is a finite graph disjoint from H_0, $f_0(H_0)$ and H_1. If we define $g'_1 : H_1 \xrightarrow{\subseteq} G$ as $g'_1 = (g'_0)^{-1} \circ g_1$, for every $v \in V^{H_0}$, we see that $f_0(v) =$

$$= ((g'_0)^{-1} \circ g'_0 \circ f_0)(v) = ((g'_0)^{-1} \circ g_0 \circ f'_0)(v) = ((g'_0)^{-1} \circ g_1 \circ f_1)(v) = (g'_1 \circ f_1)(v)$$

□

Conceiving $G \models \mathsf{RG}$ as a Fraïssé makes it stand out as one of the few countable graphs (the only one, up to isomorphism, if we consider the richness property)

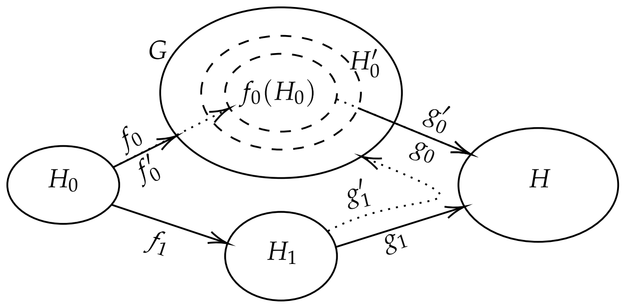

Figure 3.3: Diagram for the proof that $G \models \mathsf{RG}$ is rich with respect to $\mathcal{K}$

which contains an isomorphic copy of every finite graph. In the next chapter, we will see some more specific variations of RG, as well as other approaches to graphs from model theory.

3.2 Finite orders

The process for defining and addressing other structures built around finite relational languages (in particular, consisting of a single element) does not distance itself much from the previous example. Finite orders are no exception, as they can be characterized by transitive, irreflexive relations: from these, one can deduce their asymmetry. For instance, we will show that the Fraïssé limit of the class of finite lineal orders is the (countable) dense, linear order without endpoints, which we know to be ω-categorical thanks to Cantor's back-and-forth argument. Let us first introduce some notation:

Definition 3.12. *Let E be a binary predicate and $L_{\mathsf{Order}} = \{<\}$ the language of orders.*[7] TOSet*, the theory of linear (or total) orders, consists of the following axioms:*

- $\forall x(\neg x < x)$ *(irreflexivity)*
- $\forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)$ *(transitivity)*
- $\forall x \forall y(\neg x \doteq y \rightarrow x < y \vee y < x)$ *(linearity/totality)*

Given a finite set X, a structure $P = (X, <)$ is called a finite linear order *if $P \models \mathsf{TOSet}$.*

[7] As usual, we shall employ the notation $x \leq y$ whenever $x < y$ or $x \doteq y$, as well as $x > y$ if $y < x$.

Proposition 3.13. *The class $\mathcal{K}$ of finite linear orders has the Joint Embedding Property.*

Proof. As in proposition 3.8, for any two structures $P_0 = (X_0, <_0), P_1 = (X_1, <_1) \in \mathcal{K}$, we can create a copy P_1' of P_1 with $X_1' \cap X_0 = \emptyset$ and embed P_0 and P_1' into a structure of universe $X_0 \cup X_1'$ through inclusion. However, we need to properly define a relation $<$ in the union so that it extends $<_0, <_1'$, and check that $(X_0 \cup X_1', <)$ is a model of TOSet.

Set $< := <_0 \cup <_1' \cup (X_0 \times X_1')$: that is, for every $x, y \in X_0 \cup X_1'$, $x < y$ exactly if $x <_0 y$, $x <_1' y$ or $(x, y) \in X_0 \times X_1'$. Now for the axioms, irreflexivity holds because no element is related to itself in P_0 or P_1'. Every element of P_0 is smaller than any in P_1', hence the linearity. To prove that $<$ is transitive, note that if $x, z \in X_0$ and $x < z$, any $y \in X_0 \cup X_1'$ such that $x < y < z$ will be $y \in X_0$, so the transitivity of $<_0$ applies (and similarly for any $x, z \in X_1'$): then, we only need to check the cases $x < y < z$ with $x \in X_0$ and $z \in X_1'$. Regardless of whether $y > x, y < z$ belongs to X_0 or X_1', our definition of $<$ ensures $x < z$. □

The method of generating disjoint copies is not as convenient when proving the **AP**. Instead of simply defining arbitrarily $P_0 \times P_1 \subseteq <$, we will use the elements of the intersection to mediate between the structures, as hinted at in [Eva94].

Lemma 3.14. *The class $\mathcal{K}$ of finite linear orders satisfies the Amalgamation Property.*

Proof. We start with three orders $Q, P_0, P_1 \in \mathcal{K}$ such that the intersection $Y := X_0 \cap X_1$ is the universe of $Q \subseteq P_0, P_1$, i.e., it is a common substructure with $<_0 \restriction Y = <_1 \restriction Y$. If we construct a structure in $X_0 \cup X_1$ such that its relation $<$ extends the relations $<_0, <_1$, theorem 3.6 will guarantee the **AP** holds (setting the required g_0 and g_1 as the identities).

We define $<$: in the first place, let $<_0, <_1 \subseteq <$. It only remains to compare any $x \in X_0 \setminus Y$ with $x' \in X_1 \setminus Y$. We first give an enumeration of $Y = \{y_0, \ldots, y_n\}$ such that $y_i < y_j$ whenever $i < j$.[8] It is enough to order the elements of $X_0 \cup X_1 \setminus Y$ within each of the intervals $\{x < y_0\}$, $\{x \mid y_0 < x < y_1\}, \ldots$, $\{x > y_n\}$ they belong to: for instance, take $x < x'$ for each $x \in X_0$, $x' \in X_1$, in any fixed region. So, for $<$ to be linear, we set $z < z'$ for every $z, z' \in X_0 \cup X_1$ whenever there exists some $i \leq n$ such that $z < y_i < z'$.

This respects the properties of $<_0$ and $<_1$ in $<$. Irreflexivity of $<$ is inferred directly from that of $<_0$ and $<_1$. To prove that $<$ is transitive, consider any $x, y, z \in X_0 \cup X_1$ such that $x < y < z$. Due to the construction of $<$, x and z can be distributed either within a same interval ($x, z < y_0$, $x, z > y_n$, or $y_i < x, z < y_{i+1}$ for some $i \leq n-1$) or $x < y_i < z$ for some $i \leq n$. The latter is already defined

[8] We allow the abuse of the notation $\leq$ for natural numbers with their respective order.

to imply $x < z$. For the former case, we have several possibilities: if $x, z \in X_j$, for $j \in \{0,1\}$, y must also belong to X_j, so the result follows from the transitivity of $<_j$; if $x \in X_0$ and $z \in X_1$, we previously defined $x < z$ for any of the cases. Finally, note that $x \in X_1$ with $z \in X_0$ would imply $z < x$, contradicting our construction. □

We can now introduce a candidate for the Fraïssé limit of $\mathcal{K}$. We should bear in mind that theorem 2.18 provides criteria for ω-categoricity, but the discussion of upper cardinalities is far beyond the reach of this work.

Definition 3.15. *We call an L-structure* $P = (X, <)$ *a* dense linear order without endpoints *if it satisfies the L-theory* DLO*, which consists of:*

- *The axioms of* TOSet*.*
- $\forall x \forall y \exists z (x < y \rightarrow x < z < y)$ *(X is dense regarding* $<$*)*
- $\forall z \exists x \exists y (x < z < y)$ *(without endpoints)*

This, in particular, entails any order satisfying DLO to be infinite. To stay in the countable case, we shall consider the ordered rational numbers for the next result.

Proposition 3.16. *The order of the rational numbers,* $Q = (\mathbb{Q}, <) \models \mathsf{DLO}$*, is the Fraïssé limit of the class* $\mathcal{K}$ *of finite linear orders.*

Proof. We prove that Q is rich with respect to $\mathcal{K}$ using an argument analogous to that of proposition 3.11, based on [Ber15]: we start with orders $P_0, P_1 \in \mathcal{K}$ and embeddings $f_0 : P_0 \overset{\subseteq}{\to} f_0(P_0) \subseteq Q$, $f_1 : P_0 \overset{\subseteq}{\to} P_1$, and find –due to the **AP**– $P_2 \in \mathcal{K}$ and embeddings $g_0 : f_0(P_0) \overset{\subseteq}{\to} P_2$, $g_1 : P_1 \overset{\subseteq}{\to} P_2$, such that $g_0 \circ f_0 = g_1 \circ f_1$. Then, $g_1' := g_0^{-1} \circ g_1 : P_1 \overset{\subseteq}{\to} f_0(P_0)$ satisfies, for every element $x \in P_0$, $f_0(x) = (g_1' \circ f_1)(x)$.

It remains to be shown that $\mathrm{Age}(Q) = \mathcal{K}$: any element of $\mathrm{Age}(Q)$ is a finite substructure of Q, so it is a linear order as well. On the other hand, any order $P \in \mathcal{K}$ must be isomorphic to some finite substructure of Q: we denote $P = \{x_i\}_{i<|P|}$, select an arbitrary $y_0 \in \mathbb{Q}$, and construct $P' \subseteq Q$ by extending $h_0 : \{x_0\} \overset{\sim}{\to} \{y_0\} \subseteq P'$ to an isomorphism $h : P \overset{\sim}{\to} P'$.

For any $i < |P|$, having established $h_i : P_i \overset{\sim}{\to} P_i'$, we define $P_{i+1} = P_i \cup \{x_{i+1}\}$ and $P_{i+1}' = P_i' \cup \{y_{i+1}\}$ in the following way: due to the linearity of P, we have $x_{i+1} < x_j$ for every $j \leq i$, $x_{i+1} > x_j$ for every $j \leq i$, or $x_j < x_{i+1} < x_{j'}$ for some $j, j' \leq i$. Then, since Q is linear, dense and without endpoints, we can accordingly find some $y_{i+1} \in \mathbb{Q}$ such that $y_{i+1} < y_j$ for every $j \leq i$, $y_{i+1} > y_j$ for every $j \leq i$, or $y_j < y_{i+1} < y_{j'}$ for some $j, j' \leq i$. Then $h := \bigcup_{i<|P|} h_i$ is an isomorphism between $P = P_{|P|-1}$ and $P' = P_{|P|-1}'$. □

Once having explored the class of finite linear orders, we can infer the existence of a Fraïssé limit for the case where totality is not required. Let us call POSet the theory of finite *partial* (non-total) *orders*, and $\mathcal{K}$ their class. The **JEP** is proved following the reasoning in 3.8, as the lack of linearity eases some of the restrictions needed for TOSet, such as having to relate the elements of the new structure. However, the Amalgamation Property is slightly more intricate and requires a previous result.

Remark 3.17. For every binary relation R in A, there exists some transitive $R' \subseteq A \times A$ such that $R \subseteq R'$ and, for every transitive $S \subseteq A \times A$ with $R \subseteq S$, $R' \subseteq S$.

Proof. Given two binary relations S, T in A, define the relation $S \mid T$ as

$$\{(x,z) \mid \exists y\,((x,y) \in S \wedge (y,x) \in T)\}.$$

If we denote recursively $R_0 := R$ and, for every $n < \omega$, $R_{n+1} := R_n \cup (R_n \mid R_n)$, it is easy to see that a transitive $R' \supseteq R$ can be defined as $R' = \bigcup_{n<\omega} R_n$. To show that it is minimal, given a relation $S \supseteq R$, we check by induction that every $R_i \subseteq S$: by definition, $R_0 \subseteq S$; and, if $R_i \subseteq S$ holds, any $(x,z) \in R_{i+1} \setminus R_i$ must be generated by some $(x,y), (y,z) \in R_i$, so it is also contained in S. ◻

Definition 3.18. *Let A be a set and $R \subseteq A \times A$ be a binary relation. The* transitive closure *R' of R is defined as the smallest transitive relation in A such that $R \subseteq R'$.*

Lemma 3.19. *Let $Q_0, P_0 = (X_0, <_0), P_1 = (X_1, <_1)$ be three finite partial orders, with $Q_0 = (X_0 \cap X_1, <_0 \cap <_1)$. The relation $<$, defined as the transitive closure of $<_0 \cup <_1$, is compatible with P_0, P_1 in the sense that, for every $x, z \in X_i$, $i \in \{0,1\}$, $x < z$ if and only if $x <_i z$.*

Proof. The leftward direction is trivial. Let us prove the converse by contradiction. Assume –without loss of generality– that $x < z$, but $x \not<_0 z$, for some $x, z \in X_0$. Then, there must exist $\{x_i \in X_0 \cup X_1 \mid i \leq n\}$ such that $x_0 = x$, $x_n = z$ and $x_i < x_{i+1}$ for every $i < n$: this way, the transitivity of $<$ will imply $x < z$, but that of $<_0$ will not, at first – we will search for a subsequence of the x_i in X_0 so that the transitivity of $<_0$ implies $x <_0 z$.

The fact that some given $a \in X_0 \setminus X_1$, $c \in X_1 \setminus X_0$ satisfy $a < c$ implies that there exists some $b \in X_0 \cup X_1$ such that $a < b < c$, because of the way the transitive closure is constructed. Eventually, we can express the relation of the closure in terms of $<_0$ and $<_1$, finding $\{a_i\}_{i<N}$ and $\{c_i\}_{i<M}$ so that

$$a <_0 a_0 <_0 \ldots <_0 a_N <_0 b <_1 c_0 <_1 \ldots <_1 c_M <_1 c.$$

Thus, we may assume that for every $x_i \in X_0 \setminus X_1$ and $x_j \in X_1 \setminus X_0$ (respectively $X_1 \setminus X_0$, $X_0 \setminus X_1$) with $i < j \leq n$, there exists some $y \in X_0 \cap X_1$ such that $x_i <_0 y$ and $y <_1 x_j$ (resp. $<_1, <_0$).

Furthermore, we can decompose $\{x_i \in X_0 \cup X_1 \mid i \leq n\}$ into several sequences $\{x_0, \ldots, x_{i_0}\} \subseteq X_0$, $\{x_{i_0}, \ldots, x_{i_1}\} \subseteq X_1, \ldots$, $\{x_{i_m}, \ldots, x_n\} \subseteq X_0$ delimited by elements x_{i_j} from $X_0 \cap X_1$. By the transitivity of $<_0$ and $<_1$, we can state $x_0 <_0 x_{i_0} <_1 x_{i+1} <_0 \ldots <_1 x_{i_m} <_0 x_n$. Equivalently, since P_0 and P_1 agree on their intersection, $x_0 <_0 x_{i_0} <_0 x_{i_1} <_0 \ldots <_0 x_{i_m} <_0 x_n$. But $<_0$ is transitive, so $x_0 <_0 x_n$ leads to a contradiction. $\square$

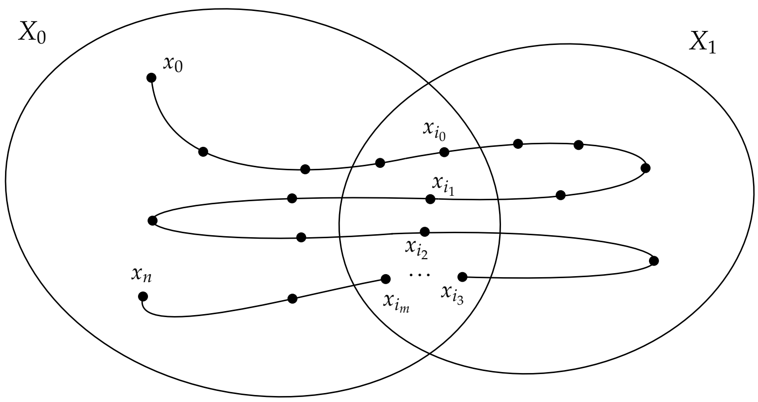

Figure 3.4: We can break down lemma 3.19's sequence of x_i into several segments.

This lemma allows us to prove the **AP** by defining a new relation in the union of the universes. Notice that this distinction was not necessary in TOSet, since we already explicitly constructed an order which addressed the difficulties which involve a common substructure. Therefore,

Proposition 3.20. *The class $\mathcal{K}$ of finite partial orders has a Fraïssé limit.*

Besides Fraïssé's construction, Albert and Burris [AB86] characterize the limit in a finite axiomatization by introducing terminology regarding model companions and existential closure, summarized below. Additionally, the authors provide a method to create these limit structures from finite orders.

Definition 3.21. *Let $P = (X, \leq)$ be a partial order. Two elements x, y are said to be* incomparable *($x \parallel y$) if $x \not\leq y$ and $y \not\leq x$. A set $A \subseteq X$ is an* antichain *if every two of its elements which are different are incomparable. We use the following notation: $\uparrow A := \{x \in X \mid \exists a \in A(a \leq x)\}$ and $\downarrow A := \{x \in X \mid \exists a \in A(x \leq a)\}$.*

Definition 3.22. *Let $P = (X, \leq)$ be a partial order and K, M, N be finite subsets of X with cardinalities k, m, n, respectively. The tuple (K, M, N) is a (k, m, n)-*configuration *if it satisfies all the conditions:*

1. *M and N are antichains.*
2. *For every $x \in M$ and every $z \in N$, $x < z$.*
3. *$K \cap \downarrow M = K \cap \uparrow N = \emptyset$.*

Theorem 3.23 (Albert, Burris)**.** *Let $P = (X, \leq)$ be a partial order and POSet^{ec} the theory which models the Fraïssé limit of the class of finite partial orders. Let ${}_k\phi^n_m$ be a first order sentence which states the following:*

For any (k, m, n)-configuration (K, M, N), there is $y \parallel K$ such that $M < y < N$.

Then, POSet^{ec} is finitely axiomatizable as $\mathsf{POSet} \cup \{\bigwedge_{k \leq 1, n \leq 2, m \leq 2}({}_k\phi^n_m)\} \cup \{{}_2\phi^1_1\}$.

Observe that, rather than being a detached structure, the order which we have just constructed translates and generalizes the notion of density (as we defined it for dense linear orders without endpoints) into the context of partial orders.

3.3 Finite vector spaces over finite fields

We now abandon finite relational languages and begin to consider function symbols, with the aid of [Hod93]. Handling some of the properties will demand more attention to detail as to the size of substructures, because they well surpass the sets which they are build upon. If the classes which concern us are uniformly locally finite, showing they have a countable amount of isomorphism types will become a more accessible task, as well as proving their theory is ω-categorical:

Definition 3.24. *Let $\mathcal{K}$ be a class of structures over a same language. We call $\mathfrak{A} \in \mathcal{K}$* locally finite *if every finitely generated substructure of $\mathfrak{A}$ is finite. Furthermore, suppose there exists a function $g : \omega \to \omega$ such that, for every structure $\mathfrak{A} \in K$, any substructure $\mathfrak{B} \subseteq \mathfrak{A}$ generated by at most n elements has $|\mathfrak{B}| \leq g(n)$. Then, $\mathcal{K}$ is said to be* uniformly locally finite*.*

Theorem 3.25. *Let L be a finite language and $\mathcal{K}$ a uniformly locally finite class of finitely generated L-structures with* **HP**, **JEP** *and* **AP***, closed under isomorphism and with a countable amount of isomorphism types. If $\mathfrak{M}$ is the Fraïssé limit of $\mathcal{K}$, $\mathrm{Th}(\mathfrak{M})$ is ω-categorical and has quantifier elimination.*

Definition 3.26. *Let $A = \{\alpha_i \mid i \leq n\}$ be a finite field. We denote the language of vector spaces over A by $L_{\mathsf{Vector}_A} = \{0, +, f_{\alpha_0}, \dots, f_{\alpha_n}\}$, where 0 is a constant, + a binary operator and, for every $i \leq n$, f_{α_i} is a unary operator. The theory $\mathsf{FVector}_A$ of vector spaces over a finite field A consists of the following axioms:*

- *Sum properties:*
  - $\forall u \forall v \forall w (u + (v + w) \doteq (u + v) + w)$ *(associativity of +)*
  - $\forall u \forall v (u + v \doteq v + u)$ *(commutativity of +)*
  - $\forall v (0 + v \doteq v + 0 \doteq v)$ *(neutral element of +)*
  - $\forall v \exists w (v + w \doteq 0)$ *(opposed element $(-u) := w$ for +)*
- *Scalar properties: given $\alpha, \beta \in A$ and $1 \in A$ the neutral element with respect to multiplication in A,*
  - $\forall v (f_{\alpha\beta}(v)) \doteq f_\alpha(f_\beta(v)))$ *(associativity with scalars)*
  - $\forall v (f_1 v \doteq v)$ *(neutral element of the product of scalars)*
  - $\forall u \forall v (f_\alpha(u + v) \doteq f_\alpha u + f_\alpha v)$ *(distributivity of vector sum)*
  - $\forall v (f_{\alpha+\beta}(v) \doteq f_\alpha(v) + f_\beta(v)$ *(distributivity of scalar sum)*

Let V be a set finitely generated by A. The structure $E = (V, (Z^E)_{Z \in L})$ is called a finitely generated vector space over A *if $E \models \mathsf{FVector}_A$.*

Note that we use the same notation for the vector sum and scalar sum, as well as for the product with and between scalars, given that they can be corresponded thanks to properties like distributivity. Here the function symbols f_α can be interpreted as the operator which assigns to any vector its product with the scalar α. We will develop the following proofs employing basic linear algebra notions from [Cla71], such as the representation of vector spaces by means of bases.

Proposition 3.27. *The class $\mathcal{K}$ of finitely generated vector spaces over a finite field $A = \{\alpha_i\}_{i \leq n}$ is uniformly locally finite and, thus, has a countable amount of isomorphism types.*

Proof. Let $F \in \mathcal{K}$ be a vector space generated by a base $\{v_1, \dots, v_m\}$. By the definition of base, any linear combination $\sum_{j \leq m} f_{\alpha_{i_j}}(v_j)$ will equal 0^E exactly when $\alpha_{i_1} = \dots = \alpha_{i_m} = 0$. Hence, any two elements of F will be different if their representation in terms of elements of the base is different. This implies that any subspace $E \subseteq F$ generated by $\{v_{i_1}, \dots, v_{i_k}\}$ has exactly $n^k = |A|^k$ elements, so we can choose the function $g(k) = n^k$.

From here, we can show that any pair of vector spaces E, E' generated by bases B, B' of cardinality k are isomorphic. There exists a map $h : E \to E'$ which assigns the i-th element of B', v'_i, to the i-th of B, v_i, and can factor out the product by a scalar, namely

$$h(f_\alpha(v_i) + f_\beta(v_j)) = f_\alpha(v'_i) + f_\beta(v'_j), \text{ for any } \alpha, \beta \in A.$$

This naturally defines a map $h(f_\alpha(u) + f_\beta(v)) = f_\alpha(h(u)) + f_\beta(h(u))$, for every $u, v \in V^E$, which is injective due to the aforementioned property of linear combinations, and is exhaustive since $|E| = |E'| = n^k$. □

We may no refer to $\mathcal{K}$ as the class of finite vector spaces over A. The Hereditary Property is also substantiated with the fact that any subspace is generated from a subset of generators. Specifically, it is constructed by taking an embedding which sends the new base to that of the original structure. Let us now prove the **JEP** and the **AP**, employing theorem 3.6 for the latter.

Proposition 3.28. *The class $\mathcal{K}$ of finite vector spaces over A satisfies the Joint Embedding Property.*

Proof. Consider E, E' vector spaces generated by the bases $B = \{v_i \mid i \leq k\}$, $B' = \{v'_i \mid i \leq k'\}$. As in the previous examples, we may presume $B \cap B' = \emptyset$, and now consider the vector space F over A generated by $E \cup E'$. F is indeed finitely generated of cardinality $n^{k+k'}$ and embeds through the inclusion E and E', mapping their generators into $B \cup B'$. The neutral element 0^F exists in F as the only common element in the intersection of E and E', and F naturally extends the interpretations of each f_α from the original spaces. The sum operator can be extended to F by letting $F = E \oplus E'$, so that the resulting structure satisfies the axioms of $\mathsf{FVector}_A$. □

Lemma 3.29. *The class $\mathcal{K}$ of finite vector spaces over a finite field A has the Amalgamation Property.*

Proof. Let $F, E_0, E_1 \models \mathsf{FVector}_A$ be finite vector spaces such that $F = E_0 \cap E_1$ is a common substructure. E_0 and E_1 can be viewed as substructures of some space generated at least by the union of their elements. By Grassmann's Theorem, there exists an vector space $E_0 + E_1$ as an L-structure of a finite dimension $\dim(E_0) + \dim(E_1) - \dim(E_0 \cap E_1)$, thus finite. Additionally, E_0 and E_1 can be embedded into the sum by keeping fixed the elements of their bases. Therefore, for every $u = \sum_i f_{\alpha_i}(v_i) \in F$,

$$(\mathrm{Id}_{E_0+E_1 \restriction E_0} \circ \mathrm{Id}_{E_0 \restriction F})(v) = \sum_i f_{\alpha_i}(v_i) = \sum_i f_{\alpha_i}(v'_i) = (\mathrm{Id}_{E_0+E_1 \restriction E_1} \circ \mathrm{Id}_{E_1 \restriction F})(v).$$

□

This particular way to glue spaces together, and the fact that structures generated by an equal number of generators are isomorphic, point towards a distinctive significance of a space with a countably infinite base. As we will observe in further examples, this kind of limit construction will become usual for other kinds of finitely generated structures.

Proposition 3.30. *Let $E = (V^E, (Z^E)_{Z\in L}) \models \mathsf{FVector}_A$ be a countably infinite vector space over a finite A. Then, E is the Fraïssé limit of the class $\mathcal{K}$ of finite vector spaces over A. Furthermore, $\mathsf{FVector}_A$ is ω-categorical and has quantifier elimination as a consequence of theorem 3.25.*

Proof. To show that $\mathrm{Age}(E) = \mathcal{K}$, we simply observe that any finitely generated substructure of E is a finite vector space over A, which belongs to $\mathcal{K}$. And every vector space over A generated by a finite number k of elements is isomorphic to a subspace $\langle v_0, \ldots, v_{k-1}\rangle^E$ of E, where $\{v_i\}_{i\leq k-1}$ any linearly independent subset of the generators of E.

On the other hand, select some vector spaces $E_0, E_1 \in \mathcal{K}$ and embeddings $f_0 : E_0 \overset{\subseteq}{\to} E$, $f_1 : E_0 \overset{\subseteq}{\to} E_1$. Again, there exists an isomorphism $h : f_1(E_0) \overset{\sim}{\to} f_0(E_0)$, $h = f_0 \circ f_1^{-1} \restriction f_1(E_0)$, which we can extend to an embedding $g_1 : E_1 \overset{\subseteq}{\to} E$ by assigning an image to the the elements of the base of E_1 which are not in $f_0(E_0)$. Then, g_1 satisfies $f_0 = g_1 \circ f_1$. □

Consequently, in general terms, the Fraïssé theorem presents an alternative construction of a vector space which can embed any other finite vector space over the same finite field, instead of having to rely on the formalization of infinite bases. Notice as well that one could also formulate some of the proofs –which refer to the extension of bases– in terms of quotient spaces: by identifying the elements of an embedded vector space, a map between the resulting classes emerges naturally. This was suggested in [Eva94], though we opted for another path to the conclusions Evans summarized.

Chapter 4

Further Fraïssé constructions

4.1 Henson graphs

In section 3.1, we presented Rado's graph as an ultrahomogeneous structure which emerges as the limit of the class of all finite graphs. As retrieved by MacPherson in [Mac11], Henson introduced countably infinitely many more ultrahomogeneous examples, characterized by the embedding of (simple, undirected) finite graphs which satisfy some additional restrictions:

Definition 4.1. *Let $n \geq 3$ natural number. The* complete graph with n vertices, K_n, *is such that every two vertices are connected by an edge. We say that a graph is* K_n-free *if none of its subgraphs is isomorphic to* K_n.

Definition 4.2. *Let E be a binary predicate and L_{Graph} the language of graphs again. The theory of K_n-free graphs, $\overline{K_n}$-Graph, is made up of the following axioms:*

- $\forall x(\neg xEx)$ *(irreflexivity)*
- $\forall x \forall y(xEy \rightarrow yEx)$ *(symmetry)*
- $\forall x_1 \ldots \forall x_n \left(\bigwedge_{i \neq j} \neg x_i \doteq x_j \rightarrow \neg \bigwedge_{i \neq j} x_i E x_j \right)$ *(K_n-free)*

Given a finite set V^G, an L-structure $G = (V^G, E^G)$ is called a finite K_n-free graph *if* $G \models \overline{K_n}$-Graph.

We are dealing with a finite relational language once more. As a subclass of the collection of finite graphs, the class $\mathcal{K}_n$ of finite K_n-free graphs has a countable amount of isomorphism types. The proof for the **HP** is analogous to that of finite graphs, noting that any embedded structure still has a finite universe and preserves the relations which define a finite K_n-free graph. The **JEP** follows a similar proof, because we can embed the graphs in the union of universes through inclusion, which does not contain any complete graph of n vertices as no additional edge was added.

Proposition 4.3. *The class $\mathcal{K}_n$ of finite K_n-free graphs has the Amalgamation Property.*

Proof. Consider $H_0, H_1 \in \mathcal{K}_n$ graphs, and $G = H_0 \cap H_1 \in \mathcal{K}_n$ a common substructure. Their free amalgam, $H := H_0 \cup H_1$ is a finite graph whose set of vertices is the union of the universes of H_0, H_1, with exactly the same edges as H_0 and H_1. To check that H is K_n-free, assume some finite substructure $S \cup S_0 \cup S_1$ is the complete graph K_n and $S \subseteq G$, $S_0 \subseteq H_0 \setminus G$, $S_1 \subseteq H_1 \setminus G$: neither of the S_1 can be empty, since otherwise $S \cup S_1 \subseteq H_1$ or $S \cup S_0 \subseteq H_0$ would be K_n. But no vertices of S_0 and S_1 are connected, so $S \cup S_0 \cup S_1$ can not be K_n. □

Thus, any $\mathcal{K}_n$ has a unique Fraïssé limit, also known as the *generic K_n-free graph $\mathcal{H}_n$* or *n-Henson graph*. Due to the equivalence between ultrahomogeneity and algebraic ω-homogeneity for countably generated structures, Henson [Hen72] also defines it as a countable K_n-free graph (unique up to isomorphism) with age $\mathcal{K}_n$, such that any isomorphism between finite substructures of $\mathcal{H}_n$ extends to an automorphism of $\mathcal{H}_n$.

In fact, it can be shown ([Mac11]) that any countably infinite homogeneous graph (or its complement) is isomorphic to Rado's graph, some $\mathcal{H}_n$ or a disjoint union of complete graphs. Casanovas [Cas14] recalls and addresses a preeminent characterization for $\mathcal{H}_3$ which strengthens the connection between the random graph and Henson graphs.

Definition 4.4. *The* triangle-free random graph *is the countable L-structure $\mathcal{H}_3 = (V, E)$, unique up to isomorphism, which satisfies the L-theory $\overline{K_3}-\mathsf{RG}$, made up of:*

- *The axioms of $\overline{K_3}-\mathsf{Graph}$.*
- *For every $m, n < \omega$, the axiom $\forall x_0 \dots \forall x_{m-1} \forall y_0 \dots \forall y_{n-1}$*

$$\bigwedge_{i<m} \left(\bigwedge_{j<n} \neg x_i \doteq y_j \wedge \bigwedge_{k<m} \neg x_i E x_k \right) \rightarrow \exists z \left(\bigwedge_{i<m} z E x_i \wedge \bigwedge_{i<n} (\neg z E y_j \wedge \neg z \doteq y_j) \right)$$

Proposition 4.5. *The triangle-free random graph $\mathcal{H}_3 = (V, E) \models \overline{K_3}-\mathsf{RG}$ is the Fraïssé limit of the class $\mathcal{K}_3$ of finite triangle-free graphs.*

Proof. We prove the richness of $\mathcal{H}_3$ in an identical way to proposition 3.11. To check that the age of $\mathcal{H}_3$ is $\mathcal{K}_3$, we notice that any of its finite subgraphs is triangle free. Conversely, we apply the same argument as in the proof for Rado's graph: given a graph $H \in \mathcal{K}_3$, we propose an enumeration $\{v_i \mid 1 \leq i \leq |H|\}$ and show that any $H_r = \{v_i \mid i \leq r \leq |H|\}$ is isomorphic to some substructure of $\mathcal{H}_3$.

We still select an arbitrary vertex $v'_1 \in \mathcal{H}_3$ as the starting point, but the inductive step manifests a slight nuance: suppose that $h_r : H_r \xrightarrow{\sim} G_r \subseteq \mathcal{H}_3$ is already constructed and v_{r+1} induces a partition $V_1 \cup V_2$ of H_r depending on whether their vertices are connected to v_{r+1}. No pair of elements from V_1 is connected, otherwise they form a triangle along with v_{r+1}. The same applies for $h_r(V_1)$, so the premises of the last axiom of $\overline{K_3}-\mathsf{RG}$ are satisfied for $h_r(V_1)$,$h_r(V_2)$ and there exists some $v'_{r+1} \in \mathcal{H}_3$ which extends the isomorphism. □

By considering directed edges (an asymmetric relation), Henson also describes in [Hen72] 2^ω pairwise non-isomorphic countable homogeneous digraphs. We adopt the language of simple graphs (which exclusively consists of a relation E) to define a *tournament* as any digraph which satisfies either $E(a,b)$ or $E(b,a)$, for any of its vertices $a \neq b$.

Definition 4.6. *Let $\mathcal{T}$ be a class of finite tournaments with the* **HP**. *We define the class $\mathcal{C}(\mathcal{T})$ as the collection of all finite digraphs whose substructures which are tournaments are isomorphic to a digraph of $\mathcal{T}$.*

The class $\mathcal{C}(\mathcal{T})$ has again a countable amount of isomorphism types, as it is based on a finite relational language. Any tournament T_0 which appears as a substructure of a digraph G_0 embeddable into an element of $G \subseteq \mathcal{C}(\mathcal{T})$ will be isomorphic to a tournament $T \subseteq G$, and thus to a tournament of $\mathcal{T}$. This proves the Hereditary Property for $\mathcal{C}(\mathcal{T})$. The **JEP** and the **AP** follow an argument similar to the finite graph case: note that any tournament in the common substructure is preserved over the free amalgam, and that no additional tournament can appear since the new relation is defined as the union of the relations from the structures.

These general classes have a Fraïssé limit. Henson constructs a tournament T_n for each natural $n \geq 3$ by adding an extra vertex and connecting it to a fixed directed version of K_n. No embedding can be defined between any two of these digraphs, so it is possible to choose a set $\{T_i\}_{i \in I}$ and define $\mathbb{T}_I$ as the class of tournaments which embed into some T_i, with $i \in I$. The uncountably many homogeneous graphs emerge as the limit of each $\mathcal{C}(\mathbb{T}_I)$.

4.2 Finite groups

Besides some classes based on finite relational languages, the case of finite groups will present the loosest conditions that define our structures, as any other algebraic object we study –in addition– can be seen as a group. While the results in this section are somewhat more general, the fact that we need to take into account map properties keeps us from applying them to the rest of the cases. Furthermore, this generality will also prevent the Fraïssé limits from being ω-categorical.

Definition 4.7. *Let* 1 *be a constant,* $\cdot$ *a binary operator.*[1] *We denote the language of groups by* $L_{\mathsf{Group}} = \{1, \cdot\}$. *The theory* Group *of groups consists of the following axioms:*

- $\forall a \forall b \forall c (a \cdot (b \cdot c) \doteq (a \cdot b) \cdot c)$ *(associativity)*
- $\forall a (1 \cdot a \doteq a \cdot 1 \doteq a)$ *(neutral element)*
- $\forall a \exists b (a \cdot b \doteq 1)$ *(opposed element)*

The structure $G = (S, \{1^G, \cdot^G\})$ *is called a* finite group *if* S *is finite and* $G \models \mathsf{Group}$.

Observation 4.8. *Finitely generated groups are not finite in general, as in the case of* $\langle 1 \rangle^{\mathbb{Z}}$. *However, given a collection* $\{G_j\}_{j \in J}$ *of finite groups, any structure generated by a finite set of generators from the* G_j *is finite.*

As in the case of vector spaces, the Hereditary Property of the class $\mathcal{K}$ of finite groups is satisfied because any subgroup is generated by a subset of the domain of the group and the axiomatic is preserved. Cayley's representation theorem (as gathered in [Cla71]) implies that there are countably many isomorphism types in all $\mathcal{K}$, since symmetric groups on a finite amount of letters have a finite amount of subgroups.

Theorem 4.9 (Cayley)**.** *Let* G *be a finite group. There exists an* $n < \omega$ *such that* G *is isomorphic to a subgroup of* S_n*, the* symmetric group *on* n *letters.*

In fact, the Joint Embedding Property of $\mathcal{K}$ also follows directly from this theorem: any two finite groups will be isomorphic to some subgroups H, H' of S_n and S_m, respectively, so they will be embeddable into some subgroup of S_N, $N := \max(m, n)$, generated by H and H'. Neumann [Neu60] showed how to amalgamate any finite collection of groups, but his argument can be narrowed down to our particular setting. For that, let us introduce some notation:

Definition 4.10. *Let* G *be a finite group,* $H \subseteq G$ *a subgroup. For any element* g *of* G*, we define a* left coset *of* H *in* G *as* $gH = \{gh \mid h \in H\}$.

[1] Also expressed by means of juxtaposition.

Remark 4.11. Left cosets over a fixed subgroup H induce an equivalence relation on G: define two elements a,b to be equivalent if $aH = bH$, that is, if $a^{-1}b \in H$. Choosing a set of representatives of the equivalence classes (the left cosets), we obtain a partition of G and a unique decomposition for every element of G as the product of a representative and an element of H.

Definition 4.12. *Let G be a finite group, $H \subseteq G$ a subgroup. We define the* left transversal *S of H as the fixed set of* left coset representatives *(representatives of the aforementioned equivalence classes). Given an element $a \in G$ and its unique product decomposition $a = sh$ in terms of $s \in S$, $h \in H$, we denote $s = a^{\sigma}$, $h = a^{-\sigma+1}$.*

Theorem 4.13. *The class $\mathcal{K}$ of all finite groups satisfies the Amalgamation Property.*

Proof. Let $A,B,H \in \mathcal{K}$ such that $H = A \cap B$ is a common subgroup of A and B. We must show that A and B embed into some finite group through some embeddings g,g', such that $g(h) = g'(h)$ for any element h of H. To do that we will construct a group of permutations on the Cartesian product $K := A \times B \times H$ of the underlying sets, and assign to every element of $A \cup B$ a permutation. Let us choose left transversals S,T of H, for A and B respectively, and reintroduce the decompositions[2]

$$a = sh,\ s = a^{\sigma} \in S,\ h = a^{-\sigma+1} \in H;\ b = th,\ t = b^{\tau} \in T,\ h = b^{-\tau+1} \in H.$$

Firstly, we define K as the set-theoretic product $S \times T \times H$ and build permutations in it: given some $a \in A$, there is a map $\rho(a) : K \to K$ which assigns to any triplet $(s,t,h) \in K$ the element $(s',t',h') \in K$, such that $t' = t$ and $s'h' = sha$. Since we can decompose elements of A and H uniquely, this is equivalent to stating

$$(s,t,h)^{\rho(a)} = ((sha)^{\sigma}, t, (sha)^{-\sigma+1}).$$

For any element $b \in B$, we can define a map $\rho'(b)$ analogously, with $(s,t,h)^{\rho'(b)} = (s,(thb)^{\tau},(thb)^{-\tau+1})$. Additionally, the maps $\rho(h_0)$ and $\rho'(h_0)$ coincide for any element $h_0 \in H$, because $hh_0 \in H$ and the decompositions σ,τ fix the representatives:

$$(shh_0)^{\sigma} = s,\ (thh_0)^{\tau} = t \Rightarrow (s,t,h)^{\rho(h_0)} = (s,t,hh_0) = (s,t,h)^{\rho'(h_0)}$$

We can now view ρ as a map from A to a permutation group $\rho(A)$ of K, and prove that it is an isomorphism. It is exhaustive by definition, and injective because its kernel is $a = 1$: if $\rho(a) = \mathrm{Id}_K$, for all $s \in S$, $h \in H$,

$$(sha)^{\sigma} = s,\ (sha)^{-\sigma+1} = h$$

[2]Observe that the exponent notation for the maps σ, τ, etc. is purely mnemonic and no operation is defined beyond their composition.

so $sha = sh$ and necessarily $a = 1$. To show that ρ is a homomorphism, let $a_0, a_1 \in A$ and any $(s,t,h) \in K$:

$$(s,t,h)^{\rho(a_0)\rho(a_1)} = \left((sha_0)^{\sigma}, t, (sha_0)^{-\sigma+1}\right)^{\rho(a_1)} =$$
$$= \left(\left((sha_0)^{\sigma}(sha_0)^{-\sigma+1}a_1\right)^{\sigma}, t, \left((sha_0)^{\sigma}(sha_0)^{-\sigma+1}a_1\right)^{-\sigma+1}\right) =$$
$$= \left((sha_0a_1)^{\sigma}, t, (sha_0a_1)^{-\sigma+1}\right) = (s,t,h)^{\rho(a_0a_1)}$$

The third equality holds as any $g \in G$ can be factored into $g = g^{\sigma}g^{-\sigma+1}$; the rest follow from the definition of ρ. Notice how this also proves that $\rho(A)$ is a group equipped with the composition as its operator.

We can show a similar result for $\rho' : B \to \rho'(B) \in \mathrm{Sym}(K)$. Therefore, we have established that A and B embed into the symmetric group $\mathrm{Sym}(K) \in \mathcal{K}$ through the embeddings $\rho : A \xrightarrow{\subsetneq} P$, $\rho' : B \xrightarrow{\subsetneq} P$. These coincide for any element of $H = A \cap B$ and assign them some permutation in $\rho(A) \cap \rho(B)$. □

The subgroup of $\mathrm{Sym}(K)$ generated by the images $\rho(A), \rho(B)$ is sometimes referred to as a *permutation group* of A and B (for instance, in [Neu60]), and it depends on the chosen transversals of H. The class $\mathcal{K}$ now satisfies the requirements of Fraïssé's theorem, so there exists a limit which is known (and can be checked immediatly) to be Hall's universal group ([Hal59]).

Definition 4.14. *Let U be a countable, locally finite group. We say U is* Hall's universal group *if the following hold:*

- *Every finite group $G \in \mathcal{K}$ admits an embedding $G \xrightarrow{\subsetneq} U$.*
- *Given $G_0, G_1 \in \mathcal{K}$, any embeddings $f_0 : G_0 \xrightarrow{\subsetneq} U$, $f_1 : G_1 \xrightarrow{\subsetneq} U$ are conjugate by some inner automorphism of U.*

However, since Hall's group embeds any countable locally finite group, it is not of finite exponent, so it is not ω-categorical. This property will be seen in the next lemma from [Ros73], which helps us restrict our examples to classes of groups whose limits have bounded exponent:

Definition 4.15. *Let G be a group. The* order *of G is the number of elements of its underlying set. For any $a \in G$, the* order of a in G *is defined as the order of the subgroup generated by a, i.e. the least positive integer n such that $a^n = 1$, if it exists. As long as it is defined, we call the least common multiple of the orders of all the elements of G the* exponent *of G.*

Lemma 4.16. *Let G be an ω-categorical group. Then, G has finite exponent.*

Proof. We will prove the existence of some $M < \omega$ such that the order of any $a \in G$ is less than M, which implies the exponent of G is less than $M!$. Let us define the formula $\alpha_n(x)$ as "x has order greater than n", that is, $\alpha_n(x) := \bigwedge_{i \leq n} \neg x^i \doteq 1$. Since G is ω-categorical, by theorem 2.20, the collection $\{\alpha_n(x) \mid n < \omega\}$ contains only finitely many types of non-equivalent formulas in G. Specifically, we have a partition

$$T_0 \cup \ldots \cup T_N := \{\alpha_i(x) \mid 1 \leq i < n_1\} \cup \{\alpha_i(x) \mid n_1 \leq i < n_2\} \cup \ldots \cup \{\alpha_i(x) \mid n_N < i\}$$

of equivalent formulas in G, because it is not possible to have a situation like $\alpha_m(x) \equiv \alpha_{m+2}(x) \not\equiv \alpha_{m+1}(x)$.

This also induces a partition $S_0 \cup \ldots \cup S_N$ of elements of G which satisfy each T_j. Note that the equivalence of formulas in T_N ($\alpha_i(x)$ with $i > n_N$) implies that the elements of S_N do not have a finite order. We will show now that $S_N = \emptyset$: assume, otherwise, there exists some $g \in S_N$. For every $j < \omega$, we can define the formula $\phi^j(x,y) = x^j \doteq y$. But if g has infinite order, $\phi(g, g^i)$ holds in G if and only if $i = j$, so no $\phi^j, \phi^{j'}$ are equivalent in G. This contradicts the ω-categoricity of G. Therefore, $S_N = \emptyset$ and the order of the elements is bounded by n_N. □

Adopting arguments similar to the case of vector spaces, it can be shown that the class of finite *abelian* (with the axiom $\forall a \forall b(ab = ba)$) groups of exponent dividing n has the Amalgamation Property. Saracino and Wood [SW82] prove that its Fraïssé limit will be ω-categorical and have quantifier elimination, as the k generators of any group have finite order less than n, so the group has no more than n^k elements and the class is uniformly locally finite. Moreover, the limit is isomorphic to $\mathbb{Z}_n^\omega$.

4.3 Finite fields of characteristic p

Back in section 3.3, we portrayed some vector spaces as basic constructions over finite fields. Now we will do the same for finite fields themselves. Let us recall that the order of a field is its cardinality, and the order of any of its elements is the cardinality of its generated subgroup. For this purpose, we will introduce their notation and a first necessary result:

Definition 4.17. *Let $+$ and $\cdot$ be binary operators, and $0, 1$ constants. We denote the language of fields by $L_{\mathsf{Field}} = \{0, 1, +, \cdot\}$, and the sum $1 + \ldots + 1$ of $n < \omega$ times 1 by $n \cdot 1$. Given a prime number p, the theory Field_p of fields of characteristic p is made up of the following:*

- *The axioms of groups on* $\{0,+\}$ *(associativity, neutral element and opposed element)*
- *The axioms of groups on* $\{1,\cdot\}$ *applied to every non-zero element*
- $\neg 0 \doteq 1$
- $\forall a \forall b(a+b \doteq b+a)$ *(commutativity of* $+$*)*
- $\forall a \forall b(a \cdot b \doteq b \cdot a)$ *(commutativity of* $\cdot$*)*
- $\forall a \forall b \forall c(a(b+c) \doteq ab+ac$ *(distributivity of* $+$*)*
- Characteristic p*:* $p \cdot 1 \doteq 0$*; and, for any* $n < p$*,* $\neg(n \cdot 1 \doteq 0)$

Given a finite set A*, the* L*-structure* $F = (A,(Z^F)_{Z \in L})$ *is called a* finite field of characteristic p *if* $F \models \mathsf{Field}_p$*.*

Lemma 4.18. *Any finite field* F *has order* p^n*, for some* p *prime and positive integer* n*.* F *is isomorphic to any field* F' *with the same order.*

Proof. By the Pigeonhole principle, any finite field must have finite characteristic, in particular, equal to a prime number p (otherwise, 0 would be the product of some non-zero elements corresponding to its divisors). We can now consider the subfield F_0 generated by the neutral element of the product, $\{0,1,\ldots,p-1\}$. Knowing that F is a group with respect to the sum, and restricting $\cdot$ to an operation $F_0 \times F \to F$, we can check that F can be defined as a vector space over F_0. Its dimension n narrows the cardinality to p^n.

For the claim that F and F' are isomorphic, it is clear that there exists an isomorphism h between F_0 and F'_0, the subfield generated by the neutral element of the product in F'. Since both F and F'_0 have characteristic p, h can be extended by taking into account the isomorphism defined between their vector spaces. □

This lemma shows that any subfield has characteristic p, so the Hereditary Property follows necessarily for the class $\mathcal{K}$ of all finite fields of characteristic p, as all substructures are subfields. $|\mathcal{K}/\cong|$ is automatically countable, thanks to the result as well. In fact, having considered such F_0 and constructed F as a vector space, the **JEP** is checked with a straightforward reasoning, once some notions from [Cha09] are introduced.

Definition 4.19. *Let* K *be a field,* $p(X)$ *a polynomial over* K*. A field extension* L *of* K *is called the* splitting field of $p(X)$ *if* p *factors into linear factors over* L*, and any* L' *with the same property has* $L \subseteq L'$*. Given* n *a positive integer and* p *a prime number, the splitting field of* $X^{p^n} - X$ *is denoted by* $\mathbb{F}_{p^n}$*.*

Proposition 4.20. *The class $\mathcal{K}$ of all finite fields of characteristic p has the Joint Embedding Property.*

Proof. Given two positive integers m, n and a prime p, we have $\mathbb{F}_{p^m} \subset \mathbb{F}_{p^n}$ if and only if m divides n: for the rightward implication, we see that $\mathbb{F}_{p^n}$ is a vector space over $\mathbb{F}_{p^m}$, hence $|\mathbb{F}_{p^n}| = |\mathbb{F}_{p^m}|^k$ and $p^n = p^{mk}$. Conversely, $p^m - 1$ divides $p^n - 1$, so all roots of $X^{p^m-1} - 1$ are also roots of $X^{p^n-1} - 1$, which implies that $\mathbb{F}_{p^n}$ contains all the roots of $\mathbb{F}_{p^m}$.

Suppose F_0, F_1 are two finite fields of characteristic p and respective orders p^{n_0}, p^{n_1}. It is a well-known fact that $F_0 \cong \mathbb{F}_{p^{n_0}}$ and $F_1 \cong \mathbb{F}_{p^{n_1}}$. Therefore, by the previous observation, F_0 and F_1 embed into $\mathbb{F}_{p^N}$, where $N = \operatorname{lcm}(n_0, n_1)$. □

The Amalgamation Property can be deduced similarly, as the common subfield can be embedded into the amalgamated structure with the identity. We can now introduce the object (described by [KT17]) which will serve as the Fraïssé limit of $\mathcal{K}$. Much like countably infinite vector spaces, it will have an ω-categorical theory with quantifier elimination due to the uniformly local finitude of finite fields.

Definition 4.21. *Let p be a prime number. The* algebraic closure $\overline{\mathbb{F}_p} := \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$ *of a field $\mathbb{F}_p$ is the least extension of $\mathbb{F}_p$ in which any non-constant polynomial has a root.*

Proposition 4.22. *Given a prime p, $\overline{\mathbb{F}_p}$ is the Fraïssé limit of the class $\mathcal{K}$ of all finite fields of characteristic p.*

Proof. Firstly, $\operatorname{Age}(\mathcal{K}) = \overline{\mathbb{F}_p}$ because $\overline{\mathbb{F}_p}$ consists of the union of splitting fields of prime base p, and every finite field of characteristic p is isomorphic to some splitting field $\mathbb{F}_{p^n}$. Now, for any fields $F_0, F \models \mathsf{Field}_p$ and embeddings $f_0 : F_0 \overset{\subseteq}{\rightarrow} \overline{\mathbb{F}_p}$, $f_1 : F_0 \overset{\subseteq}{\rightarrow} F$, consider isomorphic copies $\mathbb{F}_{p^n} \cong^h f_0(F_0)$ and $\mathbb{F}_{p^m} \cong^{h'} F$. We can infer $n \mid m$ due to the proof of proposition 4.20, since $f_1(F_0)$ is a subfield of F. Then, $f_0 = h^{-1} \circ h' \circ f_1$ commutes for elements of F_0. □

4.4 Finite Boolean algebras

One last object of study oriented toward algebraic structures is the case of Boolean algebras. For this work, their interest relies on the capability to be seen as ordered structures, which will present resemblances with totally ordered sets when constructing their Fraïssé limit.

Definition 4.23. *Let 0 and 1 be constants, $\sqcup$ and $\sqcap$ binary operators, and $^-$ a unary operator.*[3] *We denote the language of Boolean algebras by $L_{BAlg} = \{0, 1, \sqcup, \sqcap, ^-\}$. The theory* BAlg *of Boolean algebras consists of the following axioms:*

[3] $0, 1, \sqcup, \sqcap, ^-$ often recieve the names of *bottom*, *top*, *join*, *meet* and *complement*, respectively.

- $\forall a \forall b \forall c\,(a * (b * c) \doteq (a * b) * c)$ *(associativity of* $* \in \{\sqcup, \sqcap\}$*)*
- $\forall a \forall b\,(a * b \doteq b * a)$ *(commutativity of* $* \in \{\sqcup, \sqcap\}$*)*
- $\forall a \forall b \forall c\,(a * (b \star c) \doteq (a * b) \star (a * c))$ *(distributivity of* $\{*, \star\} = \{\sqcup, \sqcap\}$*)*
- $\forall a \forall b\,(a * (a \star b) \doteq a)$ *(absorption for* $\{*, \star\} = \{\sqcup, \sqcap\}$*)*
- $\forall a\,(a \sqcup 0 \doteq a \sqcap 1 \doteq a)$ *(identity)*
- $\forall a\,(a \sqcup \overline{a} \doteq 1 \wedge a \sqcap \overline{a} \doteq 0)$ *(complement)*

Given a finite set A*, an L-structure* $\mathcal{A} = (A, \{0^{\mathcal{A}}, 1^{\mathcal{A}}, \sqcup^{\mathcal{A}}, \sqcap^{\mathcal{A}}, {}^{-\mathcal{A}}\})$ *is called a* finite Boolean algebra *if* $\mathcal{A} \models \mathsf{BAlg}$.

It is fundamental to introduce an order relation for the elements of algebras: for any $a, b \in A$, $\mathcal{A} \models \mathsf{Balg}$, we write $a \leq b$ when $a \sqcup b = b$, or equivalently $a \sqcap b = a$. In this sense, 0 and 1 are called *infimum* and *supremum*, respectively, as $0 \leq a \leq 1$ for every $a \in A$. This notion will help us express some of the properties which characterize finite Boolean algebras. For this section, we will rely on several standard results from [GH09].

Definition 4.24. *Let* $\mathcal{A}$ *be a Boolean algebra. An* $a \in A \setminus \{0\}$ *is called an* atom *of* $\mathcal{A}$ *if, for every* $b < a$, $b = 0$. $\mathcal{A}$ *is referred to as* atomic *if, for every* $b \in A \setminus \{0\}$*, there exists an atom* $a \in A$ *such that* $a \leq b$*. An algebra is said to be* atomless *if it contains no atoms.*

Remark 4.25. Every finite Boolean algebra is atomic. This is shown inductively, since every non-atom element must have some element below.

Intuitively, this relies on the fact that, as opposed to –say– topologies, Boolean algebras contain a complement for every element, which end up spanning every possible atom along with the meet operator $\sqcap$. Actually, any finite algebra $\mathcal{A}$ is determined by its atoms $\mathrm{At}(\mathcal{A})$ in the following sense: any element $a \in A$ can be expressed as the join $a' = \bigsqcup\{a_0 \leq a \mid a_0 \in \mathrm{At}(\mathcal{A})\}$ of its atoms, since otherwise $0 \neq \overline{a} \sqcap a' \leq a$ or there would exist an atom $a^* \leq a \sqcap \overline{a'} \not\leq a'$. The next result points at Boolean algebras as generalization of structures based on power sets.

Lemma 4.26. *Any finite Boolean algebra* $\mathcal{A}$ *is isomorphic to the Boolean power set algebra defined over* $\mathcal{P}(\mathrm{At}(\mathcal{A}))$*, with* 0 *being the empty set,* 1 *the universe* A *of* $\mathcal{A}$*,* $\sqcup$ *the union of sets,* $\sqcap$ *the intersection, and* ${}^{-}$ *the set-theoretic complement with respect to* A*.*

Having laid the groundwork to understand Boolean algebras as ordered structures with strong properties, we can now characterize homomorphisms through the equalities

$$h(x \sqcup y) = h(x) \sqcup h(y), \;\; h(x \sqcap y) = h(x) \sqcap h(y), \;\; h(\overline{x}) = \overline{h(x)}.$$

Additionally, these also imply $h(0) = 0$, $h(1) = 1$ and $h(x) \leq h(y)$, for any $x \leq y$. The previous lemma directly evidences there are countably many isomorphism types for finite Boolean algebras. And, since any embedded substructure is characterized by the choice of its atoms, the class $\mathcal{K}$ of finite Boolean algebras also has the Hereditary Property. The **JEP** and the **AP** are motivated by a *embedding extension criterion* in [GH09], the latter being presented by [Cas22].

Proposition 4.27. *Let $\mathcal{A}$ be a Boolean algebra generated by E and denote, for every $a \in A$, $a^1 := a$ and $a^0 := \bar{a}$. A map g from E into a Boolean algebra $\mathcal{B}$ can be extended to an embedding $\mathcal{A} \overset{\subseteq}{\rightarrow} \mathcal{B}$ just in case, for every finite $F \subseteq E$, for every function $\varepsilon : F \rightarrow \{0,1\}$,*

$$\prod_{a \in F} a^{\varepsilon(a)} = 0 \quad \textit{if and only if} \quad \prod_{a \in F} g(a)^{\varepsilon(a)} = 0$$

This easily extends lemma 4.26 to handle embeddings from atomic algebras into power set algebras, as defined with set-theoretic operators: given an atomic algebra $\mathcal{A}$ and a partition $(X_a \mid a \in \mathrm{At}(\mathcal{A}))$, there exists an embedding $\mathcal{A} \overset{\subseteq}{\rightarrow} \mathcal{P}(X)$ which extends the map $a \mapsto X_a$. Therefore, $\mathcal{K}$ satisfies the **JEP** by letting any two finite algebras $|\mathcal{B}_0| < |\mathcal{B}_1|$ embed into $\mathcal{P}(\mathrm{At}(\mathcal{B}_1))$.

Proposition 4.28. *The class $\mathcal{K}$ of finite Boolean algebras has the Amalgamation Property.*

Proof. Let $\mathcal{A}, \mathcal{B}_0, \mathcal{B}_1 \in \mathcal{K}$ such that $A = B_0 \cap B_1$ and $\mathcal{A}$ is a substructure of both $\mathcal{B}_0$ and $\mathcal{B}_1$: we will show that the $\mathcal{B}_i$ embed into some power set algebra $\mathcal{P}(X)$ through the embeddings $g_i : \mathcal{B}_i \overset{\subseteq}{\rightarrow} \mathcal{P}(X)$, with $g_0(a) = g_1(a)$ for every $a \in A$. To do so, we will construct X by relating each atom a of $\mathcal{A}$ to the set of atoms $\mathcal{B}_i$ lesser than a, and including in X whichever of the sets is larger. Thus, by proposition 4.27, the mappings we consider from $\mathrm{At}(\mathcal{B}_i)$ into $\mathcal{P}(X)$ will extend to our sought g_i. Let us define, for each $a \in \mathrm{At}(\mathcal{A})$:

- For $i \in \{0,1\}$, $B_i^a := \{x \in \mathrm{At}(\mathcal{B}_i) \mid x \leq a\}$. Every pair $B_i^{a_1}, B_i^{a_2}$ is disjoint, since otherwise any $b \in B_i^{a_1} \cap B_i^{a_2}$ would belong to A and would satisfy $0 \neq b \leq a_1 \sqcap a_2 < a_1, a_2$, contradicting the fact that $a_1, a_2 \in \mathrm{At}(\mathcal{A})$. The B_i^a also form a partition of $\mathrm{At}(\mathcal{B}_i)$, as any $b \in \mathrm{At}(\mathcal{B}_i) \setminus \mathrm{At}(\mathcal{A})$ satisfies $b \leq 1^{\mathcal{B}_i} = 1^{\mathcal{A}} = \bigsqcup \mathrm{At}(\mathcal{A})$.

- If $|B_0^a| \geq |B_1^a|$, set $B^a := B_0^a$ and choose some $a^* \in B_1^a$ if the inequality is strict. Otherwise, $B^a := B_1^a$ and select some $a^* \in B_0^a$.

- If $a \in \mathrm{At}(\mathcal{A}_0) \cap \mathrm{At}(\mathcal{B}_1)$ (thus, $B_a = B_i^a = \{a\}$ for $i \in \{0,1\}$), let h_a be the identity on $\{a\}$. On the other hand, if $|B_0^a| \geq |B_1^a|$, define an injective map $h_a : B_1^a \rightarrow B_0^a = B^a$. Lastly, if $|B_0^a| < |B_1^a|$, let $h_a : B_0^a \rightarrow B_1^a = B^a$ another injective mapping.

Now we can define X as the union $\bigcup_{a\in \mathrm{At}(\mathcal{A})} B^a$, so that it contains the largest of each B_i^a and we can define an injective map $g_0 : \mathrm{At}(\mathcal{A}_0) \to \mathcal{P}(X)$ for any element $b \in \mathrm{At}(\mathcal{A}_0) \cap B_0^a$ (for some $a \in \mathrm{At}(\mathcal{A})$) as: $g_0(b) = \{b\}$ if $|B_0^a| \geq |B_1^a|$; $g_0(b) = \{h_a(b)\}$ if $|B_0^a| < |B_1^a|$ and $b \neq a^*$; and, lastly, $g_0(b) = \{h_a(b)\} \cup (B^a \setminus h_a(B_0^a))$ if $|B_0^a| < |B_1^a|$ and $b = a^*$. For $g_1 : \mathrm{At}(\mathcal{B}_1) \to \mathcal{P}(X)$, we proceed analogously, but letting $g_1(b) = \{h_a(b)\}$ whenever $b \in \mathrm{At}(\mathcal{B}_1) \cap B_1^a$ happens for some $|B_0^a| = |B_1^a|$.

The images of the maps are indeed partitions of X, so the g_i can be extended to embeddings $g_i : \mathcal{B}_i \xrightarrow{\subseteq} \mathcal{P}(X)$ due to the observation after proposition 4.27. It remains to be seen that $g_0 \restriction A = g_1 \restriction A$. Notice that showing this for $\mathrm{At}(\mathcal{A})$, instead, implies the result, since every element in $\mathcal{A}$ can be expressed from the combination of its atoms.

- For every $a \in \mathrm{At}(\mathcal{A}_0) \cap \mathrm{At}(\mathcal{B}_1)$, $g_0(a) = g_1(a) = \{a\}$.
- Given $i \neq j$, for every $a \in \mathrm{At}(\mathcal{B}_i) \setminus \mathrm{At}(\mathcal{B}_j)$, $B_i^a = \{a\}$ and $|B_i^a| < |B_j^a|$, so $a^* = a$ and $g_i(a) = B_j^a$. On the other hand, $g_j(a) = g_j(\bigsqcup_{b\in B_j^a} b) = \bigcup_{b\in B_j^a} g_j(b) = B_j^a$.
- Finally, for every $a \notin \mathrm{At}(\mathcal{A}_0) \cup \mathrm{At}(\mathcal{B}_1)$, we distinguish two cases: if $|B_0^a| = |B_1^a|$, for every $b \in B_0^a$ we defined $g_0(b) = \{b\}$, and so $g_0(a) = g_0(\bigsqcup_{b\in B_0^a} b) = \bigcup_{b\in B_0^a} g_0(b) = B_0^a$. Likewise, we get $g_1(a) = g_1(\bigsqcup_{b\in B_1^a} b) = \bigcup_{b\in B_1^a} \{h_a(b)\} = B_0^a$.
- If, otherwise, $a \notin \mathrm{At}(\mathcal{A}_0) \cup \mathrm{At}(\mathcal{B}_1)$ and $|B_i^a| < |B_j^a|$, the maps result in $g_j(a) = g_j(\bigsqcup_{b\in B_j^a} b) = \bigcup_{b\in B_j^a} g_j(b) = \bigcup_{b\in B_j^a} \{b\} = B_j^a$. For g_i, we obtain that $g_i(a) = g_i(\bigsqcup_{b\in B_i^a} b) = \bigcup_{b\in B_i^a} g_i(b) = \{h_a(a^*)\} \cup (B_j^a \setminus h_a(B_i^a)) \cup \bigcup_{b\in B_i^a\setminus\{a^*\}} \{h_a(b)\} = B_j^a$.

This proves that $\mathcal{P}(X)$ and $g_i : \mathcal{B}_i \to \mathcal{P}(X)$ are the required structures for $\mathcal{A}, \mathcal{B}_0, \mathcal{B}_1$ to check the Amalgamation Property. □

Therefore, $\mathcal{K}$ has a Fraïssé limit $\mathfrak{M}$, which can be shown to be atomless: to prove that every element $a \in M$ finds some $b \in M \setminus \{0\}$ lesser than it, we consider the algebra $\mathfrak{A} \subsetneq_{f_0} \mathfrak{M}$ with universe $A = \{0, a, \bar{a}, 1\}$, isomorphic to $\mathcal{P}(\{2,3\})$ by the map $f_1 : a \mapsto \{3\}, \bar{a} \mapsto \{2\}$. We can embed $\mathcal{P}(\{2,3\})$ into $\mathcal{P}(\{2,3,4\})$ by mapping $f_2 : \{2\} \mapsto \{2\}, \{3\} \mapsto \{3,4\}$. Then, due to the richness of $\mathfrak{M}$, there is an embedding $g : \mathcal{P}(\{2,3,4\}) \xrightarrow{\subseteq} \mathfrak{M}$ which satisfies $f_0(x) = g \circ (f_2 \circ f_1)(x)$ for every $x \in A$, so $b = g(\{3\}) < g(\{3,4\}) = f_0(a) = a$ is smaller than a in $\mathfrak{M}$. The following result from [Poi00] substantiates the ω-categoricity of the limit (due to lemma 2.23).

Theorem 4.29. *The theory of atomless Boolean algebras is complete and admits quantifier elimination in the language* $L = \{0, 1, \sqcup, \sqcap, ^-\}$.

A Fraïssé limit of $\mathcal{K}$ can also be constructed explicitly, as described by [Ver10]:

Definition 4.30. *Let S be the set of* infinitely countable sequences *on* $\{0,1\}$ *and, for any $s \in S$, let $s(n)$ denote its n^{th} element. We set the algebra* $\mathcal{S} = (S, \{0^{\mathcal{S}}, 1^{\mathcal{S}}, \sqcup^{\mathcal{S}}, \sqcap^{\mathcal{S}}, ^{-\mathcal{S}}\})$:

- $0^{\mathcal{S}}$ *(resp.* $1^{\mathcal{S}}$*) is the sequence such that* $0^{\mathcal{S}}(n) = 0$ *(resp.* $1^{\mathcal{S}}(n) = 1$*) for every* $n < \omega$*.*
- $\sqcup^{\mathcal{S}}$ *represents the* pointwise maximum *(resp.* minimum*), that is, for any* $s, t \in S$*,* $(s \sqcup^{\mathcal{S}} t)(n) = \max\{s(n), t(n)\}$ *(resp.* $\min\{s(n), t(n)\}$*) for every* $n < \omega$*.*
- $^{-\mathcal{S}}$ *is defined, for every* $s \in S$*, as* $\bar{s}^{\mathcal{S}}(n) = 1 - s(n)$ *for every* $n < \omega$*.*

Let us now omit the $\mathcal{S}$ in the function symbols of $\mathcal{S}$, and consider the periodic sequences, that is, the elements $s \in S$ such that there exists a natural $k > 0$ with $s(n+k) = s(n)$ for every $n < \omega$. These can be represented by the elements of each period as $[x_0 \dots x_{k-1}] = x_0 \dots x_{k-1} x_0 \dots x_{k-1} \dots$ We will define an algebra on the set S_p of periodic sequences on $\{0,1\}$.

Remark 4.31. The subalgebra $\mathcal{S}_p \subseteq \mathcal{S}$ of periodic sequences on $\{0,1\}$ is countably infinite, since there exist finitely many elements in S_p for each period $k < \omega$. It is also atomless: for every $s = [x_0 \dots x_{k-1}] \in S_p$, the sequence $s' = [x_0 \dots x_{k-1} 0 \dots 0] \in S_p$ with k added zeros satisfies $s \sqcup s' = s$, so $0 < s' < s$.

Definition 4.32. *Let* S_p *be the set of periodic sequences on* $\{0,1\}$*. We define the* countable atomless Boolean algebra $\mathcal{S}_p$ *as the subalgebra of* $\mathcal{S}$ *over the universe* S_p*.*

Theorem 4.33. *The countable atomless Boolean algebra* $\mathcal{S}_p$ *is the Fraïssé limit of the class* $\mathcal{K}$ *of finite Boolean algebras. Its theory is* ω*-categorical and has quantifier elimination.*

Proof. Firstly, we will show that $\mathrm{Age}(\mathcal{S}_p) = \mathcal{K}$. It is enough to consider the power set algebras $\mathcal{P}(X)$ with $X = \{0, \dots, k\}$. Given $\mathcal{S}_p^k \subseteq \mathcal{S}_p$, the subalgebra of sequences of period k, notice that the map $h : \mathcal{P}(X) \to \mathcal{S}_p^k$, $h(X) = [\chi_X(0) \dots \chi_X(k)]$ is an isomorphism, where $\chi_X(i)$ is the characteristic function. Then, every finite algebra is isomorphic to some $\mathcal{S}_p^k$. For the converse, note that any subalgebra of $\mathcal{S}_p^k$ has a finite amount of atoms and is isomorphic to a power set algebra.

To prove that G is rich with respect to $\mathcal{K}$, consider any finite algebras $\mathcal{A} \cong \mathcal{P}(\{0, \dots, r\}), \mathcal{B} \cong \mathcal{P}(\{0, \dots, s\})$ (with N the least multiple of r not smaller than s) and the embeddings $f_0 : \mathcal{A} \overset{\subseteq}{\to} \mathcal{S}_p$, $f_1 : \mathcal{A} \overset{\subseteq}{\to} \mathcal{B}$. Note that $\mathcal{A}$ defines through f_0 a subalgebra $\mathcal{S}_p' \subseteq \mathcal{S}_p^r$ whose atoms are the images of the atoms of $\mathfrak{A}$. Also, since we can work with power set algebras, let us assume that f_1 maps the atoms of $\mathcal{A}$ to $\mathrm{At}(\mathcal{B})$, which we denote by $f_1 : a_i \mapsto b_i$ for $i < r$. The map $g : \mathrm{At}(\mathcal{B}) \to \mathcal{S}_p^N$ defined below extends to an embedding $\mathcal{B} \overset{\subseteq}{\to} \mathcal{S}_p^N$, with $f_0(a) = g \circ f_1(a)$ for every $a \in A$:

$$g(b_i) = \begin{cases} f_0(a_i), & \text{for } i < r \\ [0 \overset{(r)}{\dots} 0 x_r \dots x_{N-1}], & \text{with } x_j = 1 \Leftrightarrow j = i, \text{ for } r \leq i < s-1 \\ [0 \overset{(s-1)}{\dots} 0 1 \overset{(N-s+1)}{\dots} 1], & \text{for } i = s-1 \end{cases}$$

□

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