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ALGEBRAIC MULTIVARIATE INTERPOLATION

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Contents

Abstract

The main goal of this work is to study polynomial interpolation in several variables from an algebraic perspective. To do so, we treat linear differential operators as algebraic elements, and consider the solution space of a polynomial interpolation problem as the orthogonal space via a sesqui-linear map of an ideal of multivariate polynomials. Examples and a Mathematica code are also provided.

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Introduction

When I entered university for the very first time I was as excited as afraid of this new stage of my life and the unavoidable changes that were yet to come. I didn't even notice the guy who sat next to me, until he asked me if a had an extra pen, as he had forgotten to bring one due to the excitement of the day. That seemingly simple moment crossed our paths, which led us to become a part of each other's life, and four years later we have shared countless experiences together.

In life, tiny details in the past continuously act like seeds, which slowly turn into the emotional roots we experience today. It's not until you think about it, that you see how important this factor was in that special moment, where you first met that person, that slowly became your best friend or loved one.

In mathematics, this phenomena also exists and seems to be always present. There exists countless results, whose origins belong to a particular branch, but that seem to be naturally expressed and treated with seemingly unrelated tools from a completely different area. I like to think about branches of mathematics as independent entities, that combine their roots in order to create results, unobtainable in no other way.

I believe this project highlights perfectly this idea of knowledge combination among different areas of mathematics as a required step for the creation of new results, as the interpolation problem we are about to state can be naturally managed with what I felt were, at first, completely unrelated abstract linear algebra concepts.

Let us introduce a classic one-dimensional interpolation problem, which acts as the precedent of this whole work. Let $x_0, \ldots, x_n, y_0, \ldots, y_n, y_0^1, \ldots, y_n^1 \in \mathbb{R}$ be given values, and we want to find a polynomial $f(x) \in \mathbb{R}[x]$ such that

$$f(x_j) = y_j,\ f'(x_j) = y_j^1\ \forall j = 0, \ldots, n. \tag{1}$$

A polynomial f satisfying those conditions is known as an *unidimensional interpolation polynomial*. The main goal of this project is to extend the previous notion of interpolation for an arbitrary number of variables and including any order of partial derivatives. As seen in example 7.2, a typical multivariate interpolation problem would be to find a polynomial in 2 variables $p = p(x, y)$ such that

$$(D_{xx} + D_y)p(0, 0) = 1, \; D_x p(0, 0) = i, \; p(0, 0) = 3, \; p(2, i) = 5 + i. \tag{2}$$

A polynomial satisfying the previous system of differential equations is known as a *multidimensional interpolation polynomial.* The procedure needed for finding the solution space of a system like this requires abstract algebra tools, and hence it is known as *algebraic multivariate interpolation*. This work contains a full theoretical and practical interpolation study, which we will structure into 3 main blocks.

In the first block, consisting only of chapter 1, we dig deep into the theory behind our original problem. We will see how *Hermite's interpolation formula* is the key to this type of problem. A generalization of (1) including higher order derivatives $(f^{(2)}(x_j), f^{(3)}(x_j), etc)$ is also studied and finds an explicit solution, known as *Hermite's generalized interpolation formula.*

In the second block, formed by chapters 2 to 5, we will study the theoretical foundations that will let us approach algebraic multivariate interpolation, needed for solving systems such as (2). The main idea is that we can use *linear differential operators* (chapter 3) to express any interpolation problem as the orthogonal space of a *sesqui-linear map* (chapter 2). Let us remark that this type of mapping is a natural extension of *bilinear forms*, working now over the complex numbers and using spaces of infinite dimension, which added a very interesting extra challenge, as previous known results on linear algebra used mostly finite spaces.

The use of polynomial rings in multiple variables is pretty much induced by chapter 2, and essential properties of such rings are studied in chapter 4. The last chapter of this block is reserved to *Holonomic systems*, whose study gives an structure to orthogonal spaces, effectively leading us to the solution space we are seeking.

In the third and final block, we will study a particular type of interpolation based on holonomic systems, known as *Hermite type interpolation* (chapter 6). Assuming certain regularity among the conditions of the differential system, and using *Noetherian operators* as a theoretical tool, it is possible to create an algorithmic procedure for the computation of the solution space. Finally, we have included in chapter 7 a list of examples that illustrate the procedure of Hermite type interpolation, from a theoretical point of view.

Let us remark we have written a Mathematica program, available in appendix A, that includes a complete set of algorithms for the computation of a solution in a Hermite type interpolation problem. This has been specially useful for the examples in chapter 7, as many of the steps require the use of Gröebner basis, which are already implemented in Mathematica.

Chapter 1

Interpolation in one variable

In this chapter we will explore one-dimensional algebraic interpolation and the importance of Hermite's interpolation formula and it's generalization, as seen in article [2].

Given a set o points in the plane $\{(x_j, y_j) \in \mathbb{R}^2 \colon j = 0, \ldots, n\}$, a classic interpolation problem consists in finding a certain polynomial f such that

$$f(x_j) = y_j \;\; \forall j = 0, \ldots, n.$$

The problem we want to solve generalizes the previous idea for any field $\mathbb{K}$ (not necessarily $\mathbb{R}$) and includes conditions on the derivatives of f at the given points. Hence, the general statement of the problem can be stated as follows. Consider $x_0, \ldots, x_n$ a set of points of some field $\mathbb{K}$ and some constants $r_0, \ldots, r_n \in \mathbb{N}$. We want to find a polynomial $f \in \mathbb{K}[x]$ such that $\forall j = 0, \ldots, n$:

$$f^{(k)}(x_j) = f_j^k \in \mathbb{K} \; \forall k = 0, \ldots, r_j.$$

That is, we want the evaluation at each point x_j of the k-th derivative of f to be some predefined value $f_j^k \in \mathbb{K}$. Notice that the value of r_j indicates that we may want to impose conditions up to a different derivative order for each point.

1.1 Hermite's interpolation formula

Let's first explore the case where $k = 1$, meaning we have a set of points $x_0, \ldots, x_n, f_0, \ldots, f_n, f_0^1, \ldots, f_n^1 \in \mathbb{K}$ and we want to find a certain $f \in \mathbb{K}[x]$ satisfying $f(x_j) = f_j$ and $f'(x_j) = f_j^1 \; \forall \, 0 \leq j \leq n$. Hermite's interpolation formula provides an explicit polynomial, of degree $2n + 1$, which solves the problem stated. Specifically,

$$f(x) = \sum_{j=0}^{n} h_j(x) f_j + \sum_{j=0}^{n} \bar{h}_j(x) f_j^1$$

where

$$h_j(x) = \left(1 - \frac{q_n''(x_j)}{q_n'(x_j)}(x - x_j)\right) L_j(x)^2, \quad \bar{h}_j(x) = (x - x_j)L_j(x)^2$$

$$q_n(x) = \prod_{j=0}^{n} x - x_j, \quad L_j(x) = \frac{q_n(x)}{(x - x_j)q_n'(x_j)}.$$

Example 1.1. Suppose we want to find a polynomial such that:

x_j	$f(x_j)$	$f'(x_j)$
0	1	0
1	0	-1
-1	0	3

Then:

$$q_n(x) = x(x-1)(x+1) = x^3 - x, \quad q_n'(x) = 3x^2 - 1, \quad q_n''(x) = 6x$$

$$L_0(x) = \frac{q_n(x)}{xq_n'(0)} = -x^2 + 1 \quad L_1(x) = \frac{q_n(x)}{(x-1)q_n'(1)} = \frac{1}{2}(x^2 + x)$$

$$L_2(x) = \frac{q_n(x)}{(x+1)q_n'(-1)} = \frac{1}{2}(x^2 - x)$$

$$h_0(x) = \left(1 - \frac{q_n''(0)}{q_n'(0)}(x)\right) L_0(x)^2 = (-x^2 + 1)^2$$

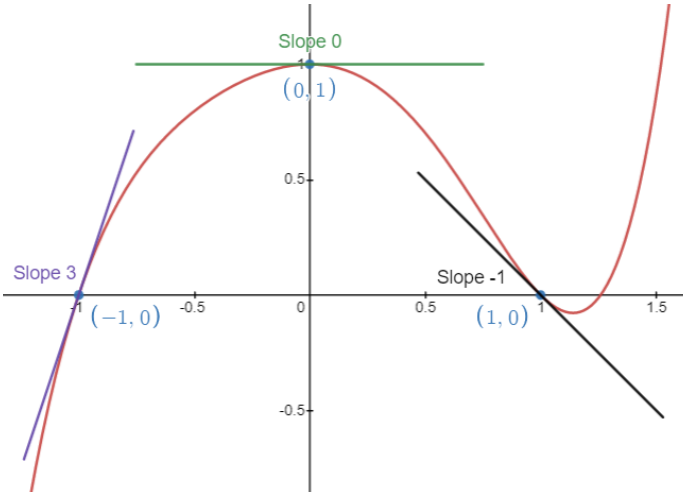
$$\bar{h}_1(x) = (x-1)L_1(x)^2 = \frac{1}{4}(x-1)(x^2 + x)^2$$

$$\bar{h}_2(x) = (x+1)L_2(x)^2 = \frac{1}{4}(x+1)(x^2 - x)^2.$$

Finally, the polynomial we seek turns out to be:

$$f(x) = (-x^2 + 1)^2 - \frac{1}{4}(x-1)(x^2 + x)^2 + \frac{3}{4}(x+1)(x^2 + x)^2 = 1 - x^2 - \frac{1}{2}x^3 + \frac{1}{2}x^5.$$

It's easy to check that this polynomial indeed satisfies all conditions listed below. As we are working in one dimension, we can also see this is the correct polynomial visually, as the values of the derivatives are also the slopes of the tangent lines at the given points.



1.2 Generalized interpolation formula

Let's now give a solution to the original problem, where we want to give values up to the k-th derivative of the polynomial, for an arbitrary value of $k \in \mathbb{N}$.

Theorem 1.2. *Suppose we are given x_j, $f_j^k \in \mathbb{K}$ and $r_j \in \mathbb{N}$, where $0 \leq j \leq n$ and $0 \leq k \leq r_j$. A polynomial that solves $f^{(k)}(x_j) = f_j^k$ and has degree $n + \sum_{j=0}^{n} r_j$ and is given by*

$$f(x) = \sum_{j=0}^{n} \sum_{k=0}^{r_j} A_{jk}(x) f_j^k$$

where

$$A_{jk}(x) = p_j(x) \frac{(x - x_j)^k}{k!} \sum_{t=0}^{r_j - k} \frac{1}{t!} g_j^{(t)}(x_j)(x - x_j)^t$$

$$p_j(x) = \prod_{s \neq j}^{r_n} (x - x_s)^{r_s + 1}, \quad g_j(x) = (p_j(x))^{-1}.$$

Proof. Suppose the interpolation polynomial is of the form $f(x) = \sum\limits_{j=0}^{n} \sum\limits_{k=0}^{r_j} A_{jk}(x) f_j^k$ for some polynomials A_{jk}. Then, as $f^{(k)}(x_j) = f_j^k$, it must be that

$$A_{jk}^{(s)}(x_i) = 0 \; if \; i \neq j, \quad A_{jk}^{(s)}(x_j) = \delta_{ks} = \begin{cases} 1, \text{ if } k = s \\ 0, \text{ if } k \neq s. \end{cases} \tag{1.1}$$

Observe that each polynomial A_{jk} has degree $n + \sum_{j=0}^{n} r_j$, and from the previous conditions we know that there are polynomials R_{jk} of degree $r_j - k$ such that

$$A_{jk}(x) = p_j(x)(x - x_j)^k R_{jk}(x). \tag{1.2}$$

If we let $S_{jk}(x) = (x - x_j)^k$ and $g_j(x) = (p_j(x))^{-1}$ we can rewrite the previous expression as

$$S_{jk}(x)\, R_{jk}(x) = A_{jk}(x)\, g_j(x) \tag{1.3}$$

We want to differentiate this equality $k + t$ times, and observe that

$$S_{jk}^{(k+t)} = \begin{cases} \text{k! , if t = 0} \\ \text{0 , if t > 0.} \end{cases}$$

By differentiating (1.3) k+t times we reach the expression

$$\sum_{i=0}^{k} \binom{k+t}{i} S_{jk}^{(i)}(x) R_{jk}^{(k+t-i)}(x) = \sum_{i=0}^{k+t} \binom{k+t}{i} A_{jk}^{(i)}(x) g_j^{(k+t-i)}(x). \tag{1.4}$$

We want to evaluate (1.4) for $x = x_j$. Notice that $S_{jk}^{(s)}(x_j) = 0$ for $s = 0, \ldots, k-1$ and $S_{jk}^{(k)} = k!$, and therefore we only need to consider the case $i = k$ for the left-hand side. Similarly, by (1.1) we know that $A_{jk}^{(s)}(x_j) = 0$ if $s \neq k$ and $A_{jk}^{(k)}(x_j) = 1$, and so we need to consider only $i = k$ for the right-hand side too. Therefore we have

$$\binom{k+t}{k} k!\, R_{jk}^{(t)} = \binom{k+t}{k} g_j^{(t)}(x_j).$$

This can be simplified to

$$R_{jk}^{(t)}(x_j) = \frac{1}{k!} g_j^{(t)}(x_j), \text{ for } t \le r_j - k.$$

As we know that R_{jk} is a polynomial of degree $r_j - k$, we can conclude that

$$R_{jk}(x) = \frac{1}{k!} \sum_{t=0}^{r_j-k} \frac{1}{t!} g_j^{(t)}(x_j)(x - x_j)^t. \tag{1.5}$$

Finally, if we substitute (1.5) into (1.2) we get the desired expression for the polynomials $A_{jk}(x)$ stated in the theorem

$$A_{jk}(x) = p_j(x) \frac{(x - x_j)^k}{k!} \sum_{t=0}^{r_j-k} \frac{1}{t!} g_j^{(t)}(x_j)(x - x_j)^t.$$

□

Chapter 2

Sesqui-linear maps

The goal of the next few chapters is to extend algebraic interpolation to an arbitrary number of variables, following our main reference [1]. In this chapter we introduce the notion of *sesqui-linearity*, as a tool that will help us link differential equations to abstract algebra theory.

Definition 2.1. Let Π, $\mathbb{F}$ and $\mathbb{L}$ be vector spaces over the complex field $\mathbb{C}$.
A map $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{L}$ is called *sesqui-linear* if for all a, $b \in \mathbb{C}$ and their corresponding complex conjugates $\bar{a}$, $\bar{b} \in \mathbb{C}$ the following conditions are satisfied

$$\langle ap + bq,\, f\rangle = a\langle p,\, f\rangle + b\langle q,\, f\rangle$$

$$\langle p,\, af + bg\rangle = \bar{a}\langle p,\, f\rangle + \bar{b}\langle p,\, g\rangle.$$

If $\mathbb{L} = \mathbb{C}$ then the map $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{C}$ is known as a *sesqui-linear form*.

Let's now explore some of the properties that *sesqui-linear maps* and *sesqui-linear forms* satisfy.

Definition 2.2. *Consider a sesqui-linear map $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{L}$. The orthogonal space of a subset $V \subset \mathbb{F}$ is defined as $V^{\perp} := \{p \in \Pi \colon \langle p, f\rangle = 0\ \forall f \in V\} \subset \Pi$. Similarly, the orthogonal space of a subset $I \subset \Pi$ is defined as $I^{\perp} := \{f \in \mathbb{F} \colon \langle p, f\rangle = 0\ \forall p \in I\}$. In particular, if $V \subset \mathbb{F}$, we have $V^{\perp} \subset \Pi$, and so*

$$V^{\perp\perp} = (V^{\perp})^{\perp} = \{f \in \mathbb{F} : \langle p, f\rangle = 0\ \forall p \in V^{\perp}\}.$$

Lemma 2.3. *Consider a sesqui-linear map $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{L}$ and a subset $V \subset \mathbb{F}$. Then $V^{\perp}$ is a vector subspace, $V \subseteq V^{\perp\perp}$ and $V^{\perp} = V^{\perp\perp\perp}$.*

Proof. Let's first see that $V^{\perp}$ it's a vector subspace of Π. Indeed:

- Let $f \in V$. By sesqui-linearity, $\langle 0, f\rangle = \langle 0+0, f\rangle = \langle 0, f\rangle + \langle 0, f\rangle$. Therefore, $\langle 0, f\rangle = 0 \; \forall f \in V \implies 0 \in V^{\perp} \implies V^{\perp} \neq \emptyset$.
- Let $f \in V$ and suppose $p, q \in V^{\perp} \implies \langle p, f\rangle = \langle q, f\rangle = 0$. Then:
$$\langle f, p+q\rangle = \langle f, p\rangle + \langle f, q\rangle = 0+0 = 0 \implies p+q \in V^{\perp}.$$
- Let $a \in \mathbb{C}$ and $p \in V^{\perp}$. We have $\langle ap, f\rangle = a\langle p, f\rangle = a \cdot 0 = 0 \implies a \cdot p \in V^{\perp}$.

Consider a certain $f \in V$. By definition we have $\langle p, f\rangle = 0 \; \forall p \in V^{\perp}$. Observe then that f "eliminates" all the elements of $V^{\perp}$, and therefore $f \in (V^{\perp})^{\perp} = V^{\perp\perp}$. This proves the inclusion $V \subseteq V^{\perp\perp}$, and in order to finish the proof we will need the following result:

Proposition 2.4. *Let $V, W \subset \mathbb{F}$ be two vector subspaces. Then, we have the following relation: $V \subseteq W \implies W^{\perp} \subseteq V^{\perp}$.*

Proof. Let $x \in W^{\perp}$. By definition, we must have $\langle x, f\rangle = 0 \; \forall f \in W$. As $V \subseteq W$, in particular $\langle x, f\rangle = 0 \; \forall f \in V \implies x \in V^{\perp}$. □

Finally, we must see that the equality $V^{\perp} = V^{\perp\perp\perp}$ holds. Let us remark that:

$$p \in V^{\perp} \iff \langle p, f\rangle = 0 \; \forall f \in V$$
$$p \in V^{\perp\perp\perp} \iff \langle p, f\rangle = 0 \; \forall f \in V^{\perp\perp}.$$

- $\boxed{\subset}$ Suppose $p \in V^{\perp}$. Then, $\langle p, f\rangle = 0 \; \forall f \in V^{\perp\perp}$ by the definition of the orthogonal space $V^{\perp\perp}$. By the previous remark this means $p \in V^{\perp\perp\perp}$, as we wanted.
- $\boxed{\supseteq}$ We already know that $V \subseteq V^{\perp\perp}$. By proposition 2.4, we must have that $(V^{\perp\perp})^{\perp} \subseteq (V)^{\perp} \implies V^{\perp\perp\perp} \subseteq V^{\perp}$.

□

Remark 2.5. All properties we have proven apply to a subset $V \subset \mathbb{F}$, and can be extended to a subset $I \subset \Pi$, as the roles of the spaces Π and $\mathbb{F}$ can be switched.

Definition 2.6. *A sesqui-linear form $\langle\, ,\, \rangle : \Pi \times \mathbb{F} \to \mathbb{C}$ is non-degenerate if $\Pi^{\perp} = 0$ and $\mathbb{F}^{\perp} = 0$. In other words, we say that $\langle\, ,\, \rangle$ is non-degenerate if:*

- $\langle p, f\rangle = 0 \; \forall p \in \Pi \implies f = 0$.
- $\langle p, f\rangle = 0 \; \forall f \in \mathbb{F} \implies p = 0$.

Lemma 2.7. *Any non-degenerate sesqui-linear $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{C}$ satisfies:*

(a) Let $V \subset \mathbb{F}$ be a vector subspace such that $dim_{\mathbb{C}} V$ is finite or $dim_{\mathbb{C}} \Pi/V^{\perp}$ is finite. Then $\langle\, , \rangle$ induces a non-degenerate sesqui-linear form on $\Pi/V^{\perp} \times V$, and the following properties are satisfied:

(a.i) $dim_{\mathbb{C}} V = dim_{\mathbb{C}} \Pi/V^{\perp}$.

(a.ii) $V^{\perp\perp} = V$.

(b) If $V, W \subset \mathbb{F}$ are subspaces, then $(V + W)^{\perp} = V^{\perp} \cap W^{\perp}$.

(c) If $V, W \subset \mathbb{F}$ are finite dimensional vector spaces, then $(V \cap W)^{\perp} = V^{\perp} + W^{\perp}$.

Proof. (a) Consider the quotient ring $\Pi/V^{\perp} := \{[p] \colon p \in \Pi\}$, and recall that $[p]$ denotes the class of the element $p \in \Pi$, meaning:

$$[p] = p + V^{\perp} = \{p + q \colon q \in V^{\perp}\}.$$

We define a new sesqui-linear form in terms of the previous $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{C}$ as

$$\begin{aligned} \langle\, , \rangle \colon \Pi/V^{\perp} \times V &\longrightarrow \mathbb{C} \\ (p, f) &\mapsto \langle [p], f\rangle := \langle p, f\rangle. \end{aligned}$$

Now we must see that the new form is well defined, sesqui-linear and non-degenerate. Indeed:

(1) Well - defined: Let $p, p* \in \Pi$ be elements of the same class in $\Pi/V^{\perp}$ and we want to see they have the same image through the form. Observe that:

$$[p] = [p*] \iff p - p* \in V^{\perp} \iff \langle p - p*, f\rangle = 0 \ \forall f \in V.$$

By sesqui-linearity of the original form, we have $\langle p, f\rangle = \langle p*, f\rangle$ and so the images are equal: $\langle [p], f\rangle = \langle [p*], f\rangle$.

(2) Sesqui-linear: Let $p, q \in \Pi$, $f, g \in V$ and $a, b \in \mathbb{C}$. Then:

$$\begin{aligned} (2.1)\ \langle a[p] + b[q], f\rangle &= \langle a(p + V^{\perp}) + b(q + V^{\perp}), f\rangle = \langle ap + bq + V^{\perp}, f\rangle \\ &= \langle [ap + bq], f\rangle = \langle ap + bq, f\rangle = a\langle p, f\rangle + b\langle q, f\rangle \\ &= a\,\langle [p], f\rangle + b\,\langle [q], f\rangle. \end{aligned}$$

$$\begin{aligned} (2.2)\ \langle [p], af + bg\rangle &= \langle p, af + bg\rangle = \langle p, af\rangle + \langle p, bg\rangle = \bar{a}\langle p, f\rangle + \bar{b}\langle p, g\rangle \\ &= \bar{a}\,\langle [p], f\rangle + \bar{b}\,\langle [p], g\rangle. \end{aligned}$$

(3) Non-degenerate: Consider the orthogonal spaces

$$V^{\perp} = \{[p] \in \Pi/V^{\perp} \colon \langle [p], f\rangle = 0 \ \forall f \in V\}$$

$$(\Pi/V^{\perp})^{\perp} = \{f \in V \colon \langle [p], f\rangle = 0 \ \forall [p] \in \Pi/V^{\perp}\}.$$

The form is non-degenerated if $V^{\perp} = \{[0]\}$ and $(\Pi/V^{\perp})^{\perp} = \{0\}$. Indeed:

- Let $p \in \Pi$ be such that $\langle [p], f\rangle = 0 \ \forall f \in V$, i.e. $[p] \in V^{\perp}$. By definition of the new sesqui-linear form, we have $\langle p, f\rangle = 0 \ \forall f \in V$, and therefore $p \in V^{\perp}$. Then, the class of the element p is $[p] = [0]$ in $\Pi/V^{\perp}$. This proves $V^{\perp} \subseteq \{[0]\}$ and the other inclusion is trivial.
- Let's see that if $f \in V$ is not 0, then it can't belong to $(\Pi/V^{\perp})^{\perp}$. Suppose that each $p \in \Pi$ satisfies $\langle p, f\rangle = 0$, i.e. $f \in \Pi^{\perp}$. As the original form $\langle\, , \rangle$ is non-degenerate, we know $\Pi^{\perp} = \{0\}$, and so $f = 0$, which yields a contradiction. Therefore, we know there exists some $p \in \Pi$ such that $\langle p, f\rangle \neq 0$. Hence, $\langle [p], f\rangle \neq 0 \implies f \notin (\Pi/V^{\perp})^{\perp}$. This gives us the inclusion $(\Pi/V^{\perp})^{\perp} \subseteq \{0\}$, and the opposite one is immediate.

In order to conclude section (a) let's prove the listed properties.

(a.i) Let's see that if $dim_{\mathbb{C}} V$ is finite, then $dim_{\mathbb{C}} V = dim_{\mathbb{C}} \Pi/V^{\perp}$, that in particular indicates that the other dimension is finite as well. Denote $m := dim_{\mathbb{C}} V$ and $n := dim_{\mathbb{C}} \Pi/V^{\perp}$, and suppose $m < \infty$. We take $[p_1], \ldots, [p_k] \in \Pi/V^{\perp}$ linearly independent and consider the applications

$$\begin{aligned} h_i \colon V &\longrightarrow \mathbb{C} \\ f &\mapsto h_i(f) := \overline{\langle [p_i], f\rangle}. \end{aligned}$$

Observe that these applications are $\mathbb{C}$ -linear, meaning that $\forall a, b \in \mathbb{C}$:

$$\begin{aligned} h_i(af + bg) &= \overline{\langle [p_i], af + bg\rangle} = \overline{\langle p_i, af + bg\rangle} = a\overline{\langle p_i, f\rangle} + b\overline{\langle p_i, g\rangle} \\ &= a\overline{\langle [p_i], f\rangle} + b\overline{\langle [p_i], g\rangle} = a\, h_i(f) + b\, h_i(g). \end{aligned}$$

Then, $h_1, \ldots, h_k \in V^*$ and let's see that these applications are linearly independent as well. Suppose $\sum_{i=1}^{k} a_i h_i(f) = 0 \ \forall f \in V$ for some $a_1, \ldots, a_k \in \mathbb{C}$. Then:

$$\sum_{i=1}^{k} a_i \bar{h}_i(f) = \sum_{i=1}^{k} a_i \langle [p_i], f\rangle = \langle \sum_{i=1}^{k} \bar{a}_i [p_i], f\rangle = 0 \ \forall f \in V \implies \sum_{i=1}^{k} \bar{a}_i [p_i] = 0.$$

As $[p_1], \ldots, [p_k]$ are linearly independent, it must be that each constant $a_i = 0$, as we wanted. Observe we have k linearly independent applications on V^*, and therefore $k \leq dim_{\mathbb{C}} V^* = dim_{\mathbb{C}} V = m < \infty$. In particular, this implies that the maximum number of linearly independent vectors we can take in $\Pi/V^{\perp}$ is m, and so $n \leq m$, which proves that $n = dim_{\mathbb{C}}\Pi/V^{\perp}$ is finite. We now repeat the same procedure, taking $f_1, \ldots, f_m$ linearly independent vectors in V and considering the applications

$$\begin{aligned} g_i \colon \Pi/V^{\perp} &\longrightarrow \mathbb{C} \\ [p] &\mapsto g_i([p]) := \langle [p], f_i \rangle. \end{aligned}$$

Let $a, b \in \mathbb{C}$ and let's test that each $g_i \in (\Pi/V^{\perp})^*$. Indeed:

$$\begin{aligned} g_i(a[p] + b[q]) &= \langle a[p] + b[q], f_i \rangle = \langle ap + bq, f_i \rangle = a\langle p, f_i \rangle + b\langle q, f_i \rangle \\ &= a\, g_i(p) + b\, g_i(q). \end{aligned}$$

Suppose $\sum_{i=1}^{m} a_i\, g_i([p]) = 0 \;\forall [p] \in \Pi/V^{\perp}$ for some $a_1, \ldots, a_m \in \mathbb{C}$. Then:

$$\sum_{i=1}^{m} a_i\, g_i([p]) = \sum_{i=1}^{m} a_i \langle [p], f_i \rangle = \langle [p], \sum_{i=1}^{m} \bar{a}_i f_i \rangle = 0 \implies \sum_{i=1}^{m} \bar{a}_i\, f_i = 0.$$

As $f_1, \ldots, f_m$ are linearly independent, it must be again that each $a_i = 0$. Therefore, we conclude that $m \leq dim_{\mathbb{C}}(\Pi/V^{\perp})^* = dim_{\mathbb{C}}\Pi/V^{\perp} = n$. We already knew that $n \leq m$, and so we conclude that $n = m$, as we wanted.

Remark 2.8. The proof that $dim_{\mathbb{C}}V = dim_{\mathbb{C}}\Pi/V^{\perp}$ with the initial hypothesis that $\Pi/V^{\perp}$ is finite is analogous to the previous procedure.

(a.ii) Finally, let's see that the equality $V = V^{\perp\perp}$ holds. It's already been proven in lemma 2.3 that $V \subseteq V^{\perp\perp}$, and so it is sufficient to see that $dim_{\mathbb{C}}V = dim_{\mathbb{C}}V^{\perp\perp}$. To do so, consider the application

$$\begin{aligned} \phi \colon V^{\perp\perp} &\longrightarrow (\Pi/V^{\perp})^* \\ f &\mapsto \phi(f) := \phi_f \end{aligned}$$

defined in terms of the linear one

$$\begin{aligned} \phi_f \colon \Pi/V^{\perp} &\longrightarrow \mathbb{C} \\ [p] &\mapsto \phi_f([p]) := \langle [p], f \rangle. \end{aligned}$$

We know that $dim_{\mathbb{C}}\Pi/V^{\perp}$ is finite, and so $dim_{\mathbb{C}}(\Pi/V^{\perp})^*$ must be finite as well, and therefore if we see that ϕ is monomorphism, we will have that it is actually an isomorphism. Indeed, let $f, g \in V^{\perp\perp}$ be such that $\phi_f = \phi_g$, and $p \in \Pi$. Then:

$$\phi_f([p]) = \phi_g([p]) \implies \langle [p], f \rangle = \langle [p], g \rangle \implies \langle p, f \rangle = \langle p, g \rangle.$$

As the previous expressions holds $\forall p \in \Pi$, it must be $f = g$, and so ϕ is a monomorphism. Therefore $V^{\perp\perp} \cong (\Pi/V^{\perp})^*$, which implies the desired equality

$$dim_{\mathbb{C}} V^{\perp\perp} = dim_{\mathbb{C}}(\Pi/V^{\perp})^* = dim_{\mathbb{C}}\Pi/V^{\perp} = dim_{\mathbb{C}} V.$$

(b) Let $V, W \subset \mathbb{F}$. Let's prove that $(V+W)^{\perp} = V^{\perp} \cap W^{\perp}$.

- $\boxed{\supseteq}$ Suppose $x \in V^{\perp} \cap W^{\perp}$. By definition we have $\langle x, v\rangle = 0\ \forall v \in V$ and $\langle x, w\rangle = 0\ \forall w \in W$. Then, $\langle x, v\rangle + \langle x, w\rangle = \langle x, v+w\rangle = 0$, which implies that each $z \in V + W$ satisfies $\langle x, z\rangle = 0$, and so $x \in (V+W)^{\perp}$.
- $\boxed{\subseteq}$ Suppose $x \in (V+W)^{\perp} \implies \langle x, z\rangle = 0\ \forall z \in V+W$. We know that $V \subset V+W$ and $W \subset V+W$, and so in particular $\langle x, v\rangle = 0\ \forall v \in V$ and $\langle x, w\rangle = 0\ \forall w \in W$. This means that $x \in V^{\perp}$ and $x \in W^{\perp} \implies x \in V^{\perp} \cap W^{\perp}$.

(c) Let V, $W \subset \mathbb{F}$ be finite dimensional vector spaces, and note we will use that $V = V^{\perp\perp}$, $W = W^{\perp\perp}$ and the analogous of part (b) for two subspaces I, $J \subset \Pi$, meaning $I^{\perp} \cap J^{\perp} = (I+J)^{\perp}$. Let's check the equality:

$$(V \cap W)^{\perp} = ((V^{\perp})^{\perp} \cap (W^{\perp})^{\perp})^{\perp} = ((V^{\perp} + W^{\perp})^{\perp})^{\perp} = V^{\perp} + W^{\perp}.$$

□

Lemma 2.9. *Let $\langle\,,\,\rangle\colon \Pi \times \mathbb{F} \to \mathbb{C}$ be a sesqui-linear form and $p_1, \dots, p_r$, $f_1, \dots, f_s$ be the bases of Π and $\mathbb{F}$, respectively. Then the following conditions are equivalent.*

(a) $\langle\,,\,\rangle$ is non-degenerate

(b) The bases have the same number of elements, $r = s$, and the matrix $(\langle p_i, f_j\rangle)$ is invertible.

Proof. We must see the equivalence $(a) \iff (b)$. Indeed:

$\boxed{\Rightarrow}$ We will use the same procedure as in a previous proof to show that $r = s$. We define the applications:

$$\begin{aligned} h_i\colon \mathbb{F} &\longrightarrow \mathbb{C} & g_j\colon \Pi &\longrightarrow \mathbb{C} \\ f &\mapsto \overline{\langle p_i, f\rangle} & p &\mapsto \langle p, f_j\rangle. \end{aligned}$$

We know that $p_1, \dots, p_r$ are linearly independent, and we've already proven that if $\langle\,,\,\rangle$ is non-degenerate, then $h_1, \dots, h_r$ are also linearly independent and $\mathbb{C}$ -linear. We must have then $r \leq dim_{\mathbb{C}}\mathbb{F}^* = dim_{\mathbb{C}}\mathbb{F} = s$. Similarly, we could prove that $s \leq r$ by showing that the applications $g_1, \dots, g_s$ belong to Π^* and are linearly independent. We define the matrix of the sesqui-linear form in the given basis of Π and $\mathbb{F}$ as:

$$M = (\langle p_i, f_j \rangle)_{1 \le i,j \le r} = \begin{pmatrix} \langle p_1, f_1 \rangle & \dots & \langle p_1, f_r \rangle \\ \\ \langle p_r, f_1 \rangle & \dots & \langle p_r, f_r \rangle \end{pmatrix}.$$

Consider $p = \sum\limits_{i=1}^{r} a_i\, p_i$ and $f = \sum\limits_{j=1}^{r} b_j\, f_j$ the expressions of some elements $p \in \Pi$ and $f \in \mathbb{F}$ in their respective basis. If we let $a := (a_1, \dots, a_r)$ and $b := (b_1, \dots, b_r)$, then we can express the product of p and f as:

$$\langle p, f \rangle = \langle \sum_{i=1}^{r} a_i\, p_i, \sum_{j=1}^{r} b_j\, f_j \rangle = \sum_{i=1}^{r} \sum_{j=1}^{r} a_i\, \overline{b_j} \langle p_i, f_j \rangle = a \cdot M \cdot \bar{b}^T.$$

Let's now see that if the form is non-degenerate then M is invertible. To do so, suppose $M \cdot x = 0$ for some $x = (x_1, \dots, x_r) \in \mathbb{C}^r$, and we want to see that $x = 0$. Indeed:

$$M \cdot x = \begin{pmatrix} \langle p_1, f_1 \rangle & \dots & \langle p_1, f_r \rangle \\ \\ \langle p_r, f_1 \rangle & \dots & \langle p_r, f_r \rangle \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \\ x_r \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{r} x_i \langle p_1, f_i \rangle \\ \\ \sum\limits_{i=1}^{r} x_i \langle p_r, f_i \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}$$

This means that for each $j = 1, \dots, r$ we have $\sum\limits_{i=1}^{r} x_i \langle p_j, f_i \rangle = \langle p_j, \sum\limits_{i=1}^{r} \bar{x}_i f_i \rangle = 0$. Then, for any $a_1, \dots, a_r \in \mathbb{C}$ we have

$$\sum_{j=1}^{r} a_j \langle p_j, \sum_{i=1}^{r} \bar{x}_i\, f_i \rangle = \langle \sum_{j=1}^{r} a_j\, p_j, \sum_{i=1}^{r} \bar{x}_i\, f_i \rangle = 0.$$

As the values of a_i are arbitrary and p_j form a base of Π, it must be

$$\langle p, \sum_{i=1}^{r} \bar{x}_i f_i \rangle = 0 \ \forall p \in \Pi \implies \sum_{i=1}^{r} \bar{x}_i f_i = 0, \text{ as the form is non-degenerate.}$$

Finally, as f_i form a basis of $\mathbb{F}$, we must have that the constants x_i are all 0, and therefore we have $x = 0$.

$\boxed{\Leftarrow}$ We proceed with the counter-reciprocal. Suppose the form $\langle\, , \rangle$ is degenerate. This means that there exists $f \neq 0$ such that $\langle p, f \rangle = 0\ \forall p \in \Pi$ or there exists $p \neq 0$ such that $\langle p, f \rangle = 0\ \forall f \in \mathbb{F}$.
Suppose the first case it's true and let $p = \sum_{i=1}^{r} a_i p_i$, $f = \sum_{j=1}^{r} b_j f_j$. Then we have that $aM\bar{b}^T = 0$ for all $a = (a_1, \dots, a_r)$. In particular, if we take all the vectors in

the canonic basis, $a = (1, \ldots, 0)$ until $a = (0, \ldots, 1)$, we would conclude that $M\bar{b}^t = 0$. As $f \neq 0$, some of the values $b_j \neq 0$, and therefore we have a non-zero vector is the kernel of the matrix M, and so M is not invertible.

Similarly, suppose we find ourselves in the second case. Then, we have that $aM\bar{b}^T = 0$ for any vector $b = (b_1, \ldots, b_r)$. By taking all the vectors b in the canonic basis, we would see that $aM = 0 \implies M^T a^T = 0$. We know that $p \neq 0$, and therefore some of the values $a_i \neq 0$. We have then that $a^T \neq 0$, and so M^T is not invertible $\implies$ M is not invertible either.

$\square$

Chapter 3

Linear differential operators

We want to create a link between multivariate algebraic interpolation and the notion of sesqui-linearity seen in the previous chapter. The main idea behind this relation is that any condition on the partial derivatives of a polynomial can be expressed as the product via a sesqui-linear map between two very specific vector spaces Π and $\mathbb{F}$. In order to understand how exactly this new product can be defined, let us first associate differential equations to multivariate polynomials via differential operators.

Notation: We will work over $\mathbb{C}[x_1, \ldots, x_n]$, the complex polynomial ring in n variables. Naturally, in the case $n = 2$ and $n = 3$ we will denote x, y, z the corresponding variables. We will also use standard notation for partial derivatives:

$$D_x p = \frac{\delta}{\delta x} p, \;\; D_y p = \frac{\delta}{\delta y} p, \;\; etc.$$

Definition 3.1. *Consider $p(x) = p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ a polynomial in n variables over the complex field. The linear differential operator identified with p is $p(D)$, where $D = (\frac{\delta}{\delta \xi_1}, \ldots, \frac{\delta}{\delta \xi_n})$. That is, the $k - th$ variable of the polynomial p corresponds to the partial derivative in respect to x_k of the operator $p(D)$.*

Example 3.2. Consider the polynomials $p(x, y) = x + y$ and $q(x, y) = 1 + xy$ in $\mathbb{C}[x, y]$. Their corresponding linear differential operators are:

- $p(D) = \frac{\delta}{\delta \xi_1} + \frac{\delta}{\delta \xi_2}$
- $q(D) = 1 + \frac{\delta^2}{\delta \xi_1 \delta \xi_2}$.

With this consideration, any condition on the partial derivatives of a polynomial f can be reformulated in terms of differential operators. For example, the equation $\frac{\delta f}{\delta \xi_1} + \frac{\delta f}{\delta \xi_2} = 0$ is equivalent to the differential system $p(D)f = 0$, with $p(x, y) = x + y$ as in the previous example. Then, we would like to define

$\langle p, f \rangle := p(D)f$, effectively linking sesqui-linear maps to algebraic interpolation, as f would be the solution to the differential system. Unfortunately, the previous definition does not satisfy all conditions on sesqui-linearity. This problem can be easily fixed with a small modification:

Proposition 3.3. *Let $\Pi = \mathbb{C}[x]$ and $\mathbb{F} = \mathbb{C}[[\xi]]$. The assignation $\langle\, , \,\rangle \colon \Pi \times \mathbb{F} \to \mathbb{F}$ defined as $\langle p, f \rangle := p(D)\bar{f}$ is a sesqui-linear map.*

Proof. Let $p(x), q(x) \in \Pi$, $f, g \in \mathbb{F}$ and $a, b \in \mathbb{C}$. Then:

- $$\begin{aligned}\langle ap(x) + bq(x), f \rangle &= (ap(x) + bq(x))(D) \cdot \bar{f} = ap(D)\bar{f} + bq(D)\bar{f} \\ &= a\langle p, f \rangle + b\langle q, f \rangle.\end{aligned}$$
- $\langle p(x), af + bg \rangle = p(D) \cdot (\overline{af + bg}) = \bar{a}p(D)\bar{f} + \bar{b}p(D)\bar{g} = \bar{a}\langle p, f \rangle + \bar{b}\langle p, g \rangle.$

Observe that some of the previous equalities rely on the fact that the differential operators we are using have constant complex coefficients, and therefore commutativity and associativity properties are preserved. □

Remark 3.4. Remark that the conjugate notation stands for

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \xi^\alpha \in \mathbb{F} \implies \bar{f} = \sum_{\alpha \in \mathbb{N}^n} \bar{a}_\alpha \xi^\alpha.$$

Remark 3.5. In one hand, observe we define the set of *symbols* as $\Pi = \mathbb{C}[x]$, meaning we are considering a system of differential equations that has derivatives with finite order. On the other hand, the solution f of the system doesn't necessarily need to have finite degree, and so we must consider $\mathbb{F}$ to be the $\mathbb{C}$-algebra of formal power series, meaning that $\xi = (\xi_1, \ldots, \xi_n)$ and:

$$\mathbb{F} = \mathbb{C}[[\xi]] = \left\{ \sum_\alpha a_\alpha \xi^\alpha \colon \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}, a_\alpha \in \mathbb{C} \right\}.$$

Example 3.6. Suppose we are given the following $p \in \Pi$ and $f \in \mathbb{F}$:

$$p(x) = x + y, \;\; f(\xi) = \sum_{j=0}^{\infty} \xi_1^{j-1} = 1 + \xi_1 + \xi_1^2 + \xi_1^3 + \ldots$$

Then, the product via the sesqui-linear form is

$$\langle p, f \rangle = p(D)\bar{f} = (\frac{\delta}{\delta\xi_1} + \frac{\delta}{\delta\xi_2})f = \sum_{j=1}^{\infty} j\,\xi_1^j = 1 + 2\xi_1 + 3\xi_1^2 + \ldots$$

Remark 3.7. Consider a subset of polynomials $I = \{p_1, \ldots, p_m\} \subset \Pi$ and suppose we want to find a certain $f \in \mathbb{F}$ such that $p_i(D)f = 0$ $\forall i = 1, \ldots, m$. Observe that:

$$\begin{cases} p_1(D)f = 0 \\ \quad\ldots \\ p_m(D)f = 0 \end{cases} \iff p(D)f = 0 \ \forall p \in \langle p_1, \ldots, p_m \rangle.$$

Therefore a series f is a solution to the system only if it belongs to the orthogonal space of I:

$$f \in I^{\perp} = \langle I \rangle^{\perp} = \{f \in \mathbb{F} \colon \langle p, f \rangle = 0 \ \forall p \in I.\}$$

As the solutions starting with the set $\{p_1, \ldots, p_m\}$ and the ideal $\langle p_1, \ldots, p_m \rangle$ are the same, we can always assume I to be an ideal.

Now we know that this type of differential system of equations can be expressed in terms of the previous sesqui-linear form. We devote the remaining of this chapter to a theory extension regarding this particular map, which will help us approach the interpolation task of finding a series f solution to the system.

Definition 3.8. *Let $\langle\, , \rangle \colon \Pi \times \mathbb{F} \to \mathbb{F}$ be a sesqui-linear map. We say two applications $h \colon \Pi \to \Pi$ and $h^* \colon \mathbb{F} \to \mathbb{F}$ are adjoints if for all $p \in \Pi$ and for all $f \in \mathbb{F}$ we have:*

$$\langle h(p), f \rangle = \langle p, h^*(f) \rangle.$$

Proposition 3.9. *(a) Consider $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. Then:*

$$\langle x^{\beta}, \xi^{\alpha} \rangle = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!}\xi^{\alpha-\beta} & \text{if } \alpha_i \geq \beta_i \ \forall i = 1, \ldots, n. \\ 0 & \text{otherwise.} \end{cases}$$

(b) The sesqui linear map $\langle\, ,\, \rangle$ is non-degenerate.

(c) The partial differentiation by ξ_i is adjoint to the multiplication by the dual variable x_i:

$$\langle x_i\, p(x), f \rangle = \langle p(x), D_i f \rangle \qquad \langle q(x)\, p(x), f \rangle = \langle p(x), q(D)\, f \rangle.$$

Proof. (a) Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$. Recall that the differential operator assigned to $p(x) := x^{\beta} = x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}$ is

$$p(D) = D^{\beta} = \frac{\delta^{|\beta|}}{\xi_1^{\beta_1} \cdot \ldots \cdot \xi_n^{\beta_n}}, \ |\beta| = \beta_1 + \ldots + \beta_n.$$

We will also use the notation $D_j := \frac{\delta}{\delta \xi_j}$, and so $D^\beta = D_1^{\beta_1} \cdot \ldots \cdot D_n^{\beta_n}$, as differential operators are linear and commutative. Observe that:

$$\langle x^\beta, \xi^\alpha \rangle = D^\beta \overline{\xi^\alpha} = D^\beta \xi^\alpha = D^\beta(\xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}) = D_1^{\beta_1}(\xi_1^{\alpha_1}) \cdot \ldots \cdot D_n^{\beta_n}(\xi_n^{\alpha_n})$$

If for any i we have $\alpha_i < \beta_i$, then we would take the derivative of the monomial $\xi_i^{\alpha_i}$ more than α_i times, which would nullify the product. Otherwise, taking all the derivatives we reach the expression

$$\langle x^\beta, \xi^\alpha \rangle = \frac{\alpha_1!}{(\alpha_1 - \beta_1)!} \xi_1^{\alpha_1 - \beta_1} \cdot \ldots \cdot \frac{\alpha_n!}{(\alpha_n - \beta_n)!} \xi_n^{\alpha_n - \beta_n} = \frac{\alpha!}{(\alpha - \beta)!} \xi^{\alpha - \beta}.$$

(b) We will use definition 2.6 to see the form is non-degenerate.

- Suppose $\langle p(x), f \rangle = 0 \; \forall p \in \Pi \implies p(D)\bar{f} = 0 \; \forall p \in \Pi$. In particular, this must hold for $p(x) = 1$, and so $p(D)\bar{f} = 1 \cdot \bar{f} = 0 \implies f = 0$.
- Suppose $\langle p(x), f \rangle = 0 \; \forall f \in \mathbb{F}$ and we want to see that $p = 0$. We write p as

  $$p(x) = \sum_\beta c_\beta x^\beta \text{ for some } \beta \in \mathbb{N}^n,$$

  and we will see that the values of the constants c_β are all 0. As $\langle p, f \rangle = 0$ for any f, in particular $\langle p, \xi^\alpha \rangle = 0$ for any α. Observe that for each monomial α:

  $$\langle p, \xi^\alpha \rangle = \langle \sum_\beta c_\beta x^\beta, \xi^\alpha \rangle = c_\alpha \langle x^\alpha, \xi^\alpha \rangle + \sum_{\beta \neq \alpha} c_\beta \langle x^\beta, \xi^\alpha \rangle = c_\alpha \alpha! + \sum_{\beta \neq \alpha} c_\beta \langle x^\beta, \xi^\alpha \rangle.$$

  As $\alpha \neq \beta$, each term in the sum will be either 0 (if any $\alpha_i < \beta_i$) or a non-constant polynomial on $\xi_1, \ldots, \xi_n$. As $\langle p, \xi^\alpha \rangle$ must be exactly zero, this means that $a_\alpha = 0$ and that each constant inside the sum is also zero. Repeating the argument for each monomial x^β in p, we see that all constants $c_\beta = 0$, which of course implies $p = 0$.

(c) Let's check that the adjoint conditions are satisfied.

- $\langle x_i p(x), f \rangle = (x_i\, p(x))(D)\bar{f} = p(D) D_i \bar{f} = p(D)\overline{D_i f} = \langle p(x), D_i f \rangle$.
- $\langle q(x)p(x), f \rangle = q(D)p(D)\bar{f} = p(D)\overline{q(D)f} = \langle p(x), q(D)f \rangle$.

□

Definition 3.10. *We say that a subset $V \subset \mathbb{F}$ is differentially closed, or D-closed for short, if all partial derivatives of any $f \in V$ also belong to V.*

Example 3.11.
- $V = \langle 1, \xi_1\xi_2 \rangle$ is not D-closed as $D_1(\xi_1\xi_2) = \xi_2 \notin V$.
- $V = \langle 1, \xi_1, \xi_1^2, \xi_1^3 \rangle$ is D-closed, as we can only take partial derivatives on ξ_1, and it's clear that the derivative of each generator also belongs to V.

Corollary 3.12. *Consider the previous sesqui-linear map $\langle p, f \rangle := p(D)\bar{f}$.*

(a) If $V \subset \mathbb{F}$ is a subset, then $V^\perp \subset \Pi$ is an ideal.

(b) If $I \subset \Pi$ is a subset, then $I^\perp \subset \mathbb{F}$ is a D-closed vector subspace.

Proof.

(a) Let $p, q \in V^\perp$ and $a \in \Pi$, and remember the definition of orthogonality:

$$V^\perp = \{p \in \Pi \colon \langle p, f \rangle = 0\ \forall f \in V\} = \{p \in \Pi \colon p(D)\bar{f} = 0\ \forall f \in V\}.$$

Let's test that this set is in fact an ideal of Π:

- $\langle 0, f \rangle = 0 \cdot \bar{f} = 0 \implies 0 \in V^\perp \implies V^\perp \neq \emptyset$.
- $(p+q)(D)\bar{f} = p(D)\bar{f} + q(D)\bar{f} = \langle p, f \rangle + \langle q, f \rangle = 0 \implies p + q \in V^\perp$.
- $(ap)(D)\bar{f} = a(D)p(D)\bar{f} = a(D)\langle p, f \rangle = a(D) \cdot 0 = 0 \implies ap \in V^\perp$.

(b) Consider the orthogonal space of the subset $I \subset \Pi$:

$$I^\perp = \{f \in \mathbb{F} \colon \langle p, f \rangle = 0\ \forall p \in I\} = \{f \in \mathbb{F} \colon p(D)\bar{f} = 0\ \forall p \in I\}.$$

By lemma 2.3 and remark 2.5 we already know that $I^\perp \subset \mathbb{F}$, and so let's now prove that $I^\perp$ is D-closed. Indeed, consider $f \in I^\perp \implies p(D)\bar{f} = 0\,\forall p \in I$. We want to see that the partial derivatives of f also belong to $I^\perp$, which is the equivalent to proving $q(D)f \in I^\perp$ for any polynomial $q \in \Pi$. Let $p \in I$, then:

$$\langle p, q(D)f \rangle = p(D)q(D)\bar{f} = q(D)p(D)\bar{f} = q(D) \cdot 0 = 0 \implies q(D)f \in I^\perp.$$

□

Definition 3.13. *We define now the sesqui-linear form $\langle\ ,\ \rangle_o \colon \Pi \times \mathbb{F} \to \mathbb{C}$ as the evaluation of the sesqui-linear map $\langle\ ,\ \rangle$ at 0. That is:*

$$\langle p, f \rangle_o := \langle p, f \rangle_{|_{\xi=0}} = p(D)\bar{f}(\xi)_{|_{\xi=0}}.$$

Proposition 3.14. *Consider a polynomial* $p(x) = \sum_{\beta} a_\beta x^\beta$ *and a a series* $f(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha$. *The evaluation of the sesqui-linear map can be expressed as*

$$\langle p, f \rangle_o = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} p^{(\alpha)}(0) \bar{f}^{(\alpha)}(0),$$

where we use the notation:

$$p^{(\alpha)} = \frac{\delta^{|\alpha|} p}{\delta x_1^{\alpha_1} \cdot \ldots \cdot \delta x_n^{\alpha_n}}, \quad f^{(\alpha)} = \frac{\delta^{|\alpha|} f}{\delta \xi_1^{\alpha_1} \cdot \ldots \cdot \delta \xi_n^{\alpha_n}}.$$

Proof. We can use proposition 3.9 to easily see that:

$$\langle x^\beta, \xi^\alpha \rangle_o = \begin{cases} \alpha! & \text{if } \alpha_i = \beta_i \ \forall i = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

The previous equality can be used to reach the following expression:

$$\langle p, f \rangle_o = \langle \sum_\beta a_\beta x^\beta, \sum_\alpha c_\alpha \xi^\alpha \rangle_o = \sum_\beta \sum_\alpha a_\beta \bar{c}_\alpha \langle x^\beta, \xi^\alpha \rangle_o = \sum_\alpha \alpha! a_\alpha \bar{c}_\alpha.$$

This tells us that in order to calculate the value of the sesqui-linear form, we only need to considerate the monomials that appear simultanously on p and on f. Observe then that when calculating $p^{(\alpha)}(0)$ and $f^{(\alpha)}(0)$ we only need to considerate the monomial $a_\alpha x^\alpha$ of p and $c_\alpha \xi^\alpha$ of f, as any other term will be canceled when evaluating at 0. Therefore, we have

$$p^{(\alpha)}(0) = \frac{\delta^{|\alpha|}}{\delta x^\alpha}(a_\alpha x^\alpha) = \alpha!\, a_\alpha, \quad f^{(\alpha)}(0) = \alpha!\, c_\alpha,$$

which leads to the desired expression:

$$\langle p, f \rangle_o = \sum_\alpha \alpha! a_\alpha \bar{c}_\alpha = \sum_\alpha \frac{p^{(\alpha)}(0)}{\alpha!} \frac{\bar{f}^{(\alpha)}(0)}{\cancel{\alpha!}} \cancel{\alpha!} = \sum_\alpha \frac{1}{\alpha!} p^{(\alpha)}(0) \bar{f}^{(\alpha)}(0).$$

□

Example 3.15. Consider $p = x^2 + y^3$ and $f = \xi_1^2 \xi_2^4 + i\xi_1^2 + (2+i)\xi_2^3$. We have:

$$\langle p, f \rangle = p(D)\bar{f} = 2\xi_2^4 - 2i + 24\xi_1^2 \xi_2 + 6(2-i).$$

$$\langle p, f \rangle_o = p(D)\bar{f}|_{\xi=0} = -2i + 6(2-i) = 12 - 8i.$$

If we just wanted to calculate the evaluation at $\xi = 0$, we could also use proposition 3.14 to reach the solution much faster. Observe that p and f only have the monomials $\alpha = (2, 0)$ and $\beta = (0, 3)$ in common, which correspond to the coefficients $a_\alpha = a_\beta = 1$ and $c_\alpha = i$, $c_\beta = 2 + i$. Then:

$$\langle p, f\rangle_o = \alpha! a_\alpha \bar{c}_\alpha + \beta! a_\beta \bar{c}_\beta = 2!\,(-i) + 3!\,(2-i) = 12 - 8i.$$

Corollary 3.16. *Let $p, q \in \Pi = \mathbb{C}[x]$ and $f \in \mathbb{C}[\xi]$. We have the following expressions:*

$$\langle D_i p, f\rangle_o = \langle p, \xi_i f\rangle_o \ \ , \ \ \langle q(D)p(x), f(\xi)\rangle_o = \langle p(x), q(\xi)f(\xi)\rangle_o.$$

Proof. Consider a polynomial $p = \sum_\beta a_\beta x^\beta \in \Pi$ and a series $f = \sum_\alpha c_\alpha \xi^\alpha \in \mathbb{F}$. Then $D_i p = \sum_\beta a_\beta \beta_i x^{\beta - e_i}$, where e_i it's the i-th vector in the canonic basis. Let's check the equality $\langle D_i p, f\rangle_o = \langle p, \xi_i f\rangle_o$. On one hand we have

$$\langle D_i p, f\rangle_o = \langle \sum_\beta a_\beta \beta_i x^{\beta - e_i}, \sum_\alpha c_\alpha \xi^\alpha\rangle_o = \sum_\beta a_\beta\, \beta_i\, \bar{c}_{\beta - e_i}\, \beta\,(\beta - e_i)!\ ,$$

where we only need to consider those monomials that satisfy $\beta - e_i = \alpha$. On the other hand

$$\langle p, \xi_i f\rangle_o = \langle \sum_\beta a_\beta x^\beta, \sum_\alpha c_\alpha \xi^{\alpha + e_i}\rangle_o = \sum_\beta a_\beta\, \bar{c}_{\beta - e_i}\, \beta!\ ,$$

where this time we only need to consider monomials such that $\beta = \alpha + e_i$, which is equivalent to the previous condition $\beta - e_i = \alpha$. Finally, as $\beta_i \cdot (\beta - e_i)! = \beta!$, we can clearly conclude that $\langle D_i p, f\rangle_o = \langle p, \xi_i f\rangle_o$. Let $q(x) = \sum_\gamma b_\gamma x^\gamma$ and let's see the second equality:

$$\begin{aligned}\langle q(D)p(x), f\rangle_o &= \langle \sum_\gamma b_\gamma D_1^{\gamma_1} \cdot \ldots \cdot D_n^{\gamma_n} p(x), f\rangle_o = \sum_\gamma b_\gamma \langle p(x), \xi_1^{\gamma_1} \cdot \ldots \cdot \xi_n^{\gamma_n} f\rangle_o = \\ &= \langle p(x), \sum_\gamma \bar{b}_\gamma \xi^\gamma f\rangle_o = \langle p(x), \bar{q}(\xi) f(\xi)\rangle_o.\end{aligned}$$

□

Definition 3.17. *Let $V \subset \mathbb{F}$ and $I \subset \Pi$ be two subsets. We define the orthogonal spaces of V and I with respect to the evaluation of the sesqui-linear form as:*

$$V^{\perp_o} = \{p \in \Pi \colon \langle p, f\rangle_o = 0 \ \forall f \in V\}$$

$$I^{\perp_o} = \{f \in \mathbb{F} \colon \langle p, f\rangle_o = 0 \ \forall p \in I\}$$

It's clear from the definition that $V^\perp$ and $I^\perp$ are vector subspaces of the spaces $V^{\perp_o}$ and $I^{\perp_o}$, respectively.

Proposition 3.18. *(a) If $I \subset \Pi$ is an ideal, then $I^{\perp_o} = I^{\perp}$ and these spaces are D-closed.*

(b) If a vector subspace $V \subset \mathbb{F}$ is D-closed, then $V^{\perp_o} = V^{\perp}$ and these sets are ideals.

Proof. (a) Let $f \in I^{\perp_o}$ and $p \in I$. As I is an ideal of Π we have $q(x)p(x) \in I$ for any polynomial $q \in \Pi$ and so

$$q(D)P(D)\bar{f}(0) = \langle q(x)p(x), f\rangle_o = 0 \; \forall q \in \Pi.$$

As this happens for any polynomial $q \in \Pi$, then any partial derivative of $p(D)\bar{f}(\xi)$ vanishes at 0, but this is only possible if $p(D)\bar{f} = 0$. Indeed, write

$$p(D)\bar{f}(\xi) = \sum_{\gamma} a_\gamma \xi^\gamma.$$

Then if we take $q(x) = x^\gamma$ for each monomial γ that appears we achieve:

$$0 = q(D)p(D)f(0) = \langle q(x), p(D)\bar{f}\rangle_o = \langle x^\gamma, \sum_{\gamma} a_\gamma \xi^\gamma \rangle_o = \gamma!\, \bar{a}_\gamma \;,$$

and so it must be $a_\gamma = 0$ for each $\gamma \implies p(D)\bar{f} = 0$. Finally, that last condition is equivalent to $\langle p, f\rangle = 0$, which implies $f \in I^{\perp}$. This proves the inclusion $I \subset I^{\perp}$, and the other one we already know is true, so we have the equality $I^{\perp_o} = I^{\perp}$. We've already seen that any partial derivative of an element of $I^{\perp_o}$ also cancels at 0, and so it's clear these spaces are D-closed.

(b) Let $p \in V^{\perp_o}$ and $f \in V$. We know V is D-closed and $f \in V$, and so the partial derivatives of f are also in V, i.e. $\bar{q}(D)f \in V$ for any polynomial $q \in \Pi$. Therefore,

$$q(D)p(D)\bar{f}(0) = \langle p, \bar{q}(D)f\rangle_o = 0 \; \forall q \in \Pi$$

Using the same reasoning from before we have that $\langle p, f\rangle = p(D)\bar{f}(\xi) = 0$, and so $p \in V^{\perp}$. Again, the contrary inclusion it's already given, and so we have proven $V^{\perp_o} = V^{\perp}$, as we wanted. Finally, if $p \in V^{\perp_o}$, $f \in V$ and $q \in \Pi$, we have $p(D)q(D)\bar{f}(0) = 0 \implies \langle q(x)p(x), f\rangle_o = 0 \implies q(x)p(x) \in V^{\perp_o}$, and therefore it's clear that the space $V^{\perp_o}$ it's an ideal.

□

Chapter 4

Zero-dimensional subset of $\mathbb{C}^n$

In this chapter we will study the known correspondence between polynomial ideals and algebraic varieties in the zero-dimensional case, which will lead us to two essential theoretical results for the next few chapters. Said correspondence guarantees that every ideal can be assigned to an algebraic variety and vice versa. Particularly, an ideal $I \subset \Pi$ can be matched with the variety

$$\mathcal{V}(I) = \{x \in \mathbb{C}^n \colon p(x) = 0 \ \forall p \in I\}.$$

Similarly, the corresponding ideal to the variety $V \subset \mathbb{C}^n$ is

$$\mathcal{I}(V) = \{p \in \Pi \colon p(x) = 0 \ \forall x \in V\}.$$

Let us recall some definitions and properties from book [3] that will useful throughout this chapter.

Definition 4.1. *Let I be an ideal of a ring R. The radical of I is defined as*

$$\sqrt{I} = \{p \in R \colon p^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Definition 4.2. *Let I be an ideal of a ring R. We say I is a prime ideal if*

$$f \cdot g \in I \implies f \in I \text{ or } g \in I.$$

Definition 4.3. *Let I be an ideal of a ring R. We say I is a primary ideal if*

$$f \cdot g \in I \implies f \in I \text{ or } g^n \in I, \ for\ some\ n \in \mathbb{N}.$$

Proposition 4.4. *The radical of an ideal $I \subseteq R$ it's the intersection of all prime ideals of R that contain I, that is*

$$\sqrt{I} = \bigcap \{P \mid P \subseteq R \text{ prime}\}.$$

Definition 4.5. *Let R be a ring and consider any chain of ideals* $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq R$*. Then, R is said to be a Noetherian ring if there exists* $n \in \mathbb{N}$ *such that* $I_n = I_m \ \forall m \geq n$*.*

Remark 4.6. It is proven that any field R is a Noetherian ring, and that if R is Noetherian then the polynomial ring $R[x]$ is also Noetherian (see [8]).

Definition 4.7. *Let I be an ideal of a ring R. We say that I admits a primary decomposition of ideals if there exist* $I_1, \ldots, I_r$ *primary ideals of R such that* $I = I_1 \cap \ldots \cap I_r$*. We say the decomposition is minimal if* $\sqrt{I_i} = \sqrt{I_j}$ *only if* $i = j$ *and* $\bigcap\limits_{j \neq i} I_j \not\subset I_i$*.*

Theorem 4.8. *(Lasker-Noether) Any ideal I of a Noetherian ring R[x] admits a minimal primary decomposition of ideals* $I = I_1 \cap \ldots \cap I_r$*. Moreover, each* $P_i = \sqrt{I_i}$ *is a prime ideal, and this primes are the same to the proper primes of the set* $\{\sqrt{I\colon f},\ f \in R\}$*.*

Theorem 4.9. *(Hilbert's weak Nullstellensatz) Let I be an ideal of a ring* $\mathbb{K}[x]$*, where* $\mathbb{K}$ *is an algebraically closed field. Then,*

$$V(I) = \emptyset \iff 1 \in I.$$

Theorem 4.10. *(Hilbert's Nullstellensatz) Let I be an ideal of a ring* $\mathbb{K}[x]$*, where* $\mathbb{K}$ *is an algebraically closed field. Then,*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Proposition 4.11. *Let I be an ideal of a ring* Π *and consider* $\Pi/I = \{p + I\colon p \in \Pi\}$*. The ideals of* Π/I *are in one-to-one correspondence with the ideals of* Π *containing* I*. The correspondence* $\phi\colon \{P\colon I \subset P\} \to \{\widehat{P}\colon \widehat{P} \subset \Pi/I\}$ *is given by*

$$\phi(P) = P/I = \{[p]\colon p \in P\}, \quad \phi^{-1}(\widehat{P}) = \{p\colon [p] \in \widehat{P}\}.$$

Theorem 4.12. *(Isomorphism theorem I) Let R and S be two commutative rings and let* $\phi\colon R \to S$ *be an homeomorphism. Then, the function* $f\colon R/ker(\phi) \to Im(\phi)$ *defined by* $f([r]) = \phi(r),\ r \in R$*, is an isomorphism. Therefore, we have*

$$R/ker(\phi) \cong Im(\phi).$$

Proof. Can be found in [5]. □

Theorem 4.13. *(Isomorphism theorem II) Let $I \subseteq J$ be ideals of a ring R. Then,*

$$R/J \cong (R/I)/(J/I).$$

Proof. Can be found in [5]. $\square$

With the previous considerations, let's state this chapter's first result.

Lemma 4.14. *Let $\Pi = \mathbb{C}[x]$, where $x = (x_1, \ldots, x_n)$, and consider a proper ideal $I \subset \Pi$. Then we have equivalence among the following conditions.*

(a) I is maximal (among the proper ideals).

(b) I is an ideal of a point $\theta \in \mathbb{C}^n$, meaning $I = \mathcal{I}(\theta)$.

(c) There exists a point $\theta \in \mathbb{C}^n$ such that

$$I = (x - \theta) = \sum_{j=1}^{n} (x_j - \theta_j)\Pi = \langle x_1 - \theta_1, \ldots, x_n - \theta_n \rangle.$$

Proof. $(a) \implies (b)$: Suppose I is maximal and let $X = \mathcal{V}(I)$. It can't be $\mathcal{V}(I) = \emptyset$, because then by Hilbert's theorem, as $\mathbb{C}$ is algebraically closed, we would have that $1 \in I \implies I = \Pi$, which contradicts that I is a proper ideal of Π. Therefore we have $X \neq \emptyset$ and so we can consider a point $\theta \in X$. If we have $p \in I$, then $p(\theta) = 0$, and so $p \in \mathcal{I}(\theta)$. Therefore we have the inclusion $I \subseteq \mathcal{I}(\theta)$, but I is a maximal ideal, so it must be $I = \mathcal{I}(\theta)$.

$(b) \implies (c)$: We must see that $(x - \theta) = \mathcal{I}(\theta)$.

- $\boxed{\subseteq}$ If we let $p \in (x - \theta) = \langle x_1 - \theta_1, \ldots, x_n - \theta_n \rangle$, then p can be expressed as a sum, where all of it's terms include at least one factor $x - \theta_j$ for some j, and so it's clear that $p(\theta) = p(\theta_1, \ldots, \theta_n) = 0$, and therefore $p \in \mathcal{I}(\theta)$.
- $\boxed{\supseteq}$ Let $p \in \mathcal{I}(\theta)$. Consider the Taylor expansion of the polynomial p around the point θ. That is,

  $$p(x) = A_1(x_1 - \theta_1) + \ldots + A_n(x_n - \theta_n) + R\ ,$$

  where $A_i = A_i(x_1, \ldots, x_n)$ and $R \in \mathbb{C}$. Evaluating now at $x = \theta$ we get $p(\theta) = R = 0$, and so

  $$p(x) = A_1(x_1 - \theta_1) + \ldots + A_n(x_n - \theta_n) \implies p \in (x - \theta).$$

$(c) \implies (a)$: Consider the morphism $\phi_\theta \colon \mathbb{C}[x] \to \mathbb{C}$ defined as $\phi_\theta(p(x)) := p(\theta)$. Clearly, ϕ_θ is an epimorphism with $ker(\phi_\theta) = \langle x_1 - \theta_1, \ldots, x_n - \theta_n \rangle = \mathcal{I}(\theta)$. Using the first isomorphism theorem we get that

$$\mathbb{C}[x]/ker(\phi_\theta) \cong Im(\phi_\theta) \implies \Pi/I \cong \mathbb{C}.$$

As $\mathbb{C}$ is a field and it's isomorphic to the quotient ring Π/I, I must be maximal.

□

Definition 4.15. *Let R be a ring. The Krull dimension (see [8]) of the ring R is defined as*

$$dim(R) = max\{n \mid P_o \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \subsetneq R,\ P_i \ prime\ ideals\}.$$

Meaning, the Krull dimension of R it's the lenght of the longest chain of nested different prime ideals we can obtain in R.

Example 4.16.

- In $R = \mathbb{Z}$, the prime ideals are of the form (p), for p prime. If we had q such that $(p) \subsetneq (q)$, then q could not be prime. Therefore the longest chain we can obtain is $(0) \subsetneq (p) \subsetneq \mathbb{Z}$, and therefore $dim(\mathbb{Z}) = 1$.
- In $R = \mathbb{C}[x, y]$, there are many possible nested ideal sequences, such as $(0) \subsetneq (x) \subsetneq (x, y)$ or $(0) \subsetneq (x+1) \subsetneq (x+1, y-2)$, and it can be proven there are no nested chains of lenght 3, and so $dim(\mathbb{K}[x, y]) = 2$.

Remark 4.17. We can think of Π/I both as a ring quotient or as a $\mathbb{C}$-vector space. In order to avoid confusion regarding dimensions we will use the following convention.

- $dim(\Pi/I)$ denotes the Krull dimension defined above of the quotient ring Π/I.
- $dim_{\mathbb{C}}(\Pi/I)$ denotes the dimension of Π/I as a $\mathbb{C}$-vector space.

Theorem 4.18. *Let R be a commutative ring with an identity element and let P be an ideal of R. Then, P is a prime ideal $\iff$ R/P is an integral domain.*

Proof. Can be seen in [5]. □

Proposition 4.19. *Let I be an ideal of a ring R and let J be a prime ideal such that $I \subseteq J \subset \mathbb{R}$. Then J/I is a prime ideal in the quotient ring R/I.*

Proof. We use theorem 4.18 and the second isomorphism theorem to prove this.
J is a prime ideal in R $\iff$ $R/J \cong (R/I)/(J/I)$ is an integral domain
$\iff$ J/I is a prime ideal in R/I. $\square$

Theorem 4.20. *Let I be an ideal of $\Pi = \mathbb{C}[x]$. Then:*

$$dim_{\mathbb{C}}(\Pi/I) < \infty \iff V(I) \text{ is a finite set.}$$

Proof. Can be found in [3]. Remark this result hold for any ring $\mathbb{K}[x]$ with $\mathbb{K}$ an algebraically closed field. $\square$

We will use all previous results in order to prove this chapter's second result:

Lemma 4.21. *Let $\Pi = \mathbb{C}[x]$ and consider an ideal $I \subset \Pi$. The following properties are equivalent.*

(a) $dim(\Pi/I) = 0$.

(b) *There exists a finite subset $V \subset \mathbb{C}^n$ such that $\sqrt{I} = \mathcal{I}(V)$.*

(c) *There exist non-zero polynomials $\phi_i \in \mathbb{C}[\lambda]$ such that $\phi_i(x_i) \in I$ for $i = 1, \ldots, n$.*

(d) $dim_{\mathbb{C}}(\Pi/I) < \infty$.

Proof. Let's see the chain of implications.

$(a) \implies (b)$: As $\mathbb{C}$ is a field, then $\mathbb{C}$ is a Noetherian ring and therefore the ring $\Pi = \mathbb{C}[x]$ is also Noetherian (see [8]). Then, we can consider the shortest minimal primary decomposition for the ideal $I \subset \Pi$. That is, $I = I_1 \cap \ldots \cap I_r$, with I_i primary and $\sqrt{I_i} = P_i$ prime for each $i = 1, \ldots, r$. By proposition 4.19 each P_i/I is a prime ideal in Π/I. Observe that if we had Q prime ideal such that $P_i \subseteq Q \subset \Pi$, then $P_i/I \subseteq Q_i/I \subset \Pi/I$. But as $dim(\Pi/I) = 0$, we must have each P_i/I to be maximal and so $P_i/I = Q/I \implies P_i = Q$, and so each P_i is a maximal ideal. Then, by lemma 4.14, for each P_i there must exist some point $\theta_i = (\theta_{i1}, \ldots, \theta_{in})$ such that $P_i = \langle x_1 - \theta_{i1}, \ldots, x_n - \theta_{in} \rangle = \mathcal{I}(\theta_i)$. Recall that the given A, B ideals we have the equality $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$. Therefore, the radical of the ideal $I = I_1 \cap \ldots \cap I_r$ can be expressed as

$$\sqrt{I} = \bigcap_{i=1}^{r} \sqrt{I_i} = \bigcap_{i=1}^{r} P_i = \bigcap_{i=1}^{r} \mathcal{I}(\theta_i) = \mathcal{I}(\theta_1, \ldots, \theta_r).$$

Finally, $V = \{\theta_1, \ldots, \theta_r\} \subset \mathbb{C}$ is a finite subset and satisfies $\sqrt{I} = \mathcal{I}(V)$, as we wanted.

$(b) \implies (a)$: Suppose there exists finitely many points $\theta_1, \dots, \theta_r \in \mathbb{C}^n$ such that $\sqrt{I} = \mathcal{I}(\theta_1, \dots, \theta_r) = P_1 \cap \dots \cap P_r$, where each $P_i := \mathcal{I}(\theta_i)$ is maximal by lemma 4.14. Consider P a prime ideal such that $I \subseteq P \subset \Pi$, and we want to see that it is maximal, as this implies that every prime in the quotient is also maximal and hence $dim(\Pi/I) = 0$. As $\sqrt{I}$ is the intersection of all primes that contain I, we must have $\sqrt{I} \subseteq P \implies P_1 \cap \dots \cap P_r \subseteq P$. Suppose there is no i such that $P_i \subseteq P$. Then for each $i = 1, \dots, r$ there exists $a_i \in P_i \backslash$P. Then, $a_1 \cdot \dots \cdot a_r \in (P_1 \backslash P) \cap \dots \cap (P_r \backslash P) \subseteq P_1 \cap \dots \cap P_r \subseteq P$. As P is prime, $a_1 \cdot \dots \cdot a_r \in P \implies a_i \in P$ for some i, which yields a contradiction. Hence, $P_i \subseteq P$ for some $i = 1, \dots, r$. As each P_i is maximal, we must have $P_i = P$, and so P is maximal.

$(b) \implies (c)$: Suppose there exist points $\theta_1, \dots, \theta_r \in \mathbb{C}^n$ such that

$$\sqrt{I} = \mathcal{I}(\theta_1, \dots, \theta_r), \text{ where } \theta_j = (\theta_{j1}, \dots, \theta_{jn}).$$

Observe that as $\mathcal{I}(\theta_j) = \langle x_1 - \theta_{j1}, \dots, x_n - \theta_{jn} \rangle$, we have that the polynomials $x_j - \theta_{1j}, \dots, x_j - \theta_{rj} \in \mathcal{I}(\theta_1, \dots, \theta_r)$. Consider then the polynomials $p_i(x) = (x - \theta_{1i}) \cdot \dots \cdot (x - \theta_{ri})$ that satisfy $p_i(x_i) \in \mathcal{I}(\theta_1, \dots, \theta_r)$ for each $i = 1, \dots, n$. Then, as $\sqrt{I} = \mathcal{I}(\theta_1, \dots, \theta_r)$, there must exist some $n \in N$ such that $(p_i(x_i))^n \in I$. Therefore, the polynomial $\phi_i(x) := (p_i(x))^n$ satisfies $\phi_i(x_i) \in I$ for each $i = 1, \dots, n$.

$(c) \implies (d)$: Suppose w.l.o.g. that each polynomial $\phi_i(x_i)$ is monic and denote $d_i = deg(\phi_i(x_i))$. Then,

$$\phi_i(x_i) = x_i^{d_i} + \sum_{j=0}^{d_i - 1} a_j x_i^j \in I,$$

and so each monomial $x_i^{d_i}$ is congruent modulo I to another polynomial of degree less that d_i on the variable x_i. Therefore, the quotient ring Π/I can be generated as a $\mathbb{C}$-vector space by the monomials $x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$, where $\alpha_i < d_i$ for each $i = 1, \dots, n$. As there are finitely many monomials that satisfy this condition it's clear that $dim_{\mathbb{C}}(\Pi/I) < \infty$.

$(d) \implies (b)$: Hilbert's Nullstellentzatz guarantees that $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$, and as $dim_{\mathbb{C}}(\Pi/I) < \infty$ we know by 4.20 that $V := \mathcal{V}(I)$ is a finite set. Then, $\sqrt{I} = \mathcal{I}(V)$ where V is a finite set, as we wanted to see.

□

Chapter 5

Holonomic systems

In previous chapters we have studied the procedure in which any polynomial induces a system of differential equations with constant coefficients. The theory on such type of systems has been widely extend by Ehrenpreis and Palamodov (see [4]), and we are only interested in the particular case where the solution space has finite dimension, known as an *holonomic system*, as our goal is to find interpolation polynomials.

In this chapter we want to study the structure of orthogonal spaces in the particular case of holonomic systems, as recall such space contains the solutions to the corresponding system of differential equations.

Definition 5.1. *An holonomic system (HS) is a system of differential equations with constant coefficients that has a finite-dimensional solution space.*

Remark 5.2. Let $\Pi = \mathbb{C}[x]$ and $\mathbb{F} = \mathbb{C}[[\xi]]$ and consider the sesqui-linear map $\langle\, ,\, \rangle \colon \Pi \times \mathbb{F} \to \mathbb{F}$ defined as in previous chapters: $\langle p, f \rangle = p(D)\bar{f}$. Any system of differential equations with constants coefficients can be expressed as

$$\begin{cases} \langle p_1, f \rangle = 0 \\ \quad \dots \\ \langle p_r, f \rangle = 0 \end{cases}, \text{ for some polynomials } p_1, \dots, p_r \in \Pi. \tag{5.1}$$

Theorem 5.3. *(Cayley-Hamilton) Let A be a square matrix over a commutative ring and let $p(\lambda) = a_0 + a_1\lambda + \ldots + a_n\lambda^n$ denote the characteristic polynomial of A. Then A satisfies it's own characteristic equation $p(\lambda) = 0$. That is, A satisfies the equation*

$$P(A) = a_0 \cdot I + a_1 \cdot A + \ldots + a_n \cdot A^n = 0.$$

Proof. See [6]. □

Theorem 5.4. *Let $V \subset \mathbb{F}$ be a finite dimensional vector subspace. The following statements are equivalent.*

(a) V is the solution space of an holonomic system.

(b) V is D-closed.

Proof. $\boxed{\Rightarrow}$ Suppose $f \in V$ is a solution to the holonomic system given in 5.1, that is: $\langle p_i, f\rangle = 0\ \forall i = 1, \ldots, r$. Let $q(x) \in \mathbb{C}[x]$ be an arbitrary polynomial and we want to see that $q(D)f \in V$. Indeed,

$$\langle p_i, q(D)f\rangle = p_i(D)q(D)f = q(D)\langle p_i, f\rangle = 0\ \forall i = 1, \ldots, r.$$

$\boxed{\Leftarrow}$ As V is subset, $V^\perp$ is an ideal by corollary 3.12. Consider $B = \{f_1, \ldots, f_r\}$ a basis of V as a vector space. As V is D-closed, the derivarives $D_i(\bar{f}_j)$ can be expressed in the basis B for each $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Let $f = (f_1, \ldots, f_k)^T$ and consider M_i the matrix whose columns are the coordinates of $D_i(\bar{f}_j)$ in the basis B. Then the equality $D_i\bar{f} = M_i\bar{f}$ holds. Denote by $\varphi_i \in \mathbb{C}[\lambda]$ the characterisitic polynomial of M_i and observe that $\varphi_i(D_i)\bar{f} = \varphi_i(M_i)\bar{f} = 0$, as $\varphi_i(M_i) = 0$ by Cayley-Hamilton's formula. If we consider now each φ_i as polynomials in $\mathbb{C}[x_i]$, then we have that

$$\varphi_i(D)\bar{f} = 0 \implies \langle \varphi_i(x_i), f\rangle = 0 \implies \varphi_i(x_i) \in V^\perp \text{ for each } i = 1, \ldots, n.$$

By lemma 4.21 we have $dim(\Pi/V^\perp) = 0$ and $dim_{\mathbb{C}}(\Pi/V^\perp) < \infty$. We can use proposition 3.18 to reach the equalities

$$V^{\perp_o} = V^\perp\ ,\ V^{\perp_o\perp_o} = V^{\perp\perp}. \tag{5.2}$$

Consider the sesqui-linear map $\langle\, ,\, \rangle_o \colon \Pi \times \mathbb{F} \to \mathbb{C}$ defined by $\langle p, f\rangle_o = p(D)\bar{f}(\xi)_{|_{\xi=0}}$ as in chapter 3. We are free to apply lemma 2.7 as $dim_{\mathbb{C}}(V) < \infty$ and $\langle, \rangle_o$ is non-degenerate, hence

$$dim_{\mathbb{C}}V = dim_{\mathbb{C}}(\Pi/V^\perp)\ ,\ V^{\perp_o\perp_o} = V. \tag{5.3}$$

Finally, equations (5.2) and (5.3) imply of course that $V^{\perp\perp} = V$, which tells us that V is the solution space of an holonomic system. □

Remark 5.5. Let $V \subset F$ be a finite dimensional vector space. The theorem above guarantees that $V^\perp \subset \Pi$ is an ideal such that

$$dim(\Pi/V^\perp) = 0,\ dim_{\mathbb{C}}V = dim_{\mathbb{C}}(\Pi/V^\perp) \text{ and } V^{\perp\perp} = V.$$

Recall that $V^{\perp\perp} = \{f \in \mathbb{F} \colon \langle p, f\rangle = 0\ \forall p \in V^\perp\}$. It is known that every ideal of $\Pi = \mathbb{C}[x]$ is finitely generated, i.e. there exist $p_1, \ldots, p_r \in \mathbb{C}[x]$ such that

$V^{\perp} = \langle p_1(x), \ldots, p_r(x) \rangle$. Then, $V^{\perp\perp} = V$ implies that every $f \in V$ is a solution to the system of equations $\langle p_i, f \rangle = 0$, $i = 1, \ldots, r$. Therefore, V is the solution space of the holonomic system $p(D)f = 0$, $p \in V^{\perp}$.

Recall the following classic results on commutative algebra, needed to prove theorem 5.9 below.

Proposition 5.6. *Let I and J be ideals of a ring R. Then*

$$I, J \text{ maximals} \implies I = J \text{ or } I + J = R.$$

Proof. We know that $I \subseteq I + J \subseteq R$. As I is maximal and $I + J$ is an ideal, we either have $I + J = I$ or $I + J = R$. □

Definition 5.7. *Let I, J be ideals of a ring R. We say that I and J are coprime if $I + J = R$.*

Theorem 5.8. *(Chinese Remainder Theorem) Let $I_1, \ldots, I_k$ be ideals of a ring R and let $I = I_1 \cap \ldots \cap I_k$. If for each $i \neq j$ we have I_i and I_j coprime ideals, then*

$$R/I \cong R/I_1 \times \cdots \times R/I_k.$$

Moreover, if R is a commutative ring then we also have

$$R/I = R/I_1 \oplus \cdots \oplus R/I_k$$

$$I = I_1 \cap \ldots \cap I_k = I_1 \cdot \ldots \cdot I_k.$$

Proof. Can be found in [7] and [8]. □

The following theorem is the underlying theory base needed for the construction of interpolation techniques in the next chapter.

Theorem 5.9. *Let $I \subset \Pi = \mathbb{C}[x]$ be an ideal such that $dim(\Pi/I) = 0$ and consider $I = I_1 \cap \ldots \cap I_r$ it's unique shortest primary decomposition. The following properties are satisfied:*

(a) There exist points $\theta_i = (\theta_{i1}, \ldots, \theta_{in})$ such that $\sqrt{I_i} = (x - \theta_i)$ for each $i = 1, \ldots, r$.

(b) There exist decompositions of the ideal I and quotient ring Π/I

$$I = I_1 \cdots I_r, \quad \Pi/I = \Pi/I_1 \oplus \cdots \oplus \Pi/I_r$$

and of the D-closed subspace

$$I^{\perp} = I_1^{\perp} \oplus \cdots \oplus I_r^{\perp}$$

such that $dim_{\mathbb{C}}(I_i^{\perp}) = dim_{\mathbb{C}}(\Pi/I_i) < \infty$, $I_i^{\perp\perp} = I_i$ and $I^{\perp\perp} = I$.

(c) We define the shift of I_i as

$$\tau_i := \{p(x+\theta_i) \colon p \in I_i\} \subset \Pi.$$

Then each τ_i is a primary ideal whose radical ideal is $\sqrt{\tau_i} = (x) = \sum_{j=1}^{n} x_j \Pi$, which corresponds to the origin, and $\tau_i^{\perp}$ consists in polynomials. We also have

$$I_i^{\perp} = \tau_i^{\perp} \cdot exp(\bar{\theta}_i, \xi)$$

where

$$(\bar{\theta}_i \cdot \xi) := \bar{\theta}_1 \xi_1 + \ldots + \bar{\theta}_n \xi_n.$$

Proof. Statement (a) it's proven in lemma 4.21. Let's focus in proving (b) and (c). As $dim(\Pi/I) = 0$, we know that each $\sqrt{I_i}$ is maximal by 4.21, and that $\sqrt{I_i} \neq \sqrt{I_j}$ if $i \neq j$ as the decomposition is minimal. Then, the sum $\sqrt{I_i} + \sqrt{I_j} = \Pi \ \forall i \neq j$ by proposition 5.6, and so there exist $x \in \sqrt{I_i}$ and $y \in \sqrt{I_j}$ such that $1 = x + y$. We know that $x^n \in I_i$ and $y^m \in I_j$ for some $n, m \geq 0$ by the definition of radical. By the binomial theorem

$$1 = (x+y)^{n+m} = \sum_{k=0}^{m} \binom{n+m}{k} x^{n+m-k} y^k + \sum_{k=m+1}^{n+m} \binom{n+m}{k} x^{n+m-k} y^k.$$

The left sum includes the terms $x^n, x^{n+1}, \ldots, x^{n+m} \in I_i$ multiplied by some element in Π, and so the whole sum belongs to I_i. Similarly, the sum on the right belongs to I_j as it includes the terms $y^m, y^{m+1}, \ldots, y^{n+m}$. Then,

$$1 \in I_i + I_j \implies I_i + I_j = \Pi, \text{ and so each pair } I_i, I_j \text{ is coprime.}$$

By the Chinese Remainder Theorem

$$R/I = R/I_1 \oplus \cdots \oplus R/I_r \ , \ I = I_1 \cdot \ldots \cdot I_r.$$

In particular, $I_i + I_r = \Pi$ for each $i = 1, \ldots, r-1$, which means there exist $p_i \in I_i$ and $q_i \in I_r$ such that $p_i + q_i = 1$. Then,

$$p_1 \cdot \ldots \cdot p_{r-1} = \prod_{i=1}^{r-1} (1 - q_i) = 1 - q, \text{ for some } q \in I_r.$$

The previous can be written as $p_1 \cdot \ldots \cdot p_{r-1} + q = 1$, and so given $f \in \mathbb{F}$

$$f = (p_1 \cdot \ldots \cdot p_{r-1} + q)(D)f = (p_1 \cdot \ldots \cdot p_{r-1})(D)f + q(D)f.$$

We know that $p_1 \cdot \ldots \cdot p_{r-1} \in I_1 \cap \ldots \cap I_{r-1}$ and $q \in I_r$, and so we can conclude that each element $f \in \mathbb{F}$ satisfies:

$$f \in (I_1 \cap \ldots \cap I_{r-1})(D)f + I_r(D)f. \tag{5.4}$$

Let's prove now that $I^{\perp} = (I_1 \cap \ldots \cap I_{r-1})^{\perp} + I_r^{\perp}$. Indeed:

- $\boxed{\supseteq}$ As $I_1 \cap \ldots \cap I_{r-1} \subseteq I$ and $I_r \subseteq I$, it must be $(I_1 \cap \ldots \cap I_{r-1})^{\perp} \supseteq I^{\perp}$ and $I_r^{\perp} \supseteq I^{\perp}$. Then it's clear that $I^{\perp} \supseteq (I_1 \cap \ldots \cap I_{r-1})^{\perp} + I_r^{\perp}$.
- $\boxed{\subseteq}$ Suppose $f \in I^{\perp}$ and let's prove the inclusion $I_r(D)f \subset (I_1 \cap \ldots \cap I_{r-1})^{\perp}$. Indeed, if we take $x \in I_r(D)f$, there exists $p_r \in I_r$ such that $x = p_r(D)f$. Given $g \in I_r \cap \ldots \cap I_{r-1}$ we have

  $$g(D)x = g(D)p_r(D)f = (g \cdot p_r)(D)f = 0,$$

  as $f \in I^{\perp}$ and $g \cdot p_r \in I$, which implies $x \in (I_1 \cap \ldots \cap I_{r-1})^{\perp}$. The inclusion $(I_1 \cap \ldots \cap I_{r-1})(D)f \subset I_r^{\perp}$ is analogous. Finally, using 5.4 we get the desired inclusion

  $$f \in I^{\perp} \implies f \in (I_1 \cap \ldots \cap I_{r-1})(D)f + I_r(D)f \subset (I_1 \cap \ldots \cap I_{r-1})^{\perp} + I_r^{\perp}.$$

We could repeat the previous procedure recursively to reach the equality

$$I^{\perp} = I_1^{\perp} + \ldots + I_r^{\perp}. \tag{5.5}$$

Consider the shift $\tau_i := \{p(x + \theta_i) \colon p \in I_i\} \subset \Pi$ and let's prove that $\sqrt{\tau_i} = (x)$. Observe that if $p \in I_i \subseteq \sqrt{I_i} = \mathcal{I}(\theta_i)$, there must exist $a_1, \ldots, a_n \in \Pi$ such that $p(x) = a_1(x_1 - \theta_{i1}) + \ldots + a_n(x_n - \theta_{in})$. Then

$$p(x + \theta_i) \in \langle x_1, \ldots, x_n \rangle = (x) \implies \tau_i \subseteq (x).$$

This implies that $\sqrt{\tau_i} \subseteq \sqrt{(x)} = (x)$, as $(x) = \mathcal{I}(0)$ is the maximal ideal corresponding the the origin. For the other inclusion, observe that $\sqrt{I_i} = \mathcal{I}(\theta_i)$ implies that for each $j = 1, \ldots, n$:

$$f_j(x) = x_j - \theta_{ij} \in \sqrt{I_i} \implies p_j(x) = f_j(x)^{m_j} = (x_j - \theta_{ij})^{m_j} \in I_i \text{ for some } m_j.$$

Then, $p_j(x + \theta_i) = x_j^{m_j} \in \tau_i$ and so $x_1, \ldots, x_n \in \sqrt{\tau_i}$. This proves the inclusion $\sqrt{\tau_i} \supseteq (x)$. Finally, as $\sqrt{\tau_i} = (x)$ is a maximal ideal, τ_i is a primary ideal. Now let's prove the equality

$$I_i^{\perp} = \tau_i^{\perp} \cdot exp(\bar{\theta}_i, \xi).$$

To do so, observe that if $f = g \cdot exp(\bar{\theta}_i \cdot \xi)$ then

$$p(D)\bar{f} = p(D)(\bar{g} \cdot exp(\theta_i \cdot \xi)) = exp(\theta_i \cdot \xi) \cdot p(D + \theta_i)\bar{\xi}. \tag{5.6}$$

The previous gives us the desired equality:

$$f \in I_i^{\perp} \iff g \in \tau_i^{\perp} \iff g \cdot exp(\bar{\theta}_i \cdot \xi) \in \tau_i^{\perp} \cdot exp(\bar{\theta}_i \cdot \xi).$$

We know that $\sqrt{\tau_i} = (x)$, which implies that $(x)^m \subseteq \tau_i$ for some m. Then, τ_i must include all the monomials with order greater than or equal to m, and $\tau_i^{\perp_o}$ includes all polynomials with order smaller than m. Consider $\mathbb{F}_m$ the complex vector subspace spanned by all monomials with total order less than m and consider the non-degenerate sesqui linear form

$$\langle\, , \,\rangle\colon \Pi/(x)^m \times \mathbb{F}_m \to \mathbb{C}.$$

We denote by $\perp_o^m$ the orthogonal space with respect to this sesqui-linear form. By the second isomorphism theorem 4.13:

$$\Pi/\tau_i \cong (\Pi/(x)^m)/(\tau_i/(x)^m).$$

As $\Pi/(x)^m$ has finite dimension, we can apply lemma 2.7 to see that Π/τ_i has finite dimension as well.

$$\begin{aligned} dim_{\mathbb{C}}(\Pi/\tau_i) &= dim_{\mathbb{C}}(\Pi/(x)^m)/(\tau_i/(x)^m) = dim_{\mathbb{C}} \frac{\Pi/(x)^m}{(\tau_i/(x)^m)^{\perp_o^m \perp_o^m}} \\ &= dim_{\mathbb{C}}(\tau_i/(x)^m)^{\perp_o^m} = dim_{\mathbb{C}}(\tau_i^{\perp_o}) = dim_{\mathbb{C}}(\Pi/\tau_i^{\perp_o\perp_o}) < \infty. \end{aligned}$$

Therefore we have that $\tau_i = \tau_i^{\perp_o\perp_o} = \tau_i^{\perp\perp}$ and we can check that $I_i^{\perp\perp} = I_i$:

$$\begin{aligned} I_i^{\perp\perp} &= (\tau_i \cdot exp(\bar{\theta}_i \cdot \xi))^{\perp} = \{q(x)\colon q(D)(\bar{f}(x)exp(\theta_i \cdot \bar{\xi}) = 0\ \forall f \in \tau_i^{\perp}\} \\ &= \{q(x)\colon q(D+\theta_i)\bar{f}(x) = 0\ \forall f \in \tau_i^{\perp}\} \\ &= \{p(x-\theta_i)\colon p \in \tau_i^{\perp\perp} = \tau_i\} \\ &= I_i. \end{aligned}$$

Finally, observe that so far we know that $I^{\perp} = I_1^{\perp} + \ldots + I_r^{\perp}$, but we desire to see that they are in fact direct sums. To do so, we will see that the dimension of $I^{\perp}$ is the sum of the dimensions of all $I_i^{\perp}$. By theorem 5.8 we know that

$$\Pi/I \cong \Pi/I_1 \oplus \cdots \oplus \Pi/I_r,$$

and lemma 2.7 guarantees $dim_{\mathbb{C}} I^{\perp} = dim_{\mathbb{C}} \Pi/I$ and $dim_{\mathbb{C}} I_i^{\perp} = dim_{\mathbb{C}}(\Pi/I_i)$ for each $i = 1, \ldots, r$. Then:

$$\begin{aligned} dim_{\mathbb{C}}(I^{\perp}) &= dim_{\mathbb{C}}(\Pi/I) = dim_{\mathbb{C}}(\Pi/I_1) + \ldots + dim_{\mathbb{C}}(\Pi/I_r) \\ &= dim_{\mathbb{C}}(I_1^{\perp}) + \ldots + dim_{\mathbb{C}}(I_1^{\perp}). \end{aligned}$$

□

Remark 5.10. In the previous theorem we take $I \subset \Pi$ as the ideal of the polynomials with constant coefficients that correspond to the linear operators of the system of differential equations. Moreover, we will strict ourselves to holonomic systems, where we can use the result stated in lemma 4.21.

Chapter 6

Nature of interpolation

6.1 Hermite type interpolation

In this section we will study the construction of an interpolation polynomial based on the result seen in theorem 5.9. Let $\theta_1, \ldots, \theta_r \in \mathbb{C}^n$ be a set of points and consider the interpolation problem of finding some $q \in \mathbb{C}[x]$ such that

$$(g_{ij}(D)q)(\theta_i) = b_{ij} \tag{6.1}$$

where $b_{ij} \in \mathbb{C}$ are given values and

$$G_i = \{g_{ij} \in \mathbb{C}[\xi] \colon j = 1, \ldots, s_i\}$$

is a set of linearly independent polynomials for each $i = 1, \ldots, r$. Meaning, we want not to have redundant or contradictory conditions for all points θ_i. For the sake of clarity, observe we have used x as the variable of the interpolation polynomial and the conditions will be expressed on ξ, in order to maintain the notation from the theorem on the previous section. The sesqui-linear form we are using is $\langle\, , \rangle_o : \Pi \times \mathbb{F} \to \mathbb{C}$, where $\Pi = \mathbb{C}[x]$ and $\mathbb{F} = \mathbb{C}[\xi]$, defined as usual:

$$\langle q, g\rangle_o := q(D)\bar{g}(\xi)_{|_{\xi=0}}. \tag{6.2}$$

The problem stated above it's known as *Hermite type interpolation*, and we will develop the theory using the following assumption

$$\text{Span } G_i \text{ is D-closed for each } i = 1, \ldots, r. \tag{6.3}$$

Remark that *Span G* denotes the $\mathbb{C}-$ vector space generated by the elements of a given set G, meaning:

$$Span\, G = \left\{ \sum_{i=1}^{r} a_i\, g_i \colon a_i \in \mathbb{C},\, g_i \in G,\, r \in \mathbb{N} \text{ arbitrary} \right\}.$$

Example 6.1. A valid system of equations for Hermite type interpolation is

$$(D_{x^2} + D_y)\, q(\theta_1) = D_x\, q(\theta_1) = q(\theta_1) = 0.$$

This is true because the corresponding set of conditions $G_1 = \{\xi_1^2 + \xi_2, \xi_1, 1\}$ is differentially closed.

Let us recall the definition of a monomial order.

Definition 6.2. *A monomial order $\preceq$ in $\mathbb{N}^n$ is a total order such that $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$:*

- *$(0, \ldots, 0) \preceq \alpha$*
- *$\alpha \prec \beta \implies \alpha + \gamma \prec \beta + \gamma$.*

Example 6.3. We will use the following monomials orders:

- *Lexicographic order:* $\alpha \prec_{lex} \beta \iff$ The first non-zero term found in the list $\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n$ is positive.
- *Degree Lexicographic order* $\alpha \prec_{grlex} \beta \iff$ The first non-zero term found in the list $\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \alpha_i, \beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n$ is positive.

Definition 6.4. *Given $p \in \Pi$, we denote by $LE(p)$ the leading multi-exponent of p, that is, α that maximizes the value of $|\alpha|$, and x^α is a monomial that appears in p.*

Let's go through the idea behind the construction of a polynomial that solves the system of differential equations (6.1). Consider

$$\tau_i := (Span\, G_i)^{\perp_o} = \{q \in \Pi \colon g(D)q(x)_{|_{x=0}} = 0\ \forall g \in Span\, G_i\}.$$

the orthogonal space of each G_i, which is an ideal by corollary 3.12 and proposition 3.18, and consists of solutions to the corresponding homogeneous system of equations given by the polynomials of G_i when evaluating at $x = 0$. The polynomials in $Span\, G_i$ have some bounded degree $m - 1$, and so if all terms of $q(x)$ have degree greater than $m - 1$ then the polynomial belongs to τ_i. Therefore

$$(x)^m = \langle x^\alpha \colon |\alpha| = m \rangle \subset \tau_i,$$

meaning τ_i includes a power of the maximal ideal (x) associated to the origin. Then, if $g \in (x) \implies g^m \in (x)^m \subset \tau_i \implies f \in \tau_i$. Therefore $(x) \subseteq \sqrt{\tau_i}$, and it's in fact an equality as (x) is maximal. Observe that the radical of τ_i is a maximal ideal, and so τ_i must be a primary ideal, that is associated to the origin as well.

We define I_i as the set of solutions to the homogeneous system of equations when shifting the evaluation to $x = \theta_i$, i.e. imposing $b_{ij} = 0$ in equation (6.1). That is

$$I_i := \{p(x - \theta_i) : p \in \tau_i\}.$$

With the previous notation, the set of solutions of (6.1) for all points $\theta_1, \ldots, \theta_r$ is:

$$I = I_1 \cap \ldots \cap I_r = I_1 \cdot \ldots \cdot I_r.$$

Consider $\langle\, , \rangle_o : \Pi/I \times I^\perp \to \mathbb{C}$ the non-degenerate sesqui-linear form induced by 6.2, meaning given $[q] \in \Pi/I$ and $g \in I^\perp$ we consider

$$\langle [q], g\rangle_o := q(D)\bar{g}(\xi)_{|_{\xi=0}}.$$

Consider M_I the set of multi-exponents that do not appear in $\{LE(p) : p \in I\}$, meaning we exclude leading exponents of I. Each class in Π/I can be represented (see [3]) by a unique element of the set

$$Span\{x^\alpha : \alpha \in M_I\}.$$

For any values of $b_{ij} \in \mathbb{C}$ there exists a unique element $q(x) \in Span\{x^\alpha : \alpha \in M_I\}$ such that

$$\langle q(x), g_{ij}(\xi)exp(\bar{\theta} \cdot \xi)\rangle_o = q(D)\bar{g}_{ij}(\xi)exp(\theta \cdot \xi))_{|_{\xi=0}} = \bar{b}_{ij}.$$

Observe now that

$$\langle q(x), g_{ij}(\xi)exp(\bar{\theta} \cdot \xi)\rangle_o = \overline{\langle g_{ij}(x), q(\xi)\rangle_{\theta_i}}$$

Therefore

$$g_{ij}(D)q(\xi)_{|_{\xi=\theta_i}} = b_{ij},$$

and so the polynomial $q(x)$ is a solution to the system of equations in (6.1).

Hence, we have proven *Hermite's interpolation theorem:*

Theorem 6.5. *Let $\theta_1, \ldots, \theta_r \in \mathbb{C}^n$ be points and consider $G_i = \{g_{ij} : j = 1, \ldots, s_i\}$ a set of linearly independent polynomials for each $i = 1, \ldots, r$. Suppose that each $Span\, G_i$ is D-closed. Then, there exists a unique $q(x) \in Span\{x^\alpha : \alpha \in M_I\}$ solution to the system*

$$(g_{ij}(D)q)(\theta_i) = b_{ij}, \text{ where } b_{ij} \in \mathbb{C} \text{ are given values.}$$

Remark 6.6. The previous theorem guarantees the existence of an interpolation polynomial solution to the differential system, and the proof hints an algorithmic approach to it's computation. Observe that

$$q(x) \in Span\{x^\alpha : \alpha \in M_I\},$$

and so the interpolation space is induced by the set of multi-exponents M_I. We want then to find an expression for such set, which in our case is computed as

$$I = I_1 \cap \ldots \cap I_r\,,$$

where $I_i = \tau_i^{\perp}$ is orthogonal space of the ideal

$$\tau_i = (Span\ G_i)^{\perp},\ i = 1, \ldots, r.$$

Recall that each τ_i corresponds to the solution space of the holonomic system

$$g_{ij}(D)p = 0, \text{ for a fixed value of i.} \tag{6.4}$$

Therefore, conditions on each point θ_i in (6.1) can be treated separately, and we only need to compute the solution of the corresponding homogeneous system of equations (6.4).

6.2 Noetherian operators

We conclude this chapter with an introduction to *Noetherian operators*, which serve as a tool to caracterize the ideal of symbols of an holonomic system, effectively letting us calculate the orthogonal spaces that Hermite type interpolation requires.

Definition 6.7. *Consider $I = \mathcal{I}(\theta) \subset \Pi$ a primary ideal that satisfies $dim(\Pi/I) = 0$. The differential operators $g_1(D), \ldots, g_r(D)$ are known as Noetherian operators for I if*

$$p \in I \iff (g_i(D)p)(\theta) = 0 \ \forall i = 1, \ldots, r.$$

Lemma 6.8. *Let $p \in \Pi = \mathbb{C}[x]$ and $f \in \mathbb{C}[\xi]$. For any point $\theta \in \mathbb{C}^n$ we have*

$$\langle p, f \cdot exp(\bar{\theta} \cdot \xi) \rangle_o = \overline{\langle f, p \rangle_\theta}.$$

Proof. This equality holds by (5.6):

$$\begin{aligned} \langle p, f \cdot exp(\bar{\theta} \cdot \xi) \rangle_o &= exp(\theta \cdot \xi) p(D + \theta) \bar{f}(\xi)_{|_{\xi=0}} = \langle p(x + \theta), f(\xi) \rangle_o \\ &= \overline{\langle f(x), p(\xi + \theta) \rangle_o} = \overline{\langle f, p \rangle_\theta}. \end{aligned}$$

□

The following theorem caracterizes the Noetherian operators of the ideal I in theorem 6.5, and will be essential in the computation of explicit solutions.

Theorem 6.9. *Let $I \subset \Pi$ be a primary ideal with $\sqrt{I} = \mathcal{I}(\theta)$, for some point $\theta \in \mathbb{C}^n$. Suppose that*

$$g_1 \cdot exp(\bar{\theta} \cdot \xi), \ldots, g_r \cdot exp(\bar{\theta} \cdot \xi)$$

is a basis of the D-closed vector space $I^\perp$. Then, the differential operatos $g_1(D), \ldots, g_r(D)$ are Noetherian operators for I.

Proof. As $I^{\perp\perp} = I$ and by proposition (3.18) we have that

$$p \in I \iff \langle p, g_i \cdot exp(\bar{\theta} \cdot \xi) \rangle_o = 0 \ \forall i = 1, \ldots, r.$$

Then, using the previous lemma we achieve that

$$p \in I \iff \langle g_i, p \rangle_\theta = (g_i(D)p)(\theta) = 0 \, \forall i = 1, \ldots, r.$$

Therefore, $g_1(D), \ldots, g_r(D)$ are Noetherian operators for the ideal I. □

Chapter 7

Interpolation examples

In this section we will construct the solution space of an interpolation problem of Hermite type, following the steps that have led us to theorem 6.5.

Remark 7.1. In the following examples we will solve 2-dimensional interpolation problems, meaning we seek a polynomial $p(x, y)$ that satisfies

$$(g_{ij}(D)p)(\theta_i) = b_{ij}, \text{where } b_{ij} \in \mathbb{C} \text{ and } g_{ij} \in \mathbb{C}[\eta],\ \eta = (\xi, \mu).$$

We will use standard notation for the partial derivatives of p in respect to the variables x and y, meaning $D_x p = \frac{\delta}{\delta x} p$ and $D_y p = \frac{\delta}{\delta y} p$.

Example 7.2. Suppose we want to find a polynomial $p(x,y)$ such that

$$(D_{xx} + D_y)p(0, 0) = 1,\ D_x p(0, 0) = i,\ p(0, 0) = 3,\ p(2, i) = 5 + i.$$

We will seek first the interpolation space that corresponds to $\theta = (0, 0)$ and then the one corresponding to $\theta = (2, i)$.

The space of symbols of $\theta = (0, 0)$ is $G_1 = \{\xi^2 + \mu, \xi, 1\}$. We define V_1 as the space spanned by the polynomials $\{g \cdot exp(\bar{\theta} \cdot \eta) : g \in G_1\}$, in this case this is $V_1 = Span\, G_1$, and consider the D-closed space $\tau_1 = V_1^{\perp}$. We will apply theorem 6.9 in order to find Noetherian operators for the ideal τ_1. That is, if $g_i \cdot exp(\bar{\theta} \cdot \eta)$ form a basis of $\tau_1^{\perp}$, then $g_i(D)$ are Noetherian operators for τ_1. By construction, a basis of $\tau_1^{\perp} = V_1$ is $\{\xi^2 + \mu, \xi, 1\}$, and so the corresponding Noetherian operators are $D_{xx} + D_y$, D_x and 1. Therefore,

$$p \in \tau_1 \iff (D_{xx} + D_y)p(0, 0) = D_x p(0, 0) = p(0, 0) = 0.$$

Observe now that $p(0, 0) = 0 \implies p \in \langle x, y\rangle$, as p cannot have an independent term. Similarly, $D_x p(0, 0) = 0 \implies p \in \langle x^2, y\rangle$ and so we know it must be $p \in \langle x, y\rangle \cap \langle x^2, y\rangle = \langle x^2, y\rangle$. This means that the solution polynomial can be written as $p = ax^2 + by$, where $a, b \in \mathbb{C}[x,y]$. In order to completely determine τ_1 we need to apply the remaining Noetherian operator. That is,

$$(D_{xx} + D_y)p(x, y) = (a_{xx} + a_y)x^2 + (b_{xx} + b_y)y + 2a + b$$
$$(D_{xx} + D_y)p(0, 0) = 2a(0, 0) + b(0, 0) = 0.$$

This last condition implies that the polynomial $2a(x, y) + b(x, y) \in \langle x, y\rangle$, and so there exist $r, s \in \mathbb{C}[x, y]$ such that $2a + b = r\,x + s\,y$. Isolationg b in the last expression we can write

$$p = ax^2 + (-2a + rx + sy)y = ax^2 - 2ay + rxy + sy^2 = a(x^2 - 2y) + rxy + sy^2.$$

Therefore, we can conclude that $\tau_1 = \langle x^2 - 2y, xy, y^2\rangle$. In this case, this ideal is the shift corresponding to $I_1 = \{p(x - \theta) \colon p \in \tau_1\} = \tau_1$. Now we repeat the process to find the interpolation space corresponding to $\theta = (2, i)$. Let $G_2 = \{1\}$ and $V_2 = Span\ \{g \cdot exp(\bar{\theta} \cdot \eta) \colon g \in G_2\} = Span\ \{exp(2\xi - i\mu)\}$. Consider now $\tau_2 = V_2^{\perp} = \langle x, y\rangle$, as $g(D) = 1$ is the unique operator that defines τ_2. Then we have $I_2 = \{p(x - \theta) \colon p \in \tau_2\} = \langle x - 2, y - i\rangle$. Consider now $I = I_1 \cap I_2 = I_1 \cdot I_2$. Using Groëbner basis (see [3]) we can calculate the intersection, and so we could see that I is generated by

$$x^2 - 2y + (4 - 2i)y^2,\ xy + 2iy^2,\ y^3 - y^2.$$

The leading monomials of I with respect to the graded lexicographic order are $LM(I) = \{x^2, xy, y^3\}$. Consider M_I the set of monomials smaller than each monomial in $LM(I)$ with respect to the graded lexicographic. That is $M_I = \{1, x, y, y^2\}$, and those are all the generators we need as

$$|M_I| = dim_{\mathbb{C}}\mathbb{C}[x, y]/I = \left|\{\xi^2 + \mu, \xi, 1, exp(\xi - i\mu)\}\right| = 4$$

and so the interpolation space in which the polynomial $p(x, y)$ lives is

$$Span\ M_I = Span\ \{1, x, y, y^2\}$$

Therefore, there exist $a_1, a_2, a_3, a_4 \in \mathbb{C}$ such that $p(x, y) = a_1 + a_2x + a_3y + a_4y^2$ is a solution to the interpolation problem. In order to find the constant values, let's impose the original conditions:

$$\begin{cases} p(0, 0) = a_1 = 3 \\ D_xp(0, 0) = a_2 = i \\ (D_{xx} + D_y)p(0, 0) = a_3 = 1 \\ p(2, i) = a_1 + 2a_2 + a_3i - a_4 = 5 + i \implies a_4 = -2 + 2i \end{cases}$$

Finally, the Hermite interpolation polynomial is:

$$p(x, y) = 1 + ix + y + (-2 + 2i)y^2.$$

Example 7.3. Suppose we want to find a polynomial $p(x, y)$ such that

$$(D_{xx} + D_{xy})p(0, 0) = 4i, \; D_x p(0, 0) = i, \; D_y p(0, 0) = 1 + i,$$

$$p(0, 0) = 2 + 3i, \; p(1, i) = -i, \; D_y p(1, i) = 1 - i.$$

We will proceed as in the previous example. In one hand, the space of symbols of $\theta = (0, 0)$ is $G_1 = \{\xi^2 + \xi\mu, \xi, \mu, 1\}$ which corresponds to the interpolation space

$$V_1 = Span\ \{g \cdot exp(\bar{\theta} \cdot \eta) \colon g \in G_1\} = Span\ \{\xi^2 + \xi\mu, \xi, \mu, 1\}.$$

The Noetherian operators corresponding to the ideal $\tau_1 = V_1^{\perp}$ are $D_{xx} + D_{xy}$, D_x, D_y and 1. Meaning,

$$p \in \tau_1 \iff (D_{xx} + D_{xy})p(0, 0) = D_x p(0, 0) = D_y p(0, 0) = p(0, 0) = 0.$$

The conditions regarding the Noetherian operators D_x, D_y and 1 imply that

$$p \in \langle x^2, y\rangle \cap \langle x, y^2\rangle \cap \langle x, y\rangle = \langle x^2, xy, y^2\rangle.$$

Then, we can write $p = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{C}[x, y]$. Applying the remaining Noetherian operator we get

$$(D_{xx} + D_{xy})p(0, 0) = 2a(0, 0) + b(0, 0) = 0.$$

This means that $2a + b \in \langle x, y\rangle \implies 2a + b = r\,x + s\,y$. Then,

$$p = ax^2 + (-2a + rx + sy)xy + cy^2 = a(x^2 - 2xy) + rx^2y + cy^2 + sxy^2,$$

which implies that $p \in \langle x^2 - 2xy, x^2y, y^2, xy^2\rangle = \tau_1 = I_1$. On the other hand, for $\theta = (1, i)$ we have $G_2 = \{\mu, 1\}$ and so $V_2 = \{\mu \cdot exp(\xi - i\mu), 1 \cdot exp(\xi - i\mu)\}$. The ideal $\tau_2 = V_2^{\perp}$ has D_y and 1 as Noetherian operators, and so

$$p \in \tau_2 \iff D_y p(0, 0) = p(0, 0) = 0.$$

The previous implies that $\tau_2 = \langle x, y^2\rangle \cap \langle x, y\rangle = \langle x, y^2\rangle$, which is the shift ideal of

$$I_2 = \{p(x - \theta) \colon p \in \tau_2\} = \langle x - 1, (y - i)^2\rangle.$$

A Gröebner basis for the ideal $I = I_1 \cap I_2$ is

$$iy^3 + x^2 + 3y^2, \; -y^3 + xy + 2iy^2, \; y^4 - 2iy^3 - y^2.$$

The leading monomials are $LM(I) = \{y^3, y^4\}$, and then smaller monomials are $\{1, x, x^2, x^3, y, y^2, xy, x^2y, xy^2\}$. As $|M_I| = dim_{\mathbb{C}}\mathbb{C}[x, y]/I = 6$, we conclude that

$$Span\ M_I = Span\{1, x, x^2, x^3, y, y^2\}.$$

Therefore the interpolation polynomial can be written as $p(x, y) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4y + a_5y^2$. The coefficients are calculated imposing the following conditions:

$$\begin{cases} (D_{xx} + D_{xy})p(0, 0) = 2a_2 = 4i \\ D_x p(0, 0) = a_1 = i \\ D_y p(0, 0) = a_4 = 1 + i \\ p(0, 0) = a_0 = 2 + 3i \\ p(1, i) = a_0 + 1_1 + a_2 + a_3 + ia_4 - a_5 = -i \\ D_y p(1, i) = a_4 + 2ia_5 = 1 - i. \end{cases}$$

The previous system leads to the desired interpolation polynomial:

$$p(x, y) = 2 + 3i + ix + 2ix^2 - (2 + 8i)x^3 + (1 + i)y - y^2.$$

Let us finish this section with a 3-dimensional example. We will use again the notation $D_z p = \frac{\delta}{\delta z} p$ and the differential conditions will correspond to polynomials in $\mathbb{C}[\eta]$, where $\eta = (\xi, \mu, \lambda)$.

Example 7.4. Consider the interpolation problem of finding a polynomial $p(x, y, z)$ solution of the following system of differential equations:

$$\begin{aligned} D_{xx}p(1, i, -i) = 2 + i, \;\; D_{yy}p(1, i, -i) = 4i \\ D_{zz}p(1, i, -i) = 2 + 3i, \;\; D_{xyz}p(1, i, -i) = i. \end{aligned} \tag{7.1}$$

Observe that the set $\{\xi^2, \mu^2, \lambda^2, \xi\mu\lambda\}$ does not span a differentially closed subset and so we can't proceed as in previous examples directly. In order to fix this, we must consider also the following differential operators:

$$D_{xy}, \; D_{xz}, \; D_{yz}, \; D_x, \; D_y, \; D_z, \; 1. \tag{7.2}$$

Therefore, we are considering the set $G = \{\xi^2, \mu^2, \lambda^2, \xi\mu\lambda, \xi\mu, \xi\lambda, \mu\lambda, \xi, \mu, \lambda, 1\}$. Let $V = Span\ \{g \cdot exp(\bar{\theta} \cdot \eta) \colon g \in G\}$ and $\tau = V^{\perp}$. A polynomial q belongs to τ only if $g(D)q(0, 0, 0) = 0$ for each $g \in G$. As before, the polynomial ξ^2 corresponds to the ideal $\langle x^3, y, z\rangle$ and $\xi\mu\lambda$ corresponds to $\langle x^2, y^2, z^2\rangle$, etc. Therefore, τ will be the intersection of all ideals corresponding to polynomials of G. This turns out to be

$$\tau = \langle z^3, yz^2, y^2z, y^3, xz^2, xy^2, x^2z, x^2y, x^3\rangle.$$

This corresponds to the ideal $I = \{p(x - 1, y - i, z + i) \colon p \in \tau\}$, whose leading monomials are

$$LT(I) = \left\{z^3, yz^2, y^2z, y^3, xz^2, xy^2, x^2z, x^2y, x^3\right\}.$$

The monomials that are smaller with respect to the degrelexicographic order are

$$M_I = \left\{1,\, x,\, x^2,\, y,\, xy,\, y^2,\, z,\, xz,\, yz,\, xyz,\, z^2\right\}.$$

We know that the interpolation polinomial satisfies $p \in Span\, M_I$, and so there exist some constants $a_i \in \mathbb{C}$ such that:

$$p(x,y,z) = a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 xy + a_5 y^2 + a_6 z + a_7 xz + a_8 yz + a_9 xyz + a_{10} z^2.$$

Now, in order to determine the constants a_i we must impose the conditions (7.1), and also we must give values the derivatives listed in (7.2), when evaluating at $\theta = (1, i, -i)$. Choosing all these values to be zero, i.e.

$$D_{xy} p(1,i,-i) = D_{xz} p(1,i,-i) = \ldots = p(1,i,-i) = 0,$$

the values of the constants are

$$a_0 = -4i,\ a_1 = -2,\ a_2 = 1 + \frac{i}{2},\ a_3 = 5,\ a_4 = -1,\ a_5 = 2i$$

$$a_6 = -4 + 2i,\ a_7 = 1,\ a_8 = -i,\ a_9 = i,\ a_{10} = 1 + \frac{3i}{2}.$$

Therefore, the interpolation polynomial is:

$$\begin{aligned} p(x,y,z) = &-4i - x + (1 + \frac{i}{2})x^2 + 5y - xy + 2iy^2 + \\ &(-4 + 2i)z + xz - iyz + ixyz + (1 + \frac{3i}{2})z^2. \end{aligned}$$

Appendix A

Mathematica code

The code used for all computations can be accessed via scanning the following QR code, which is linked to a Github repository. There is a Mathematica font available (in format .nb) and also a readable PDF version of the notebook.

In case the previous code does not work, please click or enter the following URL in your browser:

`https://github.com/josegimenez1999/TFG`

The algorithms found in the code generalize the procedure seen in the examples of chapter 7. We have used Mathematica as the algorithm for Gröebner basis calculations is already implemented. Mainly, we require the use of such basis for the computation of the intersection of two ideals, which is a known procedure based on the following theorem:

Theorem A.1. *Let $I = \langle f_1, \dots, f_r \rangle$ and $J = \langle g_1, \dots, g_s \rangle$ be ideals of $\mathbb{C}[x_1, \dots, x_n]$. Let $\prec$ be a monomial order satisfying $x_i \prec t$ for each variable x_i. Then, $I \cap J$ admits the following Gröebner basis:*

$$GB_{\prec}\{tf_1, \dots, tf_r, (1-t)g_1, \dots, (1-t)g_s\} \cap \mathbb{C}[x_1, \dots, x_n].$$

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