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DIRICHLET SERIES AND HARDY SPACES

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Abstract

The aim of this work is to study Dirichlet series, as well as some of the function spaces that they form. We will see the convergence properties of these series and how they differ from power series, as well as other important properties. We will also study the Hardy-Dirichlet spaces $\mathscr{H}^\infty$ and $\mathscr{H}^2$. Finally, we will study the relationship between Dirichlet series in these spaces and power series in an infinite number of variables in the poly-torus $\mathbb{T}^\infty$ and poly-disk $\mathbb{D}^\infty$.

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Contents

Introduction

One of the most famous unsolved problems in mathematics is to prove the Riemann hypothesis, that states that all the non-trivial zeros of the complex function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

lie on the line $s = 1/2 + it$, where $t \in \mathbb{R}$.

One way in which mathematicians have tried to solve the problem is to develop the study of a more general type of series, known as Dirichlet series,

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which were introduced by Peter Gustav Lejeune Dirichlet in 1837. Dirichlet and Dedekind began the study of these series for their applications in number theory, and proved some important results concerning Dirichlet series. But they only considered them for real values and the first one to consider complex valued Dirichlet series was Jensen. They were also used in 1896 by Jacques Hadamard and Charles-Jean de la Vallée Poussin to prove the Prime Number Theorem. Although the attempts to prove the Riemann hypothesis using Dirichlet series have not succeeded, they began to be studied for their own sake during the beginning of the twentieth century.

A fundamental mathematician in the development of the theory of Dirichlet series at this time was Harald Bohr[2], brother of the physics Nobel prize winner Niels Bohr. He proved a deep theorem relating the uniform convergence of these series and their boundedness, which we will see a particular case of in the third chapter (see Theorem 3.11). This theorem leads to a natural question concerning the convergence of Dirichlet series, which he could not answer, but he developed fundamental tools that finally allowed to answer the question (see Theorem 2.26). The key tool is Bohr's transformation, which will be very useful more than once in this work.

The second half of the twentieth century saw a decrease in the interest of Dirichlet series, until the publishing of the 1997 paper by Hedenmalm, Lindqvist, and Seip [12]. Several properties of Dirichlet series were brought back and new function spaces were studied, using the developments of functional analysis and infinite dimensional holomorphy theory which were not available to mathematicians at the beginning of the 1900s. This paper sparked a revival of this area of study and function spaces of Dirichlet series, which continues up to the present day.

Dirichlet series are not as general as power series in the sense that they only represent a special type of analytic functions. Nonetheless, their study has become fundamental in analytic number theory.

[2]Before receiving his PhD in mathematics, Bohr was a member of the Danish national football team, and won a silver medal in the 1908 London Olympics. In fact, he participated in the 17-1 match against France that still holds the Olympic record for most goals scored in one game. His fame as a footballer was so big that when he defended his doctoral thesis, the room was filled with more fans than mathematicians.

This paper is organised as follows.

The first chapter of this work is a brief overview of definitions and classical results for power series. We will see some basic results on their regions of convergence, the Cauchy-Hadamard formula, and representation formulas to recover the coefficients from the power series.

The second chapter will be dedicated to the study of Dirichlet series. We will see that their regions of convergence differ from those of power series. It is well known that power series converge absolutely in an open disk and uniformly in any disk with a smaller radius. Instead, Dirichlet series converge in vertical half-planes and the behaviour of their absolute and uniform convergence might be different. One can define the convergence abscissas, which are the left-most abscissa of the half-plane where the Dirichlet series converges. In particular we have the absolute and uniform convergence abscissas, respectively:

$$\sigma_a = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \text{ is absolutely convergent} \right\},$$

$$\sigma_u = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \text{ is uniformly convergent in } \overline{\mathbb{C}_{\sigma_u}} \right\}.$$

One of the objectives of this chapter will be to prove the Bohnenblust-Hille Theorem, which answers the question Bohr posed on the separation of the absolute and uniform convergence abscissas. In particular,

Theorem *If $\mathcal{D}$ is the set of all Dirichlet series which converge at some point in the complex plane, then*

$$S := \sup_{D \in \mathcal{D}} (\sigma_a(D) - \sigma_u(D)) = 1/2.$$

We will also see different properties of Dirichlet series, formulas to compute the different abscissas and the analogies and differences to power series.

In the third chapter we will study some important Dirichlet series function spaces, namely the so called Hardy-Dirichlet spaces. In particular, we will study $\mathscr{H}^\infty$ and $\mathscr{H}^2$, their properties and how they are related. They are analogous to the classic Hardy spaces in the disk $H^\infty(\mathbb{D})$ and $H^2(\mathbb{D})$. We will also define at last Bohr's lift. This transformation identifies spaces of Dirichlet series with spaces of power series in an infinite number of complex variables. Using this transformation, we can use the knowledge of power series to prove properties of Dirichlet series and vice-versa. We will do so at the end of the chapter, where we will prove the multiplier theorem, which roughly states that:

Theorem: *If m is analytic in $\mathbb{C}_{1/2}$, then $fm \in \mathscr{H}^2$ for any $f \in \mathscr{H}^2$ if and only if $m \in \mathscr{H}^\infty$.*

This theorem was proved in the seminal paper by Hedenmalm, Lindqvist and Seip in 1997.

The main references used for the theory of Dirichlet series are the classical books by E.C. Titchmarsh [16] and T.M. Apostol [1], and the preliminary version of a yet unpublished book by J.L. Férnandez [13]. The proof of the Bohnenblust-Hille Theorem is based on the 1997 paper by H.P. Boas [3]. For the Hardy-Dirichlet spaces, the main references are Queffélec and Queffélec [14] and the ever so useful book by Defant et al. [11]. The proof of the multiplier theorem is based on the 1997 paper by Hedenmalm et al. [12].

Notation and preliminaries

We will write $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ for the set of natural numbers, $\mathbb{Z}$ for the set of integers and $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers respectively. The letters s and z are usually reserved for complex numbers, using s as the complex variable for Dirichlet series and z for power series. If $z \in \mathbb{C}$, then we will write its real part as $\Re z$. In particular, for Dirichlet series we will write $s = \sigma + it$ with $\sigma = \Re s$. We will usually write the letters n, m, N and M to mean a natural number. If $x \in \mathbb{R}$, we define the integer part of x as $[x] := \max\{n \in \mathbb{Z} : n \le x\}$. The letter p will usually mean a prime number $\{2, 3, 5, 7, \dots\}$, p_n will be the n-th prime and $\mathfrak{p} = (p_1, p_2, p_3, \dots)$. We will use $\sum_p$ and $\prod_p$ to mean the sum and product over the primes, respectively. The function $\pi(n)$ is the prime counting function

$$\pi(n) = \#\{p : p \le n\},$$

where $\#$ is the cardinality of the set.

The unit open disk, the closed disk and the circle (also knwon as torus) in the complex plane are

$$\begin{aligned} \mathbb{D} &= \{z \in \mathbb{C} : \ |z| < 1\}, \\ \overline{\mathbb{D}} &= \{z \in \mathbb{C} : \ |z| \le 1\}, \\ \mathbb{T} &= \{z \in \mathbb{C} : \ |z| = 1\}, \end{aligned}$$

respectively. An open disk of arbitrary center a and radius r will be written as $D(a, r) = \{z \in \mathbb{C} : \ |z - a| < r\}$. The closure of a set A will be written as $\overline{A}$. We write a finite product of a set A as $A^N = \overbrace{A \times \cdots \times A}^{N}$. We will also use the infinite complex plane, the infinite poly-disk and the infinite poly-torus, defined as

$$\begin{aligned} \mathbb{C}^\infty &= \{z = (z_1, z_2, \dots) : \ z_i \in \mathbb{C}\}, \\ \mathbb{D}^\infty &= \{z = (z_1, z_2, \dots) : \ |z_i| < 1\}, \\ \mathbb{T}^\infty &= \{z = (z_1, z_2, \dots) : \ |z_i| = 1\} \end{aligned}$$

respectively. We will consider $\mathbb{D}^N$ as a subset of $\mathbb{D}^\infty$, and similarly $\mathbb{T}^N$ as a subset of $\mathbb{T}^\infty$, with the identification $(z_1, \dots, z_N) \to (z_1, \dots, z_N, 0, \dots)$. Though this will not be essential in this work, for the sake of completion, on $\mathbb{T}^\infty$ we will consider the probability measure ρ, which is the product of the normalized arc length measure in each $\mathbb{T}$. We will also consider Bochner integration in $\mathbb{T}^\infty$.

We will say that a topological space X is separable if there exists a countable dense subset $A \subset X$, i.e. $\overline{A} = X$ and A is countable.

As for sequence spaces, ℓ^2 is the set of sequences $(a_n)_n \subset \mathbb{C}^\infty$ such that $\sum_{n=1}^\infty |a_n|^2 < +\infty$. The space of all bounded complex sequences that converge to zero is c_0.

If ρ is a positive measure on a set A, the space $L^1(A, \rho)$ will be the space of all measurable functions $f : A \to \mathbb{C}$ such that

$$\|f\|_{L^1(A,\rho)} = \int_A |f| d\rho < +\infty.$$

Similarly, the space $L^2(A, \rho)$ will be the space of all measurable functions $f : A \to \mathbb{C}$ such that

$$\|f\|^2_{L^2(A,\rho)} = \int_A |f|^2 d\rho < +\infty.$$

Given a path $\gamma : [a, b] \to \mathbb{C}$, we will write the integral of a function f on γ as

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

In general if $A, B \in \mathbb{C}$, we write $\int_A^B$ for the integral over the segment path that starts at A and ends at B. We will also write $\int_{\rho-i\infty}^{\rho+i\infty}$ when we refer to $\lim_{T\to\infty} \int_{\rho-iT}^{\rho+iT}$, and suppose that the limit exists.

If $f(x)$ and $g(x)$ are two real valued functions, we will write $f(x) \lesssim g(x)$ if there is a constant $C \geq 0$ independent of x such that $f(x) \leq Cg(x)$. We will also use Landau's big O notation. We will write $f(x) = \mathcal{O}(g(x))$ if there exists $M \geq 0$ and x_0 such that $|f(x)| \leq Mg(x)$ for all $x \geq x_0$.

A useful tool that we will use is summation by parts, otherwise known as Abel transformation. If $(a_n)_n$ and $(b_n)_n$ are two sequences, then

$$\sum_{i=m}^{n} a_i(b_{i+1} - b_i) = a_n b_{n+1} - a_m b_m - \sum_{i=m+1}^{n} b_i(a_i - a_{i-1}).$$

A common tool that will be used often is the residue theorem, which applies to meromorphic functions.

Definition 0.1. *Let f be a function defined in an open set $\Omega \subset \mathbb{C}$. We say that f is meromorphic in Ω if it is holomorphic in Ω except on a set of isolated points, which are called the poles of the function.*

If f is a meromorphic function with a pole in z_0, it can be written as a Laurent series around z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{(z - z_0)^n}$$

Considering the Laurent series expansion, we define the residue of f at z_0 as

$$\mathrm{Res}(f, z_0) = a_{-1}.$$

Definition 0.2. *Let γ be a piece-wise continuously differentiable loop in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ a point not in the image of γ. Then we define the index of z_0 with respect to γ as*

$$\mathrm{Ind}(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0}.$$

Proposition 0.3 (Residue Theorem)**.** *Let f be a meromorphic function in Ω with poles $\{a_1, \ldots, a_n\} \subset \Omega$. Let γ be a piece-wise continuously differentiable loop in $\Omega \setminus \{a_1, \ldots, a_n\}$. Then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Ind}(\gamma, a_i)\mathrm{Res}(f, a_i).$$

We will also use the following result for infinite products.

Proposition 0.4. *Let $(z_n)_n \subset \{z \in \mathbb{C} : \Re z > 0\}$. Then the following are equivalent:*

- *$\prod_{n=1}^{\infty} z_n$ is unconditionally convergent.*

- $\sum_{n=1}^{\infty} \log z_n$ *is unconditionally convergent.*
- $\sum_{n=1}^{\infty} |\log z_n| < +\infty$.
- $\sum_{n=1}^{\infty} |1 - z_n| < +\infty$.

Proposition 0.5 (Fatou's lemma). *Let (Ω, μ) be a mesurable space and $f_n : \Omega \to [0, +\infty)$ a sequence of mesurable functions for $n \geq 1$. Then*

$$\int_\Omega \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

We will use the following convergence tests and results for series

Proposition 0.6 (M-Weierstrass test). *Let $(f_n)_n$ be a sequence of functions defined in $A \subset \mathbb{C}$. If there exists a sequence of numbers $(M_n)_n$ such that for all $n \geq 1$ and $z \in A$*

$$|f_n(z)| \leq M_n$$

and

$$\sum_{n=1}^{\infty} M_n < +\infty$$

then the series

$$\sum_{n=1}^{\infty} f_n(z)$$

converges absolutely and uniformly on A.

Proposition 0.7 (Root test). *Let $\sum_{n=1} a_n$ be a complex series. Consider*

$$L := \limsup_{n\to\infty} |a_n|^{1/n}.$$

If $L < 1$, the series converges absolutely, and if $L > 1$, the series does not converge.

Proposition 0.8 (Condensation test). *Let $(a_n)_n$ be a sequence of non-increasing non-negative real numbers. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.*

Proposition 0.9 (Comparison test). *Let $(a_n)_n$ and $(b_n)_n$ be two sequences of non-negative real numbers. If there exists $N \geq 1$ such that $0 \leq a_n \leq b_n$ for all $n \geq N$, then*

- *if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.*
- *if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.*

Proposition 0.10 (Abel-Dirichlet-Dedekind alternating series test). *Let $(a_n)_n$ be a sequence of complex numbers. If*

$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| < +\infty$$

and $\lim_{n\to\infty} a_n = 0$, then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

1 Power Series

This brief chapter recaps the main definitions and properties of power series. It will serve as a comparison of the similarities and differences that Dirichlet series have with respect to power series.

Definition 1.1. *Given a sequence of complex numbers $(a_n)_{n\geq 0}$ we define its corresponding power series centered around 0 as*

$$A(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where we will refer to a_n as the coefficients of the series.

In a similar way one can define the power series centered around a given complex number z_0

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n .$$

By means of a translation, one can always assume that $z_0 = 0$.

Definition 1.2. *Given a power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ we define its radius of convergence as:*

$$R = \sup_{r\in\mathbb{R}} \Big\{ |z| < r \ : \sum_{n=0}^{\infty} a_n z^n \text{ converges} \Big\}.$$

Similarly one can define the radius of absolute convergence R_a, uniform convergence R_u and the radius of holomorphy R_h as:

$$R_a = \sup_{r\in\mathbb{R}} \Big\{ |z| < r \ : \sum_{n=0}^{\infty} a_n z^n \text{ converges absolutely} \Big\},$$

$$R_u = \sup_{r\in\mathbb{R}} \Big\{ |z| < r \ : \sum_{n=0}^{\infty} a_n z^n \text{ converges uniformly on } D(0,r) \Big\},$$

$$R_h = \sup_{r\in\mathbb{R}} \Big\{ |z| < r \ : \sum_{n=0}^{\infty} a_n z^n \text{ is holomorphic in } D(0,r) \Big\}.$$

Theorem 1.3. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series. The four radii coincide and can be calculated using the Cauchy-Hadamard formula:*

$$R = R_h = R_a = R_u = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

More importantly

- *If $|z| < R$ then the power series converges absolutely and uniformly on $|z| \leq r$ for any $0 \leq r < R$,*
- *If $|z| > R$ then the power series does not converge.*

Proposition 1.4. *Let* $A(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with radius of convergence* $R > 0$*, we then have*

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{A(z)}{z^{n+1}} dz \quad n \geq 0;\ 0 < r < R.$$

Proof. We begin with the following integral formula for an integer k and any $r > 0$,

$$\frac{1}{2\pi i} \int_{|z|=r} z^k dz = \begin{cases} 1, & \text{if } k = -1, \\ 0, & \text{if } k \neq -1, \end{cases}$$

which is obtained doing a parametrization of the circle and using

$$\frac{1}{2\pi i} \int_0^{2\pi} e^{in\theta} d\theta = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Now $A(z)$ is holomorphic in $D(0, R)$ and is uniformly convergent in the circle $\{|z| = r\}$ if $r < R$, so we can permute integral and sum in the next expression

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{A(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int_{|z|=r} \frac{\sum_{k=0}^{\infty} a_k z^k}{z^{n+1}} dz = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{|z|=r} \frac{a_k}{z^{n-k+1}} dz = a_n.$$

□

This last formula is very useful and pretty well-known, but now we will give a not so well-known formula for the partial sums of a series, that is a corollary of this one after we perform a clever transformation.

Corollary 1.5. *Let* $A(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with radius of convergence* R*. If we define the partial sum of the coefficients* $A_N = \sum_{n=0}^{N} a_n$*, for any* $0 < r < \min(1, R)$ *we have*

$$A_N = \frac{1}{2\pi i} \int_{|z|=r} \frac{A(z)}{(1-z) z^{N+1}} dz.$$

Proof. For all $z \in D(0, \min(1, R))$ we claim that

$$B(z) := \frac{A(z)}{1-z} = \sum_{N=0}^{\infty} A_N z^N.$$

Firstly note that $a_N = A_N - A_{N-1}$ for $N \geq 1$, thus we can write the following

$$\begin{aligned} A(z) &= \sum_{N=0}^{\infty} a_N z^N = a_0 + \sum_{N=1}^{\infty} a_N z^N = A_0 z^0 + \sum_{N=1}^{\infty} (A_N - A_{N-1}) z^N \\ &= \sum_{N=0}^{\infty} A_N z^N - \sum_{N=1}^{\infty} A_{N-1} z^N = \sum_{N=0}^{\infty} A_N z^N - \sum_{N=0}^{\infty} A_N z^{N+1} = (1-z) \sum_{N=0}^{\infty} A_N z^N. \end{aligned}$$

So now we have written $B(z)$ as a power series which coefficients are the partial sums for the initial series. Using Proposition 1.4 we get the desired expression

$$A_N = \frac{1}{2\pi i} \int_{|z|=r} \frac{B(z)}{z^{N+1}} dz = \frac{1}{2\pi i} \int_{|z|=r} \frac{A(z)}{(1-z) z^{N+1}}.$$

□

2 Dirichlet Series

Definition 2.1. *Given a sequence of complex numbers* $(a_n)_{n\geq 1}$ *we define its corresponding **general Dirichlet series** as:*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

where $(\lambda_n)_{n\geq 1}$ *is an increasing sequence of non-negative real numbers which approaches* $+\infty$.

In the special case that $\lambda_n = \log n$ *we then have an **ordinary Dirichlet series**:*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We will refer to these last type of series from now on simply as Dirichlet series, without specifying that they are ordinary. These series are defined wherever they converge.

Observation 2.2. *We will use* s *as the complex variable instead of* z*, as is usual in the study of Dirichlet series. We will also write* $s = \sigma + it$ *with* $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$.

Also, note that a general Dirichlet series with $\lambda_n = n$ *is a power series with the change of variable* $z = e^{-s}$*. Indeed:* $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} = g(e^{-s})$ *with* $g(z) = \sum_{n=1}^{\infty} a_n z^n$ *a power series.*

Example 2.3.
- *The simplest Dirichlet series is when* $a_n = 1$ *for* $n \geq 1$*, and we get the famous **Riemann zeta function***

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- *A related series is the **Dirichlet eta function**, given by*

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

- *The function*

$$\Theta(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where μ *is the **Möbius function**. It is defined for* $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$*, where* p_i *are the prime factors of* n *for* $1 \leq i \leq r$*, by*

$$\mu(n) = \begin{cases} 0, & \text{if } \alpha_i \geq 2 \text{ for some } 1 \leq i \leq r, \\ (-1)^r, & \text{if } \alpha_i = 1 \text{ for } 1 \leq i \leq r, \\ 1, & \text{if } n = 1. \end{cases}$$

Now let us give two definitions for regions that we will be using throughout this section.

Definition 2.4. *Given a real number σ, we define the vertical half-plane*

$$\mathbb{C}_\sigma = \{s \in \mathbb{C} : \Re s > \sigma\}.$$

Given an angle $\beta \in [0, \pi/2)$, we define the closed cone

$$\Gamma_\beta = \left\{ s = re^{i\theta} : r \geq 0, |\theta| \leq \beta \right\} = \left\{ s \in \mathbb{C} : |s| \leq \frac{\Re s}{\cos \beta} \right\}.$$

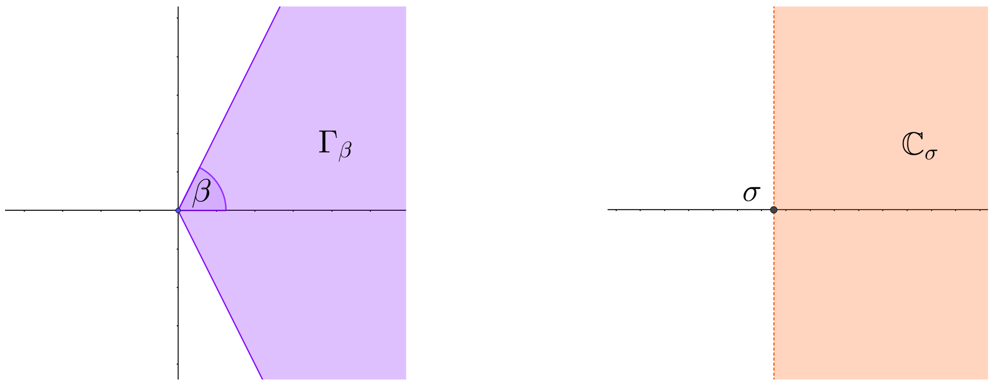

Figure 1: Visual representation of the regions in Definition 2.4.

Proposition 2.5 (Jensen's Lemma)**.** *If the general Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ converges for a certain $s_0 \in \mathbb{C}$ then $f(s)$ converges uniformly in $s_0 + \Gamma_\beta$, $\forall \beta \in [0, \pi/2)$.*

Proof. By the means of a translation, we can assume that $s_0 = 0$ without loss of generality, which implies that $\sum_{n=1}^{\infty} a_n$ converges. Indeed, if $f(s)$ converges for s_1, then $f(s + s_1)$ converges for $s = 0$.

Fix $0 \leq \beta < \pi/2$. Since $\sum_{n=1}^{\infty} a_n$ converges, for any $\epsilon > 0$ there exists $N_0 \geq 1$ such that for any $M > N \geq N_0$

$$\left| \sum_{n=N}^{M} a_n \right| < \frac{\epsilon}{2} \cos \beta.$$

Now consider $s = \sigma + it \in \Gamma_\beta$. Writing $A_k = \sum_{n=N}^{k} a_n$ for $n \geq N$ and $A_k = 0$ for $k < N$ and performing an Abel transformation we get

$$\begin{aligned}\sum_{n=N}^{M} a_n e^{-\lambda_n s} &= \sum_{n=N}^{M} A_n e^{-\lambda_n s} - \sum_{n=N}^{M} A_{n-1} e^{-\lambda_n s} = \sum_{n=N}^{M} A_n e^{-\lambda_n s} - \sum_{n=N-1}^{M-1} A_n e^{-\lambda_{n+1} s} \\ &= \sum_{n=N}^{M-1} A_n \left(e^{-\lambda_n s} - e^{-\lambda_{n+1} s} \right) + A_M e^{-\lambda_M s}.\end{aligned}$$

Hence

$$\begin{aligned}\left| \sum_{n=N}^{M} a_n e^{-\lambda_n s} \right| &\leq \sum_{n=N}^{M-1} |A_n| \left| e^{-\lambda_n s} - e^{-\lambda_{n+1} s} \right| + |A_M| \left| e^{-\lambda_M s} \right| \\ &< \frac{\epsilon}{2} \cos \beta \sum_{n=N}^{M-1} \left| e^{-\lambda_n s} - e^{-\lambda_{n+1} s} \right| + \frac{\epsilon}{2} \cos \beta \left| e^{-\lambda_M s} \right|.\end{aligned} \tag{2.1}$$

Notice that for $s \in \Gamma_\beta$, we have $\Re s \geq 0$. Also, since $(\lambda_n)_n$ is a sequence of non-negative numbers, for any $n \geq 1$ we have $|e^{-\lambda_n s}| \leq 1$. Now we can write

$$e^{-\lambda_n s} - e^{-\lambda_{n+1} s} = \int_{\lambda_n}^{\lambda_{n+1}} s e^{-st} dt.$$

Thus

$$\left|e^{-\lambda_n s} - e^{-\lambda_{n+1} s}\right| \leq |s| \int_{\lambda_n}^{\lambda_{n+1}} \left|e^{-st}\right| dt = |s| \int_{\lambda_n}^{\lambda_{n+1}} e^{-\sigma t} dt,$$

and hence

$$\sum_{n=N}^{M-1} \left|e^{-\lambda_n s} - e^{-\lambda_{n+1} s}\right| \leq \sum_{n=N}^{M-1} |s| \int\limits_{\lambda_n}^{\lambda_{n+1}} \left|e^{-st}\right| dt \leq |s| \int\limits_{\lambda_1}^{\infty} e^{-\sigma t} dt = \frac{|s|}{\sigma} e^{-\lambda_1 \sigma} \leq \frac{|s|}{\sigma} \leq \frac{1}{\cos \beta}, \tag{2.2}$$

where the last equality holds for any $s \in \Gamma_\beta$. For any $M \geq 1$ we have

$$|e^{-\lambda_M s}| \leq 1 \leq \frac{1}{\cos \beta}. \tag{2.3}$$

Finally, substituting equations (2.2) and (2.3) into equation (2.1) we get

$$\left|\sum_{n=N}^{M} a_n e^{-\lambda_n s}\right| < \epsilon,$$

which proves the result.

□

Observation 2.6. *Jensen's lemma shows that if a Dirichlet series is convergent at a point s_0 then it is convergent at the right half-plane of that point, as we can let β approach $\pi/2$. This leads us the next definition, given that the regions of convergence for Dirichlet series are half-planes instead of open disks, as they are for power series.*

Definition 2.7. *Given an ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, we define the **convergence abscissa**:*

$$\sigma_c = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \text{ is convergent} \right\}.$$

Jensen's lemma tells us that if σ_c is the convergence abscissa for a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, then:

- *If $\Re s > \sigma_c$ then the Dirichlet series converges.*
- *If $\Re s < \sigma_c$ then the Dirichlet series does not converge.*

If the series is convergent in all the complex plane, we say that $\sigma_c = -\infty$ and if it is nowhere convergent then $\sigma_c = +\infty$.

We will write $\sigma_c(f)$ should we need to make explicit that the convergence abscissa is for the corresponding Dirichlet series f.

Observation 2.8. *Note that we have not said anything about the convergence of the series on the line $\Re s = \sigma_c$. That is because, similarly to power series, the Dirichlet series can converge nowhere, on some points or on all the line.*

Using the above definition of σ_c, from Jensen's Lemma we get the following corollary.

Corollary 2.9. *Let f be a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with convergence abscissa σ_c. Then f is a holomorphic function in $\mathbb{C}_{\sigma_c}$ and its derivatives are:*

$$f^{(k)}(s) = \sum_{n=1}^{\infty} (-1)^k (\log n)^k \frac{a_n}{n^s} \quad s \in \mathbb{C}_{\sigma_c},\ k \geq 1.$$

Proof. Jensen's Lemma shows that the Dirichlet series converges uniformly for compacts in $\mathbb{C}_{\sigma_c}$ so we can thus differentiate term by term. So it is easy to show by induction that the k-th derivative of each term $a_n n^{-s}$ with respect to s is $(-1)^k a_n n^{-s} (\log n)^k$ and, applying Weierstrass's theorem, that the derivatives for f are of the form shown in the corollary and that they are continuous. $\square$

2.1 Abscissas

This section is dedicated to the study of the abscissas given in the following definition.

Definition 2.10. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be an ordinary Dirichlet series, we define the following abscissas. Firstly, the **uniform convergence abscissa**:*

$$\sigma_u = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ is uniformly convergent on } \overline{\mathbb{C}_\sigma} \right\},$$

*which was introduced by Harald Bohr. The **absolute convergence abscissa**:*

$$\sigma_a = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \text{ is absolutely convergent} \right\}.$$

*And finally, the **holomorphy abscissa**:*

$$\sigma_h = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ extends to a holomorphic function in } \mathbb{C}_\sigma \right\}.$$

And we will use the same conventions for these abscissas as we did for the convergence abscissa.

Observation 2.11. *Given these definitions, we can see easily that $\sigma_c \leq \sigma_u$ and $\sigma_c \leq \sigma_a$. As we have seen, the Dirichlet series defines an analytic function in the half-plane $\mathbb{C}_{\sigma_c}$, so $\sigma_h \leq \sigma_c$.*

Also, the absolute abscissa for a series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the convergence abscissa for the series $g(s) = \sum_{n=1}^{\infty} |a_n| n^{-s}$.

2.1.1 Examples and properties

One might ask themselves now if these different abscissas coincide as they do for power series, or if we really need these four different definitions. The answer is that they do not coincide for all Dirichlet series. We will show so in the next couple of examples, by computing some of these abscissas for the following series, and give other important properties.

Example 2.12. *The Riemann ζ function has $\sigma_c(\zeta) = \sigma_a(\zeta) = \sigma_h(\zeta) = 1$.*

It is a pretty well-known fact that the harmonic series $\sum_{n=1}^{\infty} 1/n^s$ converges if and only if $\Re s > 1$, so $\sigma_c(\zeta) = 1$. Also, all the coefficients are positive so $\sigma_a(\zeta) = 1$. The ζ function has a pole at $s = 1$ because the harmonic series diverges, and thus cannot be extended as a holomorphic function in a half-plane bigger than $\mathbb{C}_1$, hence $\sigma_h(\zeta) = 1$.

Example 2.13. *The function $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ has $\sigma_c(\eta) \leq 0$ and $\sigma_a(\eta) = 1$.*

For the absolute convergence abscissa, we have $\sigma_a(\eta) = \sigma_c(\zeta) = 1$.

Now if $\Re s > 0$, we apply the Abel-Dirichlet-Dedekind alternating series test. Indeed, the sequence $b_n = 1/n^s$ is of bounded variation because

$$\sum_{n=1}^{\infty} |n^{-s} - (n+1)^{-s}| = \sum_{n=1}^{\infty} \left| \int_n^{n+1} \frac{-s}{t^{s+1}} dt \right| \leq \int_1^{\infty} \left| \frac{-s}{t^{s+1}} \right| dt = |s| \int_1^{\infty} \frac{1}{t^{1+\Re s}} dt = \frac{|s|}{\Re s} < +\infty,$$

so $\sum (-1)^{n+1} n^{-s}$ converges and $\sigma_c(\eta) \leq 0$.

We can relate it to the zeta function. For $\Re s > 1$, where both of them converge absolutely, we can rearrange the terms by separating the even and odd terms and get:

$$\begin{aligned} \eta(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{2}{(2n)^s} \right) = \sum_{n=1}^{\infty} \frac{1 - 2^{1-s}}{n^s} = \left(1 - 2^{1-s}\right) \zeta(s). \end{aligned} \tag{2.4}$$

Lemma 2.14. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. If the sequence $(a_n)_n$ is bounded, then $\sigma_a(f) \leq 1$.*

Proof. Let $M > 0$ be such that $|a_n| \leq M$ for any $n \geq 1$, then for $\sigma > 1$, $|a_n| n^{-\sigma} \leq M n^{-\sigma}$. Hence, by the comparison test, the series $\sum_{n=1}^{\infty} |a_n| n^{-s}$ converges for $\Re s > 1$. □

A well known property of the Riemann ζ function that was famously proved by Euler is its relationship to a product concerning primes, that is stated in the next proposition.

Proposition 2.15 (Euler product). *If $\Re s > 1$, then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - p_n^{-s}} \right), \tag{2.5}$$

where p_n are the ordered list of prime numbers.

Proof. It is clear that since $\Re s > 1$, we have that $|1/p_i^s| = 1/p_i^{\Re s} < 1$, and hence we can write each factor in (2.5) as

$$\frac{1}{1 - p_n^{-s}} = \sum_{k=0}^{\infty} p_n^{-ks}.$$

Now if we take the finite product of the first N prime numbers, applying the distributive property of the product

$$\prod_{n=1}^{N} \left(\frac{1}{1 - p_n^{-s}} \right) = \sum_{j=1}^{\infty} n_j^{-s},$$

where n_j are integers which are the product of only the primes $p_1, \ldots, p_N$, and they all appear only once by the unique factorization in prime numbers. By letting N tend to ∞, the sum gets extended to all the positive integers, and we get the result. □

Observation 2.16. *Note that since a convergent product of non-zero factors is not zero, the identity* (2.5) *implies that $\zeta(s) \neq 0$ for $\Re s > 1$.*

The fact that $\sigma_h(\zeta) = 1$ does not mean that the ζ function cannot be extended in a bigger domain. In fact, we have the next proposition which extends the definition of the function.

Proposition 2.17. *The Riemann ζ function is a meromoprhic function with a pole at $s = 1$ with residue 1. It verifies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{2.6}$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$.

This is a well known result and was shown by Riemann in his 1859 seminal paper [15], and another proof can be found in [17] (see *Theorem 2.1*). Note that a consequence of this proposition, using that $\zeta(s) \neq 0$ for $\Re s > 1$, is that $\zeta(s) \neq 0$ for $\Re s < 0$, except for $s = -2n$, $n \geq 1$, which are known as the trivial zeros of the ζ function. Hence all non-trivial zeros of ζ lie in $0 < \Re s < 1$, which is known as the *critical strip.*

Example 2.18. *The absolute convergence abscissa for the Θ function is $\sigma_a(\Theta) = 1$. It is easy to see that since $\mu(n)$ is bounded, then $\sigma_a(\Theta) \leq 1$.*

To show that $\sigma_a(\Theta) \geq 1$, we need to see first that Θ is the reciprocal of the ζ function, wherever both converge and ζ does not vanish

$$\Theta(s) = \frac{1}{\zeta(s)}. \tag{2.7}$$

We will show it for $\Re s > 1$, using the reciprocal of the Euler product for the zeta function:

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

If we expand the product we get a sum which is equal to the $\Theta(s)$ function, remembering the definition of the Möbius function μ:

$$1 + \sum_{n=p_i} \left(\frac{-1}{p_i^s}\right) + \sum_{n=p_i p_j} \left(\frac{-1}{p_i^s}\frac{-1}{p_j^s}\right) + \sum_{n=p_i p_j p_k} \left(\frac{-1}{p_i^s}\frac{-1}{p_j^s}\frac{-1}{p_k^s}\right) + \ldots,$$

where each sum extends to the integers that are product of one, two, etc. primes.

Hence $\Theta(s) = \prod_p (1 - p^{-s})$. *If* $\sigma_a(\Theta) < 1$, *then by the theory of infinite products we would have that*

$$\sum_p |1 - (1 - p^{-s})| = \sum_p p^{-\Re s} < +\infty,$$

for all s *with* $\Re s > \sigma_a(\Theta)$. *In particular we would have that* $\sum_p 1/p < +\infty$, *which is a contradiction. So* $\sigma_a(\Theta) = 1$.

A lot of effort will be put in the next section on the study of the different abscissas for Dirichlet series. To remark the importance of this study and its non triviality in general, we have the next result on the convergence abscissa of the Θ function, which is an unknown quantity.

Proposition 2.19. *If* $\sigma_c(\Theta) = 1/2$, *then the Riemann hypothesis is true.*

Proof. As we have seen in equation (2.7), $\Theta(s)\zeta(s) = 1$ in $\mathbb{C}_1$, and the equality can be extended in $\mathbb{C}_{\sigma_c(\Theta)}$, considering the ζ function defined using its analytical extension and that $\sigma_c(\Theta) \leq 1$. Thus if $\sigma_c(\Theta) = 1/2$ this implies that $\zeta(s)$ does not vanish in $\mathbb{C}_{1/2}$. Given the symmetry of $\zeta(s)$ inside the critical band by Proposition 2.17, it can only vanish in the critical line, which is the statement of the Riemann hypothesis. □

2.1.2 Study of the abscissas

In the next proposition we will see the relationships between the different abscissas.

Proposition 2.20. *For a given Dirichlet series* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, *we have:*

$$\sigma_h \leq \sigma_c \leq \sigma_u \leq \sigma_a.$$

We will now prove each inequality as different lemmas, and give other important inequalities.

Lemma 2.21. *For all Dirichlet series* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, *we have:*

$$\sigma_c \leq \sigma_a \leq \sigma_c + 1.$$

Proof. By the definitions of the abscissas, $\sigma_c \leq \sigma_a$. Now let $\rho > \sigma_c$, then $\sum a_n n^{-\rho}$ converges so $a_n n^{-\rho} \to 0$. Let $M := \sup_{n\geq 1} \frac{|a_n|}{n^\rho} < +\infty$.

For all $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\rho+\epsilon+1}} \leq M \sum_{n=1}^{\infty} \frac{1}{n^{\epsilon+1}} < +\infty.$$

So $\sigma_a \leq \rho + \epsilon + 1$ for all $\epsilon > 0$, so $\sigma_a \leq \sigma_c + 1$. □

Observation 2.22. *In the Example 2.13 we have seen that* $\sigma_c(\eta) \leq 0$ *and* $\sigma_a(\eta) = 1$. *By the last lemma, we can conclude that* $\sigma_c(\eta) = 0$. *And more importantly, this example yields that the bound* $\sigma_a \leq \sigma_c + 1$ *is sharp and cannot be improved in general.*

Corollary 2.23. *For any* $t \in [0, 1]$, *there exists a Dirichlet series* f *such that* $\sigma_a(f) - \sigma_c(f) = t$.

Proof. Considering that $\sigma_c(\zeta) = \sigma_a(\zeta) = 1$ and that $\sigma_c(\eta) = 0$ and $\sigma_a(\eta) = 1$, we can produce the Dirichlet series

$$f(s) = \zeta(s+t) + \eta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+t}} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{\infty} \frac{n^{-t} + (-1)^{n+1}}{n^s}.$$

For $\Re s < 1 - t$ the series $\zeta(s+t)$ does not converge, for $1 - t < \Re s < 1$ both series converge but f does not converge absolutely because $\sigma_a(\eta) = 1$, and for $\Re s > 1$ both series converge absolutely. So $\sigma_c(f) = 1 - t$ and $\sigma_a(f) = 1$, hence $\sigma_a(f) - \sigma_c(f) = t$. $\square$

Lemma 2.24. *For all Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, we have:*

$$\sigma_u \leq \sigma_a \leq \sigma_u + 1/2.$$

Proof. For the first inequality, it is easy to see that the Dirichlet series converges uniformly where it does so absolutely. Indeed, for $\rho > \sigma_a$, and $N \geq 1$, using Weierstrass M-test,

$$\sup_{\Re s \geq \rho} \left| f(s) - \sum_{n=1}^{N} \frac{a_n}{n^s} \right| = \sup_{\Re s \geq \rho} \left| \sum_{n=N+1}^{\infty} \frac{a_n}{n^s} \right| \leq \sum_{n=N+1}^{\infty} \frac{|a_n|}{n^\rho} \xrightarrow{N \to \infty} 0,$$

since $\sum_{n=1}^{\infty} |a_n| n^{-\rho}$ converges.

Now let us prove the second inequality. Let us consider ρ and τ such that $\rho > \sigma_u$ and $\tau > \rho + 1/2$. We want to prove that $\sum_{n=1}^{\infty} |a_n| n^{-\tau} < +\infty$, so $\sigma_a \leq \tau$, and since ρ and τ were arbitrary, we will have that $\sigma_a \leq \sigma_u + 1/2$.

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\tau} = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\rho n^{\tau-\rho}} \leq \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\rho}} \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2(\tau-\rho)}} \right)^{1/2}.$$

Now since $\tau - \rho > 1/2$, the second term on the right-hand side converges, and we have to show that the first one also does to finish the proof.

We claim that the partial sums $S_N(s) = \sum_{n=1}^{N} a_n n^{-s}$ are uniformly bounded at $\Re s = \rho$, that is

$$\sup_{N \geq 1} \sup_{\Re s = \rho} |S_N(s)| \leq M, \tag{2.8}$$

for some $M \geq 0$.

Indeed, for each $N \geq 1$ we define

$$b_N := \sum_{n=1}^{N} \frac{|a_n|}{n^\rho},$$

which is a non-decreasing sequence, and satisfies that for any $N \geq 1$

$$\sup_{\Re s = \rho} |S_N(s)| \leq b_N.$$

Next, we define

$$c_N := \sup_{\Re s = \rho} |f(s) - S_N(s)|.$$

Since $c_N \to 0$, there exists $N_0 \in \mathbb{N}$ such that $c_N \leq 1$, for any $N \geq N_0$. It follows that

$$\sup_{\Re s=\rho} |f(s)| = \sup_{\Re s=\rho} |f(s) - S_{N_0}(s) + S_{N_0}(s)| \leq c_{N_0} + b_{N_0} \leq 1 + b_{N_0}.$$

Since $(b_N)_N$ is non-decreasing, we have that for all $N \leq N_0$, $\sup_{\Re s=\rho} |S_N(s)| \leq b_{N_0}$. Next, if $N \geq N_0$, since $c_N \leq 1$, we have $\sup_{\Re s=\rho} |S_N(s)| \leq \sup_{\Re s=\rho} |f(s)| + c_N \leq 2 + b_{N_0}$.

So writing $M := b_{N_0} + 2$ we have that the partial sums are uniformly bounded at $\Re s = \rho$, which proves the claim (2.8).

The final step is to see that

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\rho}} \leq M^2.$$

We have that, by equation (2.8), for any $N \geq 1$ and for any s such that $\Re s = \rho$

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right|^2 \leq M^2,$$

so for any $t \in \mathbb{R}$ and for $N \geq 1$

$$M^2 \geq \left| \sum_{n=1}^{N} \frac{a_n}{n^{\rho+it}} \right|^2 = \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\rho}} + \sum_{1\leq m<k\leq N} \frac{a_m \overline{a_k}}{(mk)^\rho} \left(\frac{k}{m} \right)^{it}. \tag{2.9}$$

We will now use the following fact: for $y \in (0, +\infty) \setminus \{1\}$, writing $y^{it} = e^{it \log y}$ we can get

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y^{it} dt = \lim_{T\to\infty} \frac{\sin(T \log y)}{T \log y} = 0,$$

to obtain by integration that the second term on the right-hand side of equation (2.9) vanishes:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{1\leq m<k\leq N} \frac{a_m \overline{a_k}}{(mk)^\rho} \left(\frac{k}{m} \right)^{it} dt = 0.$$

Thus, for any $N \geq 1$

$$\sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\rho}} \leq M^2,$$

and hence

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\rho}} \leq M^2,$$

which ends the proof. □

We have seen that $\sigma_a \leq \sigma_u + 1/2$, but we may ask ourselves if it is possible to improve on this bound, or even if both abscissas coincide for all Dirichlet series. Let us define

$$S := \sup_{D\in\mathcal{D}} (\sigma_a(D) - \sigma_u(D)), \tag{2.10}$$

where $\mathcal{D}$ is the set of all Dirichlet series that converge at some point. So the problem then is to find how big S is.

This question was first posed by Harald Bohr in 1913[3]. In [5], he proved that $S \leq 1/2$, but he was not actually able to find any Dirichlet series such that $\sigma_a \neq \sigma_u$. In fact, in the same volume of the journal, Otto Toeplitz constructed a series such that $\sigma_a - \sigma_u = 1/4$ and thus the bound was improved to $1/4 \leq S \leq 1/2$.

The problem of finding the value of S was harder than expected, and remained open for nearly twenty years. H. F. Bohnenblust and Einar Hille found a solution to the problem in 1931 in a seminal paper in *Annals of Mathematics* [4], proving that S is exactly 1/2. They approached the problem in a fashion similar to Bohr, using an ingenious transformation and translating the problem from Dirichlet series to power series in infinitely many variables. We will expand on this idea at the end of the next chapter.

A more direct and simpler proof was given by Harold P. Boas [3] much later, in 1997, which relies on these ideas. We will give this alternative proof, although we will assume a couple of technical results whose proof are out of the scope of this work. But firstly, we will need the following lemma by Bohr which relates different abscissas for a Dirichlet series.

Lemma 2.25 (Bohr). *Let $c < b < a$ and let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series that converges absolutely for $\Re s > a$ and that can be extended analytically for $\Re s > c$. If f is bounded in $\overline{\mathbb{C}_b}$, then $\sigma_u \leq b$, i.e. the Dirichlet series converges uniformly in $\overline{\mathbb{C}_{b+\delta}}$ for every $\delta > 0$.*

Proof. Since f is bounded in $\overline{\mathbb{C}_b}$, there exists $K \geq 0$ such that $\sup_{\Re s \geq b} |f(s)| \leq K$. We will see that, for $\delta > 0$ and $s \in \overline{\mathbb{C}_{b+\delta}}$, then

$$\sup_{s \in \overline{\mathbb{C}_{b+\delta}}} \left| f(s) - \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \xrightarrow{M \to \infty} 0.$$

In fact we will see that this difference is bounded by a constant times $M^{-\delta} \log M$, where the constant is independent of s and M. We will do so by integrating over a closed contour and using the residue theorem.

Let us fix $s \in \overline{\mathbb{C}_{b+\delta}}$ and $M \geq 1$. We will consider the rectangular path Γ with vertices $s + a - b \pm iM^{a-b+2}$ and $s - \delta \pm iM^{a-b+2}$, and we will label each side of the path as $\Gamma_i,\ i = 1, \ldots, 4$, starting from the right-hand side of the rectangle and following a counterclockwise order, as shown in Figure 2. In order to simplify the notation, we will write $M_{ab} := M^{a-b+2}$. We will integrate the function $g(z) = f(z)(M + 1/2)^{z-s}(z - s)^{-1}$ which only has a simple pole at $z = s$ with residue

$$\operatorname{Res}(g, s) = \lim_{z \to s} (z - s) g(z) = \lim_{z \to s} (z - s) f(z) \frac{(M + 1/2)^{z-s}}{z - s} = f(s).$$

So $\int_\Gamma g(z) dz = 2\pi i f(s)$, and moreover

$$\left| 2\pi i f(s) - \int_{\Gamma_1} g(z) dz \right| \leq \left| \int_{\Gamma_2} g(z) dz \right| + \left| \int_{\Gamma_3} g(z) dz \right| + \left| \int_{\Gamma_4} g(z) dz \right|.$$

[3] "Whether or not the number 1/2 can be replaced by a smaller number [...] I do not know"

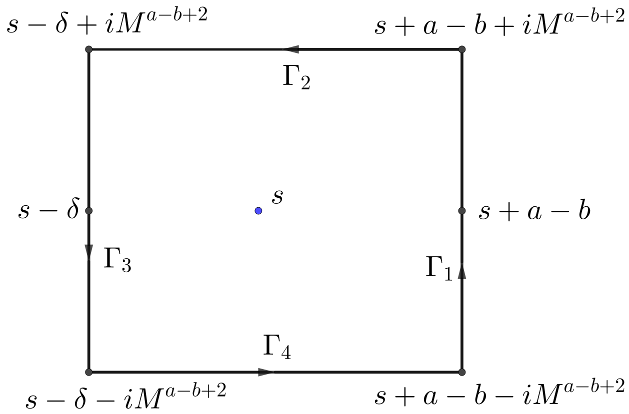

Figure 2: Rectangular path of integration Γ

For the left side of the rectangle we have

$$\begin{aligned}
\left|\int_{\Gamma_3} g(z)dz\right| &= \left|\int_{-M_{ab}}^{M_{ab}} f(s-\delta+it)\frac{(M+1/2)^{-\delta+it}}{-\delta+it}idt\right| \leq K\int_{-M_{ab}}^{M_{ab}} \frac{(M+1/2)^{-\delta}}{|-\delta+it|}dt \\
&\leq KM^{-\delta}\int_{-M_{ab}}^{M_{ab}} \frac{dt}{\sqrt{\delta^2+t^2}} = KM^{-\delta}\log\left(\frac{M_{ab}+\sqrt{\delta^2+M_{ab}^2}}{-M_{ab}+\sqrt{\delta^2+M_{ab}^2}}\right) \\
&= KM^{-\delta}\log\left(\frac{2M_{ab}^2+\delta^2+2M_{ab}\sqrt{\delta^2+M_{ab}^2}}{\delta^2}\right) = \mathcal{O}(M^{-\delta}\log M).
\end{aligned}$$

For the top we have an even better bound, using that $M_{ab} = M^{a-b+2}$

$$\begin{aligned}
\left|\int_{\Gamma_2} g(z)dz\right| &= \left|\int_{-\delta}^{a-b} f(s+t+iM_{ab})\frac{(M+1/2)^{t+iM_{ab}}}{t+iM_{ab}}dt\right| \leq KM_{ab}^{-1}\int_{-\delta}^{a-b}\left(M+\frac{1}{2}\right)^t dt \\
&= \frac{KM^{-(a-b+2)}}{\log(M+1/2)}\left((M+1/2)^{a-b}-(M+1/2)^{-\delta}\right) \\
&\leq \frac{KM^{-(a-b+2)}}{\log(M+1/2)}M^{a-b} = \mathcal{O}(M^{-2}).
\end{aligned}$$

And similarly for the bottom side we can compute

$$\left|\int_{\Gamma_4} g(z)dz\right| = \mathcal{O}(M^{-2}).$$

Hence

$$\left|2\pi i f(s) - \int_{\Gamma_1} g(z)dz\right| = \mathcal{O}(M^{-\delta}\log M). \tag{2.11}$$

Now since the path Γ_1 is inside the region of absolute convergence for f, it also converges absolutely and we can commute integral and series as follows

$$\begin{aligned}\left|2\pi i f(s) - \int_{\Gamma_1} g(z)dz\right| &= \left|2\pi i f(s) - \int_{\Gamma_1} \sum_{n=1}^{\infty} \frac{a_n}{n^z} \frac{(M+1/2)^{z-s}}{z-s} dz\right| \\ &= \left|2\pi i f(s) - \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{\Gamma_1} \left(\frac{M+1/2}{n}\right)^{z-s} \frac{dz}{z-s}\right|.\end{aligned} \tag{2.12}$$

The behaviour of the integral depends on whether $M+1/2$ is bigger or smaller than n. Thus, given that M is an integer, we will distinguish the cases $M+1 \leq n$ and $M \geq n$. In both of these cases we will perform the integral using the residue theorem applied to the function $h(z) = ((M+1/2)/n)^{z-s}(z-s)^{-1}$.

If $n \geq M+1$, we will consider the rectangular path with left side Γ_1 and right side with abscissa $R > \Re s + a - b$, as seen in Figure 3. There are no singularities inside this

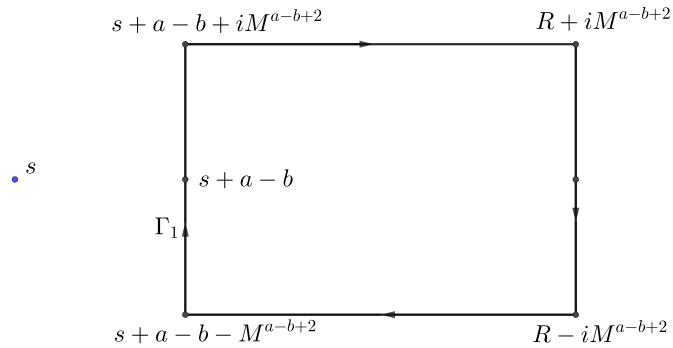

Figure 3: Rectangular path of integration used when $n \geq M+1$

region so the integral over Γ_1 is equal to minus the contributions for the other three sides. The right side vanishes as $R \to \infty$. Indeed:

$$\begin{aligned}\left|\int_{R+iM_{ab}}^{R-iM_{ab}} h(z)dz\right| &= \left|\int_{R+iM_{ab}}^{R-iM_{ab}} \left(\frac{M+1/2}{n}\right)^{z-s} \frac{dz}{z-s}\right| \\ &= \left|\int_{M_{ab}}^{-M_{ab}} \left(\frac{M+1/2}{n}\right)^{R+it-s} \frac{idt}{R+it-s}\right| \\ &\leq \frac{1}{R-s} \int_{-M_{ab}}^{M_{ab}} \left(\frac{M+1/2}{n}\right)^{R-s} dt \leq \frac{2M_{ab}}{R-s} \left(\frac{M+1/2}{n}\right)^{R-s} \xrightarrow{R\to\infty} 0,\end{aligned}$$

since $(M+1/2)/n < 1$. For the top side

$$\begin{aligned}&\left|\int_{s+a-b+iM_{ab}}^{R+iM_{ab}} h(z)dz\right| = \left|\int_{s+a-b+iM_{ab}}^{R+iM_{ab}} \left(\frac{M+1/2}{n}\right)^{z-s} \frac{dz}{z-s}\right| \\ &= \left|\int_{a-b}^{R-s} \left(\frac{M+1/2}{n}\right)^{t+iM_{ab}} \frac{dt}{t+iM_{ab}}\right| \leq M^{-(a-b+2)} \int_{a-b}^{R-s} \left(\frac{M+1/2}{n}\right)^{t} dt \\ &\leq M^{-(a-b+2)} \int_{a-b}^{\infty} \left(\frac{M+1/2}{n}\right)^{t} dt = M^{-(a-b+2)} \left(\frac{M+1/2}{n}\right)^{a-b} \frac{-1}{\log \frac{M+1/2}{n}}.\end{aligned}$$

And we can arrive at the same inequality for the bottom side. Now observe that for all $n \geq M+1$ we have

$$-\log\left(\frac{M+1/2}{n}\right) \geq -\log\left(\frac{M+1/2}{M+1}\right) = -\log\left(1-\frac{1}{2M+2}\right) > \frac{1}{2M+2}, \tag{2.13}$$

where the last inequality is given by the Taylor expansion of the logarithm function.

Hence all the terms in equation (2.12) with $n \geq M+1$ are bounded by

$$\begin{aligned}\left|\sum_{n>M}^{\infty}\frac{a_n}{n^s}\int_{\Gamma_1}h(z)dz\right| &= \left|\sum_{n>M}^{\infty}\frac{a_n}{n^s}\int_{\Gamma_1}\left(\frac{M+1/2}{n}\right)^{z-s}\frac{dz}{z-s}\right| \\ &\leq 2\sum_{n>M}^{\infty}\frac{|a_n|}{n^{\Re s}}M^{-(a-b+2)}\left(\frac{M+1/2}{n}\right)^{a-b}\frac{-1}{\log\frac{M+1/2}{n}} \\ &\leq 2(2M+2)M^{-(a-b+2)}(M+1/2)^{a-b}\sum_{n>M}^{\infty}\frac{|a_n|}{n^{\Re s+a-b}} \\ &\leq 2(2M+2)M^{-(a-b+2)}(M+1/2)^{a-b}\sum_{n>M}^{\infty}\frac{|a_n|}{n^{a+\delta}},\end{aligned}$$

and the series converges because f is absolutely convergent for $\Re s > a$. So we can write

$$\left|\sum_{n>M}^{\infty}\frac{a_n}{n^s}\int_{\Gamma_1}h(z)dz\right| = \mathcal{O}(M^{-1}). \tag{2.14}$$

Using equations (2.11) and (2.14) we get the following expression

$$\begin{aligned}\left|2\pi i f(s) - \sum_{n=1}^{M}\frac{a_n}{n^s}\int_{\Gamma_1}h(z)dz\right| &= \left|2\pi i f(s) - \int_{\Gamma_1}g(z)dz + \sum_{n=M+1}^{\infty}\frac{a_n}{n^s}\int_{\Gamma_1}h(z)dz\right| \\ &\leq \mathcal{O}(M^{-\delta}\log M) + \mathcal{O}(M^{-1}) = \mathcal{O}(M^{-\delta}\log M),\end{aligned}$$

If $n \leq M$, we will integrate around the rectangle with right side Γ_1 and left side with abscissa $-R < \Re s$, as shown in Figure 4. Now the function $h(z)$ has a pole at $z = s$ with

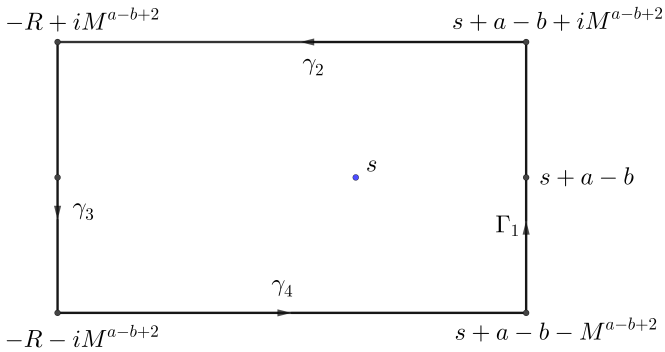

Figure 4: Rectangular path of integration used when $n \leq M$

residue 1. In consequence

$$\left|\int_{\Gamma_1}h(z)dz - 2\pi i\right| \leq \left|\int_{\gamma_2}h(z)dz\right| + \left|\int_{\gamma_3}h(z)dz\right| + \left|\int_{\gamma_4}h(z)dz\right|,$$

where γ_2, γ_3 and γ_4 are the top, left and bottom sides of the rectangle respectively.

Similarly to the previous case, for the top side we have

$$\begin{aligned}\left|\int_{\gamma_2} h(z)dz\right| &= \left|\int_{s+a-b+iM_{ab}}^{R+iM_{ab}} \left(\frac{M+1/2}{n}\right)^{z-s} \frac{dz}{z-s}\right| = \left|\int_{a-b}^{-R-s} \left(\frac{M+1/2}{n}\right)^{t+iM_{ab}} \frac{dt}{t+iM_{ab}}\right| \\ &\leq -M_{ab}^{-1} \int_{a-b}^{-R-s} \left(\frac{M+1/2}{n}\right)^{t} dt = M_{ab}^{-1} \int_{-R-s}^{a-b} \left(\frac{M+1/2}{n}\right)^{t} dt \\ &\leq M_{ab}^{-1} \int_{a-b}^{\infty} \left(\frac{M+1/2}{n}\right)^{t} dt = M_{ab}^{-1} \left(\frac{M+1/2}{n}\right)^{a-b} \frac{1}{\log \frac{M+1/2}{n}}.\end{aligned}$$

and we can get an analogous inequality for the bottom side. For the left side we have a decaying exponential:

$$\begin{aligned}\left|\int_{\gamma_3} h(z)dz\right| &= \left|\int_{-R-iM_{ab}}^{-R+iM_{ab}} \left(\frac{M+1/2}{n}\right)^{z-s} \frac{dz}{z-s}\right| = \left|\int_{-M_{ab}}^{M_{ab}} \left(\frac{M+1/2}{n}\right)^{-R+it-s} \frac{idt}{-R+it-s}\right| \\ &\leq \frac{1}{R}\int_{-M_{ab}}^{M_{ab}} \left(\frac{M+1/2}{n}\right)^{-R-\Re s} dt \leq \frac{2M_{ab}}{R} \left(\frac{M+1/2}{n}\right)^{-R-\Re s} \xrightarrow{R\to\infty} 0.\end{aligned}$$

Thus the terms in (2.12) with $n \leq M$ are bounded by

$$\begin{aligned}\left|\int_{\Gamma_1} h(z)dz - 2\pi i\right| &= \left|\int_{\Gamma_1} \left(\frac{M+1/2}{n}\right)^{z-s} \frac{dz}{z-s} - 2\pi i\right| \\ &\leq 2M^{-(a-b+2)} \left(\frac{M+1/2}{n}\right)^{a-b} \frac{1}{\log \frac{M+1/2}{n}} \\ &\leq 2\frac{M^{-(a-b+2)}}{\log M}(M+1/2)^{a-b} = \mathcal{O}(M^{-2}\log^{-1} M),\end{aligned}$$

and hence

$$\left|2\pi i f(s) - 2\pi i \sum_{n=1}^{M} \frac{a_n}{n^s}\right| = \mathcal{O}(M^{-\delta}\log M).$$

Finally

$$\left|f(s) - \sum_{n=1}^{M} \frac{a_n}{n^s}\right| = \mathcal{O}(M^{-\delta}\log M),$$

uniformly for $s \in \overline{\mathbb{C}_{b+\delta}}$, as we wanted. □

Theorem 2.26 (Bohnenblust-Hille)**.** *If $\mathcal{D}$ is the set of all Dirichlet series which converge at some point in the complex plane, then*

$$S := \sup_{D\in\mathcal{D}} (\sigma_a(D) - \sigma_u(D)) = 1/2.$$

Proof. We have already seen that $S \leq 1/2$. Now we will construct a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ which converges uniformly in $\overline{\mathbb{C}_{\delta+1/2}}$ for any $\delta > 0$ but which does not converge absolutely for $\Re s < 1$, and hence $S = 1/2$. We will assume the following results without proof, as they are out of the purpose of this work:

- Prime number theorem: It states that the number of primes less than a number x is asymptotic to $x/\log x$. We need the weaker version that states that the n-th prime p_n is bounded by
$$1/c_1 < \frac{p_n}{(n \log n)} < c_1, \tag{2.15}$$
where $c_1 > 1$ is a constant.

- Quantity of monomials: We need to know how many monomials of degree m in n variables exist, that is, how many monomials of the form $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ exist, with $\alpha_i \in \mathbb{N}$ and $|\alpha| := \sum \alpha_i = m$. The exact quantity is $\binom{n+m-1}{m}$, but we only need the lower bound
$$\frac{n^m}{m!}, \tag{2.16}$$
which is obtained by thinking of the monomial as a product of m factors with n choices for each factor, and dividing by the number of possible permutations, which is at most $m!$.

- Random polynomial modulus: A homogeneous polynomial of degree m in n variables with coefficients ± 1 is a polynomial of the form $\sum_{\alpha_1+\cdots+\alpha_n=m} \pm z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. According to the theory of random polynomials, if the signs $\pm$ are assigned at random, then the maximum modulus of the polynomial in $\mathbb{D}^n$ is bounded above with high probability by
$$c_2 n^{(m+1)/2} \sqrt{\log m}, \tag{2.17}$$
where c_2 is a constant. This bound does not work for all homogenous polynomials, but we know that there are many that satisfy it, and we will only need for one to exist for each m and n. For the proof of this result, see [11, §7.1] and the references therein.

With all these tools, we will now construct the described Dirichlet series $f(s)$ in blocks, and every coefficient will either be 0 or ± 1. For the k-th block, $k \geq 2$, consider a homogeneous polynomial with degree k in 2^k variables and with coefficients ± 1 assigned at random. Now consider the ordered list of the 2^k consecutive prime numbers starting from the 2^k-th prime, for each p we will substitute $1/p^s$ in each corresponding variable z_i, $1 \leq i \leq 2^k$:
$$z_i \longmapsto \frac{1}{(p_{2^k+i-1})^s}$$
$$\sum_{\alpha_1+\cdots+\alpha_{2^k}=k} \pm z_1^{\alpha_1} \cdots z_{2^k}^{\alpha_{2^k}} \longmapsto \sum_{\alpha_1+\cdots+\alpha_{2^k}=k} \frac{\pm 1}{\left(p_{2^k}^{\alpha_1} \cdots p_{2^{k+1}-1}^{\alpha_{2^k}}\right)^s}.$$
So the polynomial becomes a sum of $\pm 1/n^s$, where each n appears only once by the uniqueness of the factorization in prime numbers, and each n is the product of exactly k prime numbers (counted with multiplicity). If a number n does not appear in the process described, we will assign $a_n = 0$. Observe that the first coefficient which is not null is a_{49}, as 49 is the smallest product of two primes from the list starting at $p_{2^2} = 7$. The series $f(s)$ will be the sum of the different blocks, and now we need to prove that this series has the desired properties.

We claim that $\sigma_a(f) = 1$. By Lemma 2.14, since all the coefficients are bounded, we have $\sigma_a(f) \leq 1$.

Next, let us prove that $\sigma_a(f) \geq 1$. For the k-th block, by equation (2.15), the primes are bounded by
$$p_i < c_1 i \log i \leq c_1 2^{k+1} \log(2^{k+1}) < 3c_1(k+1)2^k, \tag{2.18}$$

for $2^k \leq i \leq 2^{k+1}-1$.

On another note, the number of integers n that are the product of k prime numbers from p_{2^k} to $p_{2^{k+1}-1}$ is bounded below by equation (2.16) with $n = 2^k$ and $m = k$:

$$\frac{2^{k^2}}{k^k} < \frac{2^{k^2}}{k!} < \#\left\{ n : \ n = \prod_{\substack{|\alpha|=k \\ 1\leq i\leq 2^k}} p_{2^k+i-1}^{\alpha_i} \right\}.$$

Using equation (2.18), such integers n are bounded by

$$n = \prod_{\substack{|\alpha|=k \\ 1\leq i\leq 2^k}} p_{2^k+i-1}^{\alpha_i} \leq (3c_1(k+1)2^k)^k.$$

Hence

$$\sum_{n=49}^{\infty} \frac{|a_n|}{n^{\Re s}} \geq \sum_{k=2}^{\infty} \frac{2^{k^2}/k^k}{(3c_1(k+1)2^k)^{k\Re s}} = \sum_{k=2}^{\infty} \frac{2^{k^2(1-\Re s)}}{(3c_1(k+1))^{k\Re s}k^k}.$$

Using the root test, this series diverges for $\Re s < 1$, and thus $\sigma_a(f) = 1$.

We claim that $f(s)$ converges uniformly for $\Re s \geq 1/2+\delta$. We will estimate the modulus of the each block in the series. By the definition of f, we have that the modulus of the k-th block $B_k(s)$ is

$$|B_k(s)| = \left| \sum_{|\alpha|=k} \frac{\pm 1}{\left(p_{2^k}^{\alpha_1} \cdots p_{2^{k+1}-1}^{\alpha_{2^k}}\right)^s} \right|.$$

Since that B_k is a homogeneous polynomial, using equation (2.17) with $n = 2^k$ and $m = k$, we obtain that the supremum of the modulus of each block is bounded by $c_2(2^k)^{(k+1)/2}\sqrt{\log k}$, times the modulus of each variable $|1/p_i^s| = 1/p_i^{\Re s}$:

$$|B_k(s)| \leq \frac{c_2(2^k)^{(k+1)/2}\sqrt{\log k}}{\left(p_{2^k}^{\alpha_1} \cdots p_{2^{k+1}-1}^{\alpha_{2^k}}\right)^{\Re s}}$$

In turn, each prime p_i in the k-th block is bounded by equation (2.15), and considering that $1/2 < \log 2$ we get:

$$\frac{k2^k}{2c_1} < \frac{2^k \log(2^k)}{c_1} < p_{2^k} \leq p_i, \quad 2^k \leq i \leq 2^{k+1}-1.$$

Hence

$$|B_k(s)| \leq \frac{c_2(2^k)^{(k+1)/2}\sqrt{\log k}}{\left(\frac{k2^k}{2c_1}\right)^{k\Re s}} =: M_k.$$

We want to apply the Weierstrass M-test to prove that the series $f(s)$ converges uniformly in $\Re s \geq 1/2$. Thus we have to prove that the series $\sum_{k=2}^{\infty} M_k$ converges. We will do so using the root test. We compute the following limit

$$\begin{aligned}
\lim_{k\to\infty} \left(\frac{c_2 2^{\frac{k(k+1)}{2}}\sqrt{\log k}}{\left(\frac{k2^k}{2c_1}\right)^{k\Re s}} \right)^{1/k} &= (2c_1)^{\Re s} \lim_{k\to\infty} \frac{c_2^{1/k} 2^{(k+1)/2}(\log k)^{1/(2k)}}{(k2^k)^{\Re s}} \\
&= (2c_1)^{\Re s} \lim_{k\to\infty} \frac{2^{(k+1)/2}}{(k2^k)^{\Re s}} = \sqrt{2}(2c_1)^{\Re s} \lim_{k\to\infty} \frac{2^{k(1/2-\Re s)}}{k^{\Re s}},
\end{aligned}$$

where we have used that $\lim_{k\to\infty}(\log k)^{1/(2k)} = 1$. The last limit converges to 0 if $\Re s \geq 1/2$, and goes to infinity if $\Re s < 1/2$.

So the series summed in the described blocks it converges uniformly for $\Re s \geq 1/2$ to a bounded function f, and it is analytical in $\mathbb{C}_{1/2}$. In $\Re s \geq 1$, the series converges absolutely and thus it is equal to f since we can perform the sum in any order. By Bohr's Lemma 2.25 with $c = 1/2 < b = 1/2 + \delta < a = 1$, the series converges uniformly in $\Re s \geq 1/2 + \delta$ for each $\delta > 0$. This finishes the proof. $\square$

There is no bound between the distance of the holomorphy and convergence abscissa. We can see it in the next example:

Example 2.27. *For the η function we have seen that $\sigma_c(\eta) = 0$. But we know from equation* (2.4) *that $\eta(s) = (1 - 2^{1-s})\zeta(s)$, and this can be extended to the whole plane using the functional equation for ζ (see equation* (2.6)*). We can observe that $1 - 2^{1-s}$ cancels the pole that ζ has at $s = 1$, and thus η is an entire function. So $\sigma_h(\eta) = -\infty$.*

Proposition 2.28. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with non-negative coefficients, the next equality holds*

$$\sigma_h = \sigma_c = \sigma_u = \sigma_a.$$

Proof. Since all coefficients are non-negative $\sigma_c = \sigma_a$. Now to prove that $\sigma_h = \sigma_c$ we will see that $s = \sigma_c$ is a singularity of f, and thus cannot have a holomorphic extension in $D(\sigma_c, r)$ for any $r > 0$.

Without loss of generality, we can assume that $\sigma_c = 0$. We will also assume that the series $\sum_{n=1}^{\infty} a_n n^{-s}$ extends to a holomorphic function in $D(0, r)$, for the sake of a contradiction argument. Now if f is holomorphic in $D(0, r)$, then there exists a $R > 1$ close to 1 such that f extends in $D(1, R)$ (see Figure 5). In a neighbourhood of $s = 1$ we

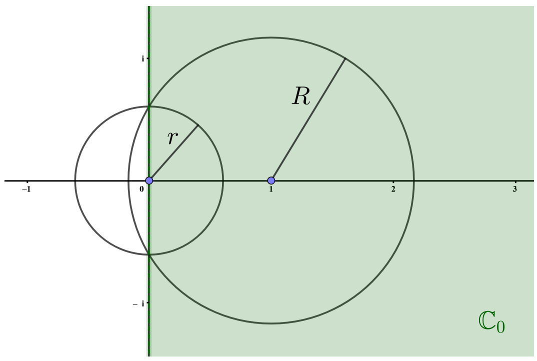

Figure 5: Diagram of the disks used in the proof.

have the series expansion

$$f(s) = \sum_{n=0}^{\infty} \frac{f^{(k)}(1)}{k!}(s-1)^k = \sum_{k=0}^{\infty} \frac{\sum_{n=1}^{\infty}(-\log n)^k a_n n^{-1}}{k!}(s-1)^k,$$

where we have used Corollary 2.9 to write the k-th derivative of f.

Now let ε be such that $1-R<-\varepsilon<0$. We then have

$$\begin{aligned} f(-\varepsilon) &= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-\log n)^k a_n}{nk!}(-\varepsilon-1)^k = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(\log n)^k a_n}{nk!}(1+\varepsilon)^k \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n}\sum_{k=1}^{\infty}\frac{(\log n)^k}{k!}(1+\varepsilon)^k = \sum_{n=0}^{\infty}\frac{a_n}{n}e^{(1+\varepsilon)\log n} = \sum_{n=0}^{\infty}\frac{a_n}{n^{-\varepsilon}}. \end{aligned}$$

Where we can permute the series because all the terms are non-negative. Hence $\sigma_c < -\varepsilon < 0$, which is a contradiction. □

Observation 2.29. *We have seen that the point $s=\sigma_c$ is always a singularity for a series with non-negative coefficients. But as the the example of the function $\eta(s)$ shows, this is not true for series with negative coefficients, because $\eta(s)$ is an entire function. Thus the intuition given by power series that they always have a singularity on their circles of convergence does not apply to Dirichlet series, which might not have any singularity on their boundary of convergence.*

Knowing all this information about the different abscissas and their relationships to one another, one important question is how to compute them for a given Dirichlet series, and if there is a formula as there is for power series. In 1894, Cahen gave a formula in his doctoral thesis [8] to compute the abscissa of convergence when the series does not converge at $s=0$. There is also a second formula given by Schnee and by Titchmarsh [16] for when the series does converge at $s=0$. These results are presented in the following theorem.

Theorem 2.30. *For a Dirichlet series $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ we have:*

- *(Cahen) If $\sum_{n=1}^{\infty}a_n$ does not converge, then*

$$\sigma_c = \limsup_{N\to\infty}\frac{\log\left|\sum_{n=1}^{N}a_n\right|}{\log N}.$$

- *(Schnee-Titchmarsh) If $\sum_{n=1}^{\infty}a_n$ converges, then*

$$\sigma_c = \limsup_{N\to\infty}\frac{\log\left|\sum_{n=N+1}^{\infty}a_n\right|}{\log N}.$$

Proof.

- Let us begin by giving a proof for Cahen's equation. The fact that $f(0)=\sum_{n=1}^{\infty}a_n$ does not converge gives us automatically that $\sigma_c\geq 0$. For $N\geq 1$, we define $A_N=\sum_{n=1}^{N}a_n$, and we write $A_0=0$. We want to prove that

$$\gamma := \limsup_{N\to\infty}\frac{\log|A_N|}{\log N} = \sigma_c.$$

We shall prove firstly that $\gamma\leq\sigma_c$ and then that $\gamma\geq\sigma_c$.

– Let $s \in \mathbb{R}, s > \sigma_c \geq 0$. Hence $f(s)$ converges. We define $b_n = a_n n^{-s}$, $B_n = \sum_{k=1}^{n} b_k$ and $B_0 = 0$. Performing an Abel transformation

$$A_N = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} b_n n^s = \sum_{n=1}^{N} (B_n - B_{n-1}) n^s = \sum_{n=1}^{N-1} B_n (n^s - (n+1)^s) + B_N N^s.$$

Since $(B_n)_n$ converges, it is bounded. Then there exists $B \geq 0$ such that $|B_n| \leq B$ for $n \geq 0$. Hence

$$|A_N| \leq B \sum_{n=1}^{N-1} ((n+1)^s - n^s) + BN^s = B(N^s - 1^s) + BN^s < 2BN^s. \tag{2.19}$$

Applying logarithms to both sides we get:

$$\log |A_N| < s \log N + \log 2B \implies \frac{\log |A_N|}{\log N} < s + \frac{\log 2B}{\log N} \implies \gamma \leq \sigma_c.$$

– Let $s > \gamma$ and we maintain the definitions from the last section. We choose ρ such that $s > \rho > \gamma$. There exists N_0 such that for any $N \geq N_0$

$$\rho \geq \frac{\log |A_N|}{\log N}$$

which yields $N^\rho > |A_N|$. Hence, there exists $C > 0$ such that $|A_N| \leq CN^\rho$, $N \geq 0$. We want to show that $(B_N)_N$ is convergent, so $s > \sigma_c$ and hence $\gamma \geq \sigma_c$. We will use Cauchy's criterion and Abel's transformation, for $M > N$:

$$\begin{aligned} B_M - B_N &= \sum_{n=N+1}^{M} \frac{a_n}{n^s} = \sum_{n=N+1}^{M} (A_n - A_{n-1}) \frac{1}{n^s} \\ &= \sum_{n=N+1}^{M-1} A_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{A_M}{M^s} - \frac{A_N}{(N+1)^s}. \end{aligned}$$

Hence

$$\begin{aligned} |B_M - B_N| &\leq \sum_{n=N+1}^{M-1} \left| A_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right| + \left| \frac{A_M}{M^s} \right| + \left| \frac{A_N}{(N+1)^s} \right| \\ &\leq C \sum_{n=N+1}^{M-1} n^\rho \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{CM^\rho}{M^s} + \frac{CN^\rho}{(N+1)^s} \\ &\leq C \sum_{n=N+1}^{M-1} n^\rho \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{2C}{N^{s-\rho}}. \end{aligned}$$

Applying the mean value theorem to the function $g(x) = 1/x^s$, there exists $c \in (n, n+1)$ such that

$$g'(c) = \frac{-s}{c^{s+1}} = \frac{\frac{1}{n^s} - \frac{1}{(n+1)^s}}{n - (n+1)} \implies \frac{s}{n^{s+1}} \geq \frac{s}{c^{s+1}} = \frac{1}{n^s} - \frac{1}{(n+1)^s},$$

and so

$$|B_M - B_N| \leq Cs \sum_{n=N+1}^{M-1} \frac{n^\rho}{n^{s+1}} + \frac{2C}{N^{s-\rho}}.$$

Now comparing with the integral

$$\sum_{n=N+1}^{M-1} \frac{n^\rho}{n^{s+1}} \leq \int_{N+1}^{\infty} n^{\rho-s-1} dn = \frac{(N+1)^{\rho-s}}{s-\rho} = \mathcal{O}(N^{\rho-s}),$$

we finally get the inequality

$$|B_M - B_N| \leq Cs\frac{(N+1)^{\rho-s}}{s-\rho} + \frac{2C}{N^{s-\rho}} = \frac{C}{N^{s-\rho}}\left(\frac{s}{s-\rho}+2\right) = \mathcal{O}(N^{\rho-s}).$$

Hence

$$\sup_{M>N} |B_M - B_N| \xrightarrow{N\to\infty} 0,$$

and $(B_N)_N$ converges, so $\gamma \geq \sigma_c$. In conclusion $\gamma = \sigma_c$.

- The proof for the Schnee-Titchmarsh formula is very similar to this one, because as one might have noticed, we have barely used the hypothesis that f does not converge at 0. Let us write $R_N = \sum_{n=N+1}^{\infty} a_n$, so we want to see that the convergence abscissa is equal to

$$\beta := \limsup_{N\to\infty} \frac{\log |R_N|}{\log N}.$$

We will firstly see that $\beta \leq \sigma_c$ and then $\beta \geq \sigma_c$, and use the same notations as before for B_N and b_n. Because f converges at 0, we know that $\beta \leq 0$ and σ_c must be negative or 0.

  - We observe first that if $\sigma_c = 0$ then we already know that $\beta \leq \sigma_c$, so we can assume $\sigma_c < 0$. So B_N converges and we have that $|B_N| \leq B$ for some B. Let $\sigma_c < s < 0$, so

$$R_N = \sum_{n=N+1}^{\infty} a_n = \sum_{n=N+1}^{\infty} b_n n^s = \sum_{n=N+1}^{\infty} (B_n - B_{n+1})n^s = \sum_{n=N}^{\infty} B_n(n^s - (n+1)^s) - B_N N^s,$$

and thus

$$|R_N| \leq B \sum_{n=N}^{\infty} (n^s - (n+1)^s) + BN^\rho = 2BN^s \xrightarrow{N\to\infty} 0,$$

which yields $\beta \leq \sigma_c$.

  - As in the Cahen formula proof, let $\beta < s < 0$ and let ρ be such that $\beta < \rho < s$. Arguing as in the Cahen's case, $\exists C \geq 0$ such that $|R_N| \leq CN^\rho$, $N \geq 1$. If we show that $\sum_{n=1}^{\infty} a_n n^{-s}$ converges, then $s \geq \sigma_c$ and $\beta \geq \sigma_c$. Again, using Cauchy's criterion for $M > N$

$$\begin{aligned} R_{M-1} - R_{N-1} &= \sum_{n=M}^{\infty} \frac{a_n}{n^s} - \sum_{n=N}^{\infty} \frac{a_n}{n^s} = \sum_{n=N+1}^{M} \frac{a_n}{n^s} = \sum_{n=N+1}^{M} \frac{R_{n-1} - R_n}{n^s} \\ &= \sum_{n=N}^{M} R_n \left(\frac{1}{(n+1)^s} - \frac{1}{n^s}\right) - \frac{R_M}{(M+1)^s} + \frac{R_N}{N^s}. \end{aligned}$$

Applying a similar argument to the one used in the Cahen formula, we arrive at

$$\left|\sum_{n=N+1}^{M} \frac{a_n}{n^s}\right| \leq Cs\frac{(N-1)^{\rho-s}}{s-\rho} + \frac{2C}{N^{s-\rho}}.$$

Finally

$$\sup_{M>N}\left|\sum_{n=N+1}^{M}\frac{a_n}{n^s}\right| \xrightarrow{N\to\infty} 0,$$

so $\sum_{n=1}^{\infty} a_n n^{-s}$ converges and $\beta \geq \sigma_c$, which ends the proof.

□

Corollary 2.31. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series, we have that:*

- *If $\sum_{n=1}^{\infty} |a_n|$ diverges, then*

$$\sigma_a(f) = \limsup_{N\to\infty} \frac{\log\left|\sum_{n=1}^{N} |a_n|\right|}{\log N}.$$

- *If $\sum_{n=1}^{\infty} |a_n|$ converges, then*

$$\sigma_a(f) = \limsup_{N\to\infty} \frac{\log\left|\sum_{n=N+1}^{\infty} |a_n|\right|}{\log N}.$$

Proof. The result follows from the observation that the absolute abscissa for a series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the convergence abscissa for the series $g(s) = \sum_{n=1}^{\infty} |a_n| n^{-s}$. □

The formulas in Theorem 2.30 are analogous to the Cauchy-Hadamard formula for power series in Theorem 1.3.

2.2 Properties

In this section we will go over some properties of Dirichlet series. One of the usual questions concerning a function is where its zeros lie. This is a difficult problem in general, as shown by the difficulty of proving the Riemann hypothesis. Nonetheless, the next proposition gives an interesting result on the topic of zeros for Dirichlet series.

Proposition 2.32. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Except for the trivial case where $a_n = 0$ for all n, there exists ρ such that $f(s) \neq 0$ for all s with $\Re s > \rho$.*

Proof. Let $N := \min\{n \in \mathbb{N} : a_n \neq 0\}$, and let us fix $\rho > \sigma_a$. We have

$$C_\rho := \sum_{n=1}^{\infty} \frac{|a_n|}{n^\rho} < +\infty.$$

For s with $\Re s \geq \rho$

$$|f(s)| = \left|\sum_{n=N}^{\infty}\frac{a_n}{n^s}\right| \geq \left|\frac{a_N}{N^s}\right| - \left|\sum_{n>N}^{\infty}\frac{a_n}{n^s}\right| \geq \frac{|a_N|}{N^{\Re s}} - \sum_{n>N}^{\infty}\frac{|a_n|}{n^{\Re s}}, \tag{2.20}$$

where we can bound the last term by

$$\sum_{n>N}^{\infty}\frac{|a_n|}{n^{\Re s}} = \sum_{n>N}^{\infty}\frac{|a_n|}{n^\rho n^{\Re s-\rho}} \leq \frac{C_\rho}{(N+1)^{\Re s-\rho}}.$$

Now since $\lim_{x\to\infty} \frac{N^x}{(N+1)^x} C_\rho (N+1)^\rho = 0$, there exists a $\tau \geq \rho$ such that

$$\frac{N^{\Re s}}{(N+1)^{\Re s}} C_\rho (N+1)^\rho \leq \frac{|a_N|}{2}$$

for all s such that $\Re s \geq \tau$. Hence

$$\frac{C_\rho (N+1)^\rho}{(N+1)^{\Re s}} = \frac{C_\rho}{(N+1)^{\Re s - \rho}} \leq \frac{|a_N|}{2N^{\Re s}},$$

for s such that $\Re s \geq \tau$. Substituting this in equation (2.20) we have that

$$|f(s)| \geq \frac{|a_N|}{N^{\Re s}} - \sum_{n>N}^{\infty} \frac{|a_n|}{n^{\Re s}} \geq \frac{|a_N|}{2N^{\Re s}} > 0,$$

and thus $f(s) \neq 0$ for s with $\Re s \geq \tau$. $\square$

A consequence of this proposition is the uniqueness of Dirichlet series.

Theorem 2.33 (Dirichlet-Dedekind). *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be two Dirichlet series with $\sigma_a(f), \sigma_a(g) < +\infty$. If there exists a sequence $(s_k)_k \subset \mathbb{C}$ with $\Re s_k \xrightarrow{k\to\infty} +\infty$ such that $f(s_k) = g(s_k)$ for any $k \in \mathbb{N}$, then $a_n = b_n$ for all $n \geq 1$ and $f \equiv g$.*

Proof. We apply the last proposition to the series $h(s) = f(s) - g(s) = \sum_{n=1}^{\infty} (a_n - b_n) n^{-s}$, and the result follows. $\square$

Proposition 2.34. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $\sigma_a < +\infty$. Then*

$$\lim_{\sigma\to\infty} \left(\sup_{\Re s \geq \sigma} |f(s) - a_1| \right) = 0,$$

and in particular

$$\lim_{\Re s\to\infty} f(s) = a_1.$$

Proof. Let $\sigma_0 > \sigma_a$ and define $A := \sum_{n=1}^{\infty} |a_n|/n^{\sigma_0} < +\infty$. Then for $s = \sigma + it$ with $\sigma > \sigma_0$ we have that

$$|f(s) - a_1| \leq \sum_{n=2}^{\infty} \frac{|a_n|}{n^\sigma} \leq \sum_{n=2}^{\infty} \frac{|a_n|}{n^{\sigma_0}} \frac{1}{n^{\sigma-\sigma_0}} \leq \frac{A}{2^{\sigma-\sigma_0}}.$$

Hence

$$\sup_{\Re s \geq \sigma} |f(s) - a_1| \leq \frac{A}{2^{\sigma-\sigma_0}},$$

and letting $\sigma \to \infty$ we get the desired result. $\square$

An interesting result on the growth of the coefficients is given in the following proposition.

Proposition 2.35. *If the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has $\sigma_c < +\infty$, then the coefficients a_n have at most polynomial growth.*

Proof. Since $\sigma_c < +\infty$, we can fix any s_0 such that $\Re s_0 > \sigma_c$, hence $f(s_0)$ converges. Then, by Lemma 2.21, the series converges absolutely for some integer $s = N > \Re s_0 + 1$, and so the general term $|a_n n^{-N}|$ tends to 0. Thus, for a big enough $n \in \mathbb{N}$, we have $|a_n n^{-N}| < 1$, which implies the polynomial growth of a_n. □

We also have an equivalent to the maximum modulus principle in half-planes for Dirichlet polynomials, which are truncated Dirichlet series.

Definition 2.36. *For $N \geq 1$, we define a **Dirichlet polynomial** as $f(s) = \sum_{n=1}^{N} a_n n^{-s}$.*

Proposition 2.37. *Let $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ be a Dirichlet polynomial. We have*

$$\sup_{\Re s > \sigma} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} \frac{a_n}{n^{\sigma + it}} \right|.$$

Proof. Assume first that $\sigma = 0$. Let us define

$$A = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right| \text{ and } B = \sup_{\Re s > 0} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right|.$$

It is clear that $A \leq B$. We now need to show that $B \geq A$. If $A = 0$ this is trivial. So suppose $A > 0$ and let us consider the function

$$g_\epsilon(s) = e^{-\epsilon\sqrt{s}} \sum_{n=1}^{N} \frac{a_n}{n^s},$$

which is a holomorphic function in $\mathbb{C}_0$, considering the principal branch of the square root in $\mathbb{C} \setminus (-\infty, 0]$, with $\sqrt{0} = 0$, and is continuous in $\overline{\mathbb{C}_0}$. For any $s = re^{i\alpha} \in \mathbb{C}_0$ we have

$$|g_\epsilon(s)| = e^{-\epsilon\sqrt{r}\cos\alpha/2} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \leq Be^{-\epsilon\sqrt{r}\cos\pi/4}.$$

Since the last expressions tends to 0 as $r \to \infty$, there exists an $R > 0$ such that for any $r \geq R$, $|g_\epsilon(re^{i\alpha})| \leq A$. Now considering the set $\tilde{D} := \overline{\mathbb{C}_0} \cap D(0, R)$, we have that for any $s \in \tilde{D}$ with $s = it,\ t \in \mathbb{R}$

$$|g_\epsilon(it)| = e^{-\epsilon\sqrt{t}\cos\pi/4} \left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right| \leq e^{-\epsilon\sqrt{t}\cos\pi/4} A \leq A.$$

By the maximum modulus principle, we have that $|g_\epsilon(s)| \leq A$ for $s \in \tilde{D}$. Hence $|g_\epsilon(s)| \leq A$ for $s \in \mathbb{C}_0$. By letting $\epsilon \to 0$ we obtain the result for $\sigma = 0$.

To prove the statement for an arbitrary σ, we apply the result to the polynomial $\sum_{n=1}^{N} (a_n/n^\sigma) n^{-s}$:

$$\begin{aligned} \sup_{\Re s > \sigma} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right| &= \sup_{\Re s > 0} \left| \sum_{n=1}^{N} \frac{a_n}{n^{s+\sigma}} \right| = \sup_{\Re s > 0} \left| \sum_{n=1}^{N} \frac{a_n}{n^\sigma} \frac{1}{n^s} \right| \\ &= \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} \frac{a_n}{n^\sigma} \frac{1}{n^{it}} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} \frac{a_n}{n^{\sigma+it}} \right| = \sup_{\Re s = \sigma} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right|. \end{aligned}$$

□

We will now give the analogous formulas to those in Propositions 1.4 and 1.5 for power series, which tell us that the coefficients and the partial sums of a Dirichlet series are uniquely determined by the function.

Proposition 2.38 (Perron's inversion formula). *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. For all $n \geq 1$ and all $\rho > \sigma_a$ the next equation holds:*

$$a_n = \lim_{T\to\infty} \frac{1}{2iT} \int_{\rho-iT}^{\rho+iT} f(s) n^s ds.$$

Proof. The proof for this proposition is similar to the one for Proposition 1.4. We claim that

$$\lim_{T\to\infty} \frac{1}{2iT} \int_{\rho-iT}^{\rho+iT} e^{\alpha s} ds = \begin{cases} 0, & \text{if } \alpha \neq 0 \\ 1, & \text{if } \alpha = 0. \end{cases} \tag{2.21}$$

If $\alpha \neq 0$, $\forall T > 0$ we have

$$\frac{1}{2iT} \int_{\rho-iT}^{\rho+iT} e^{\alpha s} ds = e^{\alpha\rho} \frac{\sin(\alpha T)}{\alpha T},$$

and taking the limit when $T \to \infty$ yields the result. If $\alpha = 0$ then the result is trivial.

Because $\rho > \sigma_a$, the series converges absolutely and it does so uniformly too by Proposition 2.24. In particular

$$c_N := \sup_{\Re s = \rho} \left| f(s) - \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \xrightarrow{N\to\infty} 0.$$

For $1 \leq m \leq N$, using equation (2.21), we have that

$$a_m = \lim_{T\to\infty} \frac{1}{2iT} \int_{\rho-iT}^{\rho+iT} \sum_{n=1}^{N} \frac{a_n}{n^s} m^s ds,$$

because $(m/s)^s = e^{s \log m/n}$, and $\log m/n = 0 \iff m = n$.

For any $s \in \mathbb{C}$ such that $\Re s = \rho$,

$$\left| f(s) - \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \leq c_N.$$

So we get

$$\left| \frac{1}{2iT} \int_{\rho-iT}^{\rho+iT} \left(f(s) m^s - \sum_{n=1}^{N} \frac{a_n}{n^s} m^s \right) ds \right| \leq c_N \left| m^\rho \right|.$$

and taking limits

$$\limsup_{T\to\infty} \left| \frac{1}{2iT} \int\limits_{\rho-iT}^{\rho+iT} \left(f(s) m^s - \sum_{n=1}^{N} \frac{a_n}{n^s} m^s \right) ds \right| = \limsup_{T\to\infty} \left| \frac{1}{2iT} \int\limits_{\rho-iT}^{\rho+iT} f(s) m^s ds - a_m \right| \leq c_N |m^\rho|.$$

And finally, if we fix m and take the limit when $N \to \infty$ we get the desired result. $\square$

We have proven an integral formula to obtain the coefficients of the series, and we now will give an expression that lets us obtain the partial sums of a series by performing an integral. But firstly, we need the following lemma:

Lemma 2.39. *Let n be a natural number, $c > 0$ and $x \in \mathbb{R} \setminus \mathbb{Z}$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w} = \begin{cases} 1, & \text{if } n < x \\ 0, & \text{if } n > x. \end{cases} \tag{2.22}$$

Equivalently,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\alpha w}\frac{dw}{w} = \begin{cases} 1, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha < 0. \end{cases} \tag{2.23}$$

Proof. It is easy to see that (2.22) and (2.23) are equivalent by writing $\log x/n = \alpha$, so we only need to prove the second integral equation.

Suppose $\alpha > 0$, we will calculate the integral using the residue theorem. Let us consider $T > 0$ and $-d < 0 < c$. The path of integration will be the rectangle with vertices $c - iT$, $c + iT$, $-d + iT$ and $-d - iT$. The only pole of the integrand is at $w = 0$ and has residue $\lim_{w\to 0} we^{\alpha w}/w = 1$, so:

$$\int_{c-iT}^{c+iT}e^{\alpha w}\frac{dw}{w} + \int_{c+iT}^{-d+iT}e^{\alpha w}\frac{dw}{w} + \int_{-d+iT}^{-d-iT}e^{\alpha w}\frac{dw}{w} + \int_{-d-iT}^{c-iT}e^{\alpha w}\frac{dw}{w} = 2\pi i.$$

In particular

$$\left|\int_{c-iT}^{c+iT}e^{\alpha w}\frac{dw}{w} - 2\pi i\right| \leq \left|\int_{c+iT}^{-d+iT}e^{\alpha w}\frac{dw}{w}\right| + \left|\int_{-d+iT}^{-d-iT}e^{\alpha w}\frac{dw}{w}\right| + \left|\int_{-d-iT}^{c-iT}e^{\alpha w}\frac{dw}{w}\right|.$$

Computing the second term in the right-hand side we have

$$\left|\int_{-d+iT}^{-d-iT}e^{\alpha w}\frac{dw}{w}\right| = \left|\int_{T}^{-T}e^{\alpha(-d+it)}\frac{idt}{-d+it}\right| \leq \int_{-T}^{T}e^{-\alpha d}\frac{dt}{|d+it|} \leq \frac{e^{-\alpha d}}{d}2T \xrightarrow{d\to\infty} 0.$$

For the other terms

$$\left|\int_{-d-iT}^{c-iT}e^{\alpha w}\frac{dw}{w}\right| = \left|\int_{-d}^{c}e^{\alpha(t-iT)}\frac{dt}{t-iT}\right| \leq \frac{1}{T}\int_{-d}^{c}e^{\alpha t}dt \leq \frac{1}{T}\int_{-\infty}^{c}e^{\alpha t}dt = \frac{1}{T}\frac{e^{\alpha c}}{\alpha},$$

and in a similar manner we get

$$\left|\int_{c+iT}^{-d+iT}e^{\alpha w}\frac{dw}{w}\right| \leq \frac{1}{T}\frac{e^{\alpha c}}{\alpha}.$$

So

$$\left|\int_{c-iT}^{c+iT}e^{\alpha w}\frac{dw}{w} - 2\pi i\right| \leq \frac{2e^{\alpha c}}{T\alpha} \xrightarrow{T\to\infty} 0, \tag{2.24}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\alpha w}\frac{dw}{w} = 1$$

as we wanted.

Now if $\alpha < 0$ we will perform a similar contour integration, but now with $d > c > 0$. Now the pole of the integrand is not contained in the region and by the residue theorem

$$\int_{c-iT}^{c+iT} e^{\alpha w}\frac{dw}{w} + \int_{c+iT}^{d+iT} e^{\alpha w}\frac{dw}{w} + \int_{d+iT}^{d-iT} e^{\alpha w}\frac{dw}{w} + \int_{d-iT}^{c-iT} e^{\alpha w}\frac{dw}{w} = 0,$$

and thus

$$\left|\int_{c-iT}^{c+iT} e^{\alpha w}\frac{dw}{w}\right| \leq \left|\int_{c+iT}^{d+iT} e^{\alpha w}\frac{dw}{w}\right| + \left|\int_{d+iT}^{d-iT} e^{\alpha w}\frac{dw}{w}\right| + \left|\int_{d-iT}^{c-iT} e^{\alpha w}\frac{dw}{w}\right|.$$

Since $\alpha < 0$, for the second term on the right-hand side we now have

$$\left|\int_{d+iT}^{d-iT} e^{\alpha w}\frac{dw}{w}\right| \leq \frac{e^{\alpha d}}{d}2T \xrightarrow{d\to\infty} 0.$$

For the other terms

$$\left|\int_{d-iT}^{c-iT} e^{\alpha w}\frac{dw}{w}\right| = \left|\int_{c}^{d} e^{\alpha(t-iT)}\frac{dt}{t-iT}\right| \leq \frac{1}{T}\int_{c}^{d} e^{\alpha t}dt \leq \frac{1}{T}\int_{c}^{\infty} e^{\alpha t}dt = \frac{1}{T}\frac{e^{\alpha c}}{|\alpha|},$$

and

$$\left|\int_{c+iT}^{-d+iT} e^{\alpha w}\frac{dw}{w}\right| \leq \frac{1}{T}\frac{e^{\alpha c}}{|\alpha|}.$$

Adding up all the terms

$$\left|\int_{c-iT}^{c+iT} e^{\alpha w}\frac{dw}{w}\right| \leq \frac{2e^{\alpha c}}{T|\alpha|} \xrightarrow{T\to\infty} 0, \tag{2.25}$$

so

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha w}\frac{dw}{w} = 0$$

if $\alpha < 0$. $\square$

Proposition 2.40 (Perron-Landau formula)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with absolute convergence abscissa σ_a. Then for $x \notin \mathbb{Z}$, and for any $c \in \mathbb{R}$ and s such that $\Re s + c > \sigma_a$:*

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s+w)\frac{x^w}{w}dw.$$

Proof. Since $\Re s+c > \sigma_a$, the series converges absolutely and uniformly in compact subsets of $\overline{\mathbb{C}_{\Re s+c}}$, so we can permute sum and integral in the following fashion for any $T > 0$

$$\begin{aligned}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}dw &= \int_{c-iT}^{c+iT}\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\frac{x^w}{w}dw = \sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &= \sum_{n<x}\frac{a_n}{n^s}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w} + \sum_{n>x}\frac{a_n}{n^s}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w}.\end{aligned} \tag{2.26}$$

The first sum is a finite one, so we can simply take the limit term by term when $T \to \infty$, and by Lemma 2.39, given that $n < x$, we have that each integral is $2\pi i$. Now we will show that

$$\sum_{n>x}\frac{a_n}{n^s}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w} \xrightarrow{T\to\infty} 0,$$

which will complete the proof.

By Lemma 2.39, each term tends to zero, but to show that the series is convergent we must know at which rate they tend to zero. We will use the bound in equation (2.25), and in our case $\alpha = \log(x/n)$. We have

$$\left| \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} \right| \leq \frac{2e^{\alpha c}}{T|\alpha|} = \frac{2e^{c\log(x/n)}}{T|\log(x/n)|} = \frac{2\left(\frac{x}{n}\right)^c}{T\log(n/x)}. \tag{2.27}$$

Hence, using that $n \geq [x] + 1$,

$$\left| \sum_{n>x} \frac{a_n}{n^s} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} \right| \leq \sum_{n>x} \frac{|a_n|}{n^{\Re s}} \frac{2}{T} \left(\frac{x}{n}\right)^c \frac{1}{\log(n/x)} \leq \frac{2}{T} \frac{x^c}{\log\left(\frac{[x]+1}{x}\right)} \sum_{n>x} \frac{|a_n|}{n^{\Re s + c}},$$

which tends to zero as $T \to \infty$ because the series converges. □

Corollary 2.41. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $\sigma_a < c$, $c > 0$. Then for $x \notin \mathbb{Z}$:*

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(w) \frac{x^w}{w} dw.$$

Proof. The result follows from applying the Perron-Landau formula for $s = 0$. □

3 Hardy-Dirichlet spaces

This chapter is dedicated to the study of two Hardy-Dirichlet spaces: $\mathscr{H}^\infty$ and $\mathscr{H}^2$. We will develop a theory similar to that of the classical Hardy spaces in the disk, and see, among other results, that the multipliers of $\mathscr{H}^2$ are $\mathscr{H}^\infty$, which is a characterization analogous to the one for $H^2(\mathbb{D})$. We will study some properties of these spaces, their abscissas and how they relate to one another. In fact, the Hardy-Dirichlet spaces $\mathscr{H}^p$ are well defined for $1 \leq p \leq \infty$, though we will only study the cases for $p = 2$ and $p = \infty$.

We will need the following definitions.

Definition 3.1. *We define $H^\infty(\mathbb{C}_0)$ as the set of bounded analytic functions in $\mathbb{C}_0$, and we will equip it with the norm $\|f\|_\infty = \sup_{s\in\mathbb{C}_0} |f(s)|$.*

For $N \geq 1$ we define $H^\infty(\mathbb{D}^N)$ as the set of holomorphic and bounded functions in $\mathbb{D}^N$. We will also equip it with the supremum norm, $\|f\|_\infty = \sup_{s\in\mathbb{D}^N} |f(s)|$.

$\mathscr{H}^\infty$ is the subspace of functions in $H^\infty(\mathbb{C}_0)$ that can be represented as a convergent Dirichlet series in some subspace of $\mathbb{C}_0$. We will also equip it with the norm $\|f\|_\infty = \sup_{s\in\mathbb{C}_0} |f(s)|$. We have $\mathscr{H}^\infty = H^\infty(\mathbb{C}_0) \cap \mathcal{D}$.

$\mathscr{H}^2$ is the set of Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ which have square-summable coefficients, that is $\sum_{n=1}^\infty |a_n|^2 < +\infty$. We will equip it with the norm $\|f\|_2 = \left(\sum_{n=1}^\infty |a_n|^2\right)^{1/2}$.

Observation 3.2. *Both $\mathscr{H}^\infty$ and $\mathscr{H}^2$ can be seen as the completion spaces of Dirichlet polynomials for the respective norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$. Hence, Dirichlet polynomials are dense in both $\mathscr{H}^\infty$ and $\mathscr{H}^2$.*

3.1 Properties of $\mathscr{H}^\infty$

One of the objectives of this section is to obtain global estimates for the abscissas of the functions in $\mathscr{H}^\infty$. We will also see that the the space $\mathscr{H}^\infty$ equipped with the supremum norm is a Banach algebra. But firstly, we need the following lemma, which in the case of the disk $H^\infty(\mathbb{D})$ is an immediate consequence of Cauchy's integral formula.

Lemma 3.3. *If $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathscr{H}^\infty$, then for any $n \geq 1$ we have that*

$$|a_n| \leq \|f\|_\infty .$$

Proof. Since $f(s) \in \mathscr{H}^\infty$, we have that $\sigma_c < +\infty$. Given that $\sigma_a \leq \sigma_c + 1$, there exists $\rho > 0$ such that $\sum_{n=1}^\infty |a_n| n^{-\rho} < +\infty$. Using Perron's formula in Proposition 2.38 and Cauchy's integral theorem applied to a rectangle with vertices $\rho \pm iT$ and $\epsilon \pm iT$ with $0 < \epsilon < \rho$, we have

$$a_n = \lim_{T\to\infty} \frac{1}{2iT} \left(\int_{\epsilon+iT}^{\rho+iT} f(s) n^s ds + \int_{\epsilon-iT}^{\epsilon+iT} f(s) n^s ds + \int_{\rho-iT}^{\epsilon-iT} f(s) n^s ds \right).$$

The top side of the rectangle is bounded by

$$\left| \int_{\epsilon+iT}^{\rho+iT} f(s) n^s ds \right| \leq \|f\|_\infty \int_\epsilon^\rho |n^{t+iT}| dt \leq \|f\|_\infty n^\rho \rho,$$

and similarly for the bottom side

$$\left| \int_{\rho-iT}^{\epsilon-iT} f(s) n^s ds \right| \leq \|f\|_\infty n^\rho \rho.$$

The left side is bounded by

$$\left|\int_{\epsilon-iT}^{\epsilon+iT} f(s)n^s ds\right| \le \|f\|_\infty \int_{-T}^{T} n^\epsilon dt = 2T\,\|f\|_\infty\, n^\epsilon.$$

Hence

$$|a_n| \le \lim_{T\to\infty} \frac{1}{2T}\left(2\,\|f\|_\infty\, n^\rho \rho + 2T\,\|f\|_\infty\, n^\epsilon\right) = \|f\|_\infty\, n^\epsilon.$$

Letting ϵ tend to 0, we obtain the result. □

We have seen in Lemma 2.14 that if the coefficients of the Dirichlet series are bounded, then $\sigma_a(f) \le 1$. Hence, the above result gives immediately the corollary:

Corollary 3.4. *If $f \in \mathscr{H}^\infty$, then $\sigma_a(f) \le 1$.*

Theorem 3.5. *$(\mathscr{H}^\infty, \|\cdot\|_\infty)$ is a Banach algebra. Moreover, its invertible elements are those functions f for which there exists $\delta > 0$ such that $|f(s)| \ge \delta$, for any $s \in \mathbb{C}_0$.*

Proof. We need to show that the product of functions in $\mathscr{H}^\infty$ is in $\mathscr{H}^\infty$ and that every Cauchy sequence in $\mathscr{H}^\infty$ has limit function in $\mathscr{H}^\infty$.

Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$, $g(s) = \sum_{n=1}^\infty b_n n^{-s} \in \mathscr{H}^\infty$. Given that both f and g are bounded and analytic in $\mathbb{C}_0$, the product function $h := fg$ is also bounded and analytic in $\mathbb{C}_0$. The function h is equal to the series

$$\sum_{n=1}^\infty \left(\sum_{km=n} a_k b_m\right)\frac{1}{n^s}, \tag{3.1}$$

wherever it converges, and given that $\sigma_a(f), \sigma_a(g) \le 1$ by the last corollary, it converges absolutely for $\Re s = \sigma > 1$ since

$$\sum_{n=1}^\infty \left(\sum_{km=n} |a_k b_m|\right)\frac{1}{n^\sigma} = \left(\sum_{k=1}^\infty \frac{|a_k|}{k^\sigma}\right)\left(\sum_{m=1}^\infty \frac{|b_m|}{m^\sigma}\right) < +\infty.$$

But fg extends the series definition in equation (3.1), and hence $h \in \mathscr{H}^\infty$.

Now let $(f_j)_j \subset \mathscr{H}^\infty$ be a Cauchy sequence with $f_j(s) = \sum_{n=1}^\infty a_n^j n^{-s}$ and $K := \sup_j \|f_j\|_\infty < +\infty$. We know that $\mathscr{H}^\infty$ is a subset of $H^\infty(\mathbb{C}_0)$, which is a Banach space with the supremum norm, thus there exists an $f \in H^\infty(\mathbb{C}_0)$ such that $\lim_{j\to\infty} \|f_j - f\|_\infty = 0$. We need to prove that $f \in \mathscr{H}^\infty$. By Lemma 3.3, for any $n, k, l \ge 1$ we have that $|a_n^l - a_n^k| \le \|f_l - f_k\|_\infty$. In particular, the sequence $(a_n^l)_l$ is Cauchy for each n, and hence converges to $\lim_{l\to\infty} a_n^l = a_n$. Again, by Lemma 3.3, we have that $|a_n^l| \le \|f_l\|_\infty \le K$. Passing to the limit we obtain that every coefficient a_n is bounded by K, hence the series $\sum_{n=1}^\infty a_n n^{-s}$ is absolutely convergent for $\Re s > 1$. We must check that the function f is equal to the series $\sum_{n=1}^\infty a_n n^{-s}$ in $\mathbb{C}_1$. From $\lim_j f_j = f$ and $\lim_j a_n^j = a_n$ we have that for any $\epsilon > 0$ and any $N \ge 1$ there exists a $k \ge 1$ such that for any $s \in \mathbb{C}_1$, $j \ge k$ and $1 \le n \le N$

$$|f_j(s) - f(s)| < \epsilon \quad \text{and} \quad |a_n^j - a_n| < \epsilon.$$

From the fact that every f_j is a convergent Dirichlet series in $\mathbb{C}_1$ and that the series $\sum_{n=1}^\infty a_n n^{-s}$ is absolutely convergent for $\Re s > 1$, we get that there exists $N \ge 1$ such that

$$\left|f_j(s) - \sum_{n=1}^N \frac{a_n^j}{n^s}\right| < \epsilon \quad \text{and} \quad \left|\sum_{n=1}^\infty \frac{a_n}{n^s} - \sum_{n=1}^N \frac{a_n}{n^s}\right| < \epsilon.$$

Hence if $j \geq k$ is fixed

$$\left| f(s) - \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leq |f(s) - f_j(s)| + \left| f_j(s) - \sum_{n=1}^{N} \frac{a_n^j}{n^s} \right| + \left| \sum_{n=1}^{N} \frac{a_n^j}{n^s} - \sum_{n=1}^{N} \frac{a_n}{n^s} \right|$$
$$+ \left| \sum_{n=1}^{N} \frac{a_n}{n^s} - \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leq \epsilon + \epsilon + \epsilon \sum_{n=1}^{\infty} \frac{1}{n^{\Re s}} + \epsilon,$$

which tends to 0 when $\epsilon \to 0$.

Next, if $f \in \mathscr{H}^\infty$ is invertible, then there exists a $g \in \mathscr{H}^\infty$ such that for any $s \in \mathbb{C}_0$, $f(s)g(s) = 1$. Hence $1 = |f(s)g(s)| \leq |f(s)| \, \|g\|_\infty$ so $|f(s)| \geq 1/\|g\|_\infty$.

Reciprocally, if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty$ with $|f(s)| \geq \delta > 0$, we have that $1/f \in H^\infty(\mathbb{C}_0)$. By Proposition 2.34, we get $|a_1| \geq \delta$ by letting $\Re s = \sigma \to \infty$. Hence there exists $\sigma > 0$ such that $\sum_{n=2}^{\infty} \frac{|a_n|}{|a_1|} n^{-\sigma} < 1$. This allows us to expand the following fraction as a geometric series:

$$\frac{1}{1 + \sum_{n=2}^{\infty} \frac{a_n}{a_1} n^{-s}} = \sum_{k=1}^{\infty} \left(- \sum_{n=2}^{\infty} \frac{a_n}{a_1} n^{-s} \right)^k,$$

which in turn can be expanded as a Dirichlet series. Indeed, the series on the right-hand side is absolutely convergent so each $\left(- \sum_{n=2}^{\infty} \frac{a_n}{a_1} n^{-s} \right)^k$ can be rearranged, so for each n^{-s} there exists a unique and finite coefficient b_n and we can write it as

$$\frac{1}{1 + \sum_{n=2}^{\infty} \frac{a_n}{a_1} n^{-s}} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}.$$

Thus we can write

$$\left(\frac{1}{f} \right)(s) = \frac{1}{\sum_{n=1}^{\infty} a_n n^{-s}} = \frac{1}{a_1} \frac{1}{1 + \sum_{n=2}^{\infty} \frac{a_n}{a_1} n^{-s}} = \sum_{n=1}^{\infty} \frac{b_n}{a_1} n^{-s},$$

and $(1/f) \in \mathscr{H}^\infty$. ☐

We will see that $\mathscr{H}^\infty$ is non-separable. But firstly we will need the following lemmas.

Lemma 3.6. *If a metric space (X, d) contains an uncountable family of points $\{x_i\}_{i \in I}$ such that*

$$\delta := \inf_{\substack{i,j \in I \\ i \neq j}} d(x_i, x_j) > 0,$$

then X is not separable.

Proof. Let Y be a dense set in X, and consider for each $j \in I$ the set $B_j := B(x_j, \delta/2) = \{x \in X : \, d(x, x_j) < \delta/2\}$. Since Y is dense and B_j are open sets, there exists $y_j \in B_j \cap Y$. Given that B_j are pairwise disjoint, we have that for $i \neq j$, $y_j \neq y_i$. But if I is uncountable, so is Y, and X is non-separable. ☐

Corollary 3.7. *The space $H^\infty(\mathbb{D})$ is non-separable.*

Proof. By the last lemma, we just need to find an uncountable family $\{f_\zeta\}_{\zeta\in\mathbb{T}} \subset H^\infty(\mathbb{D})$ such that $\|f_\zeta - f_\xi\|_\infty \geq 1$ for any $\zeta, \xi \in \mathbb{T}$, $\zeta \neq \xi$. Consider the function

$$f(z) = \exp\frac{z+1}{z-1}$$

which is holomorphic in $\mathbb{C}\setminus\{1\}$. For $z \in \mathbb{D}$ we have

$$|f(z)| = \exp\left(-\frac{1-|z|^2}{|z-1|^2}\right) < 1,$$

so $f \in H^\infty(\mathbb{D})$. Its radial boundary limit at $\zeta \in \mathbb{T}$ is

$$f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta) = \begin{cases} f(\zeta) \in \mathbb{T}, & \text{if } \zeta \neq 1 \\ 0, & \text{if } \zeta = 1. \end{cases}$$

Finally we define the set of functions $\{f_\zeta\}_{\zeta\in\mathbb{T}}$ for $\zeta \in \mathbb{T}$ as $f_\zeta(z) := f(\overline{\zeta}z)$. Its radial boundary values are

$$f_\zeta^*(\xi) = \begin{cases} f^*(\overline{\zeta}\xi) \in \mathbb{T}, & \text{if } \xi \in \mathbb{T}\setminus\{\zeta\} \\ 0, & \text{if } \xi = \zeta. \end{cases}$$

Now for $\zeta, \xi \in \mathbb{T}$ and $\zeta \neq \xi$ we have that

$$\|f_\zeta - f_\xi\|_\infty \geq |f_\zeta^*(\zeta) - f_\xi^*(\zeta)| = |f_\xi^*(\zeta)| = 1.$$

This proves the result. □

Lemma 3.8. *The space $\mathscr{H}^\infty$ is non-separable.*

Proof. We will see that there exists an isometry between $(H^\infty(\mathbb{D}), \|\cdot\|_\infty)$ and a subspace of $(\mathscr{H}^\infty, \|\cdot\|_\infty)$, hence $\mathscr{H}^\infty$ is non-separable.

If $g(z) = \sum_{n=0}^\infty b_n z^n \in H^\infty(\mathbb{D})$, we define the mapping

$$\begin{aligned} \Delta\colon H^\infty(\mathbb{D}) &\longrightarrow \mathscr{H}^\infty \\ g &\longmapsto \Delta g \end{aligned}$$

with

$$\Delta g(s) = g(2^{-s}) = \sum_{n=0}^\infty b_n (2^{-s})^n = \sum_{n=0}^\infty \frac{b_n}{(2^n)^s}.$$

Indeed Δg is an element of $\mathscr{H}^\infty$, since it is a Dirichlet series and it is in $H^\infty(\mathbb{C}_0)$ given that for $s \in \mathbb{C}_0$, $2^{-s} \in \mathbb{D}$ and g is analytic and bounded in $\mathbb{D}$. It is an isometry since

$$\|g\|_\infty = \sup_{z\in\mathbb{D}} |g(z)| = \sup_{s\in\mathbb{C}_0} |g(2^{-s})| = \|\Delta g\|_\infty,$$

where we have used that $\{2^{-s} : s \in \mathbb{C}_0\} = \mathbb{D}$. □

Observation 3.9. *We remark that this isometry is just a particular case of the Bohr's correspondence, which we will see later, for power series in one complex variable.*

We have seen that if $f \in \mathscr{H}^\infty$, then $\sigma_a(f) \leq 1$. Now, we want to give global estimates for the abscissas of convergence and of uniform convergence for the elements of $\mathscr{H}^\infty$. To do so, the next result is needed.

Theorem 3.10 (Balasubramanian-Calado-Queffélec). *Let $f = \sum_{n=1}^\infty a_n n^{-s} \in \mathscr{H}^\infty$ and $S_N(s) = \sum_{n=1}^N a_n n^{-s}$ for $N \geq 2$. Then there exists $C \geq 0$ such that*

$$\|S_N\|_\infty \leq C \|f\|_\infty \log N.$$

Proof. We will use steps from the proof of the Perron-Landau formula (Proposition 2.40) and Lemma 2.39. If $x < n$, from the deduction of equation (2.24), writing $\alpha = \log(x/n)$ we can get

$$\int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} = 2\pi i + \mathcal{O}\left(\frac{(x/n)^c}{T\log(x/n)}\right).$$

And if $x > n$, from the deduction of equation (2.25) we get

$$\int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} = \mathcal{O}\left(\frac{(x/n)^c}{T\log(x/n)}\right).$$

So for $x \notin \mathbb{Z}$, $c \in \mathbb{R}$ and $s \in \mathbb{C}_0$ such that $\Re s + c > \sigma_a(f)$ we can rewrite equation (2.26) as

$$\begin{aligned}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}dw &= \sum_{n<x} \frac{a_n}{n^s}\left(2\pi i + \mathcal{O}\left(\frac{(x/n)^c}{T\log(x/n)}\right)\right) + \sum_{n>x}\frac{a_n}{n^s}\left(\mathcal{O}\left(\frac{(x/n)^c}{T\log(x/n)}\right)\right)\\ &= 2\pi i \sum_{n<x}\frac{a_n}{n^s} + \sum_{n=1}^\infty \frac{a_n}{n^s}\mathcal{O}\left(\frac{(x/n)^c}{T\log(x/n)}\right)\end{aligned}$$

and in turn, rearrange it into

$$\left|\sum_{n<x}\frac{a_n}{n^s}\right| \leq \frac{1}{2\pi}\left|\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}dw\right| + \mathcal{O}\left|\sum_{n=1}^\infty \frac{a_n}{n^s}\frac{(x/n)^c}{T\log(x/n)}\right|. \tag{3.2}$$

We will choose $c = 2$ and $x = N + 1/2$ for some $N \geq 2$. Then, using that $|a_n| \leq \|f\|_\infty$ from Lemma 3.3, we get that the error term is controlled by

$$\left|\sum_{n=1}^\infty \frac{a_n}{n^s}\frac{(x/n)^c}{T\log(x/n)}\right| \leq \frac{x^2}{T}\sum_{n=1}^\infty \frac{|a_n|}{n^{\Re s+2}|\log(x/n)|} \leq \frac{x^2\|f\|_\infty}{T}\sum_{n=1}^\infty \frac{1}{n^{\Re s+2}|\log(x/n)|}.$$

We will see that $|\log x/n| \geq 1/4(N+1/2)$.

We will use the following fact: if $y \geq 1$, we have that $(1+1/2y)^{2y} \geq e^{1/2}$. Indeed, the inequality holds for $y = 1$ and $(1+1/2y)^{2y}$ is an increasing function. This is equivalent to $\log(1+1/2y)^{2y} \geq 1/2$ and $\log\left(\frac{y+1/2}{y}\right) \geq \frac{1}{4y}$. Hence, for $n > x = N + 1/2 > N$

$$\left|\log\frac{x}{n}\right| = \log\frac{n}{x} \geq \log\frac{N+1}{N+1/2} \geq \frac{1}{4(N+1/2)},$$

and for $n < x$ we have

$$\left|\log\frac{x}{n}\right| = \log\frac{x}{n} = \log\frac{N+1/2}{n} \geq \log\frac{N+1/2}{N} \geq \frac{1}{4N} \geq \frac{1}{4(N+1/2)}.$$

And in consequence, since $x = N + 1/2$

$$\left|\sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{(x/n)^c}{T\log(x/n)}\right| \le \frac{x^2 \|f\|_\infty}{T} \sum_{n=1}^{\infty} \frac{4(N+1/2)}{n^{\Re s+2}} \le K\frac{x^3}{T} \|f\|_\infty \le K\|f\|_\infty\,,$$

where the last inequality is achieved by choosing $T = x^3$, and K is some positive constant. Now to estimate the first term in the right-hand side of equation (3.2), we will use Cauchy's integral theorem on the rectangle with vertices $2 \pm iT$ and $\epsilon \pm iT$:

$$\int_{2-iT}^{2+iT} f(s+w)\frac{x^w}{w}dw = \int_{\epsilon-iT}^{\epsilon+iT} f(s+w)\frac{x^w}{w}dw + \int_{\epsilon+iT}^{2+iT} f(s+w)\frac{x^w}{w}dw + \int_{2-iT}^{\epsilon-iT} f(s+w)\frac{x^w}{w}dw.$$

For the first term we have

$$\begin{aligned}\left|\int_{\epsilon-iT}^{\epsilon+iT} f(s+w)\frac{x^w}{w}dw\right| &\le \|f\|_\infty \int_{-T}^{T} \frac{|x^{\epsilon+it}|}{|\epsilon+it|}dt \le \|f\|_\infty \int_{-T}^{T} \frac{x^\epsilon}{\sqrt{\epsilon^2+t^2}}dt \\ &= 2\|f\|_\infty x^\epsilon \int_0^{T/\epsilon} \frac{du}{\sqrt{u^2+1}} = 2\|f\|_\infty x^\epsilon \log\left(\frac{T}{\epsilon} + \sqrt{\frac{T^2}{\epsilon^2}+1}\right).\end{aligned}$$

The second term is bounded by

$$\left|\int_{\epsilon+iT}^{2+iT} f(s+w)\frac{x^w}{w}dw\right| \le \|f\|_\infty \int_\epsilon^2 \frac{|x^{t+iT}|}{|t+iT|}dt \le \|f\|_\infty \int_\epsilon^2 \frac{x^t}{T}dt \le \frac{2\|f\|_\infty x^2}{T} = \frac{2\|f\|_\infty}{x},$$

and similarly for the third term

$$\left|\int_{2-iT}^{\epsilon-iT} f(s+w)\frac{x^w}{w}dw\right| \le \frac{2\|f\|_\infty}{x}.$$

Substituting in (3.2) we get

$$\left|\sum_{n<x} \frac{a_n}{n^s}\right| \le \frac{1}{2\pi}\left(\frac{4\|f\|_\infty}{x} + 2\|f\|_\infty x^\epsilon \log\left(\frac{T}{\epsilon} + \sqrt{\frac{T^2}{\epsilon^2}+1}\right)\right) + K\|f\|_\infty\,.$$

Adjusting $\epsilon = 1/\log x$ so that $x^\epsilon = e$ and $T/\epsilon = x^3 \log x$, the desired inequality is achieved, for some constant C independent of N

$$\left|\sum_{n<x} \frac{a_n}{n^s}\right| \le \frac{1}{2\pi}\left(\frac{4\|f\|_\infty}{x} + 2\|f\|_\infty e \log\left(x^3 \log x\right)\right) + K\|f\|_\infty \le C\|f\|_\infty \log N.$$

□

Theorem 3.11 (Bohr)**.** *Let $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^\infty$, then $\sigma_u(f) \le 0$ and $\sigma_c(f) \le 0$.*

Proof. We will see that $\sum_{n=1}^{\infty} a_n n^{-s-\epsilon}$ converges uniformly in $\mathbb{C}_0$ for any $\epsilon > 0$. We will perform an Abel transformation with $S_N(s) = \sum_{n=1}^{N} a_n n^{-s}$ where we write $S_0(s) \equiv 0$:

$$\begin{aligned}\sum_{n=1}^{N} \frac{a_n}{n^{s+\epsilon}} &= \sum_{n=1}^{N} \frac{S_n(s) - S_{n-1}(s)}{n^\epsilon} = \sum_{n=1}^{N} \frac{S_n(s)}{n^\epsilon} - \sum_{n=1}^{N} \frac{S_{n-1}(s)}{n^\epsilon} \\ &= \sum_{n=1}^{N} \frac{S_n(s)}{n^\epsilon} - \sum_{n=0}^{N-1} \frac{S_n(s)}{(n+1)^\epsilon} = \sum_{n=1}^{N-1} S_n(s)\left(\frac{1}{n^\epsilon} - \frac{1}{(n+1)^\epsilon}\right) + \frac{S_N(s)}{N^\epsilon}.\end{aligned}$$

We will see that the right-hand side is normally convergent, that is

$$\sum_{n=1}^{N-1} \sup_{s\in\mathbb{C}_0} \left| S_n(s) \left(\frac{1}{n^\epsilon} - \frac{1}{(n+1)^\epsilon} \right) \right| + \sup_{s\in\mathbb{C}_0} \left| \frac{S_N(s)}{N^\epsilon} \right| \tag{3.3}$$

converges, which implies the uniform convergence of the Dirichlet series. Indeed, the partial sum is bounded by

$$\sum_{n=1}^{N-1} \sup_{s\in\mathbb{C}_0} \left| S_n(s) \left(\frac{1}{n^\epsilon} - \frac{1}{(n+1)^\epsilon} \right) \right| \leq \sum_{n=1}^{N-1} \frac{C\epsilon \left\| f \right\|_\infty \log n}{(n+1)^{1+\epsilon}},$$

where we have used Theorem 3.10 and the mean value theorem. The remaining term is bounded by

$$\sup_{s\in\mathbb{C}_0} \left| \frac{S_N(s)}{N^\epsilon} \right| \leq \frac{C \left\| f \right\|_\infty \log N}{N^\epsilon}.$$

Hence (3.3) converges for any $\epsilon > 0$. This implies the uniform convergence of $f(s+\epsilon)$ in $\mathbb{C}_0$ for any $\epsilon > 0$ since

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^{s+\epsilon}} - f(s+\epsilon) \right| = \left| \sum_{n=N+1}^{\infty} \frac{a_n}{n^{s+\epsilon}} \right| \leq \sum_{n=N+1}^{\infty} \left| \frac{a_n}{n^{s+\epsilon}} \right| \leq \sum_{n=N+1}^{\infty} \sup_{s\in\mathbb{C}_0} \left| \frac{a_n}{n^{s+\epsilon}} \right| \xrightarrow{N\to\infty} 0,$$

and the last bound is independent of s. This means that $\sigma_u(f) \leq 0$ which implies that $\sigma_c(f) \leq 0$. $\square$

Observation 3.12. *This theorem could have also been proved as a consequence of Lemma 2.25 instead of a result of the control of the partial sums.*

Observation 3.13. *Notice that Bohr's Theorem shows that if $f \in \mathscr{H}^\infty$, then the Dirichlet series which it represents converges to f in $\mathbb{C}_0$, which is not obvious from the definition of $\mathscr{H}^\infty$ since, a priori, the convergence was only for a smaller half-plane. This means that f can be written as a Dirichlet series in the whole $\mathbb{C}_0$ and, in particular, $\left\| f \right\|_\infty = \sup_{s\in\mathbb{C}_0} \left| \sum_{n=1}^\infty a_n n^{-s} \right|$.*

We will now see that $\mathscr{H}^\infty$ is a strict subset of $H^\infty(\mathbb{C}_0)$, but we firstly need the following lemma. It is similar to Perron's formula in the sense that it lets us retrieve the coefficients from the limit function, but it is recursive because we need the first $n-1$ coefficients to compute the n-th coefficient.

Lemma 3.14. *Let $f = \sum_{n=1}^\infty a_n n^{-s} \in \mathscr{H}^\infty$, then for any $N \geq 1$*

$$a_N = \lim_{\Re s\to\infty} N^s \left(f(s) - \sum_{n=1}^{N-1} \frac{a_n}{n^s} \right).$$

Proof. Choose $s \in \mathbb{C}_0$ with $\Re s = \sigma > 2$. Then, using Lemma 3.3,

$$\left| N^s \left(f(s) - \sum_{n=1}^{N-1} \frac{a_n}{n^s} \right) - a_N \right| = \left| \sum_{n=N+1}^{\infty} a_n \left(\frac{N}{n} \right)^s \right| \leq \left\| f \right\|_\infty \sum_{n=N+1}^{\infty} \left(\frac{N}{n} \right)^{\sigma-2} \left(\frac{N}{n} \right)^2.$$

Now given that $n \geq N+1$ and $\sigma > 2$ we have $(N/n)^{\sigma-2} \leq (\frac{N}{N+1})^{\sigma-2}$ and then

$$\begin{aligned} \left| N^s \left(f(s) - \sum_{n=1}^{N-1} \frac{a_n}{n^s} \right) - a_N \right| &\leq \|f\|_\infty \left(\frac{N}{N+1} \right)^{\sigma-2} \sum_{n=N+1}^{\infty} \left(\frac{N}{n} \right)^2 \\ &\leq \frac{\pi^2}{6} N^2 \left(\frac{N}{N+1} \right)^{\sigma-2} \|f\|_\infty \xrightarrow{\sigma\to\infty} 0, \end{aligned}$$

which proves the claim. □

Proposition 3.15. *The set $\mathscr{H}^\infty$ is a strict subset of $H^\infty(\mathbb{C}_0)$.*

Proof. We just need to find $f \in H^\infty(\mathbb{C}_0) \setminus \mathscr{H}^\infty$. Let $f(s) = e^{-s} \in H^\infty(\mathbb{C}_0)$, and suppose it can be represented as a Dirichlet series so $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathbb{C}_0$. Then by Lemma 3.14, we have that

$$a_1 = \lim_{\Re s\to\infty} e^{-s} = 0, \quad \text{and} \quad a_2 = \lim_{\Re s\to\infty} 2^s e^{-s} = 0,$$

but $|a_3| = \lim_{\Re s\to\infty} |3^s e^{-s}| = \infty$, hence $f \notin \mathscr{H}^\infty$. □

Observation 3.16. *Note that this proposition shows an essential difference between Dirichlet series and power series, in the sense that holomorphicity does not imply the expansion of the function as a Dirichlet series, as it does with power series. We have seen this in the last example, where $f(s) = e^{-s}$ is an entire function, but does not have a representation as Dirichlet series in any half-plane.*

3.2 Properties of $\mathscr{H}^2$

Proposition 3.17. *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^2$, then $\sigma_a(f) \leq 1/2$.*

Proof. Applying Cauchy-Schwarz's inequality with $\sigma = \Re s$,

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} \right)^{1/2} = \|f\|_2 \, \zeta(2\sigma)^{1/2},$$

which converges if $\sigma \geq 1/2$. □

Observation 3.18. *This bound is sharp. For example, it is attained by the series $f(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} \frac{1}{n^s}$. Indeed, $f \in \mathscr{H}^2$ since*

$$\sum_{n=2}^{\infty} \frac{1}{|\sqrt{n}\log n|^2} = \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < +\infty.$$

We have $\sigma_a(f) = 1/2$ due to the fact that

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} \frac{1}{n^\sigma} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2+\sigma} \log n}$$

converges if $\sigma > 1/2$ by the comparison test with the ζ function, and diverges by Cauchy's condensation test for $\sigma = 1/2$.

Proposition 3.19. *The space $\mathscr{H}^2$ is a Hilbert space with the scalar product*

$$\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n},$$

for $f = \sum_{n=1}^{\infty} a_n n^{-s}$, $g = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathscr{H}^2$.

Proof. The result follows from noticing that $\mathscr{H}^2$ is formally just ℓ^2 with the isometry

$$\begin{aligned} \phi\colon \quad \ell^2 &\longrightarrow \mathscr{H}^2 \\ (a_n)_n &\longmapsto \sum_{n=1}^{\infty} a_n n^{-s} \end{aligned},$$

and that ℓ^2 is a Hilbert space with the product $\langle (a_n)_n, (b_n)_n \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$. ☐

Since $\mathscr{H}^2$ is a Hilbert space, it is a natural question to find its reproducing kernel. The reproducing kernel and its estimates are a key tool when studying problems in operator theory. For $\mathscr{H}^2$ we will see that it is related to the Riemann's ζ function.

We recall that if $s \in \mathbb{C}_0$, the point evaluation functional $\phi_s : \mathscr{H}^2 \to \mathbb{C}$ defined as $\phi_s(f) = f(s)$ is bounded by Proposition 3.17. Then, since $\mathscr{H}^2$ is a Hilbert space, by Riesz's representation theorem, there exists a function $k_s \in \mathscr{H}^2$ such that $f(s) = \langle f, k_s \rangle$ for all $f \in \mathscr{H}^2$. The reproducing kernel is then defined as $K(s,a) = \overline{k_s(a)}$, and has the reproducing property

$$\langle f(s), K(s,a) \rangle = f(a).$$

Moreover, if $\{e_n(s)\}_n$ is an orthonormal basis in $\mathscr{H}^2$, then

$$K(s,a) = \sum_{n=1}^{\infty} e_n(s) \overline{e_n(a)}.$$

Proposition 3.20. *The reproducing kernel of $\mathscr{H}^2$ is $K_{\mathscr{H}^2}(s,a) = \zeta(s + \overline{a}) \in \mathscr{H}^2$, with $a \in \mathbb{C}_{1/2}$.*

Proof. An orthonormal basis in $\mathscr{H}^2$ is given by $e_n(s) = n^{-s}$ for $n \geq 1$. Hence we have that the reproducing kernel is

$$K_{\mathscr{H}^2}(s,a) = \sum_{n=1}^{\infty} n^{-s} \overline{n^{-a}} = \sum_{n=1}^{\infty} n^{-(s+\overline{a})} = \zeta(s + \overline{a}).$$

It is easy to see that the reproducing property is verified. If $f \in \mathscr{H}^2$ then

$$\langle f(s), \zeta(s + \overline{a}) \rangle = \sum_{n=1}^{\infty} a_n \overline{\frac{1}{n^{\overline{a}}}} = \sum_{n=1}^{\infty} \frac{a_n}{n^a} = f(a).$$

☐

Proposition 3.21 (Carlson's identity)**.** *If $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^2$, then*

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma + it}} \right|^2 dt,$$

for any $\sigma > \sigma_u$. This identity is also valid for $f \in \mathscr{H}^\infty$ and any $\sigma > 0$.

Proof. For $\sigma > \sigma_u$, we have that

$$|f(\sigma+it)|^2 = \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+it}} \sum_{m=1}^{\infty} \frac{\overline{a_m}}{m^{\sigma-it}} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + \sum_{n\neq m} \frac{\overline{a_m}a_n}{n^\sigma m^\sigma}\left(\frac{m}{n}\right)^{it},$$

and since the series is absolutely and uniformly convergent for all t, we can integrate term by term:

$$\int_{-T}^{T} |f(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + \sum_{n\neq m} \frac{\overline{a_m}a_n}{n^\sigma m^\sigma} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt.$$

From the proof of Proposition 2.38, using equation (2.21) we know that

$$\int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt = \frac{\sin(T\log(m/n))}{T\log(m/n)},$$

which is bounded for all m, n and T. Thus, the double series

$$\sum_{n\neq m} \frac{\overline{a_m}a_n}{n^\sigma m^\sigma} \frac{\sin(T\log(m/n))}{T\log(m/n)}$$

converges uniformly with respect to T. Hence we can take the limit term by term and obtain the result:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + \sum_{n\neq m} \frac{\overline{a_m}a_n}{n^\sigma m^\sigma} \lim_{T\to\infty} \frac{1}{2T} \frac{\sin(T\log(m/n))}{T\log(m/n)} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}.$$

The proof for $f \in \mathscr{H}^\infty$ follows the same steps. □

Observation 3.22. *The norm of a function $f \in \mathscr{H}^2$ is computed using the coefficients of the series, but can also expressed directly from the function itself if $\sigma_u(f) < 0$. Indeed, we can use Carlson's identity with $\sigma = 0$ and then*

$$\|f\|_2^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^2 dt.$$

This is analogous to the fact that the norm of a function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ in $H^2(\mathbb{D})$ can be computed both as

$$\|f\|_2^2 = \lim_{r\to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

and

$$\|f\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2.$$

Theorem 3.23. *We have the inclusion $\mathscr{H}^\infty \subset \mathscr{H}^2$. And in particular, if $f \in \mathscr{H}^\infty$ then*

$$\|f\|_2 \leq \|f\|_\infty.$$

Proof. Let $f \in \mathscr{H}^\infty$. Then for any $\sigma > 0$, f converges uniformly in $\mathbb{C}_\sigma$, and then for any $\epsilon > 0$ there exists $N_0 \geq 1$ such that for any $N \geq N_0$

$$\sup_{\Re s = \sigma} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} - f(s) \right| < \epsilon,$$

and in consequence

$$\sup_{\Re s = \sigma} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \leq \|f\|_\infty + \epsilon.$$

Using Carlson's identity (Proposition 3.21):

$$\begin{aligned} \left(\sum_{n=1}^{N} \left| \frac{a_n}{n^\sigma} \right|^2 \right)^{1/2} &= \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} \frac{a_n}{n^{\sigma+it}} \right|^2 dt \right)^{1/2} \\ &\leq \left(\lim_{T\to\infty} \frac{1}{2T} (\|f\|_\infty + \epsilon)^2 2T \right)^{1/2} \leq \|f\|_\infty + \epsilon. \end{aligned}$$

Finally, letting σ and ϵ tend to zero, and N to infinity, the inequality is proved. Hence, $f \in \mathscr{H}^2$, and consequently $\mathscr{H}^\infty \subset \mathscr{H}^2$. □

Observation 3.24. *The inclusion is strict. We have seen in Observation 3.18 that the series* $f(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} \frac{1}{n^s}$ *has* $\sigma_a(f) = 1/2$. *Since all its coefficients are non-negative, using Proposition 2.28, we get that* $\sigma_c(f) = 1/2$. *Hence* f *cannot be in* $\mathscr{H}^\infty$, *because if it was, by Bohr's Theorem we would have* $\sigma_c(f) \leq 0$.

Observation 3.25. *Notice the analogy with Hardy spaces in the unit disk, where* $H^\infty(\mathbb{D}) \subsetneq H^2(\mathbb{D})$.

For the next proposition and later in the chapter, we need the following theorem that we will assume without proof:

Theorem 3.26 (Kronecker)**.** *If* $p_1, \ldots, p_N$ *are prime numbers, then* $\left\{(p_1^{it}, \ldots, p_N^{it}) : t \in \mathbb{R}\right\}$ *is dense in* $\mathbb{T}^N$.

One of the most interesting properties of Dirichlet series that we have yet to mention is that they are *almost* periodic in the vertical direction. In fact, they are *uniformly almost* periodic, which is a stronger property, but as a brief insight we just prove the following weaker version.

Proposition 3.27. *If* $f \in \mathscr{H}^2$ *and* $\sigma_u(f) < 0$, *then* f *is* ***almost-periodic*** *in the vertical direction. That is, for any* $\epsilon > 0$ *there exists a* $T > 0$ *such that*

$$\|f(s + iT) - f(s)\|_2 < \epsilon. \tag{3.4}$$

Proof. Notice that it is enough to prove the result for Dirichlet polynomials, as we can approximate uniformly any function $g \in \mathscr{H}^2$ by Dirichlet polynomials. Let $f(s) = \sum_{n=1}^{N} a_n n^{-s} \in \mathscr{H}^2$ be a Dirichlet polynomial.

By Kronecker's Theorem, we have that if $p_1, \ldots, p_{\pi(N)}$ are prime numbers, then for any $\epsilon > 0$ there exists $T > 0$ such that

$$d(T\log(p_i), 2\pi\mathbb{Z}) < \epsilon \text{ for } i = 1, \ldots, \pi(N).$$

So if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, with $\alpha_i \in \mathbb{N}$, we have

$$|n^{-iT} - 1| = |e^{-iT \log n} - 1| = |\exp\left(-iT(\alpha_1 \log p_1 + \cdots + \alpha_k \log p_k)\right) - 1|.$$

Using the Taylor expansion for the exponential and that $T \log p_i$ is arbitrarily close to $2\pi\mathbb{Z}$, we have for any $i = 1, \ldots, k$

$$|1 - e^{-iT\alpha_i \log p_i}| = |i\alpha_i\epsilon + \mathcal{O}(\epsilon^2)| = \alpha_i\epsilon + \mathcal{O}(\epsilon^2).$$

Hence, for each $1 \leq n \leq N$ there exists $K(n) \geq 0$ such that

$$\begin{aligned}|n^{-iT} - 1| &= |e^{-iT(\alpha_1 \log p_1 + \cdots + \alpha_k \log p_k)} - 1| = |e^{-iT(\alpha_1 \log p_1)} \cdots e^{(-iT(\alpha_k \log p_k)} - 1| \\ &= |(1 + \alpha_1\epsilon + \mathcal{O}(\epsilon^2)) \cdots (1 + \alpha_k\epsilon + \mathcal{O}(\epsilon^2)) - 1| = (\alpha_1 + \cdots + \alpha_k)\epsilon + \mathcal{O}(\epsilon^2) \leq K(n)\epsilon.\end{aligned}$$

Moreover, for any $1 \leq n \leq N$ we have that $|n^{-iT} - 1| \leq K\epsilon$ with $K = \max_{1\leq n\leq N} K(n)$. So using Carlson's identity, we have

$$\begin{aligned}\|f(s + iT) - f(s)\|_2 &= \left(\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \left|\sum_{n=1}^{N} \frac{a_n}{n^{i(t+T)}} - \sum_{n=1}^{N} \frac{a_n}{n^{it}}\right|^2 dt\right)^{1/2} \\ &= \left(\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \left|\sum_{n=1}^{N} \frac{a_n}{n^{it}} \left(n^{-iT} - 1\right)\right|^2 dt\right)^{1/2} \\ &\leq K\epsilon \left(\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \left|\sum_{n=1}^{N} \frac{a_n}{n^{it}}\right|^2 dt\right)^{1/2} \lesssim \epsilon,\end{aligned}$$

which proves the result. □

3.3 Bohr's point of view

Now, we will describe Bohr's transformation, that he proposed around 1913 [5, 6]. He noticed that, using the prime decomposition of natural numbers in a Dirichlet series, each prime behaved as an independent variable. So, Bohr proposed a mapping that lets us transform a Dirichlet series into a power series in an infinite number of variables and vice-versa. To make this transformation explicit, we will firstly need a few definitions.

Definition 3.28. *A* ***multi-index*** $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots)$ *is an element of* $\mathbb{N}^{\mathbb{N}}$*, which are the sequences of non-negative integers that are* 0 *except for a finite number of entries. We will write* $|\alpha| := \sum_{n=1}^{\infty} \alpha_n$.

We also define the product by a scalar $k \in \mathbb{N}$ *as* $k\alpha := (k\alpha_1, \ldots, k\alpha_n, 0, 0, \ldots)$*, and the addition of multi-indices* $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n, 0, 0, \ldots)$.

If $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\infty}$ *and* α *is a multi-index, we write* $z^{\alpha} := \prod_{n=1}^{\infty} z_n^{\alpha_n}$*. We will also define* $|z| = (|z_1|, |z_2|, \ldots)$ *and* $|z|^{\alpha} = \prod_{n=1}^{\infty} |z_n|^{\alpha_n}$.

With a similar notation, if $n \in \mathbb{N}$*, then its factorization in prime numbers is unique and hence there exists a multi-index* $\alpha = \alpha(n)$ *such that*

$$\mathfrak{p}^{\alpha} := \prod_{n=1}^{\infty} p_n^{\alpha_n} = p_1^{\alpha_1} \cdots p_{\pi(n)}^{\alpha_{\pi(n)}} = n,$$

where $\pi(n)$ *is the function that counts the number of primes less than or equal to* n.

Definition 3.29. *A* ***power series in infinitely many variables*** $z = (z_1, z_2, \dots) \in \mathbb{C}^\infty$ *is a formal object of the form*

$$P(z) = \sum_{\alpha \in \mathbb{N}^\mathbb{N}} c_\alpha z^\alpha, \quad c_\alpha \in \mathbb{C}.$$

We will define $\mathcal{P}$ as the space of power series P in an arbitrary (may be infinite) number of variables.

Observation 3.30. *Notice that since a multi-index only has a finite number of non-zero entries, each of the terms in the power series contains a finite number of variables.*

Now let $P(z) = \sum_{\alpha \in \mathbb{N}^\mathbb{N}} c_\alpha z^\alpha$ be a power series in infinitely many variables. Considering the decomposition of each integer n into primes, we can associate each coefficient c_α in the power series to a unique $n = \mathfrak{p}^\alpha$, and in turn write $c_\alpha = a_{\mathfrak{p}^\alpha} = a_n$. Hence we can write the correspondence, which we will call Bohr's transformation $\mathfrak{B}$, as

$$\begin{aligned} \mathfrak{B}\colon \quad & \mathcal{P} \xrightarrow{\qquad\qquad\qquad} \mathcal{D} \\ & \sum_{\alpha \in \mathbb{N}^\mathbb{N}} c_\alpha z^\alpha \xmapsto{c_\alpha = a_{\mathfrak{p}^\alpha} = a_n} \sum_{n=1}^{\infty} a_n n^{-s}, \end{aligned}$$

Roughly, this transformation can be seen as evaluating the polynomial in the sequence $z = (1/p_1^s, 1/p_2^s, 1/p_3^s, \dots) = 1/\mathfrak{p}^s \in \mathbb{C}^\infty$:

$$\mathfrak{B}(P)(z) = P(1/\mathfrak{p}^s) = \sum_{\alpha \in \mathbb{N}^\mathbb{N}} c_\alpha \left(\frac{1}{\mathfrak{p}^s}\right)^\alpha = \sum_{n=1}^{\infty} a_n n^{-s} = f(s).$$

Note that if $s \in \mathbb{C}_0$, then $z = 1/\mathfrak{p}^s \in \mathbb{D}^\infty$. We will refer to the inverse correspondence as Bohr's lift $\mathfrak{L} = \mathfrak{B}^{-1}$:

$$\begin{aligned} \mathfrak{L}\colon \quad & \mathcal{D} \xrightarrow{\qquad\qquad\qquad} \mathcal{P} \\ & \sum_{n=1}^{\infty} a_n n^{-s} \xmapsto{c_\alpha = a_{\mathfrak{p}^\alpha} = a_n} \sum_{\alpha \in \mathbb{N}^\mathbb{N}} c_\alpha z^\alpha. \end{aligned}$$

This is the main idea behind Bohr's transformation $\mathfrak{B}$ and lift $\mathfrak{L}$. Notice that we have treated $P(z)$ and $f(s)$ as formal objects, without explicitly detailing where both converge and make sense. Our aim now is to give a precise statement of this approach. We will not study power series on their own, but the image of a Dirichlet series by Bohr's lift, hence we will only need to know where the image of a Dirichlet series converges.

In particular, we want to give isometric identifications for $\mathscr{H}^2$ and $\mathscr{H}^\infty$. To do so, we will need to define a couple of new function spaces and some propositions.

Proposition 3.31. *If $f \in \mathscr{H}^2$, then its image by Bohr's lift $\mathfrak{L}f$ is well defined for $z \in \mathbb{D}^\infty \cap \ell^2$.*

Proof. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $z \in \mathbb{D}^\infty \cap \ell^2$. Using Cauchy-Schwarz's inequality, we have that

$$\begin{aligned} |\mathfrak{L}f(z)|^2 &= \left| \sum_{\alpha \in \mathbb{N}^\mathbb{N}} a_{\mathfrak{p}^\alpha} z^\alpha \right|^2 \le \left(\sum_{\alpha \in \mathbb{N}^\mathbb{N}} |a_{\mathfrak{p}^\alpha}|^2 \right) \left(\sum_{\alpha \in \mathbb{N}^\mathbb{N}} |z|^{2\alpha} \right) \\ &= \left(\sum_{n=1}^{\infty} |a_n|^2 \right) \left(\sum_{\alpha \in \mathbb{N}^\mathbb{N}} |z|^{2\alpha} \right) = \|f\|_2^2 \left(\sum_{\alpha \in \mathbb{N}^\mathbb{N}} |z|^{2\alpha} \right) \end{aligned} \tag{3.5}$$

Next, observe that, wherever the following product is inconditionally convergent, we have the equality

$$\prod_{i=1}^{\infty}\left(\sum_{n=1}^{\infty}|z_i|^{2n}\right) = (1+|z_1|^2+|z_1|^4+\dots)(1+|z_2|^2+|z_2|^4+\dots)\dots = \sum_{\alpha\in\mathbb{N}^{\mathbb{N}}}|z|^{2\alpha}$$

since, by a rearrangement of the left-hand side, we can get any product $\prod |z_i|^{2\alpha_i} = |z|^{2\alpha}$.

Now, given that $|z_i| < 1$, we can write explicitly the result of the geometric series, so

$$\prod_{i=1}^{\infty}\left(\sum_{n=1}^{\infty}|z_i|^{2n}\right) = \prod_{i=1}^{\infty}(1-|z_i|^2)^{-1},$$

and using this equality in equation (3.5)

$$|\mathfrak{L}f(z)|^2 \leq \|f\|_2^2\prod_{i=1}^{\infty}(1-|z_i|^2)^{-1}$$

This infinite product converges since $\sum_{m=1}^{\infty}|z_m|^2 < +\infty$. Hence, the point evaluation of $\mathfrak{L}f(z)$ is bounded if $z \in \mathbb{D}^{\infty}\cap\ell^2$. $\square$

Definition 3.32. *We define $H^2(\mathbb{D}^{\infty}) \subset \mathcal{P}$ as the function space of power series P in $\mathbb{D}^{\infty}$*

$$P(z) = \sum_{\alpha\in\mathbb{N}^{\mathbb{N}}} a_{\mathfrak{p}^{\alpha}}z^{\alpha} \quad \text{with} \quad \|P\|^2_{H^2(\mathbb{D}^{\infty})} := \sum_{n=1}^{\infty}|a_n|^2 < +\infty.$$

Observation 3.33. *In fact, if $\mathfrak{L}f$ is finitely bounded, which we will define below, then the point evaluation is well defined in a bigger set, namely $\mathbb{D}^{\infty}\cap c_0$. See [12] and the references therein for the proof.*

Definition 3.34. *We define the m-th section for $z = (z_1, z_2, \dots) \in \mathbb{D}^{\infty}$ as*

$$z^{(m)} = (z_1, z_2, \dots, z_m, 0, \dots).$$

Then we say that $\mathfrak{L}f$ is ***finitely bounded*** *in $\mathbb{D}^{\infty}$ if $\mathfrak{L}(z^{(m)})$ converges absolutely on $\mathbb{D}^{\infty}$ for each $m \geq 1$ and there exists some constant $C \geq 0$, independent of m, such that $|\mathfrak{L}(z^{(m)})| \leq C$.*

The set of functions $\mathfrak{L}f$ that are finitely bounded is defined as $H^{\infty}(\mathbb{D}^{\infty})$, and the point evaluation is defined as

$$\mathfrak{L}f(z) = \lim_{m\to\infty}\mathfrak{L}f(z^{(m)}),$$

where the limit makes sense for $z \in \mathbb{D}^{\infty}\cap c_0$ (again, see [12] and the references therein). We will also define the norm

$$\|\mathfrak{L}f\|_{H^{\infty}(\mathbb{D}^{\infty})} = \sup_{\substack{z\in\mathbb{D}^{\infty}\\ m\geq 1}}|\mathfrak{L}f(z^{(m)})|.$$

Definition 3.35. *If $f \in L^2(\mathbb{T}^{\infty}, \rho)$ and $\alpha \in \mathbb{Z}^{\mathbb{N}}$, then the α-th* ***Fourier coefficient*** *for f is defined as*

$$\hat{f}(\alpha) = \int_{\mathbb{T}^{\infty}} f(z)z^{-\alpha}d\rho(z).$$

Recall that ρ is the probability measure in $\mathbb{T}^{\infty}$.

The function f can be uniquely determined by its Fourier coefficients and its ***Fourier series***

$$\sum_{\alpha\in\mathbb{N}^{\mathbb{N}}} \hat{f}(\alpha) z^{\alpha}.$$

We define the Hardy space $H^2(\mathbb{T}^\infty)$ as the subset of $L^2(\mathbb{T}^\infty)$ which have null Fourier coefficients outside the narrow cone, i.e.

$$H^2(\mathbb{T}^\infty) = \{f \in L^2(\mathbb{T}^\infty) : \hat{f}(\alpha) = 0 \quad \text{if} \quad \alpha \notin \mathbb{N}^{\mathbb{N}}\},$$

with norm

$$\|f\|^2_{H^2(\mathbb{T}^\infty)} := \int_{\mathbb{T}^\infty} |f(z)|^2 d\rho(z).$$

Similarly, we define $H^\infty(\mathbb{T}^\infty)$ as the subset of $L^\infty(\mathbb{T}^\infty)$

$$H^\infty(\mathbb{T}^\infty) = \{f \in L^\infty(\mathbb{T}^\infty) : \hat{f}(\alpha) = 0 \quad \text{if} \quad \alpha \notin \mathbb{N}^{\mathbb{N}}\},$$

with norm

$$\|f\|_{H^\infty(\mathbb{T}^\infty)} := \lim_{p\to\infty} \left(\int_{\mathbb{T}^\infty} |f(z)|^p d\rho(z) \right)^{1/p}.$$

Now we can give the triple identification which will be useful at the end of the chapter. Proofs of this result can be found in [11], in particular Theorem 3.8 and 5.1, and in the paper [9, Th 11.2].

Theorem 3.36. *There exist three unique isometric, linear bijections between the spaces $\mathscr{H}^2$, $H^2(\mathbb{T}^\infty)$ and $H^2(\mathbb{D}^\infty)$ by means of the identification of Dirichlet, Fourier and power series coefficients, respectively:*

$$\begin{tikzcd}
\mathscr{H}^2 \arrow[r, "\mathfrak{P}"] \arrow[rr, bend left, "\mathfrak{L}"] & H^2(\mathbb{T}^\infty) \arrow[r, "\mathfrak{E}"] & H^2(\mathbb{D}^\infty) \\
\sum_{n=1}^{\infty} a_n n^{-s} \arrow[r, mapsto, "a_n=a_{\mathfrak{p}^\alpha}=\hat{f}(\alpha)"] \arrow[rr, mapsto, bend right, "a_n=a_{\mathfrak{p}^\alpha}=c_\alpha"'] & \sum_{\alpha\in\mathbb{N}^{\mathbb{N}}} \hat{f}(\alpha) z^\alpha \arrow[r, mapsto, "\hat{f}(\alpha)=c_\alpha"] & \sum_{\alpha\in\mathbb{N}^{\mathbb{N}}} c_\alpha z^\alpha
\end{tikzcd}$$

Moreover, the isometry holds for the corresponding subspaces $\mathscr{H}^\infty$, $H^\infty(\mathbb{T}^\infty)$ and $H^\infty(\mathbb{D}^\infty)$, that is:

$$\begin{tikzcd}
\mathscr{H}^\infty \arrow[r, "\mathfrak{P}"] \arrow[rr, bend left, "\mathfrak{L}"] & H^\infty(\mathbb{T}^\infty) \arrow[r, "\mathfrak{E}"] & H^\infty(\mathbb{D}^\infty) \\
\sum_{n=1}^{\infty} a_n n^{-s} \arrow[r, mapsto, "a_n=a_{\mathfrak{p}^\alpha}=\hat{f}(\alpha)"] \arrow[rr, mapsto, bend right, "a_n=a_{\mathfrak{p}^\alpha}=c_\alpha"'] & \sum_{\alpha\in\mathbb{N}^{\mathbb{N}}} \hat{f}(\alpha) z^\alpha \arrow[r, mapsto, "\hat{f}(\alpha)=c_\alpha"] & \sum_{\alpha\in\mathbb{N}^{\mathbb{N}}} c_\alpha z^\alpha
\end{tikzcd}$$

Observation 3.37. *The proof of this theorem is outside of the scope of this work due to its length, but we can give some rough ideas for the proof. A known fact in harmonic analysis is that the Hardy space $H^\infty(\mathbb{D})$ can be identified isometrically with $H^\infty(\mathbb{T})$ via the identification of monomial and Fourier coefficients. This can be extended to finite dimensions $H^\infty(\mathbb{D}^N) \equiv H^\infty(\mathbb{T}^N)$ and finally to infinite dimensions as is the statement of the theorem. Again, see [11] for details.*

For the isometries $\mathscr{H}^\infty \equiv H^\infty(\mathbb{D}^\infty)$ and $\mathscr{H}^2 \equiv H^2(\mathbb{T}^\infty)$, we can prove the two following theorems which give the isometry for finite dimensions, and then the result for infinite dimensions follows from the fact that $\mathscr{H}^\infty$ and $\mathscr{H}^2$ are completion spaces.

Observation 3.38. *Given that the isometries hold, to simplify the notation we will simply write $\|\cdot\|_{H^2(\mathbb{T}^\infty)}$ and $\|\cdot\|_{H^2(\mathbb{D}^\infty)}$ as $\|\cdot\|_2$. Similarly we will also write $\|\cdot\|_{H^\infty(\mathbb{T}^\infty)}$ and $\|\cdot\|_{H^\infty(\mathbb{D}^\infty)}$ as $\|\cdot\|_\infty$.*

Theorem 3.39 (Bohr's fundamental lemma). *If $f(s) = \sum_{n=1}^N a_n n^{-s}$ is a Dirichlet polynomial, then*

$$\|f\|_\infty = \sup_{s\in\mathbb{C}_0} \left| \sum_{n=1}^N \frac{a_n}{n^s} \right| = \sup_{z\in\mathbb{D}^{\pi(N)}} \left| \sum_{\substack{\alpha\in\mathbb{N}^{\pi(N)} \\ 1\leq \mathfrak{p}^\alpha \leq N}} a_{\mathfrak{p}^\alpha} z^\alpha \right| = \|\mathfrak{L}(f)\|_\infty .$$

For the proof of this theorem we will firstly need a couple of previous lemmas, namely the version of Kronecker's theorem that we have seen in the last section (see Theorem 3.26), and the distinguished maximum principle for analytic functions in $\mathbb{D}^N$.

Lemma 3.40 (Distinguished maximum principle). *Let f be a continuous function in $\overline{\mathbb{D}}^N$ and holomorphic in $\mathbb{D}^N$. Then*

$$\sup_{z\in\mathbb{D}^N} |f(z)| = \sup_{z\in\mathbb{T}^N} |f(z)|.$$

Proof. We will prove it by induction. The case for $N = 1$ is a result of de maximum modulus principle for holomorphic functions in one complex variable.

Suppose the result is true for N, and that f is a continuous function in $\overline{\mathbb{D}}^{N+1}$ and holomorphic in $\mathbb{D}^{N+1}$. Fix $w \in \mathbb{T}$ and consider the function $f_w : \mathbb{D}^N \to \mathbb{C}$ defined as $f_w(z) = f(z, w)$. Notice that the function f_w is continuous in $\overline{\mathbb{D}}^N$, but it may not be holomorphic since $w \in \mathbb{T}$. But since f is uniformly continuous on $\overline{\mathbb{D}}^{N+1}$, then f_w is the uniform limit on $\mathbb{D}^N$ of the sequence $f_N(z) = f(u, \frac{N-1}{N} w)$. Hence f_w is holomorphic in $\mathbb{D}^N$ and continuous in $\overline{\mathbb{D}}^N$. Applying the distinguised maximum principle to f_w:

$$\begin{aligned}\sup_{z\in\mathbb{D}^{N+1}} |f(z)| &= \sup_{u\in\mathbb{D}^N} \left(\sup_{z_{N+1}\in\mathbb{D}} |f(u, z_{N+1})| \right) = \sup_{u\in\mathbb{D}^N} \left(\sup_{w_{N+1}\in\mathbb{T}} |f(u, w_{N+1})| \right) \\ &= \sup_{w_{N+1}\in\mathbb{T}} \left(\sup_{u\in\mathbb{D}^N} |f(u, w_{N+1})| \right) = \sup_{w_{N+1}\in\mathbb{T}} \left(\sup_{u\in\mathbb{T}^N} |f(u, w_{N+1})| \right) = \sup_{w\in\mathbb{T}^{N+1}} |f(w)|.\end{aligned}$$

□

Proof (Bohr's fundamental lemma). Notice that using Proposition 2.37, we have

$$\|f\|_\infty = \sup_{s\in\mathbb{C}_0}\left|\sum_{n=1}^{N}\frac{a_n}{n^s}\right| = \sup_{t\in\mathbb{R}}\left|\sum_{n=1}^{N}a_n n^{it}\right|.$$

By Kronecker's theorem (see Theorem 3.26), we have

$$\sup_{t\in\mathbb{R}}\left|\sum_{n=1}^{N}a_n n^{it}\right| = \sup_{t\in\mathbb{R}}\left|\sum_{\substack{\alpha\in\mathbb{N}^{\pi(N)}\\ 1\le\mathfrak{p}^\alpha\le N}} a_{p^\alpha}(p_1^{it})^{\alpha_1}\cdots(p_{\pi(N)}^{it})^{\alpha_{\pi(N)}}\right| = \sup_{z\in\mathbb{T}^N}\left|\sum_{\substack{\alpha\in\mathbb{N}^{\pi(N)}\\ 1\le\mathfrak{p}^\alpha\le N}} a_{\mathfrak{p}^\alpha}z^\alpha\right|.$$

And finally, using Lemma 3.40

$$\sup_{z\in\mathbb{T}^N}\left|\sum_{\alpha\in\mathbb{N}^N} a_{\mathfrak{p}^\alpha}z^\alpha\right| = \sup_{z\in\mathbb{D}^N}\left|\sum_{\alpha\in\mathbb{N}^N} a_{\mathfrak{p}^\alpha}z^\alpha\right| = \|\mathfrak{L}(f)\|_\infty,$$

which proves the equality in the statement of the theorem. □

Observation 3.41. *Since $\mathscr{H}^\infty$ is the completion space of polynomials under the $\|\cdot\|_\infty$ norm, using Bohr's fundamental lemma we get that $\mathscr{H}^\infty$ and $H^\infty(\mathbb{D}^\infty)$ are isometric.*

Theorem 3.42. *If $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ is a Dirichlet polynomial, then*

$$\|f\|_2 = \|\mathfrak{P}(f)\|_2 = \left(\int_{\mathbb{T}^{\pi(N)}} |\mathfrak{P}(f)(z)|^2 dz\right)^{1/2}.$$

Proof. We have

$$\mathfrak{P}(f)(z) = \sum_{\substack{\alpha\in\mathbb{N}^{\pi(N)}\\ 1\le\mathfrak{p}^\alpha\le N}} a_{\mathfrak{p}^\alpha}z^\alpha.$$

Its norm in $H^2(\mathbb{T}^{\pi(N)})$ is

$$\begin{aligned}\|\mathfrak{P}(f)\|_2^2 &= \int\limits_{\mathbb{T}^{\pi(N)}} |\mathfrak{P}(f)(z)|^2 dz = \int\limits_{\mathbb{T}^{\pi(N)}}\left|\sum_{\substack{\alpha\in\mathbb{N}^{\pi(N)}\\ 1\le\mathfrak{p}^\alpha\le N}} a_{\mathfrak{p}^\alpha}z^\alpha\right|^2 dz\\ &= \int\limits_{\mathbb{T}^{\pi(N)}}\left(\sum_{\substack{\alpha\in\mathbb{N}^{\pi(N)}\\ 1\le\mathfrak{p}^\alpha\le N}} |a_{\mathfrak{p}^\alpha}|^2 z^\alpha\overline{z}^\alpha + \sum_{\substack{\alpha\neq\beta\\ 1\le\mathfrak{p}^\alpha,\mathfrak{p}^\beta\le N}} a_{\mathfrak{p}^\alpha}\overline{a_{\mathfrak{p}^\beta}}z^\alpha\overline{z}^\beta\right)dz = \sum_{n=1}^{N}|a_n|^2 = \|f\|_2^2,\end{aligned}$$

where we have used that, by Fubini's theorem,

$$\int_{\mathbb{T}^{\pi(N)}} z^\alpha\overline{z}^\beta dz = \begin{cases}1, & \text{if } \alpha=\beta\\ 0, & \text{if } \alpha\neq\beta.\end{cases}$$

□

3.4 Multipliers of $\mathscr{H}^2$

As we will see from the following theorem, the space $\mathscr{H}^2$ is not a Banach algebra, that is, the product of two functions in $\mathscr{H}^2$ might not be an element of $\mathscr{H}^2$. This leads to the study of the multipliers of $\mathscr{H}^2$, which are defined as follows:

Definition 3.43. *We say that an analytic function m in $\mathbb{C}_{1/2}$ is a* ***multiplier*** *of $\mathscr{H}^2$ if $mf \in \mathscr{H}^2$ for any $f \in \mathscr{H}^2$. We will denote the set of all multipliers of $\mathscr{H}^2$ as $\mathscr{M}(\mathscr{H}^2)$.*

We define the operator norm for $m \in \mathscr{M}(\mathscr{H}^2)$ as

$$\|m\|_{\mathscr{M}} = \sup_{f\in\mathscr{H}^2} \{\|mf\|_2 : \|f\|_2 = 1\}.$$

The problem of characterising $\mathscr{M}(\mathscr{H}^2)$ was solved in a seminal paper by Hedenmalm, Lindqvist and Seip in 1997 in *Duke Mathematical Journal* [12]. This paper revived the analytic theory of Dirichlet series and started a renaissance of this area of study. It used heavily once again Bohr's lift and the correspondence between Dirichlet series and power series in an arbitrary number of variables. The result can be summed up in the following theorem.

This result is analogous to Schur's theorem for Hardy spaces in the unit disk, that is $\mathscr{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$. In fact, the proof follows similar steps but it is more complex. Though notice the difference that in our case, the multipliers are defined in a bigger set than the functions in $\mathscr{H}^2$, namely $\mathbb{C}_0$ instead of $\mathbb{C}_{1/2}$.

Theorem 3.44 (Hedenmalm-Lindqvist-Seip)**.** *Let m be an analytic function in $\mathbb{C}_{1/2}$. Then m is a multiplier of $\mathscr{H}^2$ if and only if $m \in \mathscr{H}^\infty$. In other words,*

$$\mathscr{M}(\mathscr{H}^2) = \mathscr{H}^\infty.$$

Moreover, if $m \in \mathscr{M}(\mathscr{H}^2)$ then $\|m\|_{\mathscr{M}} = \|m\|_\infty$.

Proof. • We will firstly prove the inclusion $\mathscr{H}^\infty \subset \mathscr{M}(\mathscr{H}^2)$ and that $\|m\|_{\mathscr{M}} \leq \|m\|_\infty$.

Let $m(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathscr{H}^\infty$ and $f(s) = \sum_{n=1}^\infty b_n n^{-s} \in \mathscr{H}^2$. We want to see that the product $(mf)(s) = \sum_{n=1}^\infty c_n n^{-s}$ with $c_n = \sum_{jk=n} a_j b_k$ is in $\mathscr{H}^2$.

For $N \geq 1$, we define the polynomial $f_N(s) = \sum_{n=1}^N b_n n^{-s}$ and the product

$$(mf_N)(s) = \sum_{n=1}^\infty \frac{c_n^{(N)}}{n^s} \quad \text{with} \quad c_n^{(N)} = \sum_{\substack{jk=n \\ k\leq N}} a_j b_k.$$

Applying Carlson's identity (see Proposition 3.21) for $\sigma > 0$ to the function $mf_N(s) \in \mathscr{H}^\infty$,

$$\begin{aligned}\sum_{n=1}^\infty \frac{|c_n^{(N)}|^2}{n^{2\sigma}} &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T |m(\sigma+it)|^2 |f_N(\sigma+it)|^2 dt \\ &\leq \|m\|_\infty^2 \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T |f_N(\sigma+it)|^2 dt = \|m\|_\infty^2 \sum_{n=1}^N \frac{|b_n|^2}{n^{2\sigma}}\end{aligned}$$

and letting $\sigma \to 0$, using Fatou's lemma, we get

$$\sum_{n=1}^{\infty} |c_n^{(N)}|^2 \leq \|m\|_{\infty}^2 \, \|f\|_2^2 \,. \tag{3.6}$$

Now observing that

$$c_n^{(N)} = \sum_{\substack{jk=n \\ k \leq N}} a_j b_k \xrightarrow{N \to \infty} \sum_{jk=n} a_j b_k = c_n.$$

and letting N tend to infinity in (3.6) we get, once again by Fatou's lemma,

$$\|mf\|_2^2 = \sum_{n=1}^{\infty} |c_n|^2 \leq \|m\|_{\infty}^2 \, \|f\|_2^2 \,,$$

which implies that $mf \in \mathscr{H}^2$ and $\|m\|_{\mathscr{M}} \leq \|m\|_{\infty}$.

- Now we will see that $\mathscr{M}(\mathscr{H}^2) \subset \mathscr{H}^{\infty}$ and that $\|m\|_{\infty} \leq \|m\|_{\mathscr{M}}$. Recall that by Theorem 3.36, we have the following commutative isometric diagram:

$$\begin{tikzcd}
\mathscr{H}^2 \arrow[r, "\mathfrak{P}"] \arrow[dr, "\mathfrak{L}"'] & H^2(\mathbb{T}^{\infty}) \arrow[d, "\mathfrak{C}"] \\
 & H^2(\mathbb{D}^{\infty})
\end{tikzcd}$$

Let $m \in \mathscr{M}(\mathscr{H}^2)$. Since $1 \in \mathscr{H}^2$, we have that $1 \cdot m = m \in \mathscr{H}^2$. So m can be written as a convergent Dirichlet series in $\mathbb{C}_{1/2}$:

$$m(s) = \sum_{n=1}^{\infty} b_n n^{-s} \quad \text{with} \quad \sum_{n=1}^{\infty} |b_n|^2 < \infty.$$

We want to see that $\mathfrak{P}(m^j) = (\mathfrak{P}m)^j \in L^1(\mathbb{T}^{\infty}, \rho)$ for $j \geq 1$. To prove it, we will show the two following claims.

– $\underline{\left\|m^j\right\|_2 \leq \|m\|_{\mathscr{M}}^j}$: We will prove it by induction. By the definition of the operator norm, for $f \in \mathscr{H}^2$ we have that

$$\|mf\|_2 \leq \|m\|_{\mathscr{M}} \, \|f\|_2 \,. \tag{3.7}$$

For $j = 1$ we consider $f = 1$ in equation (3.7) and get $\|m\|_2 \leq \|m\|_{\mathscr{M}}$. Now suppose that $\left\|m^j\right\|_2 \leq \|m\|_{\mathscr{M}}^j$ for some $j \geq 1$, in particular we have that $m^j \in \mathscr{H}^2$. So $mm^j \in \mathscr{H}^2$ and considering $f = m^j$ in (3.7) we get that $\left\|m^{j+1}\right\|_2 \leq \|m\|_{\mathscr{M}} \left\|m^j\right\|_2 \leq \|m\|_{\mathscr{M}}^{j+1}$.

– $\underline{\mathfrak{P}(mf) = \mathfrak{P}m\mathfrak{P}f}$ in $L^1(\mathbb{T}^{\infty}, \rho)$ if $f \in \mathscr{H}^2$. Let $f \in \mathscr{H}^2$, then $\mathfrak{P}m, \mathfrak{P}f \in H^2(\mathbb{T}^{\infty})$. By the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\mathfrak{P}m\mathfrak{P}f\|_1 &= \int\limits_{\mathbb{T}^{\infty}} |\mathfrak{P}m\mathfrak{P}f| d\rho \leq \left(\int_{\mathbb{T}^{\infty}} |\mathfrak{P}m|^2 d\rho\right)^{1/2} \left(\int_{\mathbb{T}^{\infty}} |\mathfrak{P}f|^2 d\rho\right)^{1/2} \\
&= \|\mathfrak{P}m\|_2 \, \|\mathfrak{P}f\|_2 < \infty,
\end{aligned}$$

hence $\mathfrak{P}m\mathfrak{P}f \in L^1(\mathbb{T}^\infty, \rho)$. Also, since m is a multiplier, we have that $mf \in \mathscr{H}^2$ so $\mathfrak{P}(mf) \in H^2(\mathbb{T}^\infty) \subset L^1(\mathbb{T}^\infty, \rho)$.
Since $\mathfrak{P}$ is a linear and continuous operator, we only need to prove the equality for a multiplier and an element of a basis in $\mathscr{H}^2$. A basis in $\mathscr{H}^2$ is $\{n^{-s}\}_n$, so let $f(s) = k^{-s}$ for some $k \geq 1$. Then

$$(mf)(s) = \left(\sum_{n=1}^{\infty} b_n n^{-s}\right)(k^{-s}) = \sum_{n=1}^{\infty} \frac{b_n}{(nk)^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where we define $a_n = b_{n/k}$ if n is divisible by k and $a_n = 0$ otherwise. Then $(\mathfrak{P}m)(z) = \sum b_n z^{\alpha(n)}$ and $(\mathfrak{P}f)(z) = z^{\alpha(k)}$. Hence

$$(\mathfrak{P}m)(\mathfrak{P}f)(z) = z^{\alpha(k)} \sum b_n z^{\alpha(n)} = \sum b_n z^{\alpha(n)+\alpha(k)}.$$

On the other hand, $\mathfrak{P}(mf)(z) = \sum a_m z^{\alpha(m)}$. We can see that $\mathfrak{P}(mf) = \mathfrak{P}m\mathfrak{P}f$ since $\sum a_m z^{\alpha(m)} = \sum b_n z^{\alpha(n)+\alpha(k)}$. Indeed, $a_m = b_n$ if and only if

$$m = k \cdot n \Longleftrightarrow \mathfrak{p}^{\alpha(m)} = \mathfrak{p}^{\alpha(k)}\mathfrak{p}^{\alpha(n)} = \mathfrak{p}^{\alpha(k)+\alpha(n)} \Longleftrightarrow \alpha(m) = \alpha(n) + \alpha(k),$$

as we wanted. The result for an infinite Dirichlet series f is obtained by the approximation of f by Dirichlet polynomials, since they are dense in $\mathscr{H}^2$.

All this implies that $\mathfrak{P}(m^j) = (\mathfrak{P}m)^j \in L^1(\mathbb{T}^\infty, \rho)$, so

$$\left(\int_{\mathbb{T}^\infty} |\mathfrak{P}m|^{2j} d\rho\right)^{1/2j} = \left\|(\mathfrak{P}m)^j\right\|_{H^2(\mathbb{T}^\infty)}^{1/j} = \left\|\mathfrak{P}(m^j)\right\|_{H^2(\mathbb{T}^\infty)}^{1/j} = \left\|m^j\right\|_2^{1/j} \leq \|m\|_{\mathscr{M}},$$

where we have used the first claim in the last inequality. Since $\lim_{j\to\infty} \left\|(\mathfrak{P}m)^j\right\|_{H^2(\mathbb{T}^\infty)}^{1/j} = \|\mathfrak{P}m\|_\infty$ the above inequality yields

$$\|\mathfrak{P}m\|_\infty \leq \|m\|_{\mathscr{M}}.$$

Thus, $\mathfrak{P}m \in H^\infty(\mathbb{T}^\infty)$.

Next, by Theorem 3.36, the map

$$\mathfrak{E}_{|H^\infty(\mathbb{T}^\infty)} : H^\infty(\mathbb{T}^\infty) \longrightarrow H^\infty(\mathbb{D}^\infty) \tag{3.8}$$

is an isometry for the $\|\cdot\|_\infty$ norm. Hence $\mathfrak{L}m = \mathfrak{E}\mathfrak{P}m \in H^\infty(\mathbb{D}^\infty)$ and $\|\mathfrak{L}m\|_\infty = \|\mathfrak{P}m\|_{H^\infty(\mathbb{T}^\infty)}$. We have

$$\mathfrak{L}m(z) = \sum_{\alpha\in\mathbb{N}^\mathbb{N}} b_{\mathfrak{p}^\alpha} z^\alpha, \; z \in \mathbb{D}^\infty \cap \ell^2.$$

For $s \in \mathbb{C}_{1/2}$, consider $z_s := (2^{-s}, 3^{-s}, 5^{-s}, 7^{-s}, \dots) = \mathfrak{p}^{-s} \in \mathbb{D}^\infty$. We have that $m(s) = \mathfrak{L}m(z_s)$. Indeed, the point evaluation is well defined as $z_s \in \mathbb{D}^\infty \cap \ell^2$ and

$$\mathfrak{L}m(z_s) = \sum_{\alpha\in\mathbb{N}^\mathbb{N}} b_{\mathfrak{p}^\alpha}(\mathfrak{p}^{-s})^\alpha = \sum_{\alpha\in\mathbb{N}^\mathbb{N}} b_{\mathfrak{p}^\alpha}(\mathfrak{p}^\alpha)^{-s} = \sum_{n=1}^{\infty} b_n n^{-s} = m(s).$$

Since $\mathfrak{L}m$ is bounded and holomorphic in $\mathbb{D}^\infty \cap c_0$ and z_s depends analytically on s and $z_s \in \mathbb{D}^\infty \cap c_0$ for $s \in \mathbb{C}_0$, we have that $\mathfrak{L}m(z_s)$ is bounded and analytic in $\mathbb{C}_0$, and so is $m(s)$. Hence $m \in \mathscr{H}^\infty$. Moreover, $\|m\|_\infty = \|\mathfrak{L}m\|_{H^\infty(\mathbb{D}^\infty)} = \|\mathfrak{P}m\|_{H^\infty(\mathbb{T}^\infty)} \leq \|m\|_{\mathscr{M}}$.

□

This theorem can be extended to the multipliers of $\mathscr{H}^p$ for $1 \leq p < +\infty$. Indeed $\mathscr{M}(\mathscr{H}^p) = \mathscr{H}^\infty$ and if $m \in \mathscr{M}(\mathscr{H}^p)$ then $\|m\|_{\mathscr{M}} = \|m\|_p$. See [2] for further reading.

4 Conclusions

An extensive study of ordinary Dirichlet series has been done. Different important results on the convergence regions of these type of series have been given. Most notably, the Bohnenblust-Hille theorem has been proven. Classic formulas for the convergence abscissas and the computation of the coefficients from the series are also given. We have also studied different properties of Dirichlet series, giving uniqueness theorems, limits and growth results.

Hardy-Dirichlet spaces have also been studied extensively, in particular $\mathscr{H}^\infty$ and $\mathscr{H}^2$. The global properties of the series in each space have been studied, as well as properties of the function spaces themselves. More importantly, an overview of Bohr's transformation is given which connects Dirichlet series and power series in an infinite number of variables. This tool is fundamental in the multiplier theorem, and is useful more than once in this work. Moreover, this tool allows the proof of results for Dirichlet series using the knowledge of power series and vice-versa. Bohr's transformation might be useful in future work in the research of Dirichlet series, as well as power series on the poly-disk.

On a personal note, the objectives I had for this work have been more than achieved. At first I wanted to study Dirichlet series as a means to have an understanding on one of the approaches used to solve the Riemann hypothesis. But as I began to read books and articles, and thanks to my advisor, I became more interested in the functional analysis part of the topic. So much so that I have been able to understand results that I deemed impossible when I first started working on this paper, such as the Bohnenblust-Hille theorem, and specially the multiplier theorem.

5 References

[1] Apostol, T.M., *Introduction to Analytic Number Theory*, Springer New York, 1976.

[2] Bayart, F., *Hardy spaces of Dirichlet series and their composition operators*, Monatshefte für Mathematik **136** (2002), no. 3, 203-236.

[3] Boas, H.P., *The football player and the infinite series*, Notices of the AMS **44** (1997), no. 11, 1430-1435.

[4] Bohnenblust, H.F. and Hille, E., *On the Absolute Convergence of Dirichlet Series*, Annals of Mathematics **32** (1931), no. 3, 600.

[5] Bohr, H., *Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der Dirichletschen Reihen* $\sum a_n/n^s$, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1913** (1913), 441-488.

[6] Bohr, H., *A theorem concerning power series*, Proceedings of the London Mathematical Society **s2-13** (1914), no. 1, 1-5.

[7] Bueno Contreras, J.J., *The Cesàro space of Dirichlet series*, Ph.D. thesis, Universidad de Sevilla, 2018.

[8] Cahen, E., *Sur la fonction* $\zeta(s)$ *de Riemann et sur des fonctions analogues*, Annales scientifiques de l'École Normale Supérieure **3e série, 11** (1894), 75-164.

[9] Cole, B.J. and Gamelin, T.W., *Representing Measures and Hardy Spaces for the Infinite Polydisk Algebra*, Proceedings of the London Mathematical Society **s3-53** (1986), no. 1, 112-142.

[10] Conway, J.B., *Functions of One Complex Variable I*, Springer New York, 1995.

[11] Defant, A., García, D. and Maestre, M., *Dirichlet Series and Holomorphic Functions in High Dimensions*, Cambridge University Press, 2019.

[12] Hedenmalm, H., Lindqvist, P. and Seip, K., *A Hilbert space of Dirichlet series and systems of dilated functions in* $L^2(0,1)$, Duke Mathematical Journal **86** (1997), no. 1, 1-36.

[13] Fernández Pérez, J.L., *Variable Compleja II*, Preliminary version, January 2021.

[14] Queffélec, H. and Queffélec, M., *Diophantine Approximation and Dirichlet Series*, Springer Singapore, 2020.

[15] Riemann, B., *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Ges. Math. Werke und Wissenschaftlicher Nachlaß **2** (1859), 145-155.

[16] Titchmarsh, E.C., *The Theory of Functions*, Oxford University Press, 1932.

[17] Titchmarsh, E.C., *The Theory of the Riemann Zeta-Function*, Oxford University Press, 1987.