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On the Range of Holomorphic Functions: from Landau to Picard's Theorems

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Abstract

The range of a holomorphic function is a classical topic in complex variable. Throughout this project, we will give several results on the size of this range. Among them we mention: Landau's Injective Theorem, Landau's Covering Theorem, Bloch's Theorem and Picard's Theorems.
The first two results analyse the uniqueness in Schwarz's Lemma giving a precise estimate on the size of both the biggest disc covered by the function and the biggest disc where the function is injective, in terms of $|f'(0)|$. Bloch's Theorem is also a covering result with fewer hypotheses on the function, and it is a key tool in the proof of Picard's Theorem. Finally, we prove both Picard's Little and Great theorems. The first states that any entire function that omits at least two values is constant, and the second one, which holds for meromorphic functions, can be viewed as a generalisation of the Casorati-Weierstraß' theorem.
Finally, along the third and fourth chapters, we will see which consequences derive from these two classical results.

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Contents

0.1 Introduction

The range of a holomorphic function is a classic object of study in holomorphic function theory. There are many elementary results regarding this topic. The first one we should mention is the open mapping theorem, which states that any non constant holomorphic function on a region is open, that is, the image of any open set is open. We can also mention a simple but powerful result, Schwarz's Lemma, which in particular states that any holomorphic function on the unit disc, $\mathbb{D}$, that fixes the origin and has unimodular derivative at the origin is a rotation. Hence, the image through this function of $\mathbb{D}$ is itself.
Other basic results on the size of the can be mentioned are Liouville's theorem, which states that any bounded entire function must be constant and Casorati-Weierstraß' Theorem, which affirms that, given a function that is holomorphic on a punctured neighbourhood of z_0 and has an essential singularity at z_0, has dense image in $\mathbb{C}$.
After having cited these results, some natural questions arise, such as: can we give a quantitative version of the open mapping theorem? what can be said of Schwarz's Lemma if we weaken its hypothesis? Can we give estimates for the radius of the disc on where the function would be injective? Can we weaken the boundedness hypothesis on Liouville's Theorem? And what else can be said about the image of f in Casorati-Weierstraß' Theorem? Our objective will be to give answer to these questions through several results and explain how do these actually make us go a step forward in this line of study.

Our first result are two **Landau's[2] Theorems** that will give some answers to the question on the uniqueness of Schwarz's Lemma.

Theorem 0.1.1. *(Landau's Theorems) Given $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$, then*

a. f is an injective function on $D(0, \rho(|f'(0)|))$, where $\rho(t) = \frac{t}{1+\sqrt{1-t^2}}$. Moreover, for any $t \in (0,1)$ there exists a function $f \in H(\mathbb{D})$ satisfying that $f(0) = 0$, $f(\mathbb{D}) \subset \mathbb{D}$, $f'(0) = t$ and $f'(\rho(t)) = 0$. In particular f is not injective on any disc whose radius is bigger than $\rho(t)$.

b. $D(0, \eta(|f'(0)|)) \subset f(\mathbb{D})$, where η is the inverse of $\mu(r) = \frac{2r\log(\frac{1}{r})}{1-r}$.

To prove the first statement of these results we use the estimates derived from Schwarz's Lemma. For the second one, we also use some basic properties of the disc's automorphisms.

[2]**Edmund Landau** was a german mathematician born in 1877, with jewish background, whose work focused on number theory and complex analysis. He studied in Berlin and moved to Palestine in 1927. Later, in 1939, he died after moving back to Berlin.

We observe that the first statement shows in particular that, if $|f'(0)|$ is close to 1, then f is injective on a disc centred at the origin and whose radius is also close to 1. The second statement gives, in particular, that f is surjective on a disc whose radius is close to 1 if $|f'(0)|$ is close to 1.
A natural question that arises is: can we, as we have already done with Schwarz's Lemma, weaken the hypothesis on Landau's Theorem? The obtained result is called **Bloch's**[3] **Theorem**. Bloch's theorem gives an answer to the question derived from the second statement of Landau's Theorems.

Theorem 0.1.2. *(Bloch's Theorem) Let $f \in H(\mathbb{D})$, with $f'(0) \neq 0$. Then, $f(\mathbb{D})$ contains a disc whose radius is $\frac{|f'(0)|}{4}$.*

We will deduce Schottky's and Picard's Theorems from this result.
The second part of this project is devoted to the study of the range of entire functions. The Fundamental's Algebra Theorem states that for any non-constant polynomial function of degree n, any point in $\mathbb{C}$ has exactly n antiimages. But we know, as the exponential function shows, that there are non constant entire functions that omit some values. A natural question arises from these observations: what can be said on the omitted values of non constant entire functions? We also observe that a consequence of Casorati-Weierstraß' Theorem shows that if f is an entire function which is not a polynomial (i.e. f is an entire transcendental function), then $f(\mathbb{C})$ is dense in $\mathbb{C}$.
A classical result related to the topic of this second part is Liouville's Theorem, it states that if the omitted set is the complementary of a disc, the function is constant. The next question is whether an entire function needs to be bounded for us to be able to prove that is constant. Which hypotheses, weaker than being bounded, can prove that the function is constant?

Theorem 0.1.3. *(Picard's Little Theorem)Let f be an entire function that omits two values. Then f is constant.*

Picard's original proof uses elliptical modular functions. The proof we present follows proceeds as Landau's did, expressing the function as an exponential of a Jukowski's transformation of a function that omits a δ-dense sets, where, given a $\delta > 0$, E is defined as a δ-set if any point in $\mathbb{C}$ is at distance at most δ from some points in E.
It also requires the use of a generalisation of Liouville's Theorem and a result known as **Schottky's Tower**.
After having proved this result, we prove how it is equivalent to other statements and derive some applications of it.

[3] **André Bloch** was a french mathematician born in 1893 with jewish background. His work focused on complex analysis. He started his studies in 1913 at *l'École Polytechnique* until the great war began. He spent most of his adult live in an asylum and died in 1948.

Finally, we generalise what Casorati and Weierstraß' Theorem says about the behaviour of an holomorphic function on a punctured domain around an essential singularity. To do so, we find **Picard's**[4] **Great Theorem**.

Theorem 0.1.4. *(Picard's Great Theorem) Let $f \in H(D(a,r) \setminus \{a\})$ where $a \in \mathbb{C}$ and $r > 0$. If f has an essential singularity at $z = a$, then f attains every value in $\mathbb{C}$ an infinite amount of times, except, at most, an exceptional value.*

Hence, now we know that, under the assumptions of the theorem, not only is the image dense in $\mathbb{C}$, but it is the actual $\mathbb{C}$ or $\mathbb{C}$ without, at most, one exceptional value and any value in $\mathbb{C}$ is attained infinitely many times, except at most one.
Eventually, we find a corollary from Picard's Great Theorem that allows us to go further than what is stated Picard's Little Theorem.

Corollary 0.1.5. *Let f be an entire non-constant function. Then there are two options for f:*

- *either f is a polynomial and hence, takes every value in $\mathbb{C}$ a finite number of times,*
- *or f is not a polynomial and takes every value in $\mathbb{C}$ an infinite amount of times, except at most one.*

The first part asserts a consequence of Algebra's Fundamental Theorem.
The second is a consequence of Picard's Great Theorem applied to $f(\frac{1}{z})$, which, since $f(z)$ is not a polynomial, has an essential singularity at zero.

[4]**Charles Émile Picard** was a french mathematician born in 1856. He studied at *l'École Normale Supérieur*. In 1924, he became the 15th fellow of the Royal Society who occupied the seat 1 in the *Académie Française*. He died in 1941, in Paris.

0.2 Structure of the Memory

The memory of this project is organised as follows:

In Chapter 1, we study an extension of Schwarz's Lemma with the use of non-euclidean distances on the unit disc.

Along Chapter 2, we study and prove Landau's and Bloch's Theorems.

In Chapter 3, using the iterate elevation and composition of functions, we prove both Picard's Little Theorem and Schottky's Theorem.

Finally, in Chapter 4, we state and prove Picard's Great Theorem and how both Picard's Theorems and Algebra's Fundamental Theorem are related via a corollary.

0.3 Preliminaries

In this introductory section, we aim to establish the base of results that we will use along the development of the project. We will state elementary results in complex analysis and prove the most used or important.

Theorem 0.3.1. *(Maximum Modulus Principle) Let $f \in H(\Omega)$, where Ω is a domain (open and connected). If there exists a point $z_0 \in \Omega$ such that it is a maximum of f, then f is constant.*

Lemma 0.3.2. *(Schwarz's Lemma) Let $f \in H(\mathbb{D})$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|f(z)| \leq |z| \text{ for any } z \in \mathbb{D}, \text{and} |f'(0)| \leq 1.$$

Moreover, if there exists $z_0 \in \mathbb{D}$ such that $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then there exists $\theta \in \mathbb{R}$ such that

$$f(z) = ze^{i\theta}, \text{ for any } z \in \mathbb{D}.$$

Proof. Let us define the auxiliary function g as follows:

$$g(z) = \begin{cases} \frac{f(z)}{z} & if \quad z \neq 0 \\ f'(0) & if \quad z = 0 \end{cases}$$

For any $r < 1$ and $|z| \leq r$, we get that $|g(z)| \leq \max_{|z|=r} |g(z)| < \frac{1}{r}$. Next, taking $r \to 1$, we obtain that $|g(z)| \leq 1$ and this our two first statements. As for the third one, if the equality holds for a certain z_0, by the maximum modulus principle, g is constant, i.e. there exists $\theta \in \mathbb{R}$ such that we can write $g(z) = k = e^{i\theta}$, since $|g(z)| = 1$. Hence, $f(z) = ze^{i\theta}$. □

From Cauchy's Integral formula we derive the next result.

Lemma 0.3.3. *(Cauchy's Inequalities) Let Ω be an open set such that, $f \in H(\Omega)$. If $\overline{D(a,r)} \subseteq \Omega$. Then,*

$$|f^{(n)}(a)| \leq \frac{n! \max_{\overline{D(a,r)}} |f(z)|}{r^n}. \tag{1}$$

Theorem 0.3.4. *(Liouville's Theorem) Let f be an entire and bounded function. Then, f is constant.*

Theorem 0.3.5. *(Open Mapping Theorem) Let Ω be a connected open set and $f \in H(\Omega)$ non-constant. Then f is open: for any open set $U \subseteq \Omega$, $f(U)$ is an open set.*

Lemma 0.3.6. *(Isolated Singularities' Classification) Let $a \in \mathbb{C}$ and assume $f \in H(D(a,r) \setminus \{a\})$. The singularity at $z = a$ is*

- ***Avoidable*** *if and only if* $\lim_{z\to a}(z-a)f(z) = 0$.
- ***a Pole*** *if and only if* $\lim_{z\to a} |f(z)| = \infty$.
- ***Essential*** *if and only if* $\lim_{z\to a} |f(z)|$ *does not exist.*

Now we state and prove the theorem that we will later compare with Picard's Great Theorem, in order to see how much of an improvement this last one is.

Theorem 0.3.7. *(Casorati-Weierstraß' Theorem) Let $f \in H(D(a,r) \setminus \{a\})$ where $z = a$ is an essential singularity. Thus, $f(D(a,r) \setminus \{a\})$ is dense in $\mathbb{C}$, i.e. $\overline{f(D(a,r) \setminus \{a\})} = \mathbb{C}$.*

Proof. Assume that it is indeed not dense in $\mathbb{C}$: there exists $c \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$|f(z) - c| \geq \varepsilon \text{ for any } z : \ 0 < |z-a| < r.$$

Hence, we have $\left|\frac{1}{f(z)-c}\right| \leq \frac{1}{\varepsilon}$ for any $0 < |z-a| < r$. From this, we do get that

$$\lim_{z\to a}(z-a)\frac{1}{f(z)-c} = 0.$$

Applying then Lemma 0.3.6, we deduce that $g(z) = \frac{1}{f(z)-c}$ has an avoidable singularity in $z = a$. Thus, $\lim_{z\to a} \frac{1}{f(z)-c} := L$ does exist. By writing this deduction in terms of f we get

$$\lim_{z\to a} f(z) = \lim_{z\to a} \frac{1}{g(z)} + c = \frac{1}{L} + c.$$

If $L \neq 0$, f would be bounded in $0 < |z-a| < \varepsilon$ and thus it has an avoidable singularity. Otherwise, if $L = 0$, $\lim_{z\to a} |f(z)| = \infty$. Hence, f has a pole in $z = a$. Both cases lead to contradiction. □

Theorem 0.3.8. *(Rouché's Theorem) Let Ω be a bounded domain whose border, γ, is piecewise-differentiable. Let $f, g \in H(\overline{\Omega})$ such that neither vanishes on γ. If*

$$|f(z) - g(z)| < |g(z)| \quad \textit{for any } z \in \gamma^*,$$

then $\sum_{a\in Z(f)} m(f,a) = \sum_{b\in Z(g)} m(g,b)$, where $m(f,z_0)$ is the multiplicity of z_0 as a zero of f.

Chapter 1

Schwarz's Lemma. Hyperbolic Distance. Applications

The first chapter of this project will depart from Schwarz's Lemma, that has been proved in the introductory part. Our goal is twofold: on the one hand we want to study generalisations and applications of Schwarz's Lemma and, on the other hand, we will state and prove the results derived from this lemma that will be needed in the following chapters. We will follow the references [3] and [4].
In order to expand the type of functions to which we can apply Schwarz's Lemma, we will study how the action of the elements of $Aut(\mathbb{D})$ does affect on the hypothesis that f originally satisfies.
We will obtain some applications of Schwarz's Lemma: **Lindelöf's Subordination Principle**. This will lead us to introduce the **hyperbolic derivative** of f: $D_h f$. It will provide us some useful estimates for the type of functions that we will be working with.

1.1 Schwarz's Lemma Consequences

Our first result in this chapter gives estimate for $|f|$ in terms of a finite number of zeros of f on $\mathbb{D}$.

Lemma 1.1.1. *(Schwarz's Lemma with zeros) Let $f \in H(\mathbb{D})$, a function such that $f : \mathbb{D} \longrightarrow \mathbb{D}$. Let $a_1, \dots, a_n \in \mathbb{D}$ such that $a_j \in Z(f)$, $j = 1, \dots, n$ (eventually some of the a_j can coincide). Then,*

$$|f(z)| \leq \prod_{j=1}^{n} \left| \frac{z - a_j}{1 - z\bar{a}_j} \right|, \ \forall z \in \mathbb{D}.$$

In particular, $|f(0)| \leq \prod_{j=1}^{n} |a_j|$.

Proof. Let us denote $B(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-z\bar{a_j}}$, $\forall z \in \mathbb{D}$, the finite Blasche product with zeros a_j. Since $f(a_j) = B(a_j) = 0$, for any $j = 1, \ldots, n$, the function $g(z) = \frac{f(z)}{B(z)}$ is an holomorphic function on $\mathbb{D}$.
For each $r \in (0, 1)$, we define $g_r(z) := g(rz)$. Hence, $g_r \in H(\overline{\mathbb{D}})$, $0 < r < 1$. Given $r \in (0, 1)$, we define $m(r) = \min_{|z|=r} |B(z)|$. Since $B(z)$ is continuous on $\overline{\mathbb{D}}$ and $|B(e^{i\theta})| = 1$, $\lim_{r\to 1^-} m(r) = 1$.
Therefore, if $z \in \partial\mathbb{D}$:

$$|g_r(z)| = \frac{|f(rz)|}{|B(rz)|} \leq \frac{|f(rz)|}{m(r)} \leq \frac{1}{m(r)}.$$

Hence, applying the Maximum Modulus Principle, $|g(rz)| \leq \frac{1}{m(r)}$ for each $r \in \mathbb{D}$. Lastly, let us fix $z \in \mathbb{D}$ and let $r \to 1^-$.What we do now obtain is $|g_1(z)| = |g(z)| \leq 1$ which in terms of f is equivalent to what we wanted to prove $\square$

Not only we can obtain new results by changing the hypotheses on f in Schwarz's Lemma, but we can also use it to give alternative proofs of classical results in complex analysis. Let us give an example.

Lemma 1.1.2. *(Liouville's Theorem) Let f be a bounded entire function. Then, f is a constant.*

Proof. Assuming the hypothesis on f, let $M = \sup |f(z)|$ and we fix $a \in \mathbb{C}$ and $R > 0$. Thus we define an holomorphic function g as:

$$g(z) = \frac{f(a + Rz) - f(a)}{2M}$$

Since f is bounded by M, $g(\mathbb{D}) \subset \overline{\mathbb{D}}$ and by definition $g(0) = \frac{f(a)-f(a)}{2M} = 0$. Consequently, we apply Schwarz's Lemma to obtain $|g'(0)| \leq 1$. But

$$|g'(z)| = \left| \frac{Rf'(a + Rz)}{2M} \right|.$$

Which taking $z = 0$ gives $|g'(0)| = |\frac{Rf'(a)}{2M}| \leq 1$.
Thereby, $|f'(a)| \leq \frac{2M}{R}$, for each $R > 0$ and $a \in \mathbb{C}$. Therefore, if $R \to \infty$: $|f'(a)| = 0$. Hence, $f'(a) = 0$ for each $a \in \mathbb{C}$ and $f' \equiv 0$, that is, f is constant. $\square$

Related to Lemma 1.1.1, we obtain this corollary that will be useful further on.

Corollary 1.1.3. *Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$ (Schwarz's lemma hypothesis). Assume we have a point $q \in \mathbb{D}$ and a finite collection of points $z_1, \ldots, z_n \in \mathbb{D}$ (which can eventually repeat, if $f(z) - q$ has a zero of order k in $z = z_i$, $1 \leq i \leq n$) such that $f(z_i) = q$, for $1 \leq i \leq n$. Then,*

$$|q| \leq \prod_{i=1}^{n} |z_i|.$$

Proof. Firstly, if $q \in \mathbb{D}$, let us recall the definition of the Möbius transformation S_q:

$$S_q(z) = \frac{z - q}{1 - z\bar{q}}.$$

We define a new function g by composing S_q with f, i.e., $g = S_q \circ f$. Both S_q and f do send $\mathbb{D}$ into $\mathbb{D}$; therefore, so does g. Lastly, $f(z_i) = q$, for $1 \leq i \leq n$ and thus, $g(z_i) = 0$ for $1 \leq i \leq n$ and $|g(0)| = |q|$.
Applying now Lemma 1.1.1 to g for $z = 0$ we obtain:

$$|q| = |g(0)| \leq \prod_{i=1}^{n} |z_i|.$$

□

1.1.1 Schwarz-Pick's Lemma

The next result we are going to study is the **Schwarz-Pick's Lemma**. The objective is to give a variation of Schwarz's Lemma without imposing that $f(0) = 0$. The prove of this lemma involves the use **Möbius Transformations**. The following question arises naturally now: Why are these transformation required in this lemma? They are very important because by including them in our results, we can weaken an important and really restrictive hypothesis from the previous lemmas: $f(0)$ will not longer need to be 0.
Assume now that the only data that we have of our holomorphic function on $\mathbb{D}$ is that $f(a) = b \in \mathbb{D}$ and that $f(\mathbb{D}) \subset \mathbb{D}$. One first try to obtain a function that maps 0 to 0 would be to use a translation to transform our function f into another one, g. We could have defined $g(z) = f(z + a) - b$ and it will send 0 to itself. Nevertheless, $g(\mathbb{D}) \not\subset \mathbb{D}$. We can conclude that this linear transformation is not the solution we require.
The appropriate answer will be given by the Möbius transformations: T_a and S_b. With these two transformations will be able to freely move points in $\mathbb{D}$ without altering the image. Instead of the linear translation: $z \to z + a$ we will now use

$$T_a(z) = \frac{z + a}{1 + z\bar{a}}, \tag{1.1}$$

and substituting the linear translation: $z \to z - b$ we will use

$$S_b(z) = \frac{z - b}{1 - z\bar{b}}. \tag{1.2}$$

Some interesting and useful properties, regarding these two transformations are found in Annex A.1.

Lemma 1.1.4. *(Schwarz-Pick's Lemma) Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$. Then,*

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}, \quad \text{for any } a \in \mathbb{D} \tag{1.3}$$

and

$$\left|\frac{f(z)-f(a)}{1-f(z)\overline{f(a)}}\right| \leq \left|\frac{z-a}{1-z\bar{a}}\right|. \tag{1.4}$$

Moreover, any of both equalities hold if and only if f is a Möbius Transformation that form a subgroup of $Aut(\mathbb{D}) = \{\varphi \in H(\mathbb{D}) : \varphi(\mathbb{D}) = \mathbb{D}\}$.

This lemma is indeed a more general version of the classic Schwarz's Lemma. It can be easily checked that by substituting $a = 0$ and $f(a) = 0$, what we obtain is nothing but the original Schwarz's Lemma.

Proof. Let us assume $f(a) = b$ where $a \in \mathbb{D}$. We also define the function $g = S_b \circ f \circ T_a$. Since S_b and $T_a \in Aut(\mathbb{D})$, $g(\mathbb{D}) \subset \mathbb{D}$.
By its definition and since $f(a) = b$ we also have that $g \in H(\mathbb{D})$ and $g(0) = 0$. Applying now Schwarz's Lemma to the function g, we obtain $|g'(0)| \leq 1$, which in terms of f can be rewritten as:

$$1 \geq |S_b'(b)||f'(a)||T_a'(0)| = \frac{1}{1-|b|^2} \cdot |f'(a)| \cdot (1-|a|^2)$$

applying properties of the Möbius transformations. Thus, (1.3) is proven. Moreover,

$$|g(\omega)| = |S_b(f(T_a(\omega)))| \leq |\omega|, \ \forall \omega \in \mathbb{D}.$$

But then taking $z \in \mathbb{D}$ such that $\omega = S_a(z)$, the previous inequality can be rewritten as $|S_b(f(z))| \leq |S_a(z)|$ (since S_a is the inverse function of T_a) and by S_b, equivalently for S_a, definition we get

$$\left|\frac{f(z)-b}{1-f(z)\bar{b}}\right| \leq \left|\frac{z-a}{1-z\bar{a}}\right|.$$

For the second part of the lemma, assume

$$|f'(a)| = \frac{1-|f(a)|^2}{1-|a|^2} \ \text{ or } \ \left|\frac{f(z)-f(a)}{1-f(z)\overline{f(a)}}\right| = \left|\frac{z-a}{1-z\bar{a}}\right|$$

if and only if $|g'(0)| = 1$ or $|g(z)| = |z|$. Therefore, for a real θ given,

$$(S_b \circ f \circ T_a)(z) = g(z) = e^{i\theta} z.$$

From this expression, f can be seen as a composition of 3 Möbius transformations: $f = T_b(e^{i\theta} S_a(z))$, for any $z \in \mathbb{D}$. $\square$

1.2 Lindelöf's Subordination Principle

We do now proceed with another application of Schwarz's Lemma: **Lindelöf's Subordination Principle**. This will provide us a new partial order between holomorphic functions on $\mathbb{D}$.

Definition 1.2.1. *Let $f, g \in H(\mathbb{D})$. We will say that f **is subordinated to** g if there exists $\omega \in H(\mathbb{D})$, such that $\omega(\mathbb{D}) \subset \mathbb{D}$, $\omega(0) = 0$ and $f = g \circ \omega$. We will refer to the function ω as **subordinate** and this relation will be denoted by $f \prec g$.*

Proposition 1.2.2. *(Subordination Principle) Let $f, g \in H(\mathbb{D})$ such that $f \prec g$. Then,*

$$|f'(0)| \leq |g'(0)|, \ \ f(D(0,r)) \subset g(D(0,r)), \ \textit{for any}\ r \in (0,1).$$

Proof. Since $f \prec g$, there exists $\omega \in H(\mathbb{D})$, $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f = g \circ \omega$. Then, applying Schwarz's Lemma,

$$|f'(0)| = |g'(0)| \cdot |\omega'(0)| \leq |g'(0)|$$

Likewise,

$$f(D(0,r)) = g(\omega(D(0,r))) \subset g(D(0,r)).$$

□

From the previous proposition we can trivially deduce that if f, g also satisfy that $|f'(0)| = |g'(0)|$, then there exists $\theta \in \mathbb{R}$ such that for any $z \in \mathbb{D}$, $f(z) = g(ze^{i\theta})$. That is, f is a rotation followed by the function g.
Our next goal is to give an invariant version of the subordination principle. We begin with a reformulation of Schwarz-Pick's lemma in terms of the so called pseudo-hyperbolic distance.
This quantity has repeatedly appeared in Lemma 1.1.4 and its reciprocal result.

Definition 1.2.3. *The **pseudo-hyperbolic distance** is defined as*

$$\sigma(a,b) = \left| \frac{a-b}{1-a\bar{b}} \right|, \ \textit{for any}\ a, b \in \mathbb{D}.$$

Observe that $\sigma(a,b) = \sigma(b,a)$. Also, $\sigma(a,b) = 0$ if and only if $a = b$. In fact, σ also satisfies the triangular inequality (see [4]).
Hence, we now rewrite Schwarz-Pick's Lemma in terms of this pseudo-hyperbolic distance:

Lemma 1.2.4. *(Schwarz-Pick's Lemma Reformulated) Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$. Then,*

$$\sigma(f(a), f(b)) \leq \sigma(a,b), \ \textit{for any}\ a, b \in \mathbb{D}.$$

In terms of σ, Lemma 1.2.4 states that any f satisfying the required hypothesis is a **Lipschitz** function for the pseudohyperbolic distance σ, with its Lipschitz constant equal to 1.
In order to get a better understanding of how this new pseudo-hyperbolic distance works, we are going to study the most basic elements on a topology derived from a distance: **the open balls**. With this new distance, $B_\sigma(a,r) = \{z : \sigma(a,z) < r\}$. How are this new types of open sets from our euclidean point of view? That is what we will try to understand in the following lemma. Observe that if $a = 0$, $B_\sigma(0,r) = B(0,r)$.

Lemma 1.2.5. *Let us take $a \in \mathbb{D}$ and $r \in (0,1)$. Then,*

$$B_\sigma(a,r) \cap D\left(0, \frac{|a|-r}{1-|a|r}\right) = \emptyset, \quad B_\sigma(a,r) \subset D\left(0, \frac{|a|+r}{1+|a|r}\right).$$

Before we prove Lemma 1.2.5, let us do some observations. If we do take $z \in B_\sigma(a,r)$, by its definition: $\left|\frac{z-a}{1-z\bar{a}}\right| < r$. Going back to contemplate this term as if we were in the euclidean metric, we apply the triangular inequality to both the numerator and the denominator to obtain the following inequality:

$$\frac{||a|-|z||}{1+|z||a|} \leq r,$$

which rearranging the terms gives

$$\frac{|a|-r}{1+|a|r} \leq |z| \leq \frac{|a|+r}{1-|a|r}. \tag{1.5}$$

These inequalities are not as precise as the ones announced in the lemma. The following figure gives a description of the pseudo-hyperbolic ball.

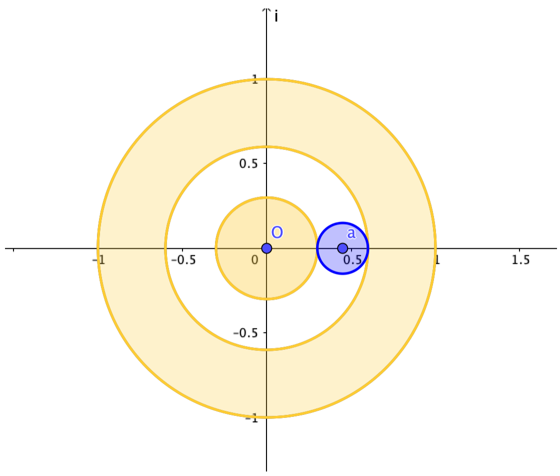

Figure 1.1: $B_\sigma(a,r)$ in the euclidean $\mathbb{D}$.

Proof of Lemma 1.2.5. We will assume, without any loss of generality, that $a \in [0,1)$. The two statements of the lemma are directly deduced from the fact that $B_\sigma(a,r) = T_a(D(0,r))$ is an euclidean disc centred in a point in $(-1,1)$ that passes through the real boundary points of $D(0,r)$, r and $-r$. Hence $B_\sigma(a,r)$ passes through both:

$$T_a(r) = \frac{r+a}{1+ra}, \quad T_a(-r) = \frac{a-r}{1-ra}.$$

□

The following corollary regarding the pseudo-hyperbolic distance will be helpful in the next chapter.

Corollary 1.2.6. *Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$. Then,*

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}.$$

Proof. Let us take $z \in \mathbb{D}$ and write $r = |z| \in (0,1)$. Applying Lemma 1.2.4 and Lemma 1.2.5 , we have:

$$f(D(0,r)) = f(B_\sigma(0,r)) \subset B_\sigma(f(0),r).$$

Substituting $a = f(0)$ in Lemma 1.2.5, we then observe

$$B_\sigma(f(0),r) \cap D(0, \frac{|f(0)| - r}{1 - |f(0)|r}) = \emptyset \text{ and } B_\sigma(f(0),r) \subset D(0, \frac{|f(0)| + r}{1 + |f(0)|r}).$$

This, in terms of $|f(z)|$, is what we aimed to prove. □

If in addition $f(0) = 0$, simply by applying this last corollary not to the initial function f but to $g(z) = \frac{f(z)}{z}$ we obtain a similar result. This time it is not $f(0)$ that is involved in the bounds, instead it depends on $f'(0)$.

Corollary 1.2.7. *Assume $f \in H(\mathbb{D})$, $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. We then have*

$$|z|\frac{|f'(0)| - |z|}{1 - |f'(0)||z|} \leq |f(z)| \leq |z|\frac{|f'(0)| + |z|}{1 + |f'(0)||z|}.$$

After these lemmas and their corollaries, it is clear that the quantity defined earlier as pseudo-hyperbolic distance appears often in the calculations. Now we introduce another distance that is closely related to this one:

Definition 1.2.8. *The* ***hyperbolic distance*** *ρ in $\mathbb{D}$ is defined as:*

$$\rho(a,b) = \log\left(\frac{1 + |\frac{a-b}{1-b\bar{a}}|}{1 - |\frac{a-b}{1-b\bar{a}}|}\right); \quad \text{for any } a, b \in \mathbb{D}. \tag{1.6}$$

Observe that ρ is clearly symmetric and that $\rho(a,b) = 0$ if and only if $a = b$. It also satisfies the triangular inequality (see [4]).
Notice that ρ can be written in terms of σ: $\rho(a,b) = \log\left(\frac{1+\sigma(a,b)}{1-\sigma(a,b)}\right)$. Also, the real function that relates them, $\log(\frac{1+x}{1-x})$ is increasing in $(0,1)$. Then, the result of Lemma 1.2.4 holds for ρ.

Lemma 1.2.9. *(Schwarz-Pick's Lemma Reformulated 2) Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$. Then,*

$$\rho(f(a), f(b)) \leq \rho(a,b), \text{ for any } a, b \in \mathbb{D},$$

f is Lipschitz for the hyperbolic distance ρ, with its Lipschitz constant equal to 1.

Given a function $f \in H(\mathbb{D})$ we can also define another hyperbolic term related to this function,

Definition 1.2.10. *Let $f \in H(\mathbb{D})$, we define its* ***hyperbolic derivative*** *as*

$$D_h f(z) = \frac{1-|z|^2}{2}|f'(z)|; \quad \forall z \in \mathbb{D}. \tag{1.7}$$

Observe that the hyperbolic derivative is written as it is usually done with the euclidean derivative, but using ρ instead of the euclidean distance

$$D_h f(z) = \lim_{h\to 0} \frac{|f(z+h) - f(z)|}{\rho(z+h, z)}.$$

To see the equivalence of both expressions, we need to see how the different distances are related to each other.

Lemma 1.2.11. *Let $z \in \mathbb{D}$. Then,*

1. $\lim_{h\to 0} \frac{\rho(z+h,z)}{\sigma(z+h,z)} = 2$.

2. $\lim_{h\to 0} \frac{\sigma(z+h,z)}{|z+h-z|} = \frac{1}{1-|z|^2}$.

3. $\lim_{h\to 0} \frac{\rho(z+h,z)}{|z+h-z|} = \frac{2}{1-|z|^2}$.

Proof. For the first assertion, we look at the limit $\lim_{x\to 0} \frac{1}{x}\log\left(\frac{1+x}{1-x}\right) = 2$. Now we observe how the pseudo-hyperbolic distance relates to the euclidean one: if $z + h \in \mathbb{D}$ we can deduce

$$\frac{\sigma(z+h,z)}{|z+h-z|} = \frac{1}{|1-|z|^2-h\bar{z}|}$$

that, obviously, when $h \to 0$ tends to the desired result. Combining both result, we obtain the third one. $\square$

Remark 1.2.12. *From the equality* (1.4), *we deduce that Möbius transformations are a subgroup of the isometries for* σ. *Moreover, considering this and that* $\rho = \log(\frac{1+\sigma}{1-\sigma})$, *we obtain that they also are for* ρ. *Hence, given* $f \in H(\mathbb{D})$ *and* T *a Möbius transformation* $D_h(f \circ T) = D_h f(T(z))$.

It is now that we can answer what we wondered earlier before we defined what the pseudo-hyperbolic derivative is: could we still have the Subordination's Principle if we weaken the hypothesis on ω?.

Proposition 1.2.13. *(Invariant version of Subordination Principle) Let* f, g *and* $\omega \in H(\mathbb{D})$, $\omega(\mathbb{D}) \subset \mathbb{D}$ *and* $f = g \circ \omega$. *Hence, for any* $z \in \mathbb{D}$ *and any* $r > 0$,

$$f(B_\rho(z, r)) \subset g(B_\rho(\omega(z), r)) \text{ and } D_h f(z) \leq D_h g(\omega(z)).$$

Proof. The function ω is under the hypothesis of Lemma 1.2.9 and hence $\omega(B_\rho(z, r)) \subset B_\rho(\omega(z), r)$. Since $f = g \circ \omega$,

$$f(B_\rho(z, r)) = g(\omega(B_\rho(z, r))) \subset g(B_\rho(\omega(z), r)).$$

For the second part of the proof, we recall Lemma 1.1.4 for ω. Then, by (1.3) we get $|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}$. Now we take derivatives at both sides of $f = g \circ \omega$ and then absolute values to get $|f'(z)| = |\omega'(z)||g'(\omega(z))|$. Applying now the bound for $|\omega'(z)|$ and diving by 2 at both sides we then get

$$\frac{(1-|z|^2)|f'(z)|}{2} \leq \frac{(1-|\omega(z)|^2)|g'(\omega(z))|}{2} \equiv D_h f(z) \leq D_h g(\omega(z)).$$

□

We can expand this results from applying them on $\mathbb{D}$ to applying them on simply connected domains.

Proposition 1.2.14. *(Subordination on Simply Connected Domains) Let* F *be a biholomorphic function such that* $F : \mathbb{D} \to \Omega$, *where* Ω *is a simply connected domain. Let us as well have* $f \in H(\mathbb{D})$, $f(\mathbb{D}) \subset \Omega$. *Then,* $\forall z \in \mathbb{D}$,

$$(1-|z|^2)|f'(z)| \leq (1-|w|^2)|F'(w)|,$$

where $w = F^{-1}(f(z))$. *Furthermore, if* $f(0) = F(0)$ *we also have*

$$f(D(0, r)) \subset F(D(0, r)); \ \forall r \in (0, 1). \tag{1.8}$$

This first result is the most generic of the ones we will do related to simply connected domains. Naturally, since it covers a wider range of domains, the results coming in this chapter, that focus on more particular domains, will give us more accurate bounds.

Proof. Based on how both f and F are defined we have: $f(\mathbb{D}) \subset \Omega = F(\mathbb{D})$. Then, we can rewrite f in term of F in the following way: $f = F \circ \omega$, where $\omega = F^{-1} \circ f$. This new ω is a holomorphic function on $\mathbb{D}$ such that $\omega(\mathbb{D}) \subset \mathbb{D}$. We are now under the hypothesis of Proposition 1.2.13 which directly gives the bound for the hyperbolic derivatives.
If $f(0) = F(0)$, then $\omega(0) = 0$ and then, applying Proposition 1.2.2 we get $f(D(0,r)) \subset F(D(0,r))$. $\square$

Lemma 1.2.15. *Let us take $f \in H(\mathbb{D})$ such that $\operatorname{Re} f(z) > 0$, $\forall z \in \mathbb{D}$. Then follows this inequality*

$$|f'(z)|(1-|z|^2) \leq 2\operatorname{Re} f(z).$$

Moreover, if $f(0) = 1$

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

Proof. First we will prove the second statement of the lemma. Let us define F on $\mathbb{D}$ defined by: $F(z) = \frac{1+z}{1-z}$. By its definition, $F(0) = 1$.
Next, if $f(0) = 1 = F(0)$, Proposition 1.2.14 gives

$$f(D(0,r)) \subset F(D(0,r)); \text{ for any } r \in (0,1).$$

Indeed let us describe the set $F(D(0,r))$. Let us fix $r \in (0,1)$. The function $F \in Mob(\mathbb{D})$ and $F(\mathbb{R}_\infty) = \mathbb{R}_\infty$, where $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. In addition, $\partial D(0,r)$ intersects orthogonally the real axis. Since F is holomorphic, F is conformal, and then the image of $\partial D(0,r)$ must intersect orthogonally $\mathbb{R}$. Hence, $F(\partial D(0,r))$ is a circle or $\mathbb{R}_\infty$, as seen in A.1, that passes through both $F(r) = \frac{1+r}{1-r}$ and $F(-r) = \frac{1-r}{1+r}$.
Also, both images of $z = -r,\ z = r \in \partial D(0,r)$ have to orthogonally intersect $\mathbb{R}_\infty$. Hence, the only possible outcome is that $F(\partial D(0,r))$ is a circle centred in $\frac{1+r^2}{1-r^2}$. Lastly, we take $z = 0 \in D(0,r)$ and we see if $F(0)$ is inside the new circle or outside. $F(0) = 1$ and $\frac{1-r}{1+r} < 1 < \frac{1+r}{1-r}$. Therefore,

$$F(D(0,r)) \subset A_r := \{\omega : \frac{1-r}{1+r} < |\omega| < \frac{1+r}{1-r}\}.$$

And so, $\frac{1-r}{1+r} < |f(z)| < \frac{1+r}{1-r}$; $\forall z \in D(0,r)$. At last, we take the limit $r \to |z|$, since $z \in D(0,r)$, to obtain the demanded result: $\frac{1-|z|}{1+|z|} < |f(z)| < \frac{1+|z|}{1-|z|}$.
For the first statement we will use 1.2.13. Since $|F'(w)|(1-|w|^2) = 2\frac{1-|w|^2}{|1-w|^2}$, and

$$\operatorname{Re} F(w) = \operatorname{Re} \frac{1+w}{1-w} = \operatorname{Re} \frac{(1+w)(1-\bar{w})}{|1-w|^2} = \frac{1-|w|^2}{|1-w|^2},$$

we obtain $|f'(z)|(1-|z|^2) \leq |F'(w)|(1-|w|^2) = 2\operatorname{Re} F(w) = 2\operatorname{Re} f(z)$; for any $z \in \mathbb{D}$. $\square$

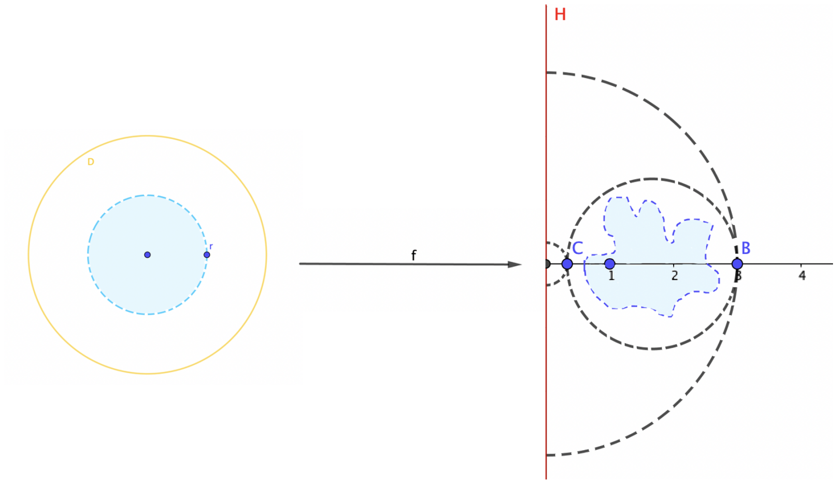

Figure 1.2: Transformation of $D(0,r)$ through f, where the points B and C are, respectively, $\frac{1+r}{1-r}$ and $\frac{1-r}{1+r}$.

We eventually get to the last result of this chapter that will provide us an estimate under certain hypothesis that will later be useful. We will now assume that 0 is not in $f(\mathbb{D})$.

Proposition 1.2.16. *Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D} \setminus \{0\}$. Then,*

$$(1-|z|^2)|f'(z)| \leq 2|f(z)| \log \frac{1}{|f(z)|}.$$

In particular,

$$|f'(0)| \leq 2|f(0)| \log \frac{1}{|f(0)|}.$$

Proof. Since $f(z) \neq 0$ for any $z \in \mathbb{D}$ and $\mathbb{D}$ is simply connected, there exists an holomorphic logarithm of f, i.e., there exists $g \in H(\mathbb{D})$ such that $f = e^g$. Since $|f(z)| < 1$, $\operatorname{Re} g(z) < 0$.
Applying now Lemma 1.2.15 to $-g$ we obtain

$$(1-|z|^2)|g'(z)| \leq -2\operatorname{Re} g(z) = 2|\operatorname{Re} g(z)|$$

that, in terms of f, is what we wanted to prove. $\square$

This estimate for the derivative cannot be improved, it is so because the function $F(z) = e^{-(\frac{1+z}{1-z})}$ verifies the equality. Let us check it.
Since $F'(z) = \frac{-2F(z)}{(1-z)^2} = \frac{-2e^{-\frac{1+z}{1-z}}}{(1-z)^2}$, $|F'(0)| = |-2e^{-1}| = 2e^{-1}$.
But $2F(0)\log\left(\frac{1}{F(0)}\right) = 2e^{-1}\log(e) = 2e^{-1}$. Hence, the bound cannot be improved.

Chapter 2

Landau and Bloch's Theorems

In this chapter we aim to start the process of achieving Picard's Great Theorem, starting from Schwarz's Lemma and its consequences derived in the previous chapter. We will follow the references [2], [4] and [10]. We will first develop a study of some versions of Landau's Theorem concerning the range of holomorphic functions on the disc.

2.1 Landau's Theorems

These Landau's Theorems analyse in a deeper and more precise way the equality in Schwarz's Lemma.
We will study what can be said of a function f satisfying the Schwarz's Lemma hypothesis, when $|f'(0)|$ is close to 1.
We recall that, applying Schwarz's Lemma, if $|f'(0)| = 1$, f is a rotation and hence, is an injective and surjective function.

2.1.1 Landau's Injective Theorem

The first result shows, in particular, that if $|f'(0)|$ is close to 1, then f is an injective function on a disc centred at 0, of radius also close to 1.

If $f'(0) \neq 0$, applying the inverse function's theorem, f is an injective on a disc centred around 0. On the other hand, the uniqueness in the Schwarz's Lemma provides us with the injectivity, in fact bijectivity, on $\mathbb{D}$ if $|f'(0)| = 1$. This version of Landau's Theorem provides a quantitative result of the qualitative version of the Inverse Function's Theorem and the uniqueness in the Schwarz's Lemma. Before we prove this result, we state a proposition.

Proposition 2.1.1. *(Characterisation of Injective-Holomorphic Functions) Let Ω be an open set in $\mathbb{C}$ and f an holomorphic function on Ω, then f is injective on a neighbourhood of $a \in \Omega$ if and only if $f'(a) \neq 0$.*

Proof.

- If $f'(a) \neq 0$, the local version of the inverse function theorem reassures the existence of a neighbourhood of a where f is an injective function.

- Assume $f'(a) = 0$, we want to prove that f is non-injective in a neighbourhood of a. We take Taylor's expansion, since f is analytic, around $z = a$

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots, \text{ where } f'(a) = 0.$$

We rearrange terms to obtain $f(z) - f(a) = \frac{f''(a)}{2!}(x-a)^2 + \dots$, the right hand side is of the order $k \geq 2$. Let us define $g(z)$ as the first non-zero term of the right hand side and $h(z)$ is the rest. We are left with

$$f(z) - f(a) - g(z) = h(z).$$

By the definition of the terms of a Taylor's expansion, let us take $r > 0$ small enough such that for any $z \in D(a,r)$: $|h(z)| < |g(z)|$. Hence, for $z \in D(a,r)$,

$$|(f(z) - f(a)) - g(z)| = |h(z)| < |g(z)|$$

and by Rouché's theorem we obtain that $\#Z(f(z)-f(a)) = \#Z(g(z))$. We know that g has at least 2 zeros and, therefore, $f(z) = f(a)$ is achieved at least twice and f is non-injective.

$\square$

In order to state the first Landau's Theorem this result we need the following lemma.

Lemma 2.1.2. *Let*

$$\rho(t) = \frac{t}{1+\sqrt{1-t^2}} \,; t \in [0,1].$$

Then, for any $t \in [0,1]$, $\rho(t) \in [0,1]$. Moreover $\rho(0) = 0$, $\rho(1) = 1$ and $\rho(t) \geq \frac{t}{2}$.

Proof.

- First we will prove that $\rho(t) \in [0,1]$, when $t \in [0,1]$. Indeed,

$$\rho'(t) = \frac{1}{(\sqrt{1-t^2}+1)\sqrt{1-t^2}} \geq \frac{1}{2} \;;\; t \in [0,1]$$

Observe that $\rho(0) = 0$ and $\rho(1) = 1$ and since ρ is an increasing function on $[0,1]$, $\rho([0,1]) \subset [0,1]$.

- If we define $s(t) = \rho(t) - \frac{t}{2} \geq 0$, we also have $s(0) = \rho(0) - 0 = 0$, and by the above, $s'(t) = \rho'(t) - \frac{1}{2} \geq 0$. Hence s is a non-decreasing function and $s(t) \geq 0$, $t \in [0, 1]$.

$\square$

Now that ρ is defined and we have given some of its basics properties, we have the tools we need in order to announce the related Landau's theorem.

Theorem 2.1.3. *(Landau's Injective Theorem) Let $f \in H(\mathbb{D})$, $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Then f is an injective function on $D(0, \rho(|f'(0)|))$.*
Moreover, for any $t \in (0, 1)$ there exists a function $f \in H(\mathbb{D})$ satisfying that $f(0) = 0$, $f(\mathbb{D}) \subset \mathbb{D}$, $f'(0) = t$ and $f'(\rho(t)) = 0$. In particular f is not injective on any disc whose radius is bigger than $\rho(t)$.

Proof. If $f'(0) = re^{i\theta}$, substituting f by $\tilde{f}(z) = e^{-i\theta} f(z)$, we may assume without loss of generality that $f'(0) = a \in [0, 1)$.
Let us now take $z_1, z_2 \in \mathbb{D}$, with $|z_1| \leq |z_2| = r < 1$ and $f(z_1) = f(z_2) = q$. We want to show that the smallest disc where this function is injective, and in particular $z_1 = z_2$, is the disc with radius $\rho(a)$. In terms of r: $r \geq \rho(a)$.
Applying Corollary 1.1.3 from the previous chapter we obtain:

$$|q| \leq |z_1||z_2| \leq r^2. \tag{2.1}$$

And applying now Corollary 1.2.7:

$$|f(z)| \geq |z|\frac{a - |z|}{1 - a|z|} \text{ for any } z \in \mathbb{D} \tag{2.2}$$

Taking $z = z_2$ in equation (2.2):

$$|q| \geq |z_2|\frac{a - |z_2|}{1 - a|z_2|} = r\frac{a - r}{1 - ar}$$

Combining both equations 2.1 and the above mentioned, we obtain $r \geq \frac{a-r}{1-ar}$, and equivalently: $ar^2 - 2r + a \leq 0$.
Once we rationalise ρ, we get that one of the solutions is $\rho(a) < 1$ and the other value is $\frac{1+\sqrt{1-a^2}}{a} \geq \frac{1}{a} > 1$. Hence, if $ar^2 - 2r + a \leq 0$, $\rho(a) \leq r \leq 1$.
This proves the first statement of the theorem.
Now we prove the second part of it:
Let us fix $t \in (0, 1)$ and consider the function f:

$$f(z) = z\frac{t - z}{1 - zt}$$

This function is clearly holomorphic on $\mathbb{D}$, $f(\mathbb{D}) \subset \mathbb{D}$, since the mapping $z \to \frac{t-z}{1-zt} \in Aut(\mathbb{D})$, and $f(0) = 0$. Moreover $f'(z) = \frac{tz^2-2z+t}{(1-tz)^2}$ and, in particular, $f'(0) = t$ and $f'(\rho(t)) = 0$.
Hence f is not injective over any $D(0, r)$ with $r > \rho(t)$. $\square$

Summarising some important details about what the theorem proves we can say:

1. The theorem shows in particular that $f' \neq 0$ on $D(0, \rho(|f'(0)|))$.
2. If $|f'(0)| = 1$, the function is injective on $\mathbb{D}$, since $\rho(1) = 1$. So we recover part of Schwarz's Lemma.
3. Since $\rho(t) \geq \frac{1}{2}$, we obtain that, in particular, the function is injective on $D(0, \frac{|f'(0)|}{2})$.

Now we proceed to the second version Landau's Theorem. This one is known as **Landau's Covering Theorem**.

2.1.2 Landau's Covering Theorem

The open mapping theorem states, in particular, that if f is a non-constant holomorphic function on $\mathbb{D}$, such that $f(0) = 0$, then $f(\mathbb{D})$ contains a disc centred at 0. A natural question arises: is it possible to give an estimate of the radius of such disc?
Our first result is a weak version of Landau's covering Theorem, which is an immediate consequence of Cauchy's estimates.

Theorem 2.1.4. *(Weak version of Landau's Theorem) Let f be a bounded holomorphic function on $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = 1$. Then, $f(\mathbb{D}) \subset \mathbb{D}$ and if $M = ||f||_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}$, $D(0, \frac{1}{6M}) \subset f(\mathbb{D})$.*

Proof. Since f is holomorphic on $\mathbb{D}$, $f(0) = 0$ and $f'(0) = 1$, its Taylor expansion is $f(z) = z + a_2 z^2 + \dots$. Let $r < 1$, then by applying Cauchy's Inequalities we have: $|a_n| \leq \frac{M}{r^n}$;$\forall n \geq 1$. In particular, $1 = |a_1| \leq M = ||f||_\infty$. Let $z \in \mathbb{D}$ such that $|z| = (4M)^{-1}$. We then have, in particular,

$$\begin{aligned} |f(z)| \geq & |z| - \sum_{n=2}^{\infty} |a_n||z|^n \geq (4M)^{-1} - \sum_{n=2}^{\infty} M(4M)^{-n} = \\ & = \frac{1}{4M} - \frac{1}{16M - 4} \geq (6M)^{-1} \end{aligned}$$

since

$$\sum_{n=2}^{\infty} \frac{M}{(4M)^n} = M \frac{1}{4M(4M-1)} = \frac{1}{16M - 4}$$

Let us now take ω such that $|\omega| < \frac{1}{6M}$, we want to show that $g(z) = f(z) - \omega$ has a zero on $\mathbb{D}$.
If $|z| = (4M)^{-1}$, then

$$|f(z) - g(z)| = |\omega| < (6M)^{-1} \leq |f(z)| \; ; \; |z| = \frac{1}{4M}.$$

Applying Rouché's Theorem we can conclude that both f and g have the same number of zeros in $B(0;\frac{1}{4M})$ (counting multiplicity). Since, $f(0)=0$, there exists $|z_0|<\frac{1}{4M}$ such that $g(z_0)=0$, i.e., $f(z_0)=\omega$. Therefore $B(0,(6M)^{-1})\subset f(\mathbb{D})$. $\square$

Observe that if $M=1$, we obtain an absolute constant $\frac{1}{6}$ such that provides $B(0,\frac{1}{6})\subset f(\mathbb{D})$. But if $|f'(0)|=1$, Schwarz's Lemma gives that f is a rotation and, in particular, $f(\mathbb{D})=\mathbb{D}$. Is it possible to obtain a radius that approaches to 1 when $|f'(0)|$ approaches to 1? An answer to this question will be given by Landau's Covering Theorem.

Lemma 2.1.5. *For $r\in(0,1)$, let*

$$\mu(r)=\frac{2r\log(\frac{1}{r})}{1-r^2}.$$

Then, the function μ is an increasing function and $\mu(r)\leq 2\sqrt{r}$.

Proof. 1. Let $\hat{\mu}=\frac{\mu}{2}$.
We first check that $\hat{\mu}$ is an increasing function on $(0,1)$.

$$\hat{\mu}'(r)=\frac{(\ln r+1)(1-r^2)+2r^2\ln r}{(1-r^2)^2}>0$$

This is equivalent to prove that, for any $r\in(0,1)$, $\ln r>\frac{r^2-1}{1+r^2}$.
Let us define,

$$\sigma(x)=\frac{x^2-1}{1+x^2}-\ln x,\quad (x\in(0,1]).$$

It is clear that $\sigma(1)=0$, if we prove that σ is decreasing in $(0,1)$ then $\sigma(x)>0=\sigma(1)$; for any $x\in(0,1)$.
Consider now the function $\hat{\sigma}(y)=\frac{y-1}{1+y}-\frac{\ln y}{2}$: observe that, in fact, $\hat{\sigma}(x^2)=\sigma(x)$.
Hence $\hat{\sigma}'(y)$ is:

$$\hat{\sigma}'(y)=\frac{-(y-1)^2}{2y(1+y)^2}<0,\ \ y\in(0,1).$$

Therefore, extending continuously μ to 0 and 1 by 0, and 1 respectively, we observe that μ is increasing in $[0,1]$.

2. Let us define $\phi(r)=2\sqrt{r}-\mu(r)$.
Rearranging the terms we can see, since $r\in(0,1)$:
$\phi(r)\geq 0\Longleftrightarrow 1-r^2+\sqrt{r}\ln r\geq 0$.
From the previous considerations, let us define $\eta(r)=r^2-\sqrt{r}\ln r$, and we want to see that 1 is an upper bound for η.

$$\eta(1)=1\ \text{ and }\ \lim_{r\to 0^+}\eta(r)=0$$

We will seek the maximum in $(0,1)$ of η: taking the derivative of η and making it equal to 0; we obtain two result numerically: $r_0 \approx 0.18704$ and $r_1 \approx 0.4465$. Applying the second derivative criteria: $\eta''(r_0) < 0$ and $\eta''(r_1) > 0$, then r_0 is a maximum: $\eta(r_0) \approx 0.76 < 1$.
Hence $\mu(r) \leq 2\sqrt{r}$.

□

Now that the Lemma is proven, we can directly deduce the existence of a unique inverse homeomorphism $\eta : [0,1] \longrightarrow [0,1]$. Another direct consequence of the lemma, is that the upper bound given for the function μ can be transformed into a lower bound for η: $\eta(s) \geq \frac{s^2}{4}$; for any $s \in [0,1]$.

Theorem 2.1.6. *(Landau's Covering Theorem) Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Then,*

$$D(0, \eta(|f'(0)|)) \subset f(\mathbb{D}).$$

In particular, since $\eta \geq \frac{s^2}{4}$,

$$D(0, \frac{(|f'(0)|)^2}{4}) \subset f(\mathbb{D}).$$

Proof. It is enough to observe that if $\omega \in \mathbb{D} \setminus f(\mathbb{D})$ then $\mu(|\omega|) \geq |f'(0)|$ since if we apply η on both sides, this will give that $|\omega| \geq \eta(|f'(0)|)$.
We will consider the disc's automorphism, taking $b \in \mathbb{D}$,

$$S_b(z) = \frac{z-b}{1-z\bar{b}} \quad \text{and} \quad S_b'(z) = \frac{1-b\bar{b}}{(1-z\bar{b})^2}.$$

So, instead of our initial function we will proceed with $g = S_\omega \circ f \in H(\mathbb{D})$. Since $\omega \notin f(\mathbb{D})$, then $g(z) \neq 0$ for any $z \in \mathbb{D}$. Moreover, $g(0) = -\omega$. Since $g(\mathbb{D}) \subset \mathbb{D} \setminus \{0\}$ and recalling the Proposition 1.2.16:

$$|g'(0)| \leq 2|g(0)| \ln \frac{1}{|g(0)|},$$

that, in terms of f is rewritten as:

$$(1-|\omega|^2)|f'(0)| \leq |S_\omega'(0)||f'(0)| \leq 2|\omega| \ln \frac{1}{|\omega|} \tag{2.3}$$

which leads us to: $|f'(0)| \leq \frac{2|\omega| \ln \frac{1}{|\omega|}}{1-|\omega|^2} = \mu(|\omega|)$. □

Proposition 2.1.7. *(Optimality of Theorem 2.1.2) The bound given is optimal: in the sense that given $t \in (0,1)$, there exists $f \in H(\mathbb{D})$ such that $f(0) = 0$, $|f'(0)| = \mu(t)$ and $dist(0, \partial f(\mathbb{D})) = t$.*

Proof. We will use, as we have seen in Proposition 1.2.16, an optimal estimate. Let us take $t \in (0,1)$ and define the following function that depends on t:

$$F_t(z) = \exp\left(-\log\left(\frac{1}{t}\right) \cdot \frac{1+z}{1-z}\right), \text{ for any } z \in \mathbb{D}.$$

The function F_t is holomorphic on $\mathbb{D}$ and $F_t(z) \neq 0$ for any $z \in \mathbb{D}$. Furthermore,

$$F_t(0) = t \Longrightarrow |F_t'(0)| = 2|F_t(0)| \log \frac{1}{|F_t(0)|} = 2t \log \frac{1}{t}$$

Now we define $f_t = S_t \circ F_t$: $f_t(\mathbb{D}) \subset \mathbb{D} \setminus \{-t\}$ and $f_t(0) = 0$. Hence,

$$|f_t'(0)| = \frac{1}{1-t^2}|F_t'(0)| = \frac{1}{1-t^2} 2t \log \frac{1}{t} = \mu(t).$$

In conclusion, $dist(0, \partial f_t(\mathbb{D})) = t = \eta(|f_t'(0)|)$. □

Now we have proven both versions of Landau's Theorem: the injective and the covering versions. We have also proven that both results provide an **optimal bound** for the radius of discs related to each version.

Theorem 2.1.8. *Let $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Hence, f covers injectively the disc $D(0, \eta(a)\rho(a))$, where $a = |f'(0)|$.*

This theorem combines the two previous versions of Landau's theorem that we have proved.

Proof. If it is required, we will apply a rotation of the form $e^{i\theta}$, where $\theta \in (0, 2\pi)$ in order to be able to denote $a := |f'(0)| > 0$.
Applying Theorem 2.1.3 we obtain that our function f is injective on $D(0, \rho(a))$.
Now we can define another function that depends on f in the following way: $g(z) = \frac{f(z\rho(a))}{\rho(a)}$. The function g is clearly holomorphic. Also, f sends $\mathbb{D}$ into itself by hypothesis. By Schwarz's Lemma we know that $|f(z)| \leq |z|$ for any $z \in \mathbb{D}$, in particular, $|f(z\rho(a))| \leq |z|\rho(a)$. In terms of g, this means $|g(z)| \leq |z| < 1$, $g(\mathbb{D}) \subset \mathbb{D}$ and this, added to the injectivity of f on $D(0, \rho(a))$, proves that g is injective on $\mathbb{D}$. We next compute its derivative:

$$g'(z) = \left(\frac{f(z\rho(a))}{\rho(a)}\right)' = \frac{1}{\rho(a)}\rho(a) f'(z\rho(a)) = f'(z\rho(a)).$$

Thus, $|g'(0)| = |f'(0 \cdot \rho(a))| = |f'(0)| = a$.
We know are under the required assumptions to apply Landau's Covering Theorem: the function g covers $D(0, \eta(a))$. This, rewritten in terms of f, is equivalent to f covers injectively the disc $D(0, \rho(a)\eta(a))$. □

Therefore the function that describes the radius of the disc that our function covers injectively is $\rho(a)\eta(a)$. Eventually, we study how this function behaves.

We do not have a formula for $\eta(a)$, but we have a lower bound that is as useful for what we demand: $\frac{a^2}{4} \leq \eta(a)$. Hence,

$$\rho(a)\eta(a) \geq \frac{a}{1+\sqrt{1-a^2}} \cdot \frac{a^2}{4} = \frac{a^3}{4(1+\sqrt{1-a^2})}.$$

Let us recall that η is the inverse homeomorphism of μ and some other properties mentioned in Lemma 2.1.5: $\eta(1) = 1 = \rho(1)$. Likewise for $a = 0$. Also, both functions are non-negative increasing on $[0,1]$ which leads to the next result.

Let us take $c, b \in [0,1]$; $c < b$. We know that $0 \leq \rho(c) < \rho(b)$ and $0 \leq \eta(c) < \eta(b)$. Thereby, $\rho(c)\eta(c) < \rho(b)\eta(b)$. Then $\rho(a)\eta(a)$ is an increasing function and this allows us to say that $\rho(a)\eta(a)$ establishes an homeomorphism between $[0,1]$ and itself.

2.2 Bloch's Theorem

In Theorem 2.1.2 we have proved that, given a holomorphic function f on $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$ contains a disc centred at 0 whose radius depends only on $|f'(0)|$.
But what happens if we weaken the hypothesis on f? What happens if $f(0)$ does not need to be 0 and $f(\mathbb{D})$ is no longer a subset of $\mathbb{D}$? The result that answers this question is called **Bloch's Theorem**.

From this result we will later deduce both Schottky's and Picard's Little Theorem, in spite of this being posterior to the other two. Before we announce and prove it, we will observe the relevance of this theorem through a comment made by J.E. Littlewood [7], on page 183.

This exceedingly odd and striking theorem resembles Theorem [2.1.2], but the condition $|f| < M$ of the later theorem is completely dropped (and nothing replaces it except that absolute constant). It can be used to prove some of the "Picard" theorems of the next section (indeed it gives proofs in which the function theory involved is of the least possible "depth")[...]

Given the existence of the theorem, and (what will become plausible in a moment) that Landau's theorem is relevant to its proof, any competent analyst should be able to find one: it is true that then oddity of the theorem is reflected in the critical step, but the step is forced, and then not difficult to make.

Theorem 2.2.1. *(Bloch's Theorem) Let $f \in H(\mathbb{D})$, $f'(0) \neq 0$. Then $f(\mathbb{D})$ contains a disc of radius $\frac{|f'(0)|}{4}$.*

In order to prove Theorem 2.2.1, we need to define:

Definition 2.2.2. *Given Ω a domain in $\mathbb{C}$, we define its **inner radius** $R(\Omega)$ as:*

$$R(\Omega) = \sup_{\omega \in \Omega} dist(\omega, \partial\Omega).$$

Equivalently, it is the supremum of the radius of all the discs included in Ω.

Observe that if $a \in \mathbb{C}$ and $\lambda > a$, then $R(a + \lambda\Omega) = \lambda R(\Omega)$. The inner radius is invariant under translations and linear for dilations.

Here are some examples of the different values that this inner radius can take depending on the domain:

- If $\Omega = \{z \in \mathbb{C} : \; Im(z) \geq 0\}$, then $R(\Omega) = \infty$.

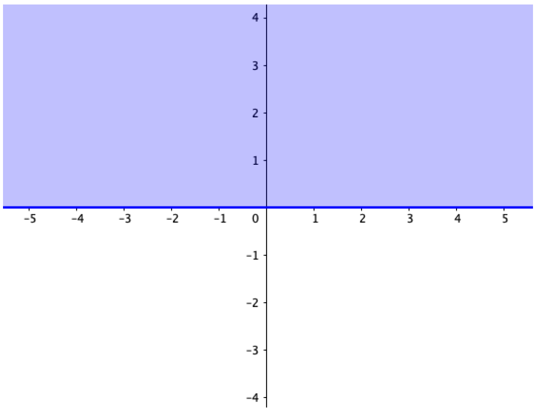

Figure 2.1: Ω_1: $\{z \in \mathbb{C} : Im(z) \geq 0\}$.

- If $\Omega = \mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$, then $R(\Omega) = \frac{\sqrt{2}}{2}$.

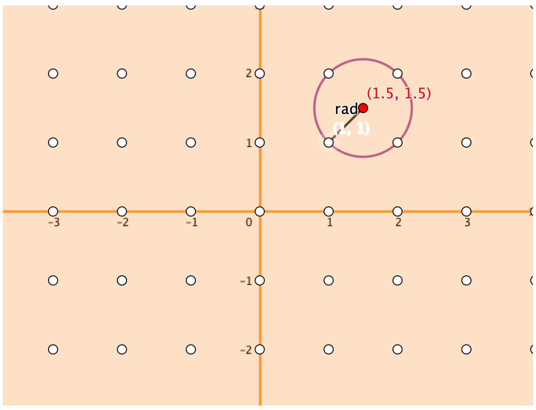

Figure 2.2: Ω_2: $\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$.

- If $\Omega = \{z \in \mathbb{C} : \ 1 \leq Im(z) \leq 5\}$, then $R(\Omega) = 2$.

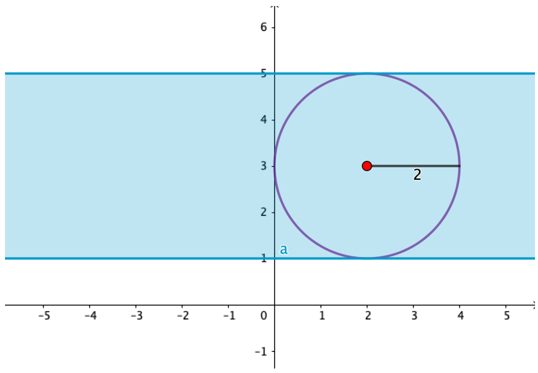

Figure 2.3: Ω_3: $\{z \in \mathbb{C} : \ 1 \leq Im(z) \leq 5\}$.

Definition 2.2.3. *Ω is a **Bloch Domain** if $R(\Omega) < \infty$.*

This is the structure that our proof of Theorem 2.2.1 is going to have:

1. First we give an invariant reformulation equivalent to Theorem 2.2.1.
2. We prove the reformulation.

Theorem 2.2.4. *(Invariant Bloch Theorem) Let $f \in H(\mathbb{D})$ and Ω a domain in $\mathbb{C}$. If $f(\mathbb{D}) \subset \Omega$, then*

$$\sup_{z \in \mathbb{D}} D_h f(z) \leq 2R(\Omega), \text{ where we recall } D_h f(z) = \frac{(1-|z|)^2}{2}|f'(z)|.$$

Proposition 2.2.5. *Theorem 2.2.1 and Theorem 2.2.4 are equivalent.*

Proof.

- Theorem 2.2.1$\Longrightarrow$Theorem 2.2.4: Assume $f(\mathbb{D})$ contains a disc of radius $\frac{|f'(0)|}{4}$. Indeed, from this assumption we deduce that for any Ω domain such that $f(\mathbb{D}) \subset \Omega$, $\frac{|f'(0)|}{4} \leq R(\Omega)$ that is $D_h f(0) \leq 2R(\Omega)$. Next let $a \in \mathbb{D}$ and let T_a be a disc's automorphism defined by $T_a(z) = \frac{z+a}{1+z\bar{a}}$. Then, $(f \circ T_a)(\mathbb{D}) = f(\mathbb{D}) \subset \Omega$ and, recalling Remark 1.2.12, we obtain $D_h(f \circ T_a)(0) = D_h f(T_a(0)) = D_h f(a)$. Since the function $f \circ T_a$ is under Bloch Theorem's hypotheses, applying the same procedure we obtain that $D_h f(a) \leq 2R(\Omega)$, for any $a \in \mathbb{D}$.

- Theorem 2.2.4$\Longrightarrow$Theorem 2.2.1: Let us apply the above observation to $f \circ T_a$ and $f \in H(\mathbb{D})$. We will denote as $\Omega = f(\mathbb{D})$. Applying Theorem 2.2.4 we obtain

$$|f'(0)| = 2D_h f(0) \leq 4R(\Omega) = 4R(f(\mathbb{D})),$$

then, $R(f(\mathbb{D})) \geq \frac{|f'(0)|}{4}$. Thus $f(\mathbb{D})$ contains a disc whose radius is as close to $\frac{|f'(0)|}{4}$ as we want but strictly smaller.

$\square$

Remark 2.2.6. *The estimate* 2 *in Theorem 2.2.4 can be substituted by a* $\delta < 2$*, that will be seen in the proof of the theorem.*

Now we prove Theorem 2.2.4.

Proof of Theorem 2.2.4. We will first consider the case where $f \in H(\overline{\mathbb{D}})$. In such case, $D_h f$ is continuous on $\mathbb{D}$ and it is identically 0 on $\partial\mathbb{D}$, and consequently it attains its maximum at a point $a \in \mathbb{D}$, that is $D_h f(a) = \max_{z\in\mathbb{D}} D_h f(z) < \infty$.
Since the hyperbolic derivative is invariant under the action of the group $Aut(\mathbb{D})$ and, as we have observed, $R(\Omega)$ is invariant under translations, we have that we can consider the function $f \circ T_a - f(a)$ instead of the initial f. So, without loss of generality, we may assume that $a = 0$ and $f(0) = 0$.
Now applying the inequality from Lemma 2.2.7: $|f(z) - f(\omega)| \leq D_h f(0)\rho(z, \omega)$, that will be proved later on, and recalling that $\max_{z\in\mathbb{D}} D_h f(z) = D_h f(0)$, we deduce that

$$|f(z)| \leq D_h f(0)\rho(0, z), \quad \text{for any } z \in \mathbb{D}. \tag{2.4}$$

(If $D_h f(0) = 0$, then $f \equiv 0$ and $D_h f(z) \equiv 0$. So we may assume that $D_h f(0) \neq 0$). Next let $r \in (0, 1)$. We define the holomorphic function g_r on $\mathbb{D}$ by $g_r(z) = \frac{f(rz)}{D_h f(0)\rho(0,r)}$. It clearly satisfies $g_r(0) = 0$ and, using (2.4) and the fact that $\rho(0, z)$ is an increasing function we have

$$|g_r(z)| \leq \frac{|f(rz)|}{D_f(0)\rho(0, r)} \leq \frac{D_f(0)\rho(0, rz)}{D_f(0)\rho(0, r)} < 1, \; g(\mathbb{D}) \subset \mathbb{D}.$$

Moreover, it verifies

$$|g_r'(0)| = \frac{2r}{\rho(0, r)}.$$

Applying now Landau's Covering Theorem, $D(0, \eta(\frac{2r}{\rho(0,r)})) \subset g_r(\mathbb{D})$. This, in terms of f, is rewritten as

$$D_h f(0)\rho(0, r)\eta\left(\frac{2r}{\rho(0, r)}\right) \leq R(\Omega). \tag{2.5}$$

Maximising the function $r \to \rho(0, r)\eta(\frac{2r}{\rho(0,r)})$, we can show that the supremum is bigger than 0.5, see Annex, which gives $R(\Omega) > \frac{1}{2}D_h f(0)$ and, equivalently, $D_h f(0) < 2R(\Omega)$ that proves the Remark 2.2.6. In particular, there exists $\delta < 2$ such that $D_h f(0) < \delta R(\Omega)$ and using that $f \circ T_a$ also

satisfies the same hypotheses, this is true for any $z \in \mathbb{D}$. This proves the theorem for $f \in H(\bar{\mathbb{D}})$.
Finally, for the general case, let $f \in H(\mathbb{D})$ and if $r \in (0,1)$, we define the radial function $h_r(z) = f(rz) \in H(\overline{\mathbb{D}})$. Clearly $h_r(\mathbb{D}) \subset f(\mathbb{D})$ and $h_r(0) = 0$. Therefore, by the previous call, we have:

$$|h_r'(z)|(1-|z|^2) = |rf'(rz)|(1-|z|^2) < 2\delta R(\Omega)$$

and now taking limit when r tends to 1^-, and divide by 2, to get:

$$D_h f(z) = \frac{(1-|z|^2)}{2}|f'(z)| \leq \delta R(\Omega).$$

□

In order to finish the proof of Bloch's Theorem, we are left to prove Lemma 2.2.7.

Lemma 2.2.7. *Under the hypothesis of Theorem 2.2.4, z, w $\in \mathbb{D}$. Then,*

$$|f(z) - f(w)| \leq \sup_{z\in\mathbb{D}} D_h f(z)\rho(z,w),$$

where ρ is the hyperbolic distance.

Proof. Since the hyperbolic derivative is invariant under Möbius transformations, we can assume $w = f(w) = 0$. Thus, the desired inequality gets simplified to:

$$|f(z)| \leq \sup_{z\in\mathbb{D}} D_h f(z) \log\left(\frac{1+|z|}{1-|z|}\right).$$

Let $B := \sup_{z\in\mathbb{D}} D_h f(z)$. By hypothesis: $|f'(z)| \leq \frac{2B}{1-|z|^2}$ and since $f(0) = 0$, writing $f(z) = \int_0^1 zf'(tz)dt$, we get:

$$|f(z)| \leq \int_0^1 |z||f'(zt)|dt \leq \int_0^1 |z|\frac{2B}{1-t^2|z|^2}dt = B\log\left(\frac{1+|z|}{1-|z|}\right).$$

To sum it up: we obtain $|f(z)| \leq \sup_{z\in\mathbb{D}} D_h f(z)\rho(0,z)$. □

The next result is a direct consequence of Lemma 2.2.7 and Theorem 2.2.4.

Corollary 2.2.8. *Let $f \in H(\mathbb{D})$ and let Ω be a domain such that $f(\mathbb{D}) \subset \Omega$. Then,*

$$|f(z) - f(w)| \leq 2R(\Omega)\rho(z,w) \text{ for any } z, w \in \mathbb{D}.$$

Observe that in particular, considering $w = 0$ and f a function that satisfies the hypothesis, we have

$$|f(z) - f(0)| \leq 2R(\Omega)\log\left(\frac{1+|z|}{1-|z|}\right).$$

This inequality that gives an estimate, provided that Ω is a Bloch domain, for the distance between $f(0)$ and $f(z)$.
This observation shows that for Bloch domains Ω, the function

$$\Phi(a, r) = a + 2R(\Omega) \log\left(\frac{1+r}{1-r}\right), \ a \geq 0 \text{ and } r \in [0, 1],$$

satisfies that for any $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \Omega$, $|f(z)| \leq \Phi(|f(0)|, r)$.

Our next goal will be to obtain an estimate of this type when $\Omega = \mathbb{C} \setminus \{0, 1\}$, whose answer will be given by **Schottky's Theorem**, and we will prove in the next chapter.

Lastly, we use Theorem 2.2.4 to prove the following statement, which is a key tool in the proof of Picard's Little Theorem.

Proposition 2.2.9. *(An in-between between Liouville and Picard's Theorem) Let f be an entire function. Assume that Ω is a Bloch domain and $f(\mathbb{C}) \subset \Omega$. We then have that f is constant.*

Proof. We fix $a \in \mathbb{C}$ and $R > 0$. Now let us define $g \in H(\mathbb{C})$:
$g(z) = f(a + Rz)$; $\forall z \in \mathbb{C}$. By applying Theorem 2.2.4 to g, we obtain:

$$R|f'(a)| = |g'(0)| \leq 4R(\Omega).$$

Since this is true for any $a \in \mathbb{C}$ and any $R > 0$, $f' \equiv 0$. Therefore, f is a constant function. ☐

2.3 Landau and Bloch's Constants

In this sections, we state some estimates that have been proven along the years for values related to both Landau's and Bloch's Theorems. Let us first start with some definitions.

Definition 2.3.1. *Let us define, for any function $f \in H(\mathbb{D})$ such that $f'(0) = 1$ the following value:*

$$\lambda(f) = \sup\{r : f(\mathbb{D}) \textit{ contains a disc of radius } r\}.$$

*Then, we define **Landau's Constant** as*

$$\boldsymbol{L} = \inf\{\lambda(f) : f \in H(\mathbb{D}), \, f'(0) = 1\}.$$

This constant has been an object of interest for many years and finding its exact value is still an open problem. Using our result, we obtain $0,25 \leq L$.

In 1938, Ahlfors proved that $0.5 \leq L$. In 2004 Chen and Shiba [1] proved that

$$0.5 + 2 \cdot 10^{-8} < L.$$

As for the upper bound, it was first obtained by R.M.Robinson from an example made by Ahlfors. Later in 1982, C. Minda, based on the same method as Ahlfors, proved through that same example that

$$L \leq \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} \approx 0.543259\ldots.$$

We conclude doing the same procedure, but now for Bloch's constant.

Definition 2.3.2. *Given any function $f \in H(\mathbb{D})$ such that $f'(0) = 1$, we define*

$$\beta(f) = \sup\{r : \textit{there exists } S \subset \mathbb{D} \textit{ on which } f \textit{ is injective and } f(S) \textit{ contains a disc of radius } r\}.$$

*Then, we define **Bloch's Constant** as*

$$\boldsymbol{B} = \inf\{\beta(f) :\ f \in H(\mathbb{D}),\ f'(0) = 1\}.$$

It is clear that $B \leq L$. The bounds for this constant came later in time compared to Landau's constant. For the lower bound, in 1998, C. Xiong proved that

$$\frac{\sqrt{3}}{4} + 3 \cdot 10^{-14} < B.$$

The upper bound for Bloch's constant is

$$B \leq \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\sqrt{1+\sqrt{3}}\Gamma(\frac{1}{4})} \approx 0.471862\ldots$$

that was obtained in 1937 by Ahlfors and Grunsky and, nowadays, it is conjectured to be the actual value of B. In 2008, R.Rettinger [9], made a theoretical algorithm to find it.

Chapter 3

Omission of Values. Picard's Little Theorem and Schottky's Theorem

3.1 Omission of Values

Up to this point, we have focused on properties of function to know its values on a precise set, typically the unit disc $\mathbb{D}$. We now will study what can be said for a holomorphic function that omits certain values. We will follow the references [2], [4], [5], [6] and [8]. We begin with some definitions.

Definition 3.1.1. *Let $f \in H(\Omega)$, where $\Omega \subset \mathbb{C}$ is a domain.*

- *Let $\omega \in \mathbb{C}$, **f omits the value** ω if $\omega \notin f(\Omega)$.*
- *Let $E \subset \mathbb{C}$, **f omits the set E** if $f(\Omega) \cap E = \emptyset$.*

*Let E be a closed set on $\mathbb{C}$. E is **δ-dense**, $\delta > 0$, if :*

$$\bigcup_{\omega \in E} \overline{D(\omega, \delta)} = \mathbb{C},$$

that is if any point in $\mathbb{C}$ is at distance at most δ from some points in E. We also define $\Delta(E) := \inf\{\delta : \text{ such that } E \text{ is } \delta\text{-dense}\}$.

Recalling an example that we have studied before, $(\mathbb{Z} + i\mathbb{Z})$, we can see that it is a $\frac{\sqrt{2}}{2}$-dense set. If there is no δ, then $\Delta(E) = \infty$. In this example, we describe the set

$$\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z}) = \bigcup_{n,m \in \mathbb{Z}} I_{n,m},$$

where $I_{n,m} = \{z = x + iy \in \mathbb{C} : \; x \in (n, n+1), \; y \in (m, m+1)\}$.

If $z \in \mathbb{C}$, then either $z \in \mathbb{Z} + i\mathbb{Z}$ or there exist n, m such that $z \in I_{n,m}$ and hence the distance to one of the vertices is at most $\frac{\sqrt{2}}{2}$. This gives that $\mathbb{Z} + i\mathbb{Z}$ is a $\frac{\sqrt{2}}{2}$-dense set and $\Delta(\mathbb{Z} + i\mathbb{Z}) = \frac{\sqrt{2}}{2}$. Another example we have already seen is $\mathbb{Z}$: in this case $\Delta(\mathbb{Z}) = \infty$.

Remark 3.1.2. *Observe that if a set $\mathbb{C}\backslash\Omega$ is δ-dense, the maximum distance from $\mathbb{C} \setminus \Omega$ to some points in $\mathbb{C}$ is δ. This can be rewritten as $R(\Omega) \leq \delta$.*

Using Remark 3.1.2 and Theorem 2.2.4, assuming $f \in H(\mathbb{D})$ such that $f(\mathbb{D}) \subset \Omega$ we get:

$$\sup_{z \in \mathbb{D}} D_h f(z) \leq 2\delta,$$

which, in terms of ρ, is equivalent to

$$\sup_{z \in \mathbb{D}} |f(z) - f(0)| \leq 2\delta\rho(0, z) = 2\delta \log\left(\frac{1 + |z|}{1 - |z|}\right).$$

Lemma 3.1.3. *Let A be a domain and B a set in $\mathbb{C}$ such that $A \cap B = \emptyset$. Then $R(B) \leq \Delta(A)$.*

Proof. Observe that $R(B)=\sup\{r :$ there exists an $a \in B$ such that $D(a, r) \subset B\}$. Then, for any $\varepsilon > 0$, $R(B) - \varepsilon < R(B)$ and thus, there exists $r > R(B) - \varepsilon$ and $a \in B$ such that $D(a, r) \subset B$. Hence, $D(a, R(B) - \varepsilon) \subset B$ that combined with the fact that $A \cap B = \emptyset$ means that $\Delta(A) \geq R(B) - \varepsilon$. Lastly, since this is true for any $\varepsilon > 0$, taking the limit when $\varepsilon \to 0$, we obtain $\Delta(A) \geq R(B)$. $\square$

So, the discussion made above in this page can be written in terms of omitted sets as: if $f \in H(\mathbb{D})$ and $f(\mathbb{D}) \cap E = \emptyset$, then

$$|f(z) - f(\omega)| \leq 2\Delta(E)\rho(z, \omega), \text{ for any } z, \omega \in \mathbb{D}.$$

Thus, Proposition 2.2.9 can be also written in terms of omitted sets as:

Proposition 3.1.4. *(Between Liouville and Picard's Theorem - Omitted) Let f be an entire function that omits a δ-dense set. Then, f is constant.*

3.1.1 Omitted Sets through Composition. Elevation of Holomorphic Functions

One of the goals of this chapter is proving Picard's Little Theorem, that we will derive from Proposition 3.1.4. This theorem states that any entire function f that omits two values is constant. The idea of the proof is based in expressing this entire function as an exponential of another entire function that omits a δ-dense set. Hence, using the required proposition, we will obtain the result.

Definition 3.1.5. *Let f be a function $f \in H(\Omega)$ that omits a set E. If f can be written as $f = \Pi \circ g$, where $g \in H(\Omega)$ and $\Pi \in H(g(\Omega))$; we will say that g is an* ***elevation*** *of f and that Π is the* ***elevator function****.*

Our aim is to obtain a suitable elevation such that the omitted set, $\Pi^{-1}(E)$, is bigger and better distributed on $\mathbb{C}$ than E. In this procedure we will use repeatedly the fact that there exists an holomorphic $\log(\cdot)$ of holomorphic functions that do not vanish on simply connected sets.
This lemma is the first look on what we aim to study in this part of the chapter. Next, we are now going to see how the omitted set is for an specific type of functions that will be useful later on.

Proposition 3.1.6. *Let $f \in H(\Omega)$ that omits $\{0, 1\}$. Then, f can be written as $f = e^{2\pi i g}$, where $g \in H(\Omega)$ that omits $\mathbb{Z}$, and reciprocally.*

Proof. We have that f omits 0 and is holomorphic on Ω. Then, there exists an holomorphic logarithm on Ω, u, such that $f = e^u$. But f not only avoids 0, also avoids 1, which forces u to omit the set $\{2\pi i k;\ k \in \mathbb{Z}\}$. This in terms of g means that g has to omit $\mathbb{Z}$.
Reciprocally, if g is holomorphic on Ω that omits $\mathbb{Z}$, then $f = e^{2\pi i g} \in H(\Omega)$ and omits $\{0, 1\}$. $\square$

Even if f only omits 0, there will still exists an holomorphic logarithm g for f. Even though, g will not necessarily omit any value.
Looking back to Proposition 3.1.6, the omitted set is $\mathbb{Z}$, which is not δ-dense. Then, we cannot apply Theorem 3.1.4. But, is it possible to eventually reach a δ-dense set if we keep doing elevations of our functions?
Let us write g, that omits $\mathbb{Z}$, through an elevation, i.e., $g = e^{2\pi i h}$. Knowing that g omits 1, h has to omit $\mathbb{Z}$. Which additional points does h omit when we take in account that g omits the whole of $\mathbb{Z}$? Take

$$e^{2\pi i z} = e^{2\pi i(x+iy)} = k \in \mathbb{Z}.$$

Then $x = \frac{j}{2}$, for $j \in \mathbb{Z}$. Eventually we are left with

$$e^{-2\pi y} = k, \text{ which leads to } y = -\frac{\log k}{2}.$$

The answer to the question that we made earlier in this paragraph is the set

$$E := \{\frac{j}{2} - i\frac{\log k}{2};\ k, j \in \mathbb{Z}, k \geq 1\}. \tag{3.1}$$

This set is δ-dense, but not on the whole $\mathbb{C}$, only in the lower half plane (due to the fact that the for any $z \in E$, $\mathrm{Im}(z) \leq 0$).
What happens if we iterate this argument? One may think that taking another logarithm, the corresponding omitted set is going to be even bigger

and it can be a δ-dense set. But this is not the case: take $\alpha \in H(\Omega)$ that omits a discrete set E such that $0 \in E$ as it is the case to the one described in (3.1). Let $r = d(0, E \setminus \{0\}) > 0$, since E is discrete. Let β be an holomorphic logarithm of α, i.e., $\alpha = e^{\beta}$. Then β omits the set $F := \{w \in \mathbb{C} : \ e^w \in E \setminus \{0\}\}$. But if $w \in F$, $|e^w| > r$, which gives that $\mathrm{Re}(w) > \log(r)$ and $F \subset \{w : \ \mathrm{Re}(w) > \log(r)\}$. Hence, F is not $\delta-$dense.
Now let $f \in H(\Omega)$ such that omits the set $\widetilde{E} := \{\frac{1}{n}; \ n > 0\} \cup \{n; \ n > 0\}$, then there exists an holomorphic logarithm of f, g, i.e., $f = e^g$. If f omits $\widetilde{E}$, g must omit both $\log(n)$ and $\log(\frac{1}{n})$, for any $n \geq 1$. Hence, g omits $\{\pm \log(n) + 2\pi i k, \ n \geq 1, \ k \in \mathbb{Z}\}$.
This next result will be useful on going from a function omitting $\{0, 1\}$ to another omitting a set with some analogies with $\widetilde{E}$.

Proposition 3.1.7. *Let Ω be a simply connected domain and let $f \in H(\Omega)$ that omits $\{-1, 1\}$. Then, there exists $g \in H(\Omega)$ that omits $\{-1, 0, 1\}$ and $f = J(g)$, where J is the **Jukowski's Transformation** given by*

$$J(z) = \frac{1}{2}(z + \frac{1}{z}), \ z \in \mathbb{C} \setminus \{0\}.$$

And reciprocally.

Proof. The function $f^2 - 1$ does not vanish on Ω, which is a simply connected domain. Hence there exists $u \in H(\Omega)$ such that $u^2 = f^2 - 1$. Observe that if we write $g = f + u$, g does not vanish since, by its own definition, $(f + u)(f - u) = f^2 - u^2 = 1$. Next,

$$J(g) = \frac{1}{2}(g + \frac{1}{g}) = \frac{1}{2}(f + u + f - u) = f,$$

as we wanted to prove. Moreover, $J(1) = 1$ and $J(-1) = -1$. Therefore, $J(g) \cap \{-1, 1\} = \emptyset$.
Reciprocally, applying the definition of J and assuming that a function g omits $\{-1, 0, 1\}$, it is easy to see that $J(g) \in H(\Omega)$ and $J(g)$ omits $\{-1, 1\}$. $\square$

Now let $f \in H(\Omega)$ such that omits $\mathbb{Z}$. In particular, it omits $\{-1, 1\}$; so we now apply Proposition 3.1.7 and we have that there exists g such that $f = J(g)$, where $g \in H(\Omega)$ and omits $\{-1, 0, 1\}$. But also, since f omits $\mathbb{Z}$ and not only $\{-1, 1\}$, g also omits $J^{-1}(\mathbb{Z} \setminus \{0\})$.
Let us describe the set $J^{-1}(\mathbb{Z} \setminus \{0\})$. We will check that

$$J^{-1}(\mathbb{Z} \setminus \{0\}) = A \cup \left(\frac{1}{A}\right) \cup (-A) \cup \left(\frac{-1}{A}\right),$$

where the set $A := \{a_n = n + \sqrt{n^2 - 1}, \ n \geq 1\}$, $\lambda A = \{\lambda a : \ a \in A\}$ and $\frac{1}{A} = \{\frac{1}{a} : \ a \in A\}$.

Observe that since we have $(n+\sqrt{n^2-1})(n-\sqrt{n^2-1}) = 1$, for any $n \geq 1$, $J(n+\sqrt{n^2-1}) = J(\frac{1}{n+\sqrt{n^2-1}}) = -J(-(n+\sqrt{n^2-1})) = -J(\frac{-1}{n+\sqrt{n^2-1}}) = k \neq 0$.
If, for instance, $a_n = n+\sqrt{n^2-1}$, then

$$J(a_n) = \frac{1}{2}(a_n + \frac{1}{a_n}) = \frac{1}{2}((n+\sqrt{n^2-1}) + (n-\sqrt{n^2-1})) = \frac{2n}{2} = n.$$

Likewise, $J(-a_n) = -n$, $J(\frac{1}{a_n}) = n$ and $J(\frac{-1}{a_n}) = -n$. Hence, $A \cup \left(\frac{1}{A}\right) \cup (-A) \cup \left(\frac{-1}{A}\right) \subset J^{-1}(\mathbb{Z} \setminus \{0\})$.
Reciprocally, if we take $w \in J^{-1}(\mathbb{Z}\setminus\{0\})$, $J(w) = k \neq 0$. It can be expressed as $w + \frac{1}{w} = 2k$, which defines w in terms of k as $w = k \pm \sqrt{k^2-1}$. Hence, $w \in A \cup \left(\frac{1}{A}\right) \cup (-A) \cup \left(\frac{-1}{A}\right)$.
Observe now that $J^{-1}(\mathbb{Z} \setminus \{0\})$ contains a sequence that tends to ∞ and another one that tends to 0, as it happened in the example $\widetilde{E}$.

Proposition 3.1.8. *Let $f \in H(\Omega)$ that omits $\{0,1\}$. Then, $f = \exp(2\pi i J(h))$, where $h \in H(\Omega)$ and omits 0 and $J^{-1}(\mathbb{Z} \setminus \{0\})$. Furthermore, the omitted set $S := \exp^{-1}(J^{-1}(\mathbb{Z} \setminus \{0\}))$ is δ-dense, where*

$$\delta = \frac{1}{2}\sqrt{\pi^2 + \log{(2+\sqrt{3})}^2} \approx 1.703 < 2.$$

Proof. Using Proposition 3.1.6, we write $f = \exp(2\pi i g)$ where $g \in H(\Omega)$ that omits $\mathbb{Z}$. Now applying the discussion made after Proposition 3.1.7, there exists $h \in H(\Omega)$ that omits 0 and $J^{-1}(\mathbb{Z} \setminus \{0\})$ and $g = J(h)$. Combining both results

$$f = \exp(2\pi i g) = \exp(2\pi i J(h)).$$

Next, let us prove that $\exp^{-1}(J^{-1}(\mathbb{Z} \setminus \{0\}))$ is δ-dense. Observe that

$$\exp^{-1}(J^{-1}(\mathbb{Z} \setminus \{0\})) = \{w : e^w \in J^{-1}(\mathbb{Z} \setminus \{0\})\}.$$

Recall that $J^{-1}(\mathbb{Z}\setminus\{0\}) = A \cup \frac{1}{A} \cup -A \cup \frac{-1}{A}$, where the elements in A are of the form $n+\sqrt{n^2-1}$, $n \geq 1$. Let us check that $S = exp^{-1}(J^{-1}(\mathbb{Z}\setminus\{0\})) = \{\pm\log{(n \pm \sqrt{n^2-1})} + i\pi k;\ k \in \mathbb{Z},\ n \geq 1\}$

- On the one hand, let $w = \log(n+\sqrt{n^2-1}) + i\pi k$. Then, $e^w = (-1)^k(n+\sqrt{n^2-1}) \in A \cup (-A)$. If $w = -\log(n+\sqrt{n^2-1}) + i\pi k$, analogously, $e^w \in \left(\frac{1}{A}\right) \cup \left(\frac{-1}{A}\right)$. Therefore, $w \in S$.
- On the other hand, if $w \in S$, then $e^w \in J^{-1}(\mathbb{Z}\setminus\{0\})$. Assume $e^w \in A$: $e^{x+iy} = n+\sqrt{n^2-1}$ for $n > 1$. Hence,

  $$y = 2\pi k,\ k \in \mathbb{Z} \text{ and } x = \log(n+\sqrt{n^2-1}).$$

  Then, w belongs to the set we wanted.

If the set is $-A$, the element k will be an odd number. In the cases where $e^w \in \left(\frac{1}{A}\right) \cup \left(\frac{-1}{A}\right)$, the logarithm will have a negative symbol.
Hence, $S = \{\pm \log\left(n \pm \sqrt{n^2-1}\right) + i\pi k;\ k \in \mathbb{Z},\ n \geq 1\}$. In this next figure, we see some of the points in the set S, for certain values of $n \geq 1$ and for $k \in \{-1, 0, 1\}$.

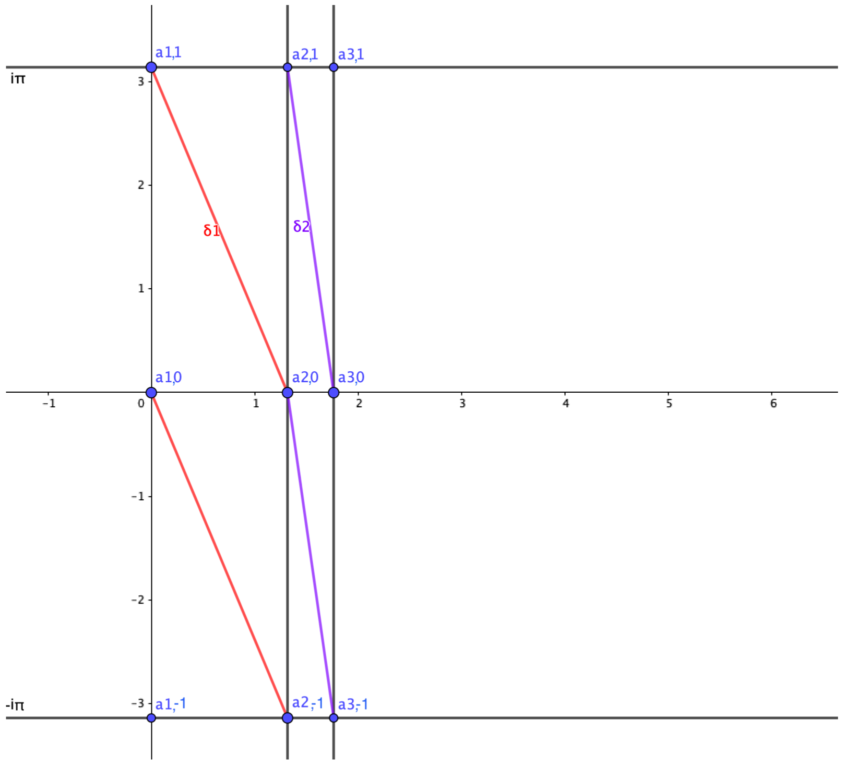

Figure 3.1: The elements of S $a_{i,j}$, where $i = n$ (n is the value in the description of S, $n \geq 1$) and $j = k$ (likewise for k, $k \in \mathbb{Z}$).

Fixed a $k \in \mathbb{Z}$, it appears that the distance between the $a_{n,k}$, whose maximum will appear in computing δ, decreases as n grows. Then we prove that the function $d(n) := \log\left(\frac{n+1+\sqrt{(n+1)^2-1}}{n+\sqrt{n^2-1}}\right)$, that describes the distance between $a_{n,k}$ and $a_{n+1,k}$, is a decreasing function.
Observe that, as shown in the figure above, when $n = 2$ (for any k), the value of $d(2)$ is greater than 0. Also,

$$\lim_{n\to\infty} \log\left(\frac{n+1+\sqrt{(n+1)^2-1}}{n+\sqrt{n^2-1}}\right) = \log 1 = 0.$$

Therefore, if we prove that $d(n)$ is a monotonous function for $n \geq 1$, we will have proved that d is decreasing.
At last, the derivative of the $d(n)$ is $\frac{1}{\sqrt{n(n+2)}} - \frac{1}{\sqrt{n^2-1}}$ and, when we find

its relative extremes, we observe that $d'(n) \neq 0$ for any $n \geq 1$. Hence, the function is decreasing.
So, δ will be half of the biggest hypotenuse, which is δ_1 (see Figure 3.1). Hence,

$$\delta = \frac{1}{2}\sqrt{\pi^2 + \log{(2+\sqrt{3})}^2} \approx 1.7311 < 2.$$

□

We have achieved what we were looking for: by composing a set of functions, we have been able to express f in terms of them. In addition, the inner function in the composition omits a 2-dense set. This can all be summed up in this result:

Theorem 3.1.9 (Schottky's Tower)**.** *Let $f \in H(\Omega)$ that omits $\{0,1\}$. Then, it can be written as*

$$f = exp(2\pi i J(e^{\phi})),$$

where $\phi \in H(\Omega)$ and omits the 2-dense set of Proposition 3.1.8.

3.2 Picard's Little Theorem

This theorem is a consequence of Theorem 3.1.9 when the simply connected domain, $\Omega = \mathbb{C}$. The case we are going to see is for the omission of $\{0,1\}$, we will later see that it still holds for any pair of values.

Theorem 3.2.1. *(Picard's Little Theorem) Let f be an entire function that omits 0 and 1. Then f is constant.*

Proof. First we have that f omits 0 and 1. Theorem 3.1.9 with $\Omega = \mathbb{C}$ gives that there exists $\phi \in H(\mathbb{C})$ such that $f = exp(2\pi i J(e^{\phi}))$ and ϕ omits a 2-dense set. Applying Proposition 3.1.4, ϕ is constant and, consequently, f is constant. □

Remark 3.2.2. *The number of omitted values has to be at least 2 to ensure that the entire function is a constant. Take $f(z) = e^z$, then, f omits $\{0\}$ and it is a non-constant function.*

In fact, we can rewrite Picard's Little Theorem in the following equivalent results.

Proposition 3.2.3. *The following results are equivalent*

1. *If $f \in H(\mathbb{C})$ and $0, 1 \notin f(\mathbb{C})$, f is constant.*
2. *If $f \in H(\mathbb{C})$ and it omits two values on $\mathbb{C}$, then f is constant.*
3. *Let $f, g \in H(\mathbb{C})$ such that $e^f + e^g = 1$, then f and g are constant.*

4. *Let $f, g \in H(\mathbb{C})$ such that they satisfy the equation $f^2 + g^2 = 1$ in $\mathbb{C}$. Then both f and g are constants if and only if $Z(f') = Z(g')$ (counting multiplicities).*

Proof. **1) $\Longleftrightarrow$ 2)** : 2) $\Longrightarrow$ 1) is immediate. Conversely, assume that 1) holds and let f such that omits any two values $\{a, b\}$. Then, we can define a new entire function $g = \frac{f-a}{b-a}$ that omits $\{0, 1\}$. Hence, g is constant and so is f.
1) $\Longleftrightarrow$ 3) :
Assume that $e^f + e^g = 1$. By definition, e^f and e^g omit 0. From the equation in the hypothesis we get: $e^f = 1 - e^g$ and since e^g omits 0, e^f also omits 1. Then, e^f omits $\{0, 1\}$ and by 1), e^f is a constant and analogously, so is e^g. Hence, both f and g are constants. Conversely, assume f omits $\{0, 1\}$. Then $1 - e^f$ is an holomorphic function that omits 0, thus, there exists an holomorphic function g such that

$$e^g = 1 - e^f.$$

Applying 3) we obtain that f, and also g, is constant.
4) $\Longrightarrow$ 1) :
Assume that 4) holds and let F an entire function that omits 0 and 1. Then, $F = e^p$ and $1 - F = e^q$, where $p, q \in H(\mathbb{C})$. Thus we can write $F = f^2$ and $1 - F = g^2$ where $f = e^{\frac{1}{2}p}$ and $g = e^{\frac{1}{2}q}$ and clearly $f^2 + g^2 = 1$. Next, by taking derivatives we get $ff' = -gg'$, we directly obtain that f' and g' have the same zeros with same multiplicity, since f and g omit 0. Recalling 4), f and g are constant and, therefore, so is F.
2) $\Longrightarrow$ 4) :
We first rewrite the equation on 4) as $(f + ig)(f - ig) = 1$. Denote $h = f + ig$ and since $h \not\equiv 0$, we can also write $h^{-1} = f - ig$. Thus, $f = \frac{h+h^{-1}}{2}$, $g = \frac{h-h^{-1}}{2i}$, which, taking derivatives, implies that

$$f' = \frac{h'}{2}\frac{h^2 - 1}{h^2}, \quad g' = \frac{h'}{2i}\frac{h^2 + 1}{h^2}. \tag{3.2}$$

Our claim is that h omits $\{-1, 1\}$. If $h^2(z_0) = 1$ for some z_0, then (3.2) gives that $f'(z_0) = 0$ and, since $Z(f') = Z(g')$, $g'(z_0) = 0$. But again using (3.2), we have that the order of z_0 as a zero of f' is bigger than its order as a zero of g', in contradiction with the hypothesis. Hence, h omits $\{-1, 1\}$ and, since 2) is equivalent to 1), h is constant and therefore, so are f and g. □

3.2.1 Consequences of Picard's Little Theorem

We have given some equivalent results to Picard's Little Theorem. Now, we want to move ahead from this result.
First we have a meromorphic version of Picard's Little Theorem.

Proposition 3.2.4. *Assume that f is a meromorphic function on $\mathbb{C}$ such that $\mathbb{C}_\infty \setminus f(\mathbb{C})$ has at least three points, then f is constant.*

Proof. First we consider the case where one of the omitted values is ∞. Then, f is entire and there are two distinct values a, b omitted by f. By Picard's Little Theorem, Theorem 3.2.1, f is constant.
Next we consider a meromorphic function that is not entire ($\infty \in f(\mathbb{C})$). By hypothesis, there exist 3 values $a, b, c \in \mathbb{C} \setminus f(\mathbb{C})$. Let us define g, an entire function, as

$$g(z) = \frac{1}{f(z) - a}, \ z \in \mathbb{C}.$$

It is entire because $a \notin f(\mathbb{C})$ and also $\frac{1}{b-a}, \frac{1}{c-a} \in \mathbb{C} \setminus g(\mathbb{C})$. Again, using Theorem 3.2.1, g is constant. Moreover, $g \neq 0$ since f is meromorphic ($f(z) \neq \infty$ for some $z \in \mathbb{C}$). Finally, we can write $f(z) = \frac{1}{g(z)} + a$ and f is clearly constant. □

Next we find a rather curious result. But we first need to recall a definition.

Definition 3.2.5. *Let f be a function and $z_0 \in \mathbb{C}$. We say that z_0 is a* ***fixed point*** *of f if $f(z_0) = z_0$.*

Proposition 3.2.6. *Let f be an entire function. Then $f \circ f$ has always a fixed point, except when f is a translation.*

Proof. Let us assume that $f \circ f$ has no fixed points and we will show that it is a translation. We have that $f \circ f$ has no fixed points, but neither does f. In fact, if there existed $z_0 \in \mathbb{C}$ such that $f(z_0) = z_0$, it would also be a fixed point of $f \circ f$. Then the function

$$g(z) = \frac{f(f(z)) - z}{f(z) - z},$$

omits $\{0, 1\}$.
Hence, by Picard's Little Theorem, g is constant, i.e. $g(z) = k \in \mathbb{C}$. In terms of f this is written as $k(f(z) - z) = f(f(z)) - z$ and taking derivatives we obtain $k(f'(z) - 1) = f'(z)f'(f(z)) - 1$.
Rearranging the terms we are left with the following equality

$$1 - k = f'(z)(f'(f(z)) - k). \tag{3.3}$$

But we have earlier stated that $g \equiv k$ cannot be neither 0 nor 1. Hence, $f'(z) \neq 0$ and $f'(f(z)) \neq k$, which means that $f' \circ f$ omits $\{0, k\}$ and, again, by Picard's Little Theorem is constant. This, combined with (3.3) proves that f' is constant as well. Thus there exist $\lambda, \mu \in \mathbb{C}$ such that $f(z) = \lambda z + \mu$. Next, if $\lambda \neq 1$, there will always exist a fixed point for f: $z_0 = \frac{\mu}{1-\lambda}$. Hence, $f(z) = z + \mu$ and μ cannot be 0 because f does not have a fixed point. Therefore, f is a translation. □

At last, we are going to use Picard's Little Theorem to do study how are the solutions of $X^n + Y^n = 1$ for meromorphic functions, for any $n \geq 3$.

Proposition 3.2.7. *Let us assume that f, g are meromorphic functions on $\mathbb{C}$ such that $f^n + g^n = 1$, for any $n \geq 3$. Then f and g are constant or they share some poles.*

Proof. Let us assume that f and g do not have mutual poles. It follows from $f^n + g^n = 1$ that none of the two functions have any pole and, therefore, $f, g \in H(\mathbb{C})$. Let us assume that $g \not\equiv 0$, hence $g^n \not\equiv 0$, and also using that $Z(f) \cap Z(g) = \emptyset$ (if they had a common 0, the equation $f^n + g^n = 1$ will not hold in that point), we can take the ratio of the initial equation over g^n to obtain

$$(\frac{f}{g})^n + 1 = \frac{1}{g^n}.$$

Since the right hand side of it is never 0, we observe that the meromorphic function $\frac{f}{g}$ omits ξ_i, for $i \in \{1, \ldots, n\}$, where the ξ_i are the n solutions to $x^n = -1$. Next, since $n \geq 3$, we apply Proposition 3.2.4 to obtain that $\frac{f}{g}$ is constant, i.e. $f = \lambda g$ (where $\lambda \neq \xi_i$, for any $i \in \{1, \ldots, n\}$). Next, we substitute this in the equation in the hypothesis to obtain

$$f^n + g^n = (\lambda g)^n + g^n = g^n(\lambda^n + 1) = 1.$$

Hence, g^n is constant and therefore, so is g. The proof for f is analogous. □

3.3 Schottky's Theorem

The result we are going to prove in this section is important since it controls the growth of a function, it also is a consequence of Theorem 3.1.9, when $\Omega = \mathbb{D}$. This result will be needed in the proof of Picard's Great Theorem.

Theorem 3.3.1. *(Schottky's Theorem) Given $f \in H(\mathbb{D})$ that omits $\{0, 1\}$. Then,*

$$|\log(f(z))| \leq C(1 + |\log|f(0)||)\left(\frac{1+|z|}{1-|z|}\right)^4, \text{ for any } z \in \mathbb{D},$$

where C is an absolute constant.

This theorem requires that f has to omit $\{0,1\}$. However, it can be given a version for a general function that omits $\{a,b\}$. Indeed, if f omits two other values, i.e., $\{a,b\}$, let us consider now the function $g=\frac{f-a}{b-a}\in H(\mathbb{D})$. The new function g does now omit $\{0,1\}$. This observation, that has been previously done, expands the range of this theorem to any holomorphic function omitting, at least, two values.

Proof. If $f\in H(\mathbb{D})$ omits $\{0,1\}$, applying Theorem 3.1.9, it can be written in terms of a function $\phi\in H(\mathbb{D})$ that omits a 2-dense set. Then we can apply Remark 3.1.2 to state that $\sup_{z\in\mathbb{D}} D_h\phi(z)\le 4$ (twice the value for which the set is δ-dense: 2) or, in terms of the distance ρ, $|\phi(z)-\phi(0)|\le 4\rho(0,z)$, for any $z\in\mathbb{D}$. From now on in this proof, we are going to undo all the elevations that we have earlier used. Consider g and $h\in H(\mathbb{D})$ such that satisfy the same properties as in Theorem 3.1.9

1. Observe that $|h|=e^{\operatorname{Re}\phi}$. Then, $|h(z)|\le|h(0)|e^{4\rho(0,z)}$ which is equal to $|h(0)|\cdot(\frac{1+|z|}{1-|z|})^4$ for any $z\in\mathbb{D}$. Likewise for $\frac{1}{|h(z)|}\le\left(\frac{1}{|h(0)|}\right)\cdot\left(\frac{1+|z|}{1-|z|}\right)^4$ for any $z\in\mathbb{D}$.

2. Let us recall that $f=e^{2\pi i g}$. If we used $\tilde{g}=g+s,\ s\in\mathbb{Z}$ instead of g, the condition that it has to satisfy $f=e^{2\pi i\tilde{g}}$ still holds. Hence, without loss of generality, we may assume that $\operatorname{Re} g(0)\in[1,2]$. Then, $1\le \mathrm{Re(g(0))}\le|\mathrm{g(0)}|\le|\mathrm{Re(g(0))}|+|\mathrm{Im(g(0))}|$ and, from the relation between f and g, $|f|=e^{2\pi\mathrm{Im(g(z))}}$. Therefore,
$$1\le|g(0)|\le 2+\frac{|\log|f(0)||}{2\pi}.$$

3. Now we express g as $J(h)$. Hence, it is easily obtained that
$$|g(z)|\le\frac{1}{2}\left(|h(0)|+\frac{1}{|h(0)|}\right)\cdot\left(\frac{1+|z|}{1-|z|}\right)^4;\ \text{for any } z\in\mathbb{D}.$$
Reconsidering the discussion made after Proposition 3.1.7 regarding Jukowski's transformation, we obtain
$$\max_{z\in\mathbb{D}}\{|h(0)|,\frac{1}{|h(0)|}\}\le|g(0)|+\sqrt{|g(0)|^2+1},$$
using the sets A defined in the same discussion. Since $|g(0)|\ge 1$, then $|g(0)|+\sqrt{|g(0)|^2+1}\le 3|g(0)|$.

Combining the 3 estimates, we have just proven
$$|\log(|f(z)|)|\le 3(6\pi+|\log(|f(0)|)|)\left(\frac{1+|z|}{1-|z|}\right)^4;\ \text{for any } z\in\mathbb{D}.$$

□

In order to see that Schottky's Theorem is stronger than Picard's Little theorem, we are going to prove this second one using Schottky's Theorem.

Proof. (Schottky's Theorem implies Picard's Little Theorem) Let $f \in H(\mathbb{C})$ such that it omits $\{0, 1\}$. If the omitted values are not these, we proceed as we have after the statement of Theorem 3.3.1.
Let us pick $r > 0$ and define the radial function $g_r(z) = f(rz)$, that still omits $\{0, 1\}$ and $g_r(0) = f(0)$. Applying Schottky's Theorem to g_r we get

$$|\log(g_r(z))| \leq C(1 + |\log|g_r(0)||) \cdot \left(\frac{1+|z|}{1-|z|}\right)^4 ;\text{ for any } z \in \mathbb{D}.$$

Now fix $z \in \mathbb{C}$ and let $r > |z|$. The previous inequality is rewritten as

$$|\log(f(z))| \leq C(1 + |\log|f(0)||) \cdot \left(\frac{1+\frac{|z|}{r}}{1-\frac{|z|}{r}}\right)^4 ;\text{ for any } |z| < r.$$

Eventually, we take $r \to \infty$. Hence $|\log|f(z)|| \leq C(1 + |\log|f(0)||)$, and $|f|$ is bounded. Recalling Liouville's Theorem, f is constant. $\square$

Chapter 4

The Picard's Great Theorem and Application of Picard's Theorems

In this last chapter we reach the final goal that we established: the proof of Picard's Great Theorem. We will follow the references [2], [4] and [8]. It is a result related to Casorati-Weierestraß' Theorem, but how is it different? First we state the Theorem and then we make the comparisons.

Theorem 4.0.1. *(Picard's Great Theorem) Let $f \in H(D(a,r) \setminus \{a\})$ where $a \in \mathbb{C}$ and $r > 0$. If f has an essential singularity at $z = a$, then f attains every value in $\mathbb{C}$ an infinite amount of times, except, at most, an exceptional value.*

In terms of sets, this can be written as follows

$$\#\{z \in \mathbb{C} : \#\{f^{-1}(\{z\})\} < \infty\} \leq 1.$$

Let us now give an example of how this cardinal can reach the value 1. We use the same function as we did to prove how precise the bound is in Picard's Little Theorem: $f(z) = e^{\frac{1}{z}}$. Clearly, $f \in \mathbb{C} \setminus \{0\}$ and it has an essential singularity at $z = 0$. Nevertheless, this function omits the value 0. Both Picard's Great Theorem and Casorati-Weierstraß' Theorem have been announced, what is it then that makes Picard's be a step forward?

- **Casorati-Weierstraß**: Given $a \in \mathbb{C}$ and $r > 0$, $\overline{f(D(a,r) \setminus \{a\})} = \mathbb{C}$.
- **Picard**: Given $a \in \mathbb{C}$ and $r > 0$, $f(D(a,r) \setminus \{a\}) = \mathbb{C}$ or there exists $b \in \mathbb{C}$ such that $f(D(a,r) \setminus \{a\}) = \mathbb{C} \setminus \{b\}$.

In fact, it says much more: it states that, perhaps except a point, f takes every value in $\mathbb{C}$ infinitely many times.

Before we begin with the proof, which will be a consequence of Schottky's Theorem, we prove the following lemma.

Lemma 4.0.2. *There exists* $s \in (0,1)$ *such that if* $z^* \in D(0, \frac{1}{2})$ *and also* $w^* = a + bi \in \mathbb{H} = \{z : \mathrm{Re(z)} > 0\}$ *satisfy* $e^{-w^*} = z^*$*, then the function* $F(z) = \exp(-a\frac{1+z}{1-z} - ib),\ z \in \mathbb{D}$ *satisfies*

1. $F(0) = z^*$.
2. $\partial(D(0, |z^*|)) \subset F(\overline{D(0, s)})$.

Proof. Let us define $F = G \circ T$, where $G(z) = exp(-az - ib) \in H(\mathbb{C})$ and $T(z) = \frac{1+z}{1-z}$ is a Möbius Transformation.
Let I be the segment

$$I := \{1 + iy;\ |y| \leq \frac{\pi}{\log 2}\} \subset \mathbb{H}.$$

Since $|e^{-w^*}| = |e^{-a-bi}| = |z^*|$ and $z^* \in D(0, \frac{1}{2})$, $e^{-a} < \frac{1}{2}$ that is $-a < \log(\frac{1}{2})$, and that makes $a > \log 2$. That gives directly that $\partial D(0, |z^*|) \subset G(I)$.
We have that $T^{-1}(w) = \frac{1-w}{1+w}$ for any $w \in \mathbb{H}$. In particular, if $w \in I$, $w = 1 + iy$ and

$$|T^{-1}(w)|^2 = \left|\frac{-iy}{2+iy}\right|^2 = \frac{y^2}{4+y^2} = 1 - \frac{4}{4+y^2} \leq 1 - \frac{4}{4 + \frac{\pi^2}{\log^2 2}} < 1.$$

Hence, we have proved that if $s = \sqrt{1 - \frac{4}{4+\frac{\pi^2}{\log^2 2}}} = \frac{\pi}{\sqrt{4\log^2 2 + \pi^2}} \in (0,1)$, then $I \subset T(D(0, s))$. Consequently, $\partial D(0, |z^*|) \subset G(I) \subset F(D(0, s))$. □

Proof of Theorem 4.0.1: We will proceed by proving that if f has at least two exceptional values, then f has a pole or an avoidable singularity at $z = a$. First, without loss of generality, we may assume that the exceptional values are 0 and 1 (just by considering $g = \frac{f-\alpha}{\beta-\alpha}$). Moreover, by a linear transformation $z \to a + rz$, we also may assume that $\Omega = \mathbb{D} \setminus \{0\}$.

So we are left with the following lemma

Lemma 4.0.3. *Let* $f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$ *be an holomorphic function. We then have that* f *has an avoidable singularity or a pole at zero.*

Let us prove the lemma.

Proof. If $\lim_{z\to 0}|f(z)| = \infty$, then $z = 0$ is a pole of f.
If the previous limit is not ∞, there exists N such that for any $\delta > 0$ there exists $0 < |z| < \delta$ and $|f(z)| \leq N$. Then we can take a subsequence $\{z_n\}_{n\geq 1} \subset \Omega = \mathbb{D}\setminus\{0\}$ such that $\lim_{n\to\infty}|z_n| = 0$ but $\{f(z_j)\}_j$ is bounded. We may assume that $\{z_n\}_n$ is a non-increasing sequence and, moreover, $|z_1| < \frac{1}{2}$.
We want to prove that $|f|$ is bounded in a neighbourhood of $z = 0$.
Since $\{f(z_j)\}_j$ is bounded, by taking a subsequence if necessary, we may assume that $\{f(z_j)\}_j$ is convergent to $\alpha \in \mathbb{C}$. Substituting f by $1 - f$ if necessary, we may assume as well that $\alpha \neq 0$. We can also assume that there exists n_0 such that for all $n \geq n_0$, we have $|f(z_n) - \alpha| < \frac{|\alpha|}{2}$, and consequently, $||f(z_n)| - |\alpha|| < \frac{|\alpha|}{2}$ and therefore $-\frac{|\alpha|}{2} < |f(z_n)| - |\alpha| < \frac{|\alpha|}{2}$. Thus, we are left with

$$\frac{|\alpha|}{2} < |f(z_n)| < \frac{3|\alpha|}{2} < 2|\alpha|, \text{ for any } n \geq 1.$$

Since $\{z_n\}_n$ is a non-increasing subsequence and $|z_1| < \frac{1}{2}$, we may apply Lemma 4.0.2 to $z^* = z_n \in D(0, \frac{1}{2})$ and to a_n and b_n such that $e^{-a_n - ib_n} = z_n$. Then, $a_n > \log 2$.
Let $F_n(z) = \exp(-a_n\frac{1+z}{1-z} - ib_n)$, given by Lemma 4.0.2. Then, $f \circ F_n$ is an entire function that omits $\{0, 1\}$. Moreover, for any $n \geq 1$,

$$\frac{|\alpha|}{2} < |f(F_n(0))| = |f(z_n)| < 2|\alpha|$$

and applying Schottky's Theorem to $f \circ F_n$ we obtain that $|\log f(F_n(z))| \leq C(1 + \log|f(F_n(0))|)\left(\frac{1+|z|}{1-|z|}\right)^4$. Now, using the bound above and taking any $z \in D(0, s)$, where s is given by Lemma 4.0.2, we get

$$|f(F_n(z))| < C(1 + \log|2\alpha|)\left(\frac{1+s}{1-s}\right)^4 = M, \text{ for any } z \in D(0, s).$$

The constant M only depends on s, α and Schottky's constant, C. Next, we recall Lemma 4.0.2 in order to obtain

$$\partial D(0, |z_n|) \subset F_n(\overline{D(0, s)})$$

which implies that $|f(z)| \leq M$, for any $z : \ |z| = |z_n|$. All this holds for any $n \geq 1$. Taking that, as we have just proved, it holds for $z : |z| = |z_1|$ and also for $z : |z| = |z_n|$, by the Maximum modulus principle we obtain $|f(z)| \leq M$ but now for any z such that $|z_n| \leq |z| \leq |z_1|$. At last, we take the limit when $n \to \infty$ to have $|f(z)| \leq M$ for any $0 < |z| \leq |z_1|$. Hence, f has an avoidable singularity at $z = 0$.
Eventually, we have proved that at most one value is never attained. If there is a complex value ω that is only attained a finite amount of times, then, by taking a small enough disk, we again obtain a punctured disk in which f does not attain two values. $\square$

To conclude this chapter, we go a step further in Picard's Little Theorem's direction as we shall see in the following corollary.

Corollary 4.0.4. *(Beyond Picard's Little Theorem) Let f be an entire non-constant function. Then there are two options for f:*

- *either f is a polynomial and hence, takes every value in $\mathbb{C}$ a finite number of times,*
- *or f is not a polynomial and takes every value in $\mathbb{C}$ an infinite amount of times, except at most one.*

Proof. Assume that f is a non-constant polynomial and let us take any $z_0 \in \mathbb{C}$. We want to know the cardinal of this set $\#\{f^{-1}(\{z_0)\}\}$. Knowing that cardinal is equivalent to know how many solutions does the equation $f(z) = z_0$ have and, applying Algebra's Fundamental Theorem, since f is a polynomial in $\mathbb{C}$, it has as many solutions as f's degree.
Conversely, if f is not a polynomial then it has an essential singularity in $z = \infty$. Applying now Picard's Great Theorem to $f(\frac{1}{z})$, that has an essential singularity at $z = 0$, it attains every value in $\mathbb{C}$ an infinite amount of times, except, at most, an exceptional value. Hence, so does $f(z)$.

$\square$

Conclusions

In this project, we have collected some of the classical theorems on the range of holomorphic functions that go further than the ones studied on the complex analysis syllabus of the degree. We have mainly focused on both holomorphic functions on $\mathbb{D}$ and on entire functions. We have proved bounds for the image of $\mathbb{D}$ via an holomorphic function using Landau's and Bloch's Theorems. Schottky's Theorem provides an upper bound for functions that omit $\{0, 1\}$ that is a key tool in the proof of Picard's Big Theorem. At last, Picard's Theorems, as we have proved, provide us with a criteria to prove if an holomorphic function is constant.
These theorems are some classical results in complex analysis and having developed their proof with the tools given at the course of complex analysis has been really fulfilling. Particularly, seeing how a simple and very well known statement as Schwarz's Lemma is used, not to obtain the last theorem in the project but, to prove some important and more advanced results, has helped me understand how to make good use of these tools that we have.
I have also confirmed that the topic of complex analysis is the one that I feel more comfortable working on. I had enjoyed the course in mathematical analysis very much as well, and not so much the fours other courses on calculus, but the one in complex analysis has been my favourite along the degree.

In conclusion, the project includes relevant results in terms of the range of holomorphic functions and gives several properties of these functions that are deduced using the mentioned theorems. Moreover, it has made me study and understand more complex results in complex analysis, which I am really grateful for.

Appendix A

Appendix

A.1 Möbius Transformations

There is a particular type of applications on $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ called the **Möbius Transformations**. This new set $\mathbb{C}_\infty$ is called the **extended complex plain**. On it, we define a topology derived from the usual on $\mathbb{C}$. We define the following neighbourhood system for any point $a \in \mathbb{C}_\infty$:

1. If $a \in \mathbb{C}$, a neighbourhood system is $\{D(a,r)\}_{r>0}$.
2. A neighbourhood system for $a = \infty$ is $\{\mathbb{C}_\infty \setminus D(0,r)\}_{r>0}$.

Let us choose the topology, whose open sets are the ones defined above, on $\mathbb{C}_\infty$. Giving to $\mathbb{S}^2$ the one derived from the usual on $\mathbb{R}^3$, we can see that, using the stereographic projection, $\mathbb{C}_\infty$ is homeomorphic to $\mathbb{S}^2$, see [3].

Definition A.1.1. *Given a, b, c and d complex numbers such that* $\mathbf{ad - bc \neq 0}$*, we define the* ***Möbius Transformation*** $\varphi : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ *as*

$$\varphi(z) = \frac{az+b}{cz+d},$$

with the following agreements

- *if $c = 0$, $\varphi(\infty) = \infty$.*
- *if $c \neq 0$, $\varphi(\infty) = \frac{a}{c}$ and $\varphi(\frac{-d}{c}) = \infty$.*

Lemma A.1.2. *The set of Möbius transformations with the composition defines a group.*

Proof. Let us choose $a, b, c, d, \alpha, \beta, \gamma$ and $\delta \in \mathbb{C}$ such that $ad - bc \neq 0$ and $\alpha\delta - \beta\gamma \neq 0$. We define ϕ as the Möbius transformations with parameters a, b, c and d and ψ as the one with parameters α, β, γ and δ.
We observe that

$$\phi(\psi(z)) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)},$$

which is also a Möbius transformation. The parameters of the composition are the same as the ones from the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Then, from this deduction and from the formula of the inverse of a $2x2$ matrix, we can observe that the inverse transformation of phi is $\psi(z) = \frac{dz-b}{-cz+a}$. This inverse satisfies that $\phi \circ \psi = \psi \circ \phi = Id_{\mathbb{C}_\infty}$. $\square$

We are now going to prove that each Möbius transformation can be obtained as a composition of these transformations:

- **Translation:** $T(z) = z + b$.
- **Homothecy:** $T(z) = az$, $a > 0$.
- **Rotation:** $T(z) = az$, where $|a| = 1$.
- **Inversion:** $T(z) = \frac{1}{z}$.

Proposition A.1.3. *Each Möbius transformation is a composition of translations, homothecies, rotations and inversions.*

Proof. If $c = 0$, we can write $\varphi(z) = \frac{a}{d}z + \frac{b}{d} = \rho e^{i\theta} z + \beta$, where $\rho = \left|\frac{a}{d}\right|$, θ is the argument of $\frac{a}{d}$ and $\beta = \frac{b}{d}$. Hence, φ is a composition of a rotation, an homothecy and a translation.
On the other hand, if $c \neq 0$, we can write $\varphi(z) = \left(\left(\frac{az+b}{cz+d}\right) - \frac{a}{c}\right) + \frac{a}{c} = \frac{bc-ad}{c(cz+d)} + \frac{a}{c}$, which is a composition of the four types of transformations mentioned above. $\square$

Let us define as $\Gamma := \{\text{lines and circles in}\,\mathbb{C}\}$. A remarkable property of these transformations is:

Proposition A.1.4. *Given φ a Möbius transformation, then $\varphi(\Gamma) \subseteq \Gamma$.*

Proof. It is clear that translations, homothecies and rotations send lines and circles into lines and circles, so let us see the remaining case: the inversion $\zeta = \frac{1}{z}$. The cartesian equation of a circle can be written as follows

$$A(x^2 + y^2) + Bx + Cy + D = 0,$$

where $|A| + |B| + |C| > 0$ and $|B| + |C| + |D| > 0$. Consider now that we write $z = x + iy$ and $\zeta = \alpha + i\beta$, hence

$$x + iy = \frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2} = \frac{\alpha}{\alpha^2 + \beta^2} - \frac{i\beta}{\alpha^2 + \beta^2}.$$

Therefore $x = \frac{\alpha}{\alpha^2+\beta^2}$ and $y = \frac{-\beta}{\alpha^2+\beta^2}$. Substituting this in the initial equation, and after simplifying and multiplying by $\alpha^2 + \beta^2$, we obtain

$$A + B\alpha - C\beta + D(\alpha^2 + \beta^2) = 0,$$

which is essentially, assuming that $D \neq 0$, the equation of a circle. If $D = 0$, it is a line. □

Now that we haven given some general properties of this type of transformations, we are going to focus on a concrete example of these:

$$\varphi_\alpha = \frac{z - \alpha}{1 - \bar{\alpha} z}, \text{ where } 1 - |\alpha|^2 \neq 0.$$

Theorem A.1.5. *Given $\alpha \in \mathbb{D}$, φ_α send $\mathbb{D}$ into $\mathbb{D}$ and α to 0. The inverse transformation of φ_α is $\varphi_{-\alpha}$, $\varphi'_\alpha(0) = 1 - |\alpha|^2$ and $\varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$.*

Proof. Our function $\varphi_\alpha \in H(\mathbb{D})$. If we take $\varphi_{-\alpha}(\varphi_\alpha(z)$, we obtain

$$\frac{\varphi_\alpha - \alpha}{1 + \alpha\bar{\varphi}_\alpha} = \frac{\frac{z-\alpha+\alpha(1-\bar{\alpha}z)}{1-\bar{\alpha}z}}{\frac{1-\bar{\alpha}z+\bar{\alpha}(z-\alpha)}{1-\bar{\alpha}z}} = z\frac{1-\alpha\bar{\alpha}}{1-\alpha\bar{\alpha}} = z.$$

Hence $\varphi_{-\alpha}$ is the inverse function of φ_α.
Let us take $t \in \mathbb{R}$, then

$$\left| \frac{e^{it} - \alpha}{1 - \bar{\alpha}e^{it}} \right| = \frac{|e^{it} - \alpha|}{|e^{-it} - \bar{\alpha}|} = \frac{|\beta|}{|\bar{\beta}|} = 1.$$

This means that $\varphi_\alpha(\partial\mathbb{D}) = \partial\mathbb{D}$ and applying the maximum modulus principle, we obtain that $\varphi_\alpha(\mathbb{D}) \subset \mathbb{D}$.
Finally, $\varphi'_\alpha(z) = \frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}$ and substituting 0 and α, we get the desired results. □

A.2 Calculations

In this part of the annex we are going to compute the value of the maximum of $f(r) = \eta(\frac{r}{\rho(0,r)})\rho(0,r)$, and that it is bigger than 0.5. We require this to obtain the bound after (2.5).
First we use the lower bound that we have for η: $\eta(s) \geq \frac{s^2}{4}$. Then,

$$f(r) \geq \frac{r^2}{4\rho^2(0,r)}\rho(0,r) = \frac{r^2}{4\rho(0,r)} = \frac{r^2}{4\log(\frac{1+r}{1-r})} = g(r) \in C^1((-1,1)).$$

Now we take the derivative of g and look for its zeros:

$$g'(r) = \frac{2r(4\log(\frac{1+r}{1-r})) - r^2\left(\frac{8}{1-r^2}\right)}{16\log^2(\frac{1+r}{1-r})}.$$

Clearly it is only zero when $8r\log(\frac{1+r}{1-r})+\frac{8r^2}{r^2-1}=0$ and since $r\neq 0$, this is equivalent to solve $\log(\frac{1+r}{1-r})+\frac{r}{r^2-1}=0$. After solving it numerically, we obtain two solutions: $r_{\pm}=\pm 0.79638835586$. But r is a positive value, so we reject the negative solution r_-.
At last, we observe that, since $g''(r_+)<0$, r_+ is indeed a maximum and it is greater than 0.5. Recalling that $f(r)\geq g(r)$, we obtain that the maximum of $f(r)$ is greater than 0.5, as we wanted to prove.

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