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p-adic groups in Quantum Mechanics

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Abstract

Number theory is being used in physics as a mathematical tool more and more. At the end of the 20th century, p-adic numbers made its appearance in quantum gravitational theories like string theory. This was motivated by the non-archimedian nature of space time at Planck scale. In this work we aim to formalize the basis of p-adic physics by exploring how to translate complex Quantum Mechanics to p-adic Quantum mechanics. This will be done using Weyl's formalism, which defines bounded operators and allows to relate different time-evolution pictures in quantum mechanics. This is done by the means of representation theory. We will be exploring the representation theory of p-adic reductive groups, specially induced, supercuspidal and projective representations. With that knowledge we will define the p-adic Heisenberg group that encodes the information on the p-adic phase space and study the Schrödinger representation. We will explain the importance of the Stone-von Neumann theorem that states uniqueness up to equivalence and we will study the Maslov indices of the group.

Resum

La teoria de nombres s'utilitza cada cop més en física com a eina matemàtica. Cap al final del segle 20, els nombres p-àdics comencen a introduir-se en teories de gravitació quàntica com la teoria de cordes. Aquesta tendència ve motivada per la natura no arquimediana del espai-temps a l'escala de Planck. L'objectiu d'aquest treball és formalitzar les bases de la física p-àdica traduint la mecànica quàntica complexa a la mecànica quàntica p-àdica. Ho farem mitjançant el formalisme de Weyl, que defineix operadors acotats i permet relacionar les diferents imatges d'evolució temporal en quàntica. Farem això a través de la teoria de representacions de grups reductius p-àdics. A partir d'això definirem el grup de Heisenberg p-àdic que codifica la informació sobre l'espai de fase p-àdic i estudiarem la representació de Schrödinger. Explicarem la importància del teorema de Stone-von Neumann que assegura la unicitat llevat d'equivalencia i estudiarem els índexs de Maslov.

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Contents

Introduction

Since the start of the 20th century, the p-adic fields $\mathbb{Q}_p$, non-archimedean completions of the rational field, have played a more and more important role in a great variety of mathematical fields. Algebraic results on number fields can be trivially extended to $\mathbb{Q}_p$ and most analytical theorems have p-adic analogous. Being able to easily extend most of mathematics to other fields has proven to be very useful, in describing hierarchies, discrete topologies, and even being used in modern fields in number theory, like the Langlands program.

But the utility of the p-adic field is not restricted to mathematics. In a paper written in 1987 [Vo1987] I.G. Volovich proposes for the first time a p-adic physical theory. In classical physics, space and time follow the archimedean axioms, and therefore it is natural to extend the measurements to the real numbers, even to the complex field, to utilize its algebraic properties. But with the arrival of modern physics, the paradigm changes and the way we understand space and time changed as well.

This paper was motivated by progress in theories such as super strings. These theories have been successful in describing many elementary particle results, and avoiding divergences by introducing profound concepts like the Kaluza-Klein approach (cf. [Ka1921]) or supersymmetry. There are still some issues that are hard to understand with the theory, such as gravitational collapse and the cosmological singularity. Although these theories do not deal with point-like objects but extended object like strings, understanding what is space-time might bring some light to these problems.

Most modern theories, space-time is considered to be a differential manifold, after Einstein's formulation of general relativity, and that principle has been used without extensive questioning since then. But theories that try to tie general relativity and quantum mechanics, as string theory, deal with very small distances, at the scale of the Planck length, where the nature of space-time as we understand it might be compromised.

The Planck length, usually denoted as L_P, is defined in terms of three fundamental physical constants: the speed of light c, the reduced Planck constant $\hbar$ and the gravitational constant G, therefore describing the scale of quantum gravitational processes. It takes the value $L_P = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \times 10^{-35}\, m$.

Quantum gravity seems to indicate that the Planck length is the shortest measurable distance, as any measurement of that scale using highly energetic particle collisions results in black hole production.

If one were to locate an elementary particle it will need to have energy of at least the Planck mass $M_P = \sqrt{\frac{\hbar c}{G}}$. In such case, the radius of the event horizon of that particle would be $r \sim 2GM_P/c^2 = 2L_P$. No information of the geometric structure of space-time is accessible at sub-Planckian scales.

The notion of space we usually use comes from our perception of the macroscopic world and is an Euclidean space. It follows that it satisfies the archimedean axiom, which states that any segment can be divided into the sum of smaller segments along the same line. This contradicts that the Planck length is the smaller measurable distance, therefore indicating that we may need to abandon the archimedean axiom.

We want to describe physics through a non-archimedean geometry. For that we need to find a field containing the rational numbers with a non-archimedean topology. Thanks to the Ostrowski's theorem, we know that every non-trivial non-archimedean absolute value on $\mathbb{Q}$ is equivalent to a p-adic absolute value, for some prime p. Thus the only non-archimedean completion of the rational field is the p-adic field $\mathbb{Q}_p$. I.G. Volovich proposes that the fundamental entities of space-time are not fundamental particles or strings, but numbers. This lead to proposing a principle of invariance of the number field.

Number Field invariance principle. *Fundamental physical laws should be invariant under the change of the number field.*

At first sight one can doubt about considering physical quantities to be p-adic and not real, because measurements are obviously not p-adic. However, notice that measurements are not real numbers either, but take values in the rational field, which is a subgroup of both $\mathbb{R}$ and $\mathbb{Q}_p$. There are two ways of developing a formalism of p-adic physics. We can either consider p-adic valued functions or complex-valued functions on the p-adic field. We will use the latter as it is more convenient to create a p-adic theory of quantum mechanics.

With the years, p-adic numbers have become an important tool in some physics subjects. In string theory, it is used to calculate Veneziano scattering amplitudes, and the worldsheet of the string is regarded not as a Riemann surface but as an object in p-adic geometry. Theories relating quantum and relativistic models, like quantum field theory (cf. [Sm1991]) and AdS/CFT (Anti de Sitter/Conformal Field Theory) (cf. [Gu2017]), have been studied in the context of p-adic numbers. Some papers even explore the possibility of p-adic matter as an alternative to dark matter. p-adic physic has found its use in new theories, but it is necessary to come back to the basics and build the theory from its fundamentals. It is of interest to give a proper formalization, starting from Quantum Mechanics, which might help bring forward a deeper understanding.

In this work, we aim to give a formalization of p-adic Quantum Mechanics. For that let's draw a general idea of the theory on the complex case.

Quantum mechanics is a non-relativistic theory where the state of a mechanical system is described by a vector ψ on a Hilbert space. States are defined up to a phase, that is ψ and $e^{i\alpha}\psi$ represent the same physical system, therefore all possible states are points in the projective space of the Hilbert space. The nature of the Hilbert space changes depending on the system: when describing position and momentum, the Hilbert space is the space of square integrable functions $L^2(\mathbb{C})$ while for the spin of a proton, the space is formed by two dimensional complex vectors, $\mathbb{C}^2$. The physical quantities are represented by observables, Hermitian and linear operators acting on the Hilbert space. The eigenvectors of the operator are called eigenstates, and their eigenvalues are the value of that observable for that state. Notice that eigenvalues are real numbers, because of the operator being Hermitian, and are the measurable values in an experiment. States are not measurable. A generic quantum state is the linear combination of eigenstates. This is known as superposition. When an observable is measured for a composite state, we cannot predict the exact value of the measurement but the probability of a certain value being observed. After the measurement, the state collapses to the eigenstate associated with the eigenvalue measured.

On the Schrödinger picture of quantum mechanics, the operators do not evolve with time, but the wave function ψ does. The equation that defines the evolution is the Schrödinger equation

$$i\hbar \frac{d}{dt}\psi(t) = H\psi(t)$$

where H is called the Hamiltonian and it is the observable corresponding to the total energy of the system. The solution to this differential equation is $\psi(t) = e^{-iHt/\hbar}\psi(0)$ where the term $U(t) = e^{-iHt/\hbar}$ is called the time-evolution operator and is unitary. Notice that the eigenstates of the Hamiltonian are time independent states. There are multiple ways to describe dynamics of a quantum state, called dynamical pictures. One of the most used other than the Schrödinger picture is the Heisenberg picture, where time evolution is linked to the operators while the states remain constant.

For this work we will focus on harmonic analysis, which is the branch of Quantum Mechanics that describes the momentum and position operators. The classical space that contains the information on these physical quantities is called the phase space, a $2n$-dimensional vector space, which elements are usually written as (q, p), where q is a generalized way of talking about position and p is the linear momentum. Classically, the phase space is identified with $\mathbb{R}^{2n}$, where n is the dimensions of the physical system (usually 3). For the purpose of formalizing p-adic quantum mechanics, we will consider that $p, q \in \mathbb{Q}_p^{2n}$ instead.

The process of translating the classical physical quantities to operators on the Hilbert space of square-integrable functions is called quantization and it is what gives birth to Quantum Mechanics. In the Schrödinger picture, the position operator $\hat{x}$ is the delta distribution δ_x and the momentum operator $\hat{p}$ is the gradient operator $-i\hbar\nabla$ (cf. [Woit2021]). The commutation relations between these two operators are of vital importance to the theory

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{1}$$

since two operators can only have the same set of eigenstates if they commute. These relations are called the canonical commutation relations.

The process of quantization is described by the representation of a p-adic Lie group, a topological group with manifold structure over $\mathbb{Q}_p$, that encodes the information of the phase space.

We want to represent the group of translations on the phase space, $\mathbb{Q}_p^{2n}$, into the Hilbert space of square-integrable functions, but we run into a problem. Since $\mathbb{Q}_p^{2n}$ is abelian, the commutator is zero, therefore the representation space would also have trivial commutator, contradicting the necessary canonical commutation relations. To solve this problem we will use Weyl formulation of quantum kinematics (cf. [We1931]), which consists of considering a projective representation T on the translation group that satisfies the canonical commutation relations. As it will be seen in Section 2.4, a projective representation T on a group G extends to an ordinary representation ϕ on the central extension of G by $S^1 \subset \mathbb{C}$ given by $T_z \to \alpha T_z$ where $z \in G$ and $\alpha \in S^1$, that is called the Heisenberg group. We use this result to give an ordinary representation of the central extension $G \times S^1$ on the space of unitary operators. While T_z describes the functions on the phase space, the parameter α in S^1 can be seen as a temporal parameter, more precisely, as the evolution operator defined classically as $U(t) = e^{-iHt/\hbar}$, with the necessary corrections when considering the p-adic case.

In Chapter 3 we develop in more details this representation, which receives the name of the Schrödinguer representation. We will see that it is an irreducible smooth and admissible representation. The Stone-von Neumann theorem (cf. [Neum1931]) states that any smooth representation that restricts to the identity in S^1 decomposes as the direct sum of irreducible representations equivalent to the Schrödinger representation. We will focus on understanding its physical and mathematical importance. Historically, the theorem is very important, since it connects two branches of quantum mechanics that were historically disconnected, the wave formulation by Schrödinger (cf. [Schr1926]) and matrix mechanics by W. Heisenberg, M. Born and P. Jordan (cf. [Hei1925]). Moreover, it proves that all the different dynamical pictures of quantum mechanics are equivalent. We will define a way to extend functions on the phase space to functions on the space of unitary operators defining the Weyl operator. Finally, we will use the Stone-von Neumann theory to define an intertwining operator between induced representations of the Heisenberg group to calculate Maslov indices.

1 Preliminaries

1.1 p-adic numbers

For any prime p, the field of p-adic numbers $\mathbb{Q}_p$ arises from the completion of the rational number field with respect to a non-archimedean absolute value. This construction starts from the concept of a valuation.

Definition 1.1. *Let K be a field and Γ a totally ordered abelian group by a relation we will denote as $\leq$. A map $v : K^* \to \Gamma$ is called a valuation of K in Γ if for all x, $y \in K^*$*

(a) $v(xy) = v(x) + v(y)$ *and*

(b) $v(x + y) \geq min(v(x), v(y))$.

It is convenient to extend this map to K. In that case we write $v(0) := +\infty$ with the usual relations

$$+\infty + (+\infty) = +\infty, \quad \gamma + (+\infty) = +\infty, \quad +\infty \geq \gamma.$$

The valuation we will be interested in is the p-adic valuation defined as follows. Consider $K = \mathbb{Q}$ and $(\mathbb{Z}, +, \leq)$ as the ordered abelian group. The valuation v_p is defined such that $v_p(x)$ is equal to the exponent to which p appears in the prime decomposition of x. A valuation defines an absolute value.

Definition 1.2. *Let K be a field, a map $|\cdot| : K \to \mathbb{R}$ is called an absolute value if it satisfies the following properties for all $x, y \in K$:*

(a) $|x| \geq 0$,

(b) $|x| = 0 \leftrightarrow x = 0$,

(c) $|xy| = |x||y|$ *and*

(d) $|x + y| \leq |x| + |y|$.

If it also satisfies $|x + y| \leq \max(|x|, |y|)$, then we say that the absolute value is non-archimedean or ultrametric. If it does not, we say it is archimedean or non-ultrametric.

The p-adic valuation v_p defines a p-adic absolute value $|\,|_{v_p} =: |\,|_p : \mathbb{Q} \to \mathbb{R}$ given by $|x|_p := p^{-v_p(x)}$. Another well known absolute value on $\mathbb{Q}$ is the usual absolute value $|\,|_\infty$ defined as $|x|_\infty = \max(-x, x)$. This absolute value is archimedean.

We say that two absolute values are equivalent when they define the same topology over K through the distance $d(x, y) := |y - x|$. A famous result for the absolute values in $\mathbb{Q}$ is the Ostrowski's theorem.

Theorem 1.3 (Ostrowski's theorem)**.** *Every archimedean absolute value on $\mathbb{Q}$ is equivalent to $|\,|_\infty$ while every non-trivial and non-archimedean absolute value is equivalent to $|\,|_p$ for some prime number p.*

The proof of this result can be found in several of our references.

Definition 1.4. *The p-adic field $\mathbb{Q}_p$ is the completion of the rational number field $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$.*

It can be seen that any p-adic number can be expressed as a convergent series of the form

$$x = \sum_{n=k}^{\infty} x_n p^n$$

where $x_n = 0, ..., p-1$ and $k \in \mathbb{Z}$.

Definition 1.5. *The ring of p-adic integers is the subset of $\mathbb{Q}_p$*

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Notice that Z_p is also the closed unitary ball. An equivalent definition is to consider $\mathbb{Z}_p$ as the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$, i.e. a pro-p group (cf. Subsection 2.2). $\mathbb{Z}_p$ is an integral domain and a principal ideal domain. More precisely, its ideals are $\{0\}$ and $p^k\mathbb{Z}_p$ for $k \in \mathbb{N}$, where $p\mathbb{Z}_p$ is the unique maximal ideal of $\mathbb{Z}_p$.

$\mathbb{Q}_p$ can be regarded as the fraction field of $\mathbb{Z}_p$, $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. Notice that the multiplicative group of invertible elements in the ring $\mathbb{Z}_p$ consists of the p-adic numbers with unit absolute value. The subset of nonzero p-adic numbers is given by $\mathbb{Q}_p^\times = \amalg_{m\in\mathbb{Z}} p^m \mathbb{Z}_p^\times$.

Definition 1.6. *Let $x \in \mathbb{Q}_p$, then the fractional part of x is $\{x\} = \sum_{n=0}^{-k-1} x_n p^{n+k} \in \mathbb{Z}[\frac{1}{p}]$.*

Notice that for any two p-adic numbers x and y, the difference $\{x+y\} - \{x\} - \{y\} \in \mathbb{Z}[1/p] \cap \mathbb{Z}_p = \mathbb{Z}$.

The p-adic distance, $d_p(x,y) := |x-y|_p$ induces a certain topology onto $\mathbb{Q}_p$. $B(a;n) = B_n(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq p^{-n}\}$ is the p-adic ball of center a and radius p^{-n}. Due to the non-archimedean properties, every point in $B(a;n)$ is a center. It can be seen that the p-adic balls are both open and closed sets, and that they are disconnected sets. Moreover, it can be shown that $\mathbb{Q}_p = \bigcup_n B(a;n)$ is a locally compact and totally disconnected topological field. This implies $\mathbb{Q}_p$ has a Haar measure, i.e. a real and positive measure dx that is invariant under translation $d(x+a) = dx$ (cf. [A-K-S2010]). The measure dx is unique if normalized such that $\int_{\mathbb{Z}_p} dx = 1$. For any $a \in \mathbb{Q}_p^\times$, notice that $d(ax) = |a|_p dx$.

Definition 1.7. *A complex additive character χ on $\mathbb{Q}_p$ is an homomorphism $\chi : \mathbb{Q}_p \to \mathbb{C}^*$ with the property*

$$\chi(x+y) = \chi(x)\chi(y).$$

The group of additive characters is isomorphic to the additive group $\mathbb{Q}_p$, given by the mapping $u \longmapsto \chi(ux) = \chi_u(x) := exp(2\pi i\{ux\})$.

We will often consider not $\mathbb{Q}_p$ but the vector space $\mathbb{Q}_p^n$, the product space of n copies of $\mathbb{Q}_p$. The absolute value can be extended to a norm by defining $|x|_p = \max_{1\leq j\leq n} |x_j|_p$

for $x = (x_1, ..., x_n) \in \mathbb{Q}_p$, which is also non-archimedean. Similarly, the Haar measure can be extended to a translation invariant measure $d^n x = dx^1 \dots dx^n$ on $\mathbb{Q}_p^n$ (cf. [A-K-S2010]). If $A : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ is a linear isomorphism with $\det A \neq 0$, then $d^n(Ax) = |\det A|_p d^n x$.

Let K be a measurable subset of $\mathbb{Q}_p$ and $L^\alpha(\mathbb{Q}_p^n)$ the set of all measurable complex-valued functions $f : K \to \mathbb{C}$ such that $\int_K |f(\alpha)|^\alpha d^n x < \infty$.

Definition 1.8. *A function $f \in L^1(\mathbb{Q}_p^n)$ is said to be integrable if the limit*

$$\lim_{N\to\infty} \int_{(B(0;-N))^n} f(x) d^n(x).$$

exists. In such a case, the limit is called the improper integral and denoted as $\int_{\mathbb{Q}_p^n} f(x) d^n(x)$.

There exists an analogous to the change of variable theorem in $\mathbb{Q}_p^n$:

Theorem 1.9. *Let $x(y)$ be an analytic diffeomorphism of a closed and open set $K_1 \subset \mathbb{Q}_p$ onto a closed and open set $K \subset \mathbb{Q}_p$, and $x'(y) \neq 0$, $y \in K_1$. Then for any $f \in L^1(K)$ we have*

$$\int_K f(x) dx = \int_{K_1} f(x(y)) |x'(y)|_p dy.$$

Proposition 1.10. *Some of the following properties will be useful.*

1. $\int_{|x|_p = p^\nu} dx = p^\nu (1 - \frac{1}{p})$,

2. $\int_{|x|_p = p^\nu} \chi(\xi x) dx = \begin{cases} p^\nu(1-\frac{1}{p}), & \text{if } |\xi|_p \leq p^{-\nu}, \\ -p^{\nu-1}, & \text{if } |\xi|_p \leq p^{-\nu+1}, \\ 0, & \text{if } |\xi|_p \leq p^{-\nu+2}. \end{cases}$

3. $\int_{\mathbb{Q}_p} f(|x|_p) \chi(\xi x) dx = (1-\frac{1}{p}) \frac{1}{|\xi|_p} \sum_{\nu \geq 0} p^{-\nu} f(\frac{1}{p^\nu |\xi|_p}) - \frac{1}{|\xi|_p} f(\frac{p}{|\xi|_p})$, $\xi \neq 0$.

4. $\int_{\mathbb{Q}_p} \chi(ax^2 + bx) dx = \frac{\lambda_p(a)}{\sqrt{|a|_p}} \chi(-\frac{-b^2}{4a})$, $p \neq 2$, *where for a non-zero p-adic number* $a = p^k(a_0 + a_1 p + ...)$,

$$\lambda_p(a) = \begin{cases} 1, & \textit{if k is even }, \\ (\frac{a_0}{p}), & \textit{if k is odd and } k \equiv 1 \,(\ mod\ 4), \\ i(\frac{a_0}{p}) & \textit{if k is odd and } k \equiv 3 \,(\ mod\ 4), \end{cases}$$

with $(\frac{a_0}{p})$ *denoting the Legendre symbol.*

5. $\int_{\mathbb{Q}_2} \chi(ax^2 + bx) dx = \frac{\lambda_2(a)}{\sqrt{|a|_2}} \chi(-\frac{-b^2}{4a})$, *where for a non-zero 2-adic number* $a = 2^k(a_0 + 2a_1 + ...)$,

$$\lambda_2(a) = \begin{cases} 1 + (-1)^{a_1} i, & \textit{if k is even} \\ (-1)^{a_1 + a_2}(1 + (-1)^{a_1} i), & \textit{if k is odd.} \end{cases}$$

Proposition 1.11. *The $\mathbb{C}$-vector space $L^2(\mathbb{Q}_p^n)$ equipped with the scalar product*

$$(f,g) = \int_{\mathbb{Q}_p^n} f(x)\bar{g}(x)d^n(x)$$

is a Hilbert space.

As in previous cases, properties from the real and complex cases extend.

Proposition 1.12. *The following properties are true*

1. *The Cauchy-Bunjakovsky inequality holds: $|(f,g)| \leq ||f||_{L^2} \cdot ||g||_{L^2}$.*
2. *The Fourier transform $F : L^2(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^n)$ acts as follows:*

$$F[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{Q}_p^n} f(x)\chi(\xi \cdot x)dx^n$$

and

$$F^{-1}[\hat{f}](x) = f(x) = \int_{\mathbb{Q}_p^n} \hat{f}(\xi)\chi(-\xi \cdot x)d^n\xi.$$

3. *The Parseval-Steklov equality holds: $(f,g) = (F[f], F(g))$. In other words, the Fourier transform is a unitary operator in $L^2(\mathbb{Q}_p^n)$.*

Definition 1.13. *A complex-valued function ψ defined over $\mathbb{Q}_p^n$ is said to be locally constant if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that*

$$\psi(x + x') = \psi(x)$$

when $x' \in B(0; -l(x))^n$. The function $l : \mathbb{Q}_p^n \to \mathbb{Z}$ is called a characteristic function associated with ψ.

Let us denote by $\epsilon(\mathbb{Q}_p^n)$ the space of locally constant functions $\psi : \mathbb{Q}_p^n \longrightarrow \mathbb{C}$.

Let the subset $\mathcal{D}(\mathbb{Q}_p^n) \subset \epsilon(\mathbb{Q}_p^n)$ be the space of compactly supported functions in $\epsilon(\mathbb{Q}_p^n)$ and $\mathcal{D}^*(\mathbb{Q}_p^n)$ the set of all complex linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$. Notice that $\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $L^2(\mathbb{Q}_p^n)$.

1.2 Symplectic vector spaces

It will be necessary to define symplectic vector spaces to generalize the definition of the groups talked about in the following sections. Our discussion of these objects is mainly based on [Me1998].

Definition 1.14. *A symplectic vector space is a pair (E, β) formed by a finite vector space E over a field K and $\beta : E \times E \to K$ is an antisymmetric and non-degenerate bilinear form, called the symplectic form.*

We will focus our discussion on a field K of characteristic 0. In this case, antisymmetry on the symplectic form implies that $\beta(v,v) = 0 \, \forall v \in E$.

Definition 1.15. *Two symplectic vector spaces* (E, β_E), (F, β_F) *are symplectomorphic if there exists an isomorphism between two vector spaces* $\phi : E \to F$ *that preserves the symplectic form, that is* $\forall u, v \in E$, $\beta_E(u, v) = \beta_F(\phi(u), \phi(v))$.

To prove the main result of this section we will need to define certain objects beforehand. From the remaining of this section, let (E, β) be a symplectic vector space.

Definition 1.16. *The symplectic orthogonal complement of any subspace* $U \subset E$ *is the subset* $U^\beta = \{v \in E |\, \beta(v, u) = 0 \,\forall u \in E\}$.

The definition of an orthogonal complement can be generalized to any bilinear form, either symmetric or antisymmetric, conserving its basic properties. Therefore some of the results known for the orthogonal complement of an inner product are also valid for the orthogonal complement of a symplectic form.

Proposition 1.17. *Let* E *be a* $K-$*vector space and* β *is a non-degenerate bilinear form. Let* U *be a subspace of* E*. The following properties are satisfied*

1. U^β is a subspace of E.
2. Let $V \subset E$ be a subspace such that $V \subset U$, then $U^\beta \subset V^\beta$,
3. $dim(U) + dim(U^\beta) = dim(E)$,
4. $U \subseteq (U^\beta)^\beta$,
5. Let U, V be two subspaces of E, then $(U \cap V)^\beta = U^\beta + V^\beta$.

Definition 1.18. *A subspace* $U \subset E$ *is called isotropic if* $U \subseteq U^\beta$*, symplectic if* $U \cap U^\beta = 0$ *or Lagrangian if* $U = U^\beta$*. The set of all Lagrangian subspaces* $Lag(E)$ *is called the Lagrangian Grassmannian.*

Lemma 1.19. *In any symplectic space there is a Lagrangian subspace.*

Proof. Any subspace of dimension 1 is isotropic, therefore, there exists L a maximal isotropic subspace. Assume L is not Lagrangian. Then there exists $v \in L^\beta \setminus L$ and $L \oplus < v >$ is an isotropic subspace that contains L. This contradicts L being maximal. □

This result justifies why we always consider symplectic vector spaces of even dimension.

Corollary 1.20. *The dimension of any symplectic vector space* E *is even.*

Proof. For any Lagrangian subspace $L \subset E$, $dim(E) = dim(L) + dim(L^\beta) = 2dim(L)$. □

For any isotropic subspace $L \subseteq E$, we can define the quotient $\pi : L^\beta \to L^\beta / L$. A symplectic form can be defined in the quotient space by $\beta_\pi([v],[u]) := \beta(v,u)$ where $v, u \in L^\beta$.

Proposition 1.21. *For any subspace $S \subseteq L^\beta$, $\pi(S^\beta) \subseteq (\pi(S))^\beta$.*

Proof. Let $v \in S^\beta$, then $\pi(v) \in \pi(S^\beta)$. For any given $s \in S$, $\beta_\pi(\pi(v), \pi(s)) = \beta(v,s) = 0$. $\square$

Lemma 1.22. *Given a Lagrangian subspace $M \subset E$, there exists another Lagrangian subspace $L \subset E$ such that $L \cap M = \{0\}$.*

Proof. Let L be maximal isotropic subset of E such that $L \cap M = \{0\}$. Suppose that L is not Lagrangian, then we consider the quotient map $\pi : L^\beta \to L^\beta / L$. There exists a subspace $F \subseteq L^\beta / L$ of dimension 1, isotropic, such that $F \cap \pi(M \cap L^\beta) = \{0\}$. This is possible because $\pi(M \cup L^\beta)$ is isometric. Indeed, by using Proposition 1.21 we can prove that $\pi(M \cap L^\beta) \subseteq \pi((M \cap L^\beta)^\beta) \subseteq \pi((M \cap L^\beta))^\beta$. Then the codimension is positive and such an F exists. Then $L' := \pi^{-1}(F)$ is an isotropic space with $L \subset L'$ and $L' \cap M = \{0\}$. This contradicts maximality, therefore L is a Lagrangian subspace. $\square$

Example 1.23. Consider the K-vector space K^{2n} with the canonical basis $\{e_i, \omega_j\}_{0<i,j\leq n}$ and $\beta_0 : K^{2n} \to K$ defined by

$$\beta_0(e_i, e_j) = 0 = \beta_0(\omega_i, \omega_j) \quad \beta_0(e_i, \omega_j) = \delta_{ij} \quad \beta_0(\omega_i, e_j) = -\delta_{ij} \; \forall\, 0 \leq i, j \leq n.$$

This form can be expressed in terms of a $2n \times 2n$ matrix

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and the usual inner product $< \cdot, \cdot >$ as

$$\beta_0(v, u) =< J_0 v, u > .$$

The pair (K^{2n}, β_0) forms a symplectic vector space called the *canonical standard vector space*. Later on we will consider the case $K = \mathbb{Q}_p$ and the symplectic vector space $(\mathbb{Q}_p^{2n}, J_0)$.

This example is key because it can be proven that any symplectic vector space is symplectomorphic to (K^{2n}, J_0). The idea behind the proof is simple: consider a symplectic space (F, β) and E a subspace of F, then $F = E \oplus E^*$. Taking $\{e_i\}$ as a basis of E and $\{\omega_i\}$ as a basis for E^*, the symplectic form satisfies $\beta((v,\omega),(v',\omega')) = \omega'(v) - \omega(v')$. If E is a Lagrangian subspace, then the rest is straightforward. Let's explore this result in more details:

Theorem 1.24 (Linear Darboux's theorem). *Any symplectic space (F,β) is symplectomorphic to the standard symplectic space (K^{2n},β_0), i.e. there exist a basis $\{e_i,\omega_j\}_{i,j=1,\ldots,n}$ of F such that β satisfies*

$$\beta(e_i,e_j) = 0 = \beta(\omega_i,\omega_j), \quad \beta(e_i,\omega_j) = \delta_{ij}, \quad \beta(\omega_i,e_j) = -\delta_{ij}, \; \forall\, 0 \leq i,\, j \leq n.$$

Such a basis is known as the symplectic (or Darboux) basis.

Proof. Let E be a subspace of F. Consider the map $\phi^\flat : E \to E^*$

$$\begin{aligned} \phi^\flat &: E \longrightarrow E^* \\ \phi^\flat(v) &\longmapsto \beta(v,\cdot) \end{aligned}$$

Given the properties of the symplectic form, it can be proven that $\phi^\flat$ is an isomorphism.

Let L and M be two Lagrangian subspaces such that $L \cap M = \{0\}$. The following composition defines an isomorphism between M and L^*

$$M \overset{i}{\hookrightarrow} E \xrightarrow{\phi^\flat} E^* \to L^*,$$

where the last arrow is the dual to the inclusion $L \to E$. If we pick $e_1,\ldots,e_n$ as a base for L and $\omega_1,\ldots,\omega_n$ as a dual base for $L^* \cong M$, the form

$$\beta : F \times F \to K$$

satisfies

$$\beta(e_i,e_j) = 0 = \beta(\omega_i,\omega_j), \quad \beta(e_i,\omega_j) = \delta_{ij}, \quad \beta(\omega_i,e_j) = -\delta_{ij}, \; \forall\, 0 \leq i,\, j \leq n.$$

Therefore (E,β) is symplectomorphic to (K^{2n},β_0). $\square$

2 Representation theory

2.1 Lie groups and Lie algebras

We will develop representation theory on topological groups in the following sections, but it is important to have in mind that the groups we will be dealing with are not only topological but carry a manifold structure. In physics the study of Lie groups and Lie algebras is essential as they describe the symmetries of the physical systems, therefore in this section we will be giving the basic definitions and results on these objects. An usual example is the group of all rotations in a 3-dimensional space, $SO(3)$. Another example is the Poincaré group formed by both translations and Lorentz transformations in space-time. It fully describes General Relativity. In quantum mechanics, it is the Heisenberg group that describes the space of momentum and position. We consider a field K that can be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$. This section is based on [Schn2011].

Let V be a K-Banach space, and a power series $F(X) := \sum_{\alpha_{\mathbb{N}^n}} = X^\alpha v_\alpha$ on a variable $(X_1, ..., X_n)$, where $X^\alpha = X_1^{\alpha_1} \cdot ... \cdot X_n^{\alpha_n}$. For any real ϵ, we say that $F(X)$ is ϵ-convergent if $\lim_{|\alpha| \to \infty} \epsilon^{|\alpha|} ||v_\alpha|| = 0$. We define the K-vector space

$$\mathcal{F}_\epsilon(K^n; V) := \text{ all the } \epsilon\text{-convergent power series } F(X).$$

Definition 2.1. *Let U be an open subset of K^n. A function $f : U \to V$ is said to be locally analytic if for any point $x_0 \in U$ there is a ball $B_\epsilon(x_0) \subseteq U$ around x_0 and a power series $F \in \mathcal{F}_\epsilon(K^n; V)$ such that*

$$f(x) = F(x - x_0) \text{ for any } x \in B_\epsilon(x_0).$$

We define the K-vector space

$$C^{an}(U, V) := \text{ all locally analytic functions } f : U \to V.$$

Let $(M, \mathcal{A})$ be a manifold over K. A function $f : M \to V$ is called locally analytic if $f \circ \varphi^{-1} \in C^{an}(\varphi(U), V)$ for any chart (U, φ) for M. We define the K-vector space

$$C^{an}(M, V) := \text{ all locally analytic functions } f : M \to E.$$

Definition 2.2. *Let M and N be two manifolds over K. A map $g : M \to N$ is called locally analytic if g is continuous and $\varphi \circ g \in C^{an}(g^{-1}(U), K^n)$ for any chart (U, φ, K^n) for N.*

Definition 2.3. *A Lie group over a field K is a locally analytic manifold over K with a group structure such that the identity, inverse and multiplication map*

$$\begin{array}{lll} \{\emptyset\} \to G, & G \to G, & G \times G \to G. \\ g \longmapsto 1, & g \longmapsto g^{-1} & (g, h) \longmapsto gh \end{array}$$

are locally analytic.

Notice that a Lie group is a topological group.

Example 2.4. The simplest example of a Lie group over K is K^n for any $n \in \mathbb{N}$. For example, the additive group $\mathbb{Q}_p^n$ is a Lie group over the non-archimedean field $\mathbb{Q}_p$. It is the same for the real or complex field.

Proposition 2.5. *$GL(n,K)$ is a Lie group and any closed subgroup G of $GL(n,K)$ is also a Lie group.* □

The most common examples for Lie groups are matrix Lie groups like $SO(n,K)$, $SU(n,K)$ or $GL(n,K)$. Other examples that are important in the context of this work are the following.

Example 2.6. The group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \text{ where } a,\ b,\ c \in K$$

is a Lie group called the 3-dimensional Heisenberg group $H(3,K)$. If $K = \mathbb{R}$, $H(3,\mathbb{R})$ is a Lie group which the topology induced by the usual norm. Same case for $K = \mathbb{Q}_p$, with the p-adic topology.

Example 2.7. The symplectic group defined by $Sp(2n,K) = \{g \in GL(2n.K) |\, gJg^t = J\}$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is a Lie group. $Sp(2n,K)$ is the set of linear transformations on a $2n-$dimensional K-vector space that preserve a symplectic form.

Definition 2.8. *Let G_1 and G_2 be two Lie groups over K. A homomorphism of Lie groups $f : G_1 \to G_2$ is a locally analytic map which is also a group homomorphism.*

Definition 2.9. *A K-vector space $\mathfrak{g}$ equipped with a K-bilinear map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that is antisymmetric and satisfies the Jacobi identity $[[v,w],z] + [[w,z],v] + [[z,v],w] = 0$ is called a Lie algebra over K. This map is called the Lie bracket.

We can define a Lie algebra for any Lie group. Given a Lie group G consider $T_e(G)$ the tangent space of the manifold G at the identity e. That is the set of all tangent vector to G at e. In this context, a tangent vector is understood as an equivalence class of a chart in the neighborhood of e and a vector at K^n.

Given two elements $v, w \in T_e(G)$ we define a map

$$\begin{aligned} [\cdot,\cdot] : T_e(G) \times T_e(G) &\to T_e(G). \\ (v,w) &\longmapsto vw - wv \end{aligned}$$

Definition 2.10. *Given a Lie group G, the pair $Lie(G) := (T_e(G), [,])$ is called the Lie algebra of G.*

The map $[,]$ is antisymmetric, K-bilinear and satisfies the Jacobi identity. Therefore the Lie algebra of G is a Lie algebra.

A Lie algebra can also be defined by any associative algebra A. Redefining the ring multiplication by $[X, Y] = XY - YX$, the vector space A with the multiplication $[\,,]$ is a Lie algebra. Consider the algebra of matrices with coefficients in any field K, $A = M_n(K)$. Changing the matrix composition by $[X, Y] = XY - YX$ gives back the Lie algebra $\mathfrak{gl}(n, K)$.

Proposition 2.11. *Let G be a closed Lie subgroup of $GL(n, K)$. The set of all $X \in Mat_n(K)$ such that $exp(tX) \subset G$ with the Lie bracket $[X, Y] := XY - YX$ is a Lie subalgebra of $\mathfrak{gl}(n, K)$.* □

Example 2.12. The set $\mathfrak{h}_n(K)$ of all matrices $X \in Mat_3(K)$ of the form

$$\begin{pmatrix} 0 & \boldsymbol{a} & c \\ 0 & 0 & \boldsymbol{b} \\ 0 & 0 & 0 \end{pmatrix} \text{ where } \boldsymbol{a}, \boldsymbol{b} \in K^3 \text{ and } c \in K$$

with the natural commutator given by $[X, Y] = XY - YX$ for any $X, Y \in Mat_3(K)$ is the Lie algebra of the 3-dimensional Heisenberg Lie group defined in Example 2.6.

Example 2.13. The set $\mathfrak{sp}(2n, F)$ of all matrices $X \in Mat_{2n}(K)$ that satisfy $XJ + J^t X = 0$, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

is the Lie algebra of $Sp(2n, K)$.

Definition 2.14. *If $(\mathfrak{g}, [,]_1)$ and $(\mathfrak{g}, [,]_2)$ are two Lie algebras over K then a homomorphism of Lie algebras $\sigma : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a K-linear map which satisfies*

$$[\sigma(x), \sigma(y)]_2 = \sigma([x, y]_1) \text{ for any } x, y \in \mathfrak{g}_1.$$

2.2 Profinite groups and locally profinite groups

Unlike the real and complex cases, p-adic Lie groups can be seen in a purely algebraic way as a profinite group. In this section we will define the concept of profinite, pro-p-groups and p-valuable groups using references [Blon2011] and [Neuk1986]. We will follow the purely algebraic approach in [Schn2011], first explored by Lazard in the 1960s, to prove that any p-adic Lie group has an open and compact subgroup that is a p-valuable group and that any p-valuable group has a natural structure of a p-adic Lie group.

The concept of a profinite group comes from the definition of the projective limit of finite groups.

Definition 2.15. *A projective system of groups over an ordered set I is a family*

$$\{G_i, f_{ij} | i,\, j \in I,\, i \leq j\}$$

of groups G_i and homomorphisms $f_{ij} : G_j \to G_i$ such that $f_{ik} = f_{ij} \circ f_{jk}$, whenever $i \leq j \leq k$. Equivalently, such that the following diagram commutes.

$$\begin{tikzcd} G_k \arrow[rr, "f_{ik}"] \arrow[dr, "f_{jk}"'] & & G_i \\ & G_j \arrow[ur, "f_{ij}"'] & \end{tikzcd}$$

Definition 2.16. *Let (G_i, f_{ij}) be a projective system. A pair $(G, \pi_i)_{i \in I}$ is called an inverse (or projective) limit of this system if the morphism $\pi_i : G \to G_i$ satisfy that $\pi_i = f_{ij} \circ \pi_j$ for all $i \leq j$ and if for any other such pair (H, ψ_i) there exists a unique morphism $u : H \to G$ that makes the following diagram commute*

$$\begin{tikzcd} & H \arrow[d, "u"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & \\ & G \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\ G_j \arrow[rr, "f_{ij}"'] & & G_i \end{tikzcd}$$

The morphisms π_i are called projections. The projective limit of (G_i, f_{ij}) is denoted as $\lim\limits_{i \in I} G_i$.

Proposition 2.17. *The projective limit $G = \varprojlim\limits_{i \in I} G_i$ of a projective system $\{G_i, f_{ij}\}_{i,j \in I}$ exists and can be realized as the group*

$$G = \{\prod_{i \in I} \sigma_i \in \prod_{i \in I} G_i |\, f_{ij}(\sigma_j) = \sigma_i \text{ if } i \leq j\}.$$

□

If G_i are topological spaces and f_{ij} are continuous maps then G is a closed subspace of the topological space $\prod_{i \in I} G_i$.

Definition 2.18. *A profinite group G is a topological group isomorphic to the projective limit of a projective system of discrete finite groups. If the finite groups are of order power of p, then G is called a pro-p group.*

Often, profinite groups are defined by their topological properties.

Proposition 2.19. *A Hausdorff and compact topological group G with a basis of open neighborhoods of the identity that consists of normal subgroups is the profinite group*

$$G \cong \varprojlim_{N} G/N$$

where N runs through all open normal subgroups of G. The converse is also true.

Notice that such a topological group is totally disconnected, i.e. every element of G is its own connected component.

Proof. Let G be a profinite group and $\{N_i, i \in I\}$ the set of its normal subgroups. Since $\bigcup_{g \in G} gN_i$ for any $i \in I$ is a open disjoint covering of G and by definition G is compact, each N_i has a finite number of cosets. Therefore $G_i = G/N_i$ is a finite group. Let say that $i \leq j$ if $N_i \supseteq N_j$, and define $f_{ij} : G_j \to G_i$ as the canonical projections. Then $\{G_i, f_{ij}\}$ is a projective system of finite groups and the homomorphism

$$f : G \to \varprojlim_{i \in I} G_i$$

that sends σ to $\prod_{i \in I} \sigma_i$, where $\sigma_i = \sigma mod N_i$, is an isomorphism and a homeomorphism. The homomorphism f is injective because the kernel of f is $\bigcap_{i \in I} N_i$ which is $\{1\}$ since G is Hausdorff.

The groups $U_S = \prod\limits_{i \notin S} G_i \times \prod\limits_{i \in S} \{1_{G_i}\}$ form a subbasis of open neighborhoods of 1 in $\prod_{i \in I} G_i$ when S runs through all the finite subsets of I. Notice that $f^{-1}(U_S \cap \varprojlim\limits_{i \in I})G_i = \bigcap\limits_{i \in S} N_i$, then f is continuous.

By using that the image of G by f is closed in $\varprojlim\limits_{i \in I} G_i$ it can be proven that $f(G)$ is dense in $\varprojlim\limits_{i \in I} G_i$, hence $f(G) = \varprojlim\limits_{i \in I} G_i$. Since G is compact, f maps closed sets into closed sets, and it is therefore an open map. With this we proved that f is an isomorphism and a homeomorphism.

To prove the converse, we consider a projective system of finite groups $\{G_i, f_{ij}\}$. The groups G_i can be considered as discrete and compact topological spaces, and $G = \lim_{\leftarrow} G_i$ is a closed subgroup of the compact Hausdorff group $\prod_{i \in I} G_i$. Then G is a Hausdorff compact topological group. We can form a basis of open neighborhoods of 1 by defining the normal subgroups $U_S \cap G$ where $U_S = \prod\limits_{i \notin S} G_i \times \prod\limits_{i \in S} H_i$ when S is a finite subset of I and H_i a normal subgroup of G_i. $\square$

Example 2.20. The additive group of the p-adic integers is a pro-p group $\mathbb{Z}_p = \varprojlim\limits_{n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$.

Definition 2.21. *A topological group G is locally profinite if it satisfies one of the equivalent following conditions:*

(a) every neighborhood of the identity in G contains a compact open subgroup,

(b) G is locally compact and totally disconnected.

Notice that compact locally profinite groups are profinite. In fact, a compact open subgroup K of a locally compact, totally disconnected, Hausdorff group is a profinite group.

Example 2.22. The p-adic field $\mathbb{Q}_p$ seen as an additive group is a locally profinite group. We can consider the neighborhoods of the identity given by $p^n\mathbb{Z}_p$, for each $n \in \mathbb{Z}$.

It is important to remember some fundamental properties of the locally compact totally disconnected topology. Let G be a locally profinite group.

(a) Closed subgroups of G are locally profinite. This is also true for quotient groups by closed normal subgroups.

(b) Any compact subgroup of G is contained in an open compact subgroup of G.

(c) Points are closed in G.

(d) The centralizer of any element of G is closed.

We will be following [Schn2011] to see that pro-p groups can be seen as a compact p-adic Lie group in a natural way when considered together with a p-valuation.

Definition 2.23. *Let G be a group. A p-valuation on G is a function $\omega : G \to \mathbb{R}^+ \cup \{\infty\}$, with $\omega(1) = \infty$, and satisfies*

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$

where the commutator is the usual $[g,h] = ghg^{-1}h^{-1}$ for g, $h \in G$.

Proposition 2.24. *The following properties are satisfied:*

(a) $\omega(g^{-1}) = \omega(g)$ *for any* $g \in G$,

(b) $\omega(ghg^{-1}) = \omega(h)$,

(c) $\omega(gh) = min(\omega(g), \omega(h))$ *if* $\omega(g) \neq \omega(h)$. □

Each p-valuation defines a unique topology over the group G. We can define two families of subgroups of G that will be useful in the future. For any real number $\nu > 0$,

$$G_\nu := \{g \in G : \omega(g) \geq \nu\} \text{ and } G_\nu^0 := \{g \in G : \omega(g) > \nu\}.$$

We can prove using the second property of Proposition 2.24 that both G_ν and G_ν^0 are normal subgroups of G.

The groups G_ν form a decreasing exhaustive and separated filtration of G with the additional properties

$$G_\nu = \bigcap_{\nu'<\nu} G_{\nu'} \text{ and } [G_\nu, G_{\nu'}] \subseteq G_{\nu+\nu'}.$$

This gives a natural topological structure to the group G. In fact, there is a unique topological group structure on G for which G_ν forms a fundamental system of open neighborhoods of the identity. We say that it is the topology defined by the valuation ω.

Example 2.25. Let E be a finite extension of the field $\mathbb{Q}_p$, o_E its ring of integers, π_E a prime element and v its additive valuation normalized by $v(p) = 1$. For any fixed $n \in \mathbb{N}$ we consider the algebra $M_{n\times n}(E)$ of $n \times n$ matrices over E. For any non-zero matrix $A = (a_{ij})$ we define a map $\omega := \min_{i,j} v(a_{ij})$ and $\omega(0) = \infty$. This map is a p-valuation on the group

$$G := \left\{ g \in GL_n(E) : \omega(g-1) > \frac{1}{p-1} \right\}.$$

Definition 2.26. *For each $\nu > 0$ consider the subquotient group*

$$gr_\nu G := G_\nu / G_\nu^0.$$

The direct sum

$$gr\, G := \bigoplus_{\nu>0} gr_\nu G$$

is called the graded abelian group. Each element $\xi \in G$ is called homogeneous of degree ν if $\xi \in G_\nu$, and an element $g \in G_\nu$ such that $\xi = gG_{\nu+}$ is called the representative of ξ.

Notice that grG is in fact a $\mathbb{F}_p$-vector space, since $p\xi = 0$ for any homogeneous element $\xi \in grG$. We can define a graded $\mathbb{F}_p$-bilinear map

$$[\ ,\ ] : grG \times grG \to grG$$

which satisfies $[\xi, \xi] = 0$ for any $\xi \in grG$ and satisfies the Jacobi identity:

$$[[\zeta, \xi], \eta] + [[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] \quad \forall \zeta,\ \xi,\ \eta \in grG.$$

Therefore $(grG, [\ ,\ ])$ is a Lie algebra over $\mathbb{F}_p$. The discussion on the structure of grG does not end here. The map

$$\begin{aligned} gr_\nu G &\longrightarrow gr_{\nu+1} G \\ gG_\nu^0 &\longmapsto g^p G_{(\nu+1)}^0 \end{aligned}$$

is well defined and $\mathbb{F}_p$-linear. The direct sum of these maps is then a $\mathbb{F}_p$-linear map

$$P : grG \longrightarrow grG.$$

We will see that grG is a graded module over the ring of polynomial functions on P with coefficients on $\mathbb{F}_p$, $\mathbb{F}_p[P]$.

Let G be a profinite group equipped with the topology defined by a p-valuation ω. Then G_ν are open normal subgroups, G/G_ν are finite, and $G = \varprojlim_\nu G/G_\nu$. Each G/G_ν is a p-group, then G is necessarily a pro-p group.

Definition 2.27. *The pair (G, ω) is called of finite rank if grG is finitely generated as an $\mathbb{F}_p[P]$-module and*

$$rank(G, \omega) := rank_{\mathbb{F}_p[P]} grG$$

is the rank of the pair.

It can be shown that the property of being of finite rank does not depend on the choice of a p-valuation.

Let $g \in G$ be any element of the group, then we can define a group homomorphism

$$\begin{aligned} c : \mathbb{Z} &\longrightarrow G \\ m &\longmapsto g^m. \end{aligned}$$

It extends uniquely to a continuous map

$$c : \mathbb{Z}_p \to \varprojlim_N \mathbb{Z}/p^{a_N}\mathbb{Z} \xrightarrow{c} \varprojlim_N G/N = G$$

for any normal subgroup N of G. For any given $g_1, ..., g_r \in G$, we can consider the continuous map

$$\begin{aligned} \mathbb{Z}_p^r &\longrightarrow G \\ (x_1, ..., x_r) &\longmapsto g_1^{x_q} \cdots g_r^{x^r}. \end{aligned}$$

This map depends on the order of the g_i therefore it is not in general a group homomorphism.

Definition 2.28. *The sequence of elements $(g_1, ..., g_r)$ in G is called an ordered basis of (G, ω) if the map defined above is a bijection (by compactness it is a homeomorphism) and*

$$\omega(g_1^{x_1} \cdots g_r^{x^r}) = \min_{1 \le i \le r} (\omega(g_i) + v(x_i)) \text{ for any } x_1, ..., x_r \in \mathbb{Z}_p,$$

where v is the p-adic valuation on $\mathbb{Q}_p$.

We introduce some new notation. For any element $g \neq 1$ in G,

$$\sigma(g) := gG^0_{\omega(g)} \in grG.$$

Proposition 2.29. *If (G, ω) is of finite rank, then for any sequence of elements $(g_1, ..., g_r) \in G \setminus \{1\}$ the following assertions are equivalent:*

(a) $(g_1, ..., g_r)$ *is an ordered basis of* (G, ω)*;*

(b) $\sigma(g_1), .., \sigma(g_r)$ *is a basis of the* $\mathbb{F}_p[P]$*-module* grG.

Proposition 2.30. *Any* (G, ω) *of finite rank has an ordered basis of length equal to* $rank(G, \omega)$.

Corollary 2.31. *The next results follow from the last proposition.*

(a) If (G, ω) *is of finite rank then* G *is topologically finitely generated.*

(b) The elements of an ordered basis of (G, ω) *cannot be* p*-th powers in* G*.*

Definition 2.32. (G, ω) *is called saturated if any* $g \in G$ *such that* $\omega(g) > \frac{p}{p-1}$ *is a* p*-th power in* G*.*

Proposition 2.33. *Let* (G, ω) *be of finite rank with an ordered basis* $(g_1, ..., g_r)$*; then it is saturated if and only if*

$$\frac{1}{p-1} < \omega(g_i) \leq \frac{p}{p-1} \text{ for any } 1 \leq i \leq r.$$

Definition 2.34. *A pro-*p *group is called* p*-valuable if there exists a* p*-valuation* ω *on* G *that defines the topology on* G *and such that the rank of* G *is finite.*

It has to be seen if p-valuable groups exist. We will see in the following results that any p-adic Lie group contains a p-valuable group.

Theorem 2.35. *Let* G *be a* p*-adic Lie group; then there exists a compact open subgroup* $G' \subseteq G$ *and an integral valued* p*-valuation* ω *on* G' *defining the topology of* G' *such that:*

(a) (G', ω) *is saturated;*

(b) rank G' = *dim* G*.*

Proof. Let d be the dimension of the p-adic Lie group G. Then we choose a chart $c = (U, \varphi, \mathbb{Q}_p^d)$ for G around the identity $e \in G$. We impose that $\varphi(e) = 0$. The multiplication in G, m_G, is continuous, therefore there is an open neighborhood $V \subset U$ of e such that $m_G(V \times V) \subseteq U$. Then $(V, \varphi_{|_V}, \mathbb{Q}_p^d)$ is also a chart around e and

$$\varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1}) : \varphi(V) \times \varphi(V) \to \varphi(U)$$

is locally analytic. We consider the expansion around 0 and find a $d-$tuple $(F_1, ..., F_d)$ of power series

$$F_i(X, Y) = \sum_{\alpha,\beta} c_{i,\alpha,\beta} X^\alpha Y^\beta$$

where $X = (X_1, ..., X_d)$ and $Y = (Y_1, ..., Y_d)$, and $c_{i,\alpha,\beta} \in \mathbb{Q}_p$. For any $n >> 0$,

(a) $p^n\mathbb{Z}_p^d \subseteq \varphi(V)$,

(b) $\lim_{|\alpha|+|\beta|\to\infty}(v(c_{i,\alpha,\beta}) + n(|\alpha| + |\beta|)) = \infty$ for any $1 \leq i \leq d$

(c) $(F_1(x,y), ..., F_d(x,y)) = \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$ for any $x \in (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$ in $p^n\mathbb{Z}_p^d$.

One finds that

$$v(c_{i,\alpha,\beta}) + n(|\alpha| + |\beta|) \geq n \text{ for any } \alpha, \beta, i.$$

This implies that for big enough n, the sets $\varphi^{-1}(p^n\mathbb{Z}_p^d)$ are compact open subsets of G and are closed by multiplication. By a similar argument over the inverse map, one can see that $\varphi^{-1}(p^n\mathbb{Z}_p^d)$ is indeed a subgroup of G. If we fix a big enough n, and replace the chart φ by $\psi := p{-}n\varphi$, the coefficients of both series expansions lie in $\mathbb{Z}_p$.

We define

$$G' := \psi^{-1}(p^2\mathbb{Z}_p^d)$$

and

$$\begin{aligned} \omega : G' \setminus \{0\} &\longrightarrow \mathbb{R} \cup \{\infty\} \\ g &\longmapsto l + \delta \text{ if } g \in \psi^{-1}(p^{l+1}\mathbb{Z}_p^d) \setminus \psi^{-1}(p^{l+2}\mathbb{Z}_p^d) \end{aligned}$$

where $\delta = 1$ for $p = 2$ and $\delta = 0$ for $p \neq 0$. The function ω is well defined because $G'_{l+\delta} = \psi^{-1}(p^{l+1}\mathbb{Z}_p^d)$ and is an integral valued p-valuation on G'.

What is left is to see that (G', ω) is saturated and that its rank is equal to the dimension of G. For any $l \in \mathbb{N}$, the operator $P : gr_{l+\delta}G' \to gr_{l+\delta+1}G'$ is an injective map between finite abelian groups of the same order. It is then bijective. This means that grG' is finitely generated by $gr_{1+\delta}G'$ as an $\mathbb{F}_p[P]$ module. We conclude by Lemma 26.10 in [Schn2011] that (G', ω) is saturated. At last, we use Proposition 26.15 in [Schn2011] to prove that

$$\begin{aligned} rankG &= \lim_{n\to\infty} \frac{v([G' : G'^{p^n}])}{n} \\ &= \lim_{n\to\infty} \frac{v([G' : G'_{n+2+\delta}])}{n} \\ &= \lim_{n\to\infty} \frac{dn}{n} \\ &= d = dimG. \end{aligned}$$

□

This result provides us with a way to generate p-valuable groups. In fact, it can be seen that it gives all of them and that any p-valuable group has a natural structure of a p-adic Lie group.

lets choose an ordered basis $(g_1, ..., g_r)$ of (G, ω) so that we can define an homeomorphism

$$\begin{aligned} c : \mathbb{Z}_p^r &\xrightarrow{\sim} G \\ (x_1, ..., x_r) &\mapsto g_1^{x_1} \cdot ... \cdot g_r^{x_r}. \end{aligned}$$

Let φ be the inverse of c, then we can see the triple $(G, \varphi, \mathbb{Q}_p^r)$ as a "global" chart for G. By equipping G with the maximal atlas containing only the global chart, we obtain the structure of an r-dimensional manifold over $\mathbb{Q}_p$ on G. We can construct the coordinate functions as

$$\begin{aligned} \varphi_i : G &\longrightarrow \mathbb{Q}_p \\ g &\longmapsto g_1^{x_1} \cdot \ldots \cdot g_i^{x_i} \text{ if } g = g_1^{x_1} \cdot \ldots \cdot g_r^{x_r}. \end{aligned}$$

By construction, they are locally analytic. Since the manifold structure on G is independent of the choice of a basis or a valuation ω, the group operation of G is a morphism of manifolds. We have proven then that any p-valuable pro-p group G carries a natural structure of compact p-adic Lie group, and dim$G$ = rank G. It is left to prove that this structure is unique.

Theorem 2.36. *Any p-valuable pro-p group G carries a unique structure of manifold over $\mathbb{Q}_p$ which makes it into a p-adic Lie group.*

Proof. Let G_1 and G_2 be two different Lie group structures for a group G. Uniqueness is proven by showing that $id_G : G_1 \to G_2$ is a Lie group isomorphism. For more details see [Schn2011]. □

2.3 Representation of locally compact and totally disconnected groups

In this section we will discuss the representation theory of locally compact and totally disconnected p-adic groups. We will use as references [Se1967] for an understanding on the key ideas, [Blon2011] for a first glance of the topic and [Ste2009] for a more complete discussion. The specific references for each concept will be indicated in each section.

2.3.1 Basic definitions and results

Let G be a topological group, that is a topological space that has continuous group operations. Notice that a p-adic field $\mathbb{Q}_p$ is a topological group with respect to the addition.

The topology on $\mathbb{Q}_p$ induced by the p-adic valuation determines the product topology over the set $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$ of $n \times n$ matrices with coefficients in the p-adic field. As a main example we will take the subgroup $GL_n(\mathbb{Q}_p)$ of $M_n(\mathbb{Q}_p)$. $GL_n(\mathbb{Q}_p)$ is the inverse image of $\mathbb{Q}_p^\times$ relative to the continuous map $det : M_n(\mathbb{Q}_p) \to \mathbb{Q}_p$, it is an open subgroup and this naturally inherits the topology of $M_n(\mathbb{Q}_p)$.

Definition 2.37. *A complex representation of a topological group G is a pair (π, V) where V is $\mathbb{C}$-vector space and π is a homomorphism from G into the group of invertible linear operators in V.*

If n is the dimension of V, we say that the representation is n-dimensional. If V is an infinite dimensional vector space, then the representation is infinite dimensional. We can refer to a representation simply by π or V without specifying the complete pair.

Definition 2.38. *We define a G-morphism between two representations (π_1, V_1) and (π_2, V_2) of G as a linear homomorphism $\phi : V_1 \to V_2$ such that for any $g \in G$, $\phi \circ \pi_1(g) = \pi_2(g) \circ \phi$. We also say that ϕ intertwines π_1 and π_2. The set of all G-morfisms that intertwine π_1 and π_2 is denoted as $\mathrm{Hom}_G(\pi_1, \pi_2)$.*

We will use the notation $\mathrm{Hom}_G(V, V')$ too depending on the context.

Definition 2.39. *Two representations (π_1, V_1) and (π_2, V_2) are equivalent (or isomorphic) if $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains a linear isomorphism. We will write $\pi_1 \simeq \pi_2$.*

Definition 2.40. *A subrepresentation of (π, V) is a pair $(\pi|_W, W)$ where W is an invariant subspace of V, i.e. $\forall w \in W$, $\pi(g)(w) \in W$ for any $g \in G$.*

The operators $\pi(g)$ define automorphisms of the quotient vector space V/W, hence a quotient representation $(\bar{\pi}, V/W)$ of (π, V). Given two subrepresentations $(\pi_{|W}, W)$ and $(\pi_{|W'}, W')$ of (π, V) with W contained in W', the quotient representation $(\bar{\pi}_{|W'}, W'/W)$ is called a subquocient representation of (π, V).

Definition 2.41. *A representation (π, V) of G is said to be finitely generated if there exists a finite subset $\{v_1, ..., v_n\}$ of V such that*

$$V = Span(\{\pi(g)v_j | g \in G, 1 \leq j \leq m\}).$$

Definition 2.42. *A representation (π, V) of a group G is said to be irreducible if it is non-zero and has no non-trivial subrepresentation. If π is not irreducible, it is said to be reducible.*

If a representation V is irreducible, $V = Span(\{\pi(g)v \,|\, g \in G\})$ for any non-zero $v \in V$. This is because $Span(\{\pi(g)v \,|\, g \in G\}) \subset V$ is a subrepresentation of V.

Definition 2.43. *A representation (π, V) of G is semisimple if V is a direct sum of irreducible subrepresentations of π.*

We are interested in studying the representation of p-adic reductive groups.

Definition 2.44. *We will define a p-adic group as a subgroup of the $GL(\mathbb{Q}_p^n)$ defined by polynomial equations in the entries of matrices.*

Definition 2.45. *A p-adic group is said to be reductive if it has a semisimple representation with finite kernel.*

An additional condition that proves to be useful when considering topological groups is the property of smoothness. This definition is skipped for finite groups because every representation is smooth in the discrete topology.

Definition 2.46. *A representation (π, V) of a topological group G is smooth if the stabilizer $\mathrm{Stab}_G(v) = \{g \in G/\pi(g)(v) = v\}$ of v in G is an open subgroup of G for every $v \in V$.*

It is easy to see that subrepresentations, quotients and subquocients of smooth representations are smooth. Notice that a representation (π, V) is smooth if and only if for each $v \in V$ there exists some open compact open subgroup K of G such that v lies in $V^k = \{v \in V | \, \pi(k)v = v \forall k \in K\}$.

Lemma 2.47. *The following are equivalent.*

1. (π, V) *is semisimple.*
2. *For every invariant subspace* $W \subset V$ *there exists an invariant subspace* $W^\perp$ *such that*

$$V = W \oplus W^\perp.$$

Proof. Given a representation V and an invariant subspace $W \subset V$, there exists a non-empty closed partially ordered set of subrepresentations $U \subset V$ such that $U \cap W = \{0\}$. Zorn's lemma implies the existence of a maximal element U_{max} of this ordered set . Suppose that $W \oplus U_{max} \neq V$. Since (π, V) is semisimple, there exists an irreducible submodule U' such that $U' \nsubseteq W \cap U_{\max}$. From the irreducibility of U' we deduce that $U' \cap (W \oplus U) = \{0\}$. This contradicts the maximality of U_{max}. Therefore $U \oplus W = V$ and $U = W^\perp$.

To finish the prove we consider all the partially ordered sums of families of irreducible subrepresentations $\oplus_\alpha W_\alpha$ and apply again Zorn's lemma. We define $W := \oplus_\alpha W_\alpha$ as the maximal family. Then by hypothesis there exists an invariant subspace U such that $W \oplus U = V$. We want to prove that the maximal family generates all the representation. Assume $U \neq \{0\}$, it can be shown that there exist subrepresentations U_1 and U_2 such that $U_2 \subset U_1 \subset U$ and U_1/U_2 is irreducible. Using the hypothesis, $W \oplus U_2$ has a complement U_3 such that

$$W \oplus U_2 \oplus U_3 = V.$$

We can think of U_1/U_2 as an irreducible subrepresentation in U_3 up to isomorphisms. If we identify U_1/U_2 with an irreducible subrepresentation U_4 of U_3, then $W \oplus U_4$ contradicts the fact that W is the sum of a maximal family of irreducible subrepresentations. Then $U = \{0\}$ and $V = \oplus_\alpha W_\alpha$. □

Lemma 2.48. *Let G be a locally compact, totally disconnected and Hausdorff topological group, and assume, in addition, that it is compact. Then*

(a) *Every smooth representation of G is semisimple.*

(b) *For any given irreducible smooth representation* (π, V) *of G, there exists an open normal subgroup N of G such that* $\pi(n) = id_V$ *for all* $n \in N$.

Proof. Let $v \in V$, then $W = Span(\{\pi(g)v | \, g \in G\})$ is finite dimensional. The set $N = \bigcap_{g \in G/N} gStab_G(v)g^{-1}$ is an open normal subgroup of G, therefore G/N is finite.

Then the restriction of π to the subrepresentation W for each $n \in G/N$ is the identity $\pi(n)|_W = id_W$. Since representations of finite groups are semisimple, W is the direct sum of irreducible submodules of V. Since we have shown that any $v \in V$ belongs to a finite-dimensional semisimple subrepresentation of V we can define a maximal subrepresentation W' of V. If we assume that $W' \neq V$, we can find another subrepresentation that contains W', contradicting maximality. Therefore $V = W'$ is semisimple.

The second part follows immediately from the fact that whenever π is irreducible, $Span(\{\pi(g)v |\, g \in G\}) = V$. □

Given a representation (π, V) of a non-compact group G, the subrepresentation on any open compact subgroup K is smooth, and by the previous lemma, is semisimple. Then V is the direct sum of K-invariant subspaces, subspaces that are irreducible representations of K. To prove the following lemma we will use the following result: If (π, V) is an irreducible smooth representation of G, then $\dim(V)$ is (at most) countable.

Theorem 2.49 (Schur's Lemma)**.** *Let (π, V) be an irreducible smooth representation of G, then $\mathrm{End}_G(V) = \mathbb{C}$. That is, every operator with intertwines π with itself π is a scalar multiple of the identity.*

Proof. Let's show first that the dimension of $\mathrm{End}_{G(V)}$ is at most countable. Let v be a non-zero vector of V. Since π is irreducible, $V = Span(\pi(g)v \,|\, g \in G)$, and then $A \in \mathrm{End}_G(V)$ is totally defined by $A(v)$. This means that the map from $\mathrm{End}_G(V)$ into V that takes $A \longmapsto A(v)$ is injective and then $dim(V)$ is countable, which implies that $\mathrm{End}_G(V)$ has also countable dimension. Notice that both $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ are subrepresentations of V. If A is non-zero, $\mathrm{Im}(A)$ is nonzero. By the irreducibility of V, $\mathrm{Im}(A) = V$ (and $\mathrm{Ker}(A)$ is zero), and then A is surjective. Therefore A is bijective and any non-zero element of $\mathrm{End}_G(V)$ is invertible. $\mathrm{End}_G(V)$ is a division ring.

For any $A \in \mathrm{End}_G(V)$, the set $\mathbb{C}(A)$ is commutative and all nonzero elements are invertible, thus $\mathbb{C}(A)$ is a field. If A is trandescendal over $\mathbb{C}$, the set $\{1/(A - c) \,|\, c \in \mathbb{C}\}$ is linearly independent and uncountable. Then $dim_{\mathbb{C}}(\mathbb{C}(A))$ is uncountable, which contradicts the countability of $\mathrm{End}_G(V)$. Then $\mathbb{C}(A)$ is an algebraic extension of $\mathbb{C}$, and $\mathbb{C}(A) = \mathbb{C}$. □

Corollary 2.50. *Let Z be the center of G and (π, V) be an irreducible and smooth representation of G, then there exists a smooth one-dimensional representation χ_π of Z such that $\pi(z) = \chi_\pi(z) I_V$, for $z \in Z$. We call χ_π the central quasicharacter or central character of π.*

Proof. Consider an element z in the center Z of G, then $\pi(gz) = \pi(zg)$ for all $g \in G$. That implies that $\pi(z) \in \mathrm{End}_G(\pi)$. Then by Schur's Lemma there exists $\chi_\pi(z) \in \mathbb{C}^\times$ such that $\pi(z) = \chi_\pi(z) I_V$ for any $z \in Z$. To see that χ_π is a one-dimensional representation it suffices to notice that the properties of π imply that $\chi_\pi(z_1 z_2) = \chi_\pi(z_1)\chi_\pi(z_2)$. It remains to see that χ_π is smooth. Let $v \in V$ be nonzero. Since (π, V) is smooth, there exists a compact open subgroup K of G such that $v \in V^K = \{v \in V |\, \pi(k)v = v \forall k \in K\}$. Then for all $k \in Z \cap K$, $\chi_\pi(k) = 1$. Then χ_π is a smooth representation of Z. □

Corollary 2.51. *If G is abelian, then every irreducible smooth representation of G is one-dimensional.*

Definition 2.52. *A smooth representation (π, V) is admissible if for every compact open subgroup K of G, the space $V^K = \{v \in V | \pi(k)v = v \forall k \in K\}$ is finite dimensional.*

2.3.2 Induced representations

Induced representations provide a way of constructing representations from a subgroup. Consider G a locally profinite group and $H \subseteq G$ a closed subgroup. Let (σ, W) be a smooth representation of H, which implies that H is also locally profinite. Then $\mathrm{Ind}_H^G(\sigma)$, the space of functions $f : G \to W$ satisfying

(a) $f(hg) = \sigma(h)f(g)$ for all $h \in H$,

(b) there exists an open compact K such that $f(gk) = f(g)$ for all $k \in K$,

defines a smooth representation of G. The action of G on $\mathrm{Ind}_H^G(\sigma)$ is given by $(g \cdot f)(x) = f(xg)$ for any $x \in G$.

We define the compact induction as the space $c - \mathrm{Ind}_H^G(\sigma)$ of functions $f : G \to W$ in $\mathrm{Ind}_H^G(\sigma)$ satisfying the additional condition that $Supp(f) \subset HC$ for some compact subset C.

The map $f : \mathrm{Ind}_H^G(W) \to W$ given by $f \mapsto f(1)$ defines an homomorphism in $\mathrm{Hom}_H(\mathrm{Ind}_H^G(\sigma), \sigma)$. The restriction of this map to $c - \mathrm{Ind}_H^G(\sigma)$ belongs to $Hom_H(c - \mathrm{Ind}_H^G(\sigma), \sigma)$.

Proposition 2.53. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Then*

(a) If H/G is compact and σ is an admissible representation, then $c - \mathrm{Ind}_H^G(\sigma) = \mathrm{Ind}_H^G(\sigma)$ is admissible.

(b) (Frobenius reciprocity) Let (π, V) be a smooth representation of G, then the map f induces a canonical isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \xrightarrow{\sim} \mathrm{Hom}_H(\pi, \sigma).$$

Proof. First, we prove (a). Assume that the representation σ is admissible and that $H \setminus G$ is compact. Choose K a compact open subgroup of G. The images in $H \setminus G$ of the double cosets HgK where g runs over a set of coset representatives for $G \setminus K$ forms a open cover. Using compactness of $H \setminus G$, this open cover has a finite subcover. This means that there are finitely many disjoint $H - K$ double cosets in G. Choose a finite set X such that $G = HXK$.

Let $f \in (Ind_H^G(W))^K$ and $x \in X$. Since $H \cap xKx^{-1}$ is a compact open subgroup of H and σ is admissible, $dim(W^{H \cap xKx^{-1}}) < \infty$. Define $W_0 = \sum_{x \in X} W^{H \cap xKx^{-1}}$. By a straightforward calculation it is shown that $f(X) \subset W_0$ for all $f \in (Ind_H^G(W))^K$. The map $f : (Ind_H^G(W))^K \to C(X, W_0)$ given by $f \longmapsto f|X$ is one to one. Since X is finite and $dim(W_0) < \infty$, we have $dimC(X, W_0) < \infty$, thus $dim((Ind_H^G(W))^K) < \infty$.

For (b), let us consider $\mathcal{A} \in Hom_G(\pi, \mathrm{Ind}_H^G(\sigma))$. The composition of $\mathcal{A}$ and $f \mapsto f(1)$ is the map from V to W that sends a vector v to $(\mathcal{A}v)(1)$. Let $h \in H$ and $v \in V$, then

$$(\mathcal{A}(\pi(h)v))(1) = (\mathrm{Ind}_H^G(\sigma))(h)\mathcal{A}v)(1) = (\mathcal{A}v)(h) = \sigma(h)((\mathcal{A}v)(1)).$$

Then the map $v \mapsto (Av)(1)$ belongs to $Hom_H(\pi, \sigma)$. Suppose that $(Av)(1) = 0$ for all $v \in V$, then

$$(\mathcal{A}v)(g) = (Ind_H^G(\sigma)(g)\mathcal{A}v)(1) = (\mathcal{A}\pi(g)v)(1) = 0$$

for all $g \in G$. Thus $\mathcal{A} = 0$. The map is injective.

To see that it is surjective consider an element $A \in \mathrm{Hom}_H(\pi, \sigma)$ and define a function A^G from V to the space of functions from G to W by $A^G(v)(g) = A(\pi(g)v)$, $v \in V$. Note that

$$A^G(v)(hg) = A(\pi(hg)v) = A(\pi(h)\pi(g)v) = \sigma(h)A(\pi(g)v) = \sigma(h)(A^G(v)(g))$$

where $h \in H, g \in G$. Let $v \in V$. Then $v \in V^K$ for some compact subgroup K of G, so

$$A^G(v)(gk) = A(\pi(gk)v) = A(\pi(g)v) = A^G(v)(g) \quad g \in G, k \in K.$$

Hence $A^G(v)$ is right K-invariant. Then $A^G(v) \in \mathrm{Ind}_H^G(W)$ for every $v \in V$. To finish we have to check that A^G maps to A. By definition of A^G, if $v \in V$, then $A^G(v)(1) = A(\pi(1)v) = A(v)$.

□

A similar result to the first point of the proposition is the following:

Lemma 2.54. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Suppose that $c - \mathrm{Ind}_H^G(\sigma)$ is admissible. Then $c - \mathrm{Ind}_H^G(\sigma) = \mathrm{Ind}_H^G(\sigma)$.*

Definition 2.55. *A representation (π, V) is said to be unitary if there exists a G-invariant inner product on V.*

Notice that we are not asking for V to be a Hilbert space, therefore strictly speaking the previous definition refers to a *unitarizable* representation. After completing V with respect to the given inner product to a Hilbert space, we can talk about unitary representations of G in a Hilbert space.

Lemma 2.56. *If (π, V) is an admissible unitary representation, then π is semisimple.*

There are two kind of induced representations that play central roles in the theory of admissible representations of reductive p-adic groups. One are representations induced from an irreducible smooth representation of open subgroups that are compact modulo the centre of the group G, where the inducing representation σ is finite. The other kind are representations induced from smooth representations of parabolic subgroups.

2.4 Projective representation

We will use [Kir1976] and [Ho-Hu1992] as references for this section.

Consider an n-dimensional vector space V over a field K. Let $P(V)$ be the corresponding projective space, that is the set of all one dimensional subspaces in V. The group of automorphisms of $P(V)$ is isomorphic to the group $PGL(n, K) = GL(n, K)/C$ where C is the center of $GL(n, K)$, the set of all scalar matrices. We denote by π the canonical projection $\pi_V : GL(V) \to PGL(V)$. We will abstain from specifying the subindex when there is no confusion of what space V is being used.

Definition 2.57. *A projective representation of a group G on a vector space V is a homomorphism of G into a group $PGL(V)$.*

Proposition 2.58. *Let P be a projective representation of G on V. There exist maps $P' : G \to GL(V)$ and $m : G \times G \to K^\times$ such that*

$$P'(g)P'(h) = m(g,h)P'(gh), \text{ for all } g, h \in G.$$

On the other hand, if there exist such P' and m maps, then there exists a unique homomorphism $P : G \to PGL(V)$ such that $P(g) = \pi P'(g)$ for all $g \in G$.

We call P' a section of P and m a Schur multiplier associated with the section P'.

Proof. Consider the set X of coset representatives of $GL(V)$ in $PGL(V)$ and define the map $P' : G \to GL(V)$ such that for each $g \in G$, $P'(g)$ is the unique element of X satisfying $\pi(P'(g)) = P(g)$. Let us take g, $h \in G$, then $P'(gh)K^\times = P'(g)P'(h)K^\times$. This implies the existence of a unique $m(g,h) \in K^\times$ such that $m(g,h)P'(gh) = P'(g)P'(h)$. Conversely, let P' and m be two maps satisfying the hypothesis. We define $P : G \to PGL(V)$ as $P = \pi P'$. The for all g, $h \in G$,

$$\begin{aligned} P(gh) &= \pi(P'(gh)) = \pi(m(g,h)^{-1}P'(g)P'(h)) \\ &= \pi(P'(g)P'(h)) = \pi(P'(g))\pi(P'(h)) = P(g)P(h), \end{aligned}$$

proving that P is a group homomorphism, i.e. a projective representation of G on V. $\square$

This proposition is implying that whenever the multiplier m is trivial, i.e $m(g,h) = 1_K \, \forall g, h \in G$, then P' is an ordinary representation of G.

Since for every $g \in G$, $P(g)$ is invertible, the multiplier satisfies $m(g,1) = 1 = m(1,g)$. The associativity of the group implies

$$m(x,yz)m(y,z) = \alpha(x,y)m(x,y)m(xy,z) \quad x,\, y,\, z \in G.$$

Any map $a : G \times G \to K^\times$ satisfying these conditions is called a 2-cocycle. Notice that the multiplier is unique after the choice of a section, but the latter depends on the choice of the set X of representatives of $GL(V)$. Given two sections P_1 and P_2 of a projective representation P, with respective multipliers m_1 and m_2, we say that m_1 and m_2 are cohomologous. We denote by $M(G)$ the abelian group formed by all the cohomology classes.

Definition 2.59. *Let V_1 and V_2 be two K-vector spaces and the homomorphisms $P_1 : G \to PGL(V_1)$ and $P_2 : G \to PGL(V)$ two projective representations of G on V_1 and V_2 respectively. We say that P_1 and P_2 are projective equivalent if there exists an isomorphism $\phi : V_1 \to V_2$ such that*

$$\phi \circ P_1(g) \circ \phi^{-1} = P_2(g),$$

for all $g \in G$.

In the case were the vector space is a Hilbert space $\mathcal{H}$ we define a projective unitary representation.

Definition 2.60. *A projective unitary representation of a group G is a homomorphism of this group into $P\tilde{U}(H) = \mathcal{U}(\mathcal{H})/C$, where $\mathcal{U}(\mathcal{H})$ is the set of unitary operators acting on the Hilbert space $\mathcal{H}$.*

All these definitions extend to topological groups by asking that the homomorphism is continuous.

Projective representations are tightly related to central extensions. First we will shortly define a central extension.

Definition 2.61. *An exact sequence of groups $\{G_i\}_{i=1,..,n}$ is a sequence of group homomorphisms $\{f_i\}$*

$$1 \to G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} ... \xrightarrow{f_n} G_n \to 1$$

such that $Im(f_{i-1}) = Ker(f_i)$ for $i = 1, ..., n$. If $n = 2$, we call it a short exact sequence.

Definition 2.62. *An extension of a group Q by a group N is a short exact sequence*

$$1 \to N \xrightarrow{f} G \xrightarrow{g} Q \to 1.$$

If $Im(f)$ is in the center of G, $Z(G)$, (or isomorphic to a subgroup of the center) then the sequence is called a central extension of the group Q. We call G the central extension of Q by N.

Since f is an injective group homomorphism, we can assume N is a subgroup of G, identified with he kernel of g. We denote this extension by (G, g). Sometimes it will not be necessary to specify the homomorphism g. If G is a finite group, then we call the sequence a finite extension of the group Q.

Example 2.63. Consider a K-vector space V. The exact sequence

$$1 \to K^\times \xrightarrow{\delta} GL(V) \xrightarrow{\pi} PGL(V) \to 1$$

is a central extension, where for every $k \in K^\times$ and $v \in V$, $\delta(k) : v \mapsto kv$ and π is the canonical projection.

Definition 2.64. *A representation group of G is a finite central extension (G^*, τ) of G such that $Ker(\tau) \subseteq (G^*)'$, where $(G*)'$ is the commutator of $G*$, and $Ker(\tau) \cong M(G)$.*

Whenever we talk about a representation group G^* for a group G, we will implicitly assume that a choice has been made for the map τ.

We will study the conditions on a central extension of a group G that ensures that every projective representation of G corresponds to a linear representation of this central extension.

Theorem 2.65. *Let G^* be a representation group for G and let $\lambda : \mathrm{Ker}(\tau) \to K^\times$ be any homomorphism. Suppose that R is a linear representation of G^* such that $R(a) = \lambda(a)I$ for each $a \in A$. If we define $P(g) = R(r(g))$ for all $g \in G$, P is a projective representation whose associated multiplier is $m(g,h) = \lambda(\phi(g,h))$ for all $g,h \in G$.*

Theorem 2.66. *Let $G*$ be a representation group of G. Given a projective (matrix) representation P of G, there is a function $\delta : G \to K^\times$ and a linear representation T of G^* such that, for all g in G,*

$$P(g) = \delta(g)R(r(g)).$$

These results can be extended to infinite-dimensional representations (cf. [Kir1976]).

2.5 Supercuspidal representations

Until now we have been mainly focused on [Ste2009], but for the case of supercuspidal representations we will contrast with other sources since the definitions vary. We will be given the definitions found in [Cass1995], although most results are easily translated.

Before giving the first definition we shall recall what a semidirect product is. Given a group G, a subgroup $H \subseteq G$ and a normal subgroup $N \lhd G$, we say that G is the semidirect product of H acting on N or $G = H \ltimes N$ if $G = NH$ and $N \cap H = \{e\}$, the identity of G.

Definition 2.67. *Let G be an arbitrary connected reductive p-adic group. We define a parabolic subgroup P of G as a subgroup of G of the form $P = M \ltimes N$ where M is a connected reductive p-adic group and N is the unipotent radical of P, that is, N is the connected nilpotent subgroup P.*

For any N, the unipotent radical of a parabolic subgroup P of G, we define

$$V(N) = Span(\{\pi(n)v - v | n \in N, v \in V\})$$

and $V_N = V/V(N)$. The representation (π, V) defines a representation (π_N, V_N) of P on V_N, with N acting trivially. This representation is called the Jacquet module of V with respect to P.

Definition 2.68. *An admissible representation (π, V) is said to be supercuspidal (or absolutely cuspidal) if for any proper parabolic subgroup $P = MN$ in G, $V = V(N)$ and hence $V_N = \{0\}$.*

We can give an equivalent definition of supercuspidal representations that does not involve parabolic subgroups.

Definition 2.69. *Let (π, V) be a smooth representation of G. A complex valued function of G of the form $g \longmapsto < \tilde{v}, \pi(g)v >$ for any fixed $v \in V$, and $\tilde{v} \in \tilde{V}$, is called a matrix coefficient of π.*

Proposition 2.70. *A representation (π, V) is supercuspidal if and only if π is an admissible representation and every matrix coefficient of π is compactly supported modulo the center of G.* □

Theorem 2.71. *(Jacquet) If (π, V) is an irreducible admissible representation of G, then there exists a parabolic subgroup $P = MN$ and an irreducible supercuspidal representation (σ, W) of M such that (π, V) is a subrepresentation of $Ind_P^G \sigma$.*

Proof. We consider the set of standard parabolic subgroups P of G such that $V_N \neq \{0\}$ and we fix a minimal element P of this set. Consider P_1 a parabolic subgroup of G that is also a proper subgroup of P, and N_1 the unipotent radical of P_1. By minimality of P, $V_{N_1} = \{0\}$. Then π_N is supercuspidal, and π finitely generated. It can be proven that there exists a representation σ of P such that $\mathrm{Hom}_M(\pi_M, \sigma) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma)$. □

From this result we prove that

Corollary 2.72. *Suppose that (π, V) is a supercuspidal representation of G, then for $v \in V$ and $\tilde{v} \in \tilde{V}$, the matrix coefficient $\pi_{v,\tilde{v}}$ has compact support on G modulo Z.* □

The other direction of the implication is also true. The following theorem gives a characterization of supercuspidal representations.

Theorem 2.73. *[Cass1995] Let (π, V) be an admissible representation of G. The following are equivalent*

(a) (π, V) is supercuspidal.

(b) for every $v \in V$, $\tilde{v} \in \tilde{V}$ the matrix coefficient $c_{v,\tilde{v}}$ has compact support module Z. □

3 Groups in quantum mechanics

3.1 Origin of the Heisenberg group in Quantum mechanics

The main references of this subsection are [Di-Var2004] and [An2010].

At its origin, quantum mechanics was divided into two distinct formalism, matrix mechanics, formulated by Heisenberg, Born and Jordan, and wave mechanics, formulated by Schrödinger. The canonical commutation rule (CCR) had a central role, mainly in matrix mechanics. For a d-dimensional case with n particles the CCR are

$$[\hat{p}_k, \hat{q}_j] = i\hbar\delta_{kj}\mathbb{1}_d \quad 1 \leq i, j \leq n$$

where $\hat{p}_j$ and $\hat{q}_j$ are the momentum and position operators and $\hbar$ the reduced Planck constant, which can and will be considered to be $\hbar = 1$ from now on. In addition to its physical meaning, the CCR imply that the Hilbert space where $\hat{p}_i$ and $\hat{q}_j$ are defined cannot be finite dimensional. Otherwise, $tr(i\hbar\mathbb{1}) = i\hbar dim(\mathcal{H}) \neq 0$ but it is known that the trace of any commutator has to be zero. Moreover, in [We1931], Hermann Weyl proves that the momentum and position operators cannot be simultaneously bounded. This shows that there is two ways of rigorously describe the CCRs. Either we express it for infinitesimal operators in a suited domain and deal with the unbounded operators or we give a different formulation that allows for bounded operators. The later leads to Weyl's formulation and it is more convenient than the first alternative in some scenarios.

Weyl's formulation idea comes from considering two families of operators

$$U_k(t) = e^{i\hat{p}_k t} \text{ and } V_j(s) = e^{i\hat{q}_j s},\ t, s \in \mathbb{R}^d.$$

Both U and V are unitary representations of $\mathbb{R}^d$ in the space of unitary operators acting on the Hilbert space, $\mathcal{U}(\mathcal{H})$. The commutation rules over $\hat{p}_j$ and $\hat{q}_j$ induce the following commutation rule for $U_j(t)$ and $V_j(s)$

$$U_j(t)V_k(s) = e^{its\delta_{jk}} V_k(s)U_j(t) \tag{WCR}$$

where WCR refers to Weyl commutation relations. Any pair of unitary representations of $\mathbb{R}$ in $\mathcal{U}(\mathcal{H})$ U and V satisfying the WCR are called a Weyl system.

The definition of a Weyl system can be regarded with greater generality.

Definition 3.1. *A Weyl System is a pair of unitary representations U, V of abelian groups A, B with a non degenerate pairing*

$$<,>: A \times B \longrightarrow S^1$$

that satisfy

$$U(a)V(b) =< a, b >^{-1} V(b)U(a) \quad a \in A, b \in B.$$

$A \times B$ is a partition of the phase space, where momentum lies in B while position is encoded in A. For that reason we will sometimes write a as $p \in \mathbb{R}^d$, not to confuse with the momentum operator, and respectively $q \in \mathbb{R}^d$ for b, as to showcase its physical meaning. In [We1931], Weyl treats only the case where A and B are finite dimensional locally compact Lie groups, therefore describes systems of a finite number of particles. Considering a theory of infinitely many degrees of freedom, such as quantum fields, means dropping the condition of locally compactness, which brings a lot of complications. We will not address that scenario here.

The fact that this formalization allows for such a generalization makes for the perfect framework in which to develop a p-adic version of quantum mechanics.

Further generalizations can be done on the definition of a Weyl system. Notice that the previous definition is associated with a particular splitting of the phase space. To get a more invariant description one should formulate the concept of a Weyl system directly on the phase space. To simplify notation we are going to consider the case of one particle, $n = 1$, removing the necessity to index the momentum and position operators $\hat{p}$ and $\hat{q}$.

Weyl noticed that the map

$$\begin{aligned} W : \mathbb{R}^{2d} &\longrightarrow \mathcal{U}(\mathcal{H}) \\ (q, p) &\longmapsto e^{\frac{i}{2} q \cdot p} U(q) V(p) \end{aligned}$$

is a projective unitary representation of the group of translations acting on the phase space. The phase factor which keeps W from being an ordinary representation encodes a symplectic structure in the sense that they are of the form $e^{i\beta(x,y)}$ with β the symplectic form. He was able to show that any affine space admits a faithful irreducible projective representation only when it is symplectic and that in such a case, the factors of the projective representation are as above, and hence determined by the symplectic structure. His final formulation of quantum kinematics is described by *a projective unitary representation of an abelian group G with symplectic structure* [Di-Var2004].

The Schrödinger model for a Weyl system in the Hilbert space $L^2(\mathbb{R}^d)$ is the operator W defined as

$$(W(q, p) f)(t) = e^{(i/2) q \cdot p} e^{i q \cdot t} f(t + p)$$

which is the same as defining the Weyl system

$$(U(q) f)(t) = e^{i q \cdot t} f(t) \quad (V(p) f)(t) = f(t + p)$$

The Schrödinger model describes quantum kinematics in the wave mechanics formulation. Proving that the Schrödinger projective representation is irreducible and a unique up to equivalence solution of the WCR would prove that the wave mechanics model and the matrix mechanics model are equivalents.

Uniqueness was proved a bit later by Stone-von Neumann in the real case.

The Heisenberg group is born from this construction. It is possible to construct a group G_m as a central extension of G where the projective representation of G defines an ordinary representation of G_m (cf. Section 2.4). We will call G_m the Heisenberg group.

If $G = \mathbb{R}^{2d}$, there exists a central extension

$$G_m := G \times S^1$$

with the multiplication

$$(x,s)(y,t) = (xy, st \cdot m(x,y)),$$

where $m(x,y)$ is a trivial multiplier. G_m is a separable locally compact group in the product topology. Given projective representation T of G, the map

$$T_m : (x,t) \longmapsto tT(x)$$

is an ordinary representation of G_m with $T(1,t) = tT(1) = t1$.

The set of functions on the phase space $\mathcal{F}(P)$ has a structure of a Lie Algebra with respect to the Poisson bracket

$$\{p_i, q_j\} = \delta_{ij}.$$

It can be shown that there exists a bijective Lie algebra homomorphism between $\mathcal{F}(P)$ and the central extension of the algebra of translations. The unitary representation T_m of the central extension of the translation group induces a unitary representation of the algebra of functions on a Hilbert space that allows to see momentum and position as operators.

3.2 General definitions of the Heisenberg group

Outside the paradigm considered in the last section, the Heisenberg group can be seen defined in a slightly different ways that are still relevant to address. We will consider directly the p-adic case, as it is a direct translation from the real case presented in [Rott2010].

Let us define the simpler version first.

Definition 3.2. *The set of matrices in $GL(n, \mathbb{Q}_p)$ of the form*

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & Id_n & b \\ 0 & 0 & 1 \end{pmatrix} : a,\, b \in \mathbb{Q}_p^n \text{ and } c \in \mathbb{Q}_p \right\}$$

is a Lie group with the matrix composition. It is called the p-adic polarized Heisenberg group.

A more useful way to define the same group is the following

Definition 3.3. *The polarized p-adic Heisenberg group H^{2n}_{pol} is the vector space $\mathbb{Q}_p^{2n+1}$ with the matrix multiplication*

$$(p,\, q,\, t) \cdot (p',\, q',\, t) := (p + p',\, q + q',\, t + t' + pq').$$

It is a Lie group with identity element $(0,0,0)$ and inverse element $(-p, -q, -t + pq)$.

A more general definition can be given in the context of symplectic vector spaces. Consider a $2n$-dimensional symplectic space (V, J) where V is a $\mathbb{Q}_p$-vector space and the symplectic form $J : V \times V \to \mathbb{Q}_p$ takes values in the p-adic field.

Definition 3.4. *The p-adic Heisenberg group $H_p(V)$ over (V, J) is defined as the group $H_p(V) := V \times \mathbb{Q}_p$ of the pairs $\{(u;t) : u \in V, t \in \mathbb{Q}_p\}$ with the composition law*

$$(u;t) \cdot (v;t') = (u + v; t + t' + \frac{1}{2}J(u,v)).$$

The polarized p-adic Heisenberg group is obtained from the symplectic case by doing a change of basis. The Darboux theorem ensures that there is an isomomorphism between both formulations. It is called polarized as a way to denote that the formulation depends on the choice of a base.

Proposition 3.5. *There exists an isomorphism between the p-adic Heisenberg group and the polarized p-adic Heisenberg group.*

Proof. Consider the Darboux basis $\{e_i, f^j\}$ (cf. Theorem 1.24) for the symplectic $\mathbb{Q}_p$-vector space (V, J), then the vector space can be seen as $\mathbb{Q}_p^{2n}$ and J can be realized as the matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The p-adic Heisenberg group can now be seen as the group (q, p, α) where $q, p \in \mathbb{Q}_p^n$ and $t \in \mathbb{Q}_p$, with the group operation

$$(q, p, t) \cdot (q', p', t) = (q + q', p + p't + t' + \frac{1}{2}(pq' - qp')).$$

The map defined by

$$\varphi : H^n_{pol} \longrightarrow H^n$$
$$(q,\ p,\ t) \longmapsto (q,\ p,\ t - \frac{1}{2}pq)$$

is an homomorphism.

Let's prove injectivity. Assume $\varphi(p, q, t) = \varphi(p', q', t')$, then $(p, q, t - \frac{1}{2}pq) = (p', q', t' - \frac{1}{2}p'q')$. Obviously, $p = p'$, $q = q'$ and $t - \frac{1}{2}pq = t' - \frac{1}{2}p'q$. Finally $t = t'$. Proving surjectiveness is straightforward. □

Notice that in this case the center of the Heisenberg group is not compact. Many problems on the Heisenberg group don't have a satisfactory answer because of this fact. Some of this problems are simplified when considering the reduced Heisenberg group (cf. [Tha1998]). This is the central extension of the translation group by S^1 instead of $\mathbb{R}$ or $\mathbb{Q}_p$, or equivalently, the quotient $H_p(V)/\Gamma$ where $\Gamma = \{(0.2\pi k) : k \in Z\}$. This is also the most common definition used in p-adic quantum mechanics (cf. [Hu-Hu2015], [Di-Var2004]).

Definition 3.6. *The p-adic Heisenberg group $H_p(V)$ over (V, J) is the central extension of V by the unit circle $S^1 \subset \mathbb{C}$. In other words, it is the group $H_p(V) := V \times S^1$ of pairs $\{(u;\alpha) : u \in V, \alpha S^1\}$ with the composition law*

$$(u;\alpha) \cdot (v;\beta) = (u + v; \alpha\beta\chi(\frac{1}{2}J(u,v)))$$

where χ is the additive character defined in section 1.1.

3.3 Schrödinger's representation and Stone-von Neumann theorem

Let $\mathcal{U}$ be the space of unitary operators acting on the space $\mathcal{D}$ equipped with the weak operator topology. A projective representation $T : V \to U$ of the symplectic space V seen as a group defines a representation on the Heisenberg group $H_p(V)$.

For simplicity, we are going to use Darboux theorem to choose a base such that we can realize (V, J) as $(\mathbb{Q}_p^{2n}, J_0)$. Let $z = (x_1, ..., x_n, y_1, ..., y_n) \in V$, $\psi \in \mathcal{D}(\mathbb{Q}_p^n)$ and $\xi \in \mathbb{Q}_p^n$. The map $T : V \to \mathcal{U}$ defined as $T_z(\psi)(\xi) = \psi(\xi + x)\prod_i \chi(y_i\xi_i + \frac{1}{2}x_iy_i)$ is a Schrödinger projective representation of V.

By analogy with Weyl's formalism, the operators U and V are defined as $U(x, y)\psi = \prod_i \chi(y_i\xi_i + \frac{1}{2}x_iy_i)\psi(\xi)$ and $V(x)\psi(\xi) = \psi(x + \xi)$.

This defines a representation on the p-adic Heisenberg group.

Definition 3.7. *We define the continuous homomorphism called the Schrödinger representation as*

$$\begin{aligned} \phi : H(V) &\longrightarrow \mathcal{U} \\ g &\longmapsto \alpha T_z \ : \mathcal{D}(\mathbb{Q}_p^n) \longrightarrow \mathcal{D}(\mathbb{Q}_p^n) \\ &\qquad\qquad \psi \longmapsto \alpha T_z[\psi] \\ &\qquad\qquad\qquad\qquad \mathbb{Q}_p^n \longrightarrow \mathbb{C} \\ &\qquad\qquad\qquad\qquad \xi \longrightarrow \alpha T_z[\psi](\xi) \end{aligned}$$

where

$$T_z[\psi](\xi) = \psi(\xi + x)\chi(\sum_i (y_i\xi_i + \frac{1}{2}x_iy_i)) = \psi(\xi + x)\prod \chi(y_i\xi_i + \frac{1}{2}x_iy_i)$$

where $\xi = (\xi, ..., \xi_n) \in \mathbb{Q}_p^n$ *and* $g = (x_1, ..., x_n, y_1, ..., y_n; \alpha)$ *when using the symplectic basis on* V.

Notice that $\phi(((x, y); \alpha)) = \alpha\phi(((x, y); 1))$.

Theorem 3.8. *The Schrödinger representation is an irreducible and admissible representation.*

Proof. The first step is to prove that the Schrödinger representation is irreducible. We will prove that it has no non-trivial invariant subspaces. Let $\mathbb{W}$ be a non-trivial subspace of $\mathcal{D}(\mathbb{Q}_p^n)$ that is invariant by the action of the Heisenberg group $H(V)$. We can choose an orthonormal basis $\{\psi_\alpha\}$ such that any element of $\mathbb{W}$ can be written as a linear combination of these elements. Let us $E_\alpha \subset \mathbb{Q}_p^n$ the support of ψ_α, i.e. $\psi_\alpha(E) \neq 0$ and $\psi_\alpha(\mathbb{Q}_p^n \setminus E) = 0$. Define $E = \bigcup E_\alpha$ and $\mathcal{D}(E) \subset \mathcal{D}(\mathbb{Q}_p^n))$ the set of all functions of $\mathcal{D}(\mathbb{Q}_p^n)$ that vanish on $\mathbb{Q}_p^n \setminus E$. For any α, $\psi_\alpha \in \mathcal{D}(E)$ by definition. Hence, $\mathbb{W} \subseteq \mathcal{D}(E)$. We want to show that $\mathbb{W} = \mathcal{D}(E)$. For that, assume there exists a non-zero function $g \in \mathcal{D}(E)$. We can define an auxiliar function for the proof: for any $\alpha \in S^1$

$$f_\alpha := T_{(0,y)}[\psi_\alpha].$$

Since $f_\alpha(\xi) = T_{(0,y)}[\psi_\alpha](\xi) = \chi(\sum_i y_i\xi_i)\psi(\xi + x)$, $f_\alpha \in \mathbb{W}$. Then

$$(f_\alpha, g) = \int_{\mathbb{Q}_p^n} f_\alpha(\xi)\overline{g(\xi)}d^n\xi = F[\psi_\alpha\overline{g}](y) = 0$$

for any $\alpha \in S^1$. This implies that $\psi_\alpha(\xi)g(\xi)$ vanishes $\forall \xi \in \mathbb{Q}_p$. Since, for $\xi \in E$, there is at least one $\alpha \in S^1$ such that $\psi_\alpha(\xi) \neq 0$, then $g(\xi) = 0 \; \forall \xi \in E$. Since $g \equiv 0$ in $\mathbb{Q}_p \setminus E$, $g \equiv 0$ in $\mathbb{Q}_p$, which contradicts the hypothesis. We have proved then that $\mathbb{W} = \mathcal{D}(E)$. Notice that any invariant subspace as $\mathbb{W}$ is invariant by translation. Indeed, let $\psi \in \mathbb{W}$, then $\phi_g(\psi) \in \mathbb{W}$ but $\phi(\psi)[\xi] = \psi(\xi + x) \prod_i \chi(y_i \xi_i + \frac{1}{2} x_i y_i) \in \mathbb{W}$, then $\psi(\xi + x) \in \mathbb{W}$. Since $\mathbb{Q}_p$ as an Haar measure space is ergodic with respect to the translation group, then $\mathbb{Q}_p^n \setminus E$ has to be a zero measure set, since E cannot. We conclude $E = \mathbb{Q}_p^n$ up to a zero mesure set, then $\mathbb{W} = \mathcal{D}(\mathbb{Q}_p^n)$ and the representation is irreducible.

It is left to prove that the Schrödinger representation is admissible. For that it is sufficient to prove that it is supercuspidal. Since the Schrödinger representation $(\phi, \mathcal{D}(\mathbb{Q}_p))$ and the induced representation $(\phi^*, (\mathcal{D}(\mathbb{Q}_p^n))^*)$ are both smooth, by the Frobenius reciprocity (cf. Theorem 2.53) we know that for any $\psi \in \mathcal{D}(\mathbb{Q}_p^n)$ and $\Psi \in \mathcal{D}(\mathbb{Q}_p^n)^*$ there exists an open compact subgroup K of $H(V)$ such that $\psi \in \mathcal{D}(\mathbb{Q}_p^n)^K = \{\psi \in \mathcal{D}(\mathbb{Q}_p^n) : \phi(g)\psi = \psi\}$, $\Psi \in (\mathcal{D}(\mathbb{Q}_p^n)^*)^K$ and $\Psi(\phi(k_1 g k_2)\psi) = \Psi(\phi(g)\psi)$ for any $k_1, k_2 \in K$ and $g \in H(V)$.

Take $g = ((x,y);1)$ and $k_1 = ((x',y');1)$ with x' and y' non-zero and $k_2 = id_{H(V)}$, then $\Psi(\phi(((x,y);))\psi)) = \Psi(\phi(((x',y');1)((x,y);1))\psi) = \Psi(\phi(((x+x',y+y');\chi(\frac{1}{2}(x'y - xy'))\psi)) = \chi(\frac{1}{2}(x'y - xy')))\Psi(\phi(((x+x',y+y');1))\psi) = \chi(\frac{1}{2}(x'y - xy'))\chi(\frac{1}{2}(x'y + xy' + x'y'))\Psi(\phi(((x,y);1))\psi) = \chi(x'y + \frac{1}{2}x'y'))\Psi(\phi(((x,y);1))\psi)$. Setting $k_1 = id_{H(V)}$ and $k_2 = ((x',y');1)$, we obtain by the same process that $\Psi(\phi(((x,y);1))\psi) = \chi(xy' + \frac{1}{2}x'y'))\Psi(\phi(((x,y);1))\psi)$. Using Theorem 2.73 we can show supercuspidality .

□

Theorem 3.9 (Non-archimedean Stone-von Neumann Theorem)**.** *Any smooth representation ρ of the Heisenberg group $H_p(V)$ into a Hilbert space $\mathcal{H}$ satisfying*

$$\rho(0,\alpha) = \alpha Id$$

decomposes into the direct sum of irreducible representations equivalent to the Schrödinger representations.

We have seen how to represent the Heisenberg group on the space of linear operators on the Hilbert space and seen that it is unique. There are different ways to define how functions over the Heisenberg group are extended to functions on the representation space. One way of doing it is with the Weyl quantization through defining the Weyl operator and its kernel function in terms of the Schrödinger representation.

We can define a symplectic Fourier transform for any $\psi \in \mathcal{D}(\mathbb{Q}_p^{2n})$ as

$$F_s[\psi](z) = \check{\psi}(z) = \int_{\mathbb{Q}_p^n} \chi(J_0(z,z'))\psi(z')d^{2n}z'.$$

This new transformation is related to the Fourier transformation via the formula $F_s[\psi](J_0 z) = F[\psi](z)$. Therefore the operator $F_s \in \mathcal{D}(\mathbb{Q}_p^{2n})$ extends into a unitary operator on $L^2(\mathbb{Q}_p^{2n})$. We can use this transform to define the Weyl operator of any function that acts on the phase space.

Definition 3.10. *The Weyl operator W_f associated with the symbol $f \in \mathbb{Q}_p^{2n}$ is defined by*

$$W_f[\psi](\xi) = \int_{\mathbb{Q}_p^n} \check{f}(z) T_{(z,1)}[\psi](\xi) d^{2n}z = \int_{\mathbb{Q}_p^n} \check{f}(z)\psi(\xi + x)\chi(\sum_i (y_i\xi_i + \frac{1}{2}x_iy_i))d^{2n}z$$

where $\psi \in \mathcal{D}(\mathbb{Q}_p^{2n})$ and $\xi \in \mathbb{Q}_p^n$.

Proposition 3.11. *W_f is a linear and continuous operator on $\mathcal{D}(\mathbb{Q}_p^n)$, namely $W_f[\psi] \in \mathcal{D}(\mathbb{Q}_p)$, namely $W_f[\psi] \in \mathcal{D}(\mathbb{Q}_p^n)$ for $\psi \in \mathcal{D}(\mathbb{Q}_p^n)$.*

Proof. To see that it has compact support notice that $\check{f} \in \mathcal{D}(\mathbb{Q}_p^n)$. Then there exists $z' = (0, ..., 0, a_1, ..., a_n) \in \mathbb{Q}_p^{2n}$ with $|a_i|_p = p^l$ where $i = 1, ..., n$ for an integer l such that $\check{f}(z) = \check{f}(z + z')$. We then see that $W_f[\psi](\xi) = \chi(\sum_i \xi_i a_i) W_f[\psi](\xi)$. If $|xi|_p > p^{-l}$, there exists an integer $k \in \{1, ..., n\}$ such that $|\xi_k a_k|_p > 1$, thus $\chi(\sum_i \xi_i a_i) \neq 1$. Then the support of $W_f[\psi]$ is a subset of $B(0; l)^n$ and thus is compact.

To show that $W_f[\psi]$ is a locally compact function, we can define a succession of locally constant operators

$$W_f^N[\psi](\xi) = \int_{(B(0;N))^{2n}} \check{f})(z)\psi(\xi + x)\chi(\sum_i (y_i\xi_i + \frac{1}{2}x_iy_i))d^{2n}z$$

for any integer N. Let $l \in \mathbb{Z}$ be the largest characteristic number associated with ψ. Then for any $\xi' \in (B(0, N))^n$ we have $W_f^N[\psi](\xi + \xi') = W_f^N[\psi](\xi)$ if $N > -l$ because $|\sum_i y_i\xi'_i|_p \leq max\{|y_i\xi_i|_p\} \leq 1$ and $\psi(\xi + x + \psi') = \psi(\xi + x)$ when $N > -l$. Then $W_f[\psi] = lim_{N\to\infty} W_f^N[\psi]$ is locally constant.

□

Proposition 3.12. *(a) $W_f[\psi](\xi) = \int_{\mathbb{Q}_p^n} K_{W_f}(\xi, \eta)\psi(\eta)d^n\eta$, where*

$$K_{W_f}(\xi, \eta) = \int_{\mathbb{Q}_p^n} f(\frac{1}{2}(\eta + \xi), y)\chi(\sum_i (\eta_i - \xi_i)y_i)d^n y$$

is called the kernel function of the Weyl operator.

(b) $||W_f[\psi]||_L^2 \leq ||F[f]||_{L_1}||\psi||_L^2$.

(c) For $f, g \in \mathcal{D}$, we have the composition law $K_{W_f \circ W_g} = K_{W_h}$, where

$$h(z) = \int_{\mathbb{Q}_p^n} f(z + z')g(z + z'')\chi(-2J_0(z', z''))d^{2n}z'd^{2n}z'' \in \mathcal{D}(\mathbb{Q}_p^n)$$

and $c = \begin{cases} 1, & p \geq 3; \\ 1/4^n, & p = 2. \end{cases}$

Proof. (a) Let $\eta = \xi + x$, then we have

$$
\begin{aligned}
W_f[\psi](\xi) &= \int_{\mathbb{Q}_p^n} \check{f}(\eta - \xi, y)\psi(\eta)\chi(\sum_i \frac{1}{2} y_i(\eta_i + \xi_i))d^n\eta d^n y. \\
&= \int_{\mathbb{Q}_p^n} f(z')\psi(\eta)\chi(\sum_i(\eta_i - \xi_i)y_i')\chi(\sum \frac{1}{2} y_i(\eta_i + \xi_i - 2x_i'))d^{2n}z' d^n\eta d^n y \\
&= \int_{\mathbb{Q}_p^n} f(z')\psi(\eta)\chi(\sum_i(\eta_i - \xi_i)y_i')\delta(\frac{1}{2} y_i(\eta_i + \xi_i - 2x_i'))d^{2n}z' d^n\eta \\
&= \int_{\mathbb{Q}_p^n} f(\frac{1}{2}(\eta + \xi), y')\psi(\eta)\chi(\sum_i(\eta_i - \xi_i)y_i')d^n y' d^n\eta
\end{aligned}
$$

where we have applied that $F[\delta] = 1$ and $F[1] = \delta$.

(b) Note that $F[f] \in L^1(\mathbb{Q}_p^{2n})$ since $F[f]$ belongs to $\mathcal{D}(\mathbb{Q}_p^{2n})$ which is dense in $L^1(\mathbb{Q}_p^{2n})$. By manipulating the Fourier transform of a Dirac distribution and expressing the kernel function as

$$
\begin{aligned}
K_{W_f}(\xi, \eta) &= \int_{\mathbb{Q}_p^n} f(k, y)\chi(\sum_i(\nu_i - \xi_i)y_i)\chi(\sum_i \frac{1}{2} x_i(2k_i - \eta_i - \xi_i))d^n k d^n y d^n x \\
&= \int_{\mathbb{Q}_p^n} F[f](x, \eta - \xi)\chi(-\sum_i \frac{1}{2} x_i(\eta_i + \xi_i))d^n x,
\end{aligned}
$$

which leads to the inequality

$$
\begin{aligned}
\int_{\mathbb{Q}_p^n} |W_f[\psi](\xi)|^2 d^n\xi &\leq \int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^n} |K_{W_f}(\xi, \eta)| d^\eta \right) \left(\int_{\mathbb{Q}_p^n} |K_{W_f}(\xi, \eta)| \cdot |\psi(\eta)|^2 d^n\eta \right) d^n\xi \\
&\leq \left(\int_{\mathbb{Q}_p^n} |F[f](\xi, \eta)| d^n\xi d^n\eta \right)^2 \int_{\mathbb{Q}_p^n} |\psi(\eta)|^2 d^n\eta.
\end{aligned}
$$

(c) One can check that $K_{W_f \circ W_g}(\xi, \eta) = \int_{\mathbb{Q}_p^n} K_{W_f}(\xi, \zeta)K_{W_g}(\zeta, \eta)d^n\xi$. Then since

$$
\begin{aligned}
&\int_{\mathbb{Q}_p^n} K_{W_f}(\epsilon, \zeta)K_{W_g}(\xi, \eta)d^n\xi \\
&= \int_{\mathbb{Q}_p^n} f(\frac{1}{2}(\xi + \zeta), x)g(\frac{1}{2}(\zeta + \eta), y)\chi(\sum_i(\zeta_i - \xi)x_i)\chi(\sum_i(\eta_i - \zeta_i)y_i)d^n x d^n y d^n\zeta,
\end{aligned}
$$

we get

$$
\begin{aligned}
h(\xi, \eta) &= \int_{\mathbb{Q}_p^n} K_{W_f \circ W_g}(\xi - \frac{1}{2}x, \xi + \frac{1}{2}x)\chi(-\sum_i \eta_i x_i)d^n x \\
&= \int_{\mathbb{Q}_p^n} f\left(\frac{1}{2}(\xi - \frac{1}{2}x + \zeta), \alpha\right) g\left(\frac{1}{2}(\xi + \frac{1}{2}x + \zeta), \beta\right) \\
&\chi\left(\sum_i(\zeta_i - \xi_i + \frac{1}{2}x)\alpha_i\right) \chi\left(\sum_i(\xi_i - \zeta_i + \frac{1}{2}x)\alpha_i\right) \chi(-\sum_i \eta_i x_i)d^n\alpha d^n\beta d^n\zeta d^n x \\
&= |-\frac{1}{4}|_p^{-n} \int_{\mathbb{Q}_p^n} f(\xi + u, \eta + \alpha')g(\xi + v, \eta + \beta')\chi(2\sum_i(v_i\alpha_i' - u_i\beta_i'))d^n\alpha' d^n\beta' d^n u d^n v,
\end{aligned}
$$

where $|-1/4|_p^{-n} = |4|_p^n$ in front of the integral comes from the variable change $u = \frac{1}{2}(-\frac{1}{2}x - \zeta + \eta)$, $v = \frac{1}{2}(\frac{1}{2}x - \xi + \eta)$, $\alpha' = \alpha - \eta$ and $\beta' = \beta - \eta$.

□

3.4 Induced representation of the Heisenberg group

Let Γ be an abelian subgroup of the Heisenberg group $H(V)$, then we have a smooth representation $\Upsilon(\Gamma, \mathcal{C})$ induced by the unitary character $\mathcal{C}$ of Γ restricting to the identity on the center S^1 of $H(V)$ and being locally constant restricted on V. Namely, $\Upsilon = \mathrm{Ind}_{\Gamma}^{H(V)}(\mathcal{C})$.

Let $\mathbb{H}(\Gamma, \mathcal{C})$ be the set of complex valued functions on $H(V)$ which are locally constant compactly supported when restricted on V and satisfy the conditions $\psi(\gamma g) = \mathcal{C}(\gamma)\psi(g)$ for any $g \in H(V)$, $\gamma \in \Gamma$. Then the induced representation is defined to be the representation of $H(V)$ defined by translations: $\Upsilon(\Gamma, \mathcal{C})(g_0)[\psi](g) = \psi(gg_0)$ for $\psi \in \mathbb{H}$, g, $g_0 \in H(V)$. Then $\Upsilon(\Gamma, \mathcal{C})(\alpha) = \alpha Id$ for $\alpha \in S^1$.

Proposition 3.13. *[Ho1980] The representation* $(\Upsilon(\Gamma, \mathcal{C}), \mathbb{H})$ *is an irreducible and admissible representation.*

The Stone-von Neumann theorem implies that every induced representation $\Upsilon(\Gamma, \mathcal{C})$ is isomorphic to the Schrödinger representation. We will be centering the following results on defining the unique unitary isomorphism $\Theta(\Gamma, \mathcal{C}, \Gamma', \mathcal{C}') : \mathbb{H}(\Gamma, \mathcal{C}) \to \mathbb{H}(\Gamma', \mathcal{C}')$. This isomorphism is determined by the property

$$\Theta(\Gamma, \mathcal{C}; \Gamma', \mathcal{C}')\Upsilon(\Gamma, \mathcal{C})\Theta^{-1}(\Gamma, \mathcal{C}; \Gamma', \mathcal{C}') = \Upsilon(\Gamma', \mathcal{C})$$

up to a unit scalar related to the representations.

Definition 3.14. *An isomorphism* $\Theta(\Gamma, \mathcal{C}, \Gamma', \mathcal{C}') : \mathbb{H}(\Gamma, \mathcal{C}) \to \mathbb{H}(\Gamma', \mathcal{C}')$ *with the previous property is called the intertwining operator between the two representations.*

Let us take $\Gamma = (L, S^1) =: \Gamma(L)$ or $\Gamma = (l, S^1) =: \Gamma(l)$ for any self-dual lattice L or Lagrangian subspace l in (V, J). For any $\psi \in \mathbb{H}(\Gamma, \mathcal{C})$, its restriction to V satisfies $\psi(z+\gamma) = \chi(\frac{1}{2}J(z,\alpha))\mathcal{C}((\gamma;1))\psi(z)$ for $z \in V$, $\gamma \in L$ or l, and $\Upsilon(\Gamma, \mathcal{C})(g_0)[\psi](z) = \alpha\chi(\frac{1}{2}J(z,w))\psi(z+w)$ for $g_0 = (w;\alpha) \in H(V)$. For any self-dual lattice L, there exist two transversal Lagrangian subspaces l and l', such that $L = l \cap L \oplus l' \cap L$. The characters $\mathcal{C}_l$, $\mathcal{C}_L$ of $\Gamma(l)$ and $\Gamma(L)$ are specified such that $\mathcal{C}_L((l \cap L;1)) = \mathcal{C}_l((l \cap L;1))$.

Proposition 3.15. *There exists a constant* $c(l, L) \in \mathbb{C}$ *of modulus one such that the intertwining operator* $\Theta(\Gamma(l), \mathcal{C}_l; \Gamma(L), \mathcal{C}_L) : \mathbb{H}(\Gamma(l), \mathcal{C}_l) \to \mathbb{H}(\Gamma(L), \mathcal{C}_L)$ *is given by*

$$\Theta(\Gamma(l), \mathcal{C}_l; \Gamma(L), \mathcal{C}_L)[\psi](g) = c(l, L) \sum_{u \in l' \cap L} \mathcal{C}_L((-u;1))\psi((u;1)\cdot g) -$$

Proof. To prove that the intertwining operator is well defined, i.e.

$$\Theta(\Gamma(l), \mathcal{C}_l; \Gamma(L)\mathcal{L})[\psi] \in \mathbb{H}(\Gamma(L), \mathcal{L}),$$

we choose a suitable symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ of V such that $L = \mathbb{Z}_p e_1 \oplus ... \oplus \mathbb{Z}_p e_n \oplus \mathbb{Z}_p f_1 \oplus ... \oplus \mathbb{Z}_p f_n$, $l = \mathbb{Q}_p e_1 \oplus ... \oplus \mathbb{Q}_p e_n$ and $l' = \mathbb{Q}_p f_1 \oplus ... \oplus \mathbb{Q}_p f_n$. For $g = ((x,y);\alpha)$ we express

$$\Phi(\Gamma(l), \mathcal{C}_l; \Gamma(L), \mathcal{C}_L)[\psi](g) = c(l, L)\alpha\mathcal{C}_l(x)\chi(-\sum_i \frac{1}{2}x_i y_i) \sum_{u \in \mathbb{Z}_p^n} \mathcal{C}_L(-u)\chi(-\sum_i u_i x_i)\psi(u+y),$$

where $\mathcal{C}_l(x) := \mathcal{C}_l(((x,0);1))$, $\mathcal{C}_L(u) := \mathcal{C}_L(((0,u);1))$ and $\psi(u+y) := \psi(((0,u+y);1))$. We can see that $\Phi(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi](g)$ is invariant under translation in $\gamma \in \Gamma(L)$ and $\Phi(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi]$ is locally constant on V, since the functions $\chi(-\sum_i u_i x_i)$ and $\psi(u+y)$ on $\mathbb{Q}_p^n$ can be taken independent of u. Moreover, there exists $a \in \mathbb{Z}_p^n$ with $|a_p| = p^l$ for some integer l such that

$$\Phi(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi](g) = \mathcal{C}_L((a,0);1)\chi(\sum_i x_i a_i)\Theta(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi](g).$$

Meanwhile

$$\{y \in \mathbb{Q}_p^n : \Theta(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi]((x,y);\alpha) \neq 0\} \subset \bigcup_{u\in\mathbb{Z}_p^n} \{y-u : \psi(((0,y);1)) \neq 0\}.$$

The latter is bounded in $\mathbb{Q}_p^n$, therefore $\Theta(\Gamma(l),\mathcal{C}_l);\Gamma(L),\mathcal{C}_L)[\psi]$ is compactly supported on V. Next we should prove $\Theta(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi]$ is compactly supported modulo V.

To show surjectiveness we define the intertwining operator $\Theta(\Gamma(L),\mathcal{C}_L,\Gamma(l),\mathcal{C}_l)$: $\mathbb{H}(\Gamma(L),\mathcal{L}) \to \mathbb{H}(\Gamma(l),\mathcal{C}_l)$ by

$$\begin{aligned}\Theta(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)[\psi](g) &= c(l,L)\int_{l/l\cap L} \mathcal{C}_l((-u;1))\psi((u;1)\cdot g)d\mu_{l/l\cap L}\\ &= c(l,L)\int_{\mathbb{Q}_p^n/\mathbb{Z}_p^n} \alpha\mathcal{C}_l(-u)\chi(\frac{1}{2}\sum_i u_i y_i)\psi(u+x,y)d^n u\\ &= c(l,L)\alpha\mathcal{C}_l(x)\chi(-\sum_i \frac{1}{2}x_i y_i)\int_{\mathbb{Q}_p^n/\mathbb{Z}_p^n} \mathcal{C}_l(-u)\chi(\frac{1}{2}\sum_i u_i y_i)\psi(u,y)d^n u,\end{aligned}$$

where u is regarded as a representative of an equivalent class in $l/l\cap L$ and $\psi(u+x,y) = \psi(((u+x,y);1))$. A similar argument implies that $\Theta(\Gamma(L),\mathcal{C}_L;\Gamma(l),\mathcal{C}_l) \in \mathbb{H}(\Gamma(l),\mathcal{C}_l)$. To show that $\Theta(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)\circ\Theta(\Gamma(L),\mathcal{C}_L;\Gamma(l),\mathcal{C}_l) \sim Id$ we calculate

$$\begin{aligned}&\Theta(\Gamma(l),\mathcal{C}_l;\Gamma(L),\mathcal{C}_L)\circ\Theta(\Gamma(L),\mathcal{C}_L;\Gamma(l),\mathcal{C}_l)[\psi](g)\\ &= c(l,L)\alpha\sum_{v\in\mathbb{Z}_p^n}\int_{\mathbb{Q}_p^n/\mathbb{Z}_p^n} \mathcal{C}_l(-u)\chi(\sum_i(u_i v_i + \frac{1}{2}u_i y_i))\psi(u+x,y)d^n u\\ &= c(l,L)\psi((x,y);\alpha).\end{aligned}$$

for any $\psi \in \mathbb{H}(\Gamma(L),\mathcal{C}_L)$. Isometry can be derived from the Parseval-Steklov equality for Fourier transformations.

□

Corollary 3.16. *Let l_1 and l_2 be two Lagrangian subspaces in (V,J), and characters $\mathcal{C}_{l_1}$, $\mathcal{C}_2$ of $\Gamma(l_1)$ and $\Gamma(l_2)$ that satisfy $\mathcal{C}_{l_1}$, then the isomorphism $\Theta(\Gamma(l_1),\mathcal{C}_l;\Gamma(l_2),\mathcal{C}_{l_2}) : \mathbb{H}(\Gamma(l_1),\mathcal{C}_{l_1}) \to \mathbb{H}(\Gamma(l_2),\mathcal{C}_{l_2})$ is given by*

$$\begin{aligned}\Theta(\Gamma(l_1),\mathcal{C}_{l_1};\Gamma(l_2),\mathcal{C}_{l_2})[\psi](g) &= c(l,L)\int_{l_2/l_1\cap l_2} \mathcal{C}_{l_2}((-u;1))\psi((u;1)\cdot g)d\mu_{l_2/l_1\cap l_2}\\ &= c(l,L)\alpha\mathcal{C}_{l_2}(x)\chi(-\sum_{i=1}^n \frac{1}{2}x_i y_i)\int_{\mathbb{Q}_p^m} \mathcal{C}_{l_2}(-u)\chi(-\sum_{i=1}^m u_i x_i)\psi(u+y)d^m u,\end{aligned}$$

where the measure $\mu_{l_2/l_1\cap l_2}$ on $l_2/l_1 \cap l_2$ has to be chosen to guarantee the unitarity of intertwining operators and the second equality is expressed in terms of the suitable symplectic basis of (V, J) with $m = n - dim_{\mathbb{Q}_p} l_1 \cap l_2$ and $\mathcal{C}_{l_2}(u) = \mathcal{C}_{l_2}(((u_1, ..., u_m, 0, ..., 0); 1)), \psi(u + y) = \psi(((u_1, ..., u_m, 0, ..., 0, y_1, ..., y_n); 1))$. □

Definition 3.17. *The isomorphism of the previous corollary, expressed in the suitable symplectic basis, is called the canonical isomorphism between the representation spaces $\mathbb{H}(\Gamma(l_1), \mathcal{C}_{l_1})$ and $\mathbb{H}(\Gamma(l_2), \mathcal{C}_2)$ and denoted by $\Theta_c(\Gamma(l_1), \mathcal{C}_{l_1}; \Gamma(l_2), \mathcal{C}_{l_2})$.*

It is known that one of the methods of studying quantizations is by investigating the topological characteristics associated with this procedure. An example of such characteristic are the Maslov indices (cf .[Zel1993]). The Stone-Von Neumann theorem and the intertwining operator gives a method of defining these indices.

Definition 3.18. *For a triple (l_1, l_2, l_3) of Lagrangian subspaces in (V, J) and the characters $\mathcal{C}^0_{l_i}((l_i; 1)) = 1$ ($i = 1, 2, 3$), we have*

$$\Theta_c(\Gamma(l_3), \mathcal{C}^0_{l_3}; \Gamma(l_1), \mathcal{C}^0_{l_1}) \circ \Theta_c(\Gamma(l_2), \mathcal{C}^0_{l_2}; \Gamma(l_3), \mathcal{C}^0_{l_3}) \circ \Theta_c(\Gamma(l_1), \mathcal{C}^0_{l_1}; \Gamma(l_2), \mathcal{C}^0_{l_2})$$

and the coefficient $\alpha(l_1, l_2, l_3) \in S^1$ is called the Maslov index associated with (l_1, l_2, l_3), or equivalently, the Maslov index is determined by

$$\Theta_c(\Gamma(l_2), \mathcal{C}^0_{l_2}; \Gamma(l_3), \mathcal{C}^0_{l_3}) \circ \Theta_c(\Gamma(l_1), \mathcal{C}^0_{l_1}; \Gamma(l_2), \mathcal{C}^0_{l_2}) = \alpha(l_1, l_2, l_3)\Theta_c(\Gamma(l_1), \mathcal{C}^0_{l_1}; \Gamma(l_3), \mathcal{C}^0_{l_3}).$$

Proposition 3.19. *Let l_1, l_2, l_3, l_4 be four Lagrangian subspaces in (V, J) then the Maslov index is given by*

$$\alpha(l_1, l_2, l_3) = \int_{v\in l_3/l_2\cap l_3,\, w\in l_2/l_1\cap l_2,\, v+w\in l_1} \chi(\frac{1}{2}J(w, v))d\mu_{l_3/l_2\cap l_3} d\mu_{l_2/l_2\cap l_1},$$

and has the following properties:

1. *(permutation relations)* $\alpha(l_1, l_2, l_3) = \overline{\alpha(l_1, l_3, l_2)}, \alpha(l_1, l_2, l_3) = \alpha(l_2, l_3, l_1)$,
2. *(cocycle relations)* $\alpha(l_1, l_2, l_3)\alpha(l_1, l_3, l_4)\alpha(l_2, l_4, l_3)\alpha(l_2, l_1, l_4) = 1$.

Proof. For any $\psi \in \mathbb{H}(\Gamma(l_1), \mathcal{C}^0_{l_1})$ and $g = (z; \alpha) \in H(V)$, we have

$$\begin{aligned}
&\Theta_c(\Gamma(l_3), \mathcal{C}^0_{l_3}; \Gamma(l_1), \mathcal{C}^0_{l_1}) \circ \Theta_c(\Gamma(l_2), \mathcal{C}^0_{l_2}; \Gamma(l_3), \mathcal{C}^0_{l_3}) \circ \Theta_c(\Gamma(l_1), \mathcal{C}^0_{l_1}; \Gamma(l_2), \mathcal{C}^0_{l_2})[\psi](g) \\
&= \int_{l_1/l_1\cap l_3} d\mu_{l_1/l_1\cap l_3} \int_{l_3/l_2\cap l_3} d\mu_{l_3/l_2\cap l_3} \int_{l_2/l_1\cap l_2} d\mu_{l_2/l_2\cap l_1} \\
&\alpha\chi(\frac{1}{2}J(w, v) + J(v + w, u) + \frac{1}{2}J(w, u + v + z))\psi(u + v + w + z). \\
&= \int_{v\in l_3/l_2\cap l_3, w\in l_2/l_1\cap l_2, v+w\in l_1} \chi(\frac{1}{2}J(w, v))d_{\mu_{l_3/l_2\cap l_3} d\mu_{l_2/l_2\cap l_1}\psi(g)},
\end{aligned}$$

where ψ has been viewed as a function on V in the first and the second equality. The following properties of the Maslov index can be easily deduced from the definition. □

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