Coherent stochastic resonance

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We consider Brownian motion on a line terminated by two trapping points. A bias term in the form of a telegraph signal is applied to this system. It is shown that the first two moments of survival time exhibit a minimum at the same resonant frequency.

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I. INTRODUCTION

The phenomena encompassed by the terminology "stochastic resonance" has been studied by many investigators, motivated by applications in both the biological and physical sciences [1]. The general subject area deals with interactions between noise and a periodic force jointly driving mainly nonlinear dynamical systems. However, there is one form of stochastic resonance, henceforth referred to as coherent stochastic resonance (CSR), whose properties were studied in Ref. [2], which has been shown to appear in linear systems. A typical example of this is diffusion along a line terminated by two traps, with the diffusing particle performing motion biased by an oscillating field. A physical system to which this applies is pulsed-field gel electrophoresis [3]. Recently, this type of analysis has been extended to the study of a similar system on a semi-infinite line in [4], in which case a field must be added in order to ensure that the particle reaches the trap with probability 1. While CSR has been demonstrated in [4], the analysis is flawed [5], but nevertheless leads to qualitatively correct results.

A parameter useful in describing the behavior of such systems is the mean survival time of the particle, or mean-first-passage time (MFPT), that is, the average time the particle diffuses until it is finally trapped. It was shown in [2] that the MFPT, considered as a function of the frequency of the bias field, $\langle T(\omega)\rangle$, exhibits a minimum at some frequency of the driving field. A similar result is that there is a bifurcation, again depending on ω, in the behavior of $\langle T\rangle$ considered as a function of the magnitude of the bias [6]. That is, for some values of ω the MFPT increases as a function of the amplitude, while for others it decreases.

The analysis necessary to derive these results was based either on the simulation of a random walk on a lattice or on a perturbation expansion of the solution to the diffusion limit to such a walk described by

$$\frac{\partial p}{\partial t}=D\frac{\partial^2 p}{\partial x^2}-v\cos(\omega t)\frac{\partial p}{\partial x} , \tag{1}$$

where D and v are constants. This is to be solved subject to the boundary conditions $p(0,t)=p(L,t)=0$. While a solution of the resulting problem might seem to be straightforward to find, this does not prove to be the case. Several variants of this type of problem has been studied in the literature of applied probability [7], with results mainly given in terms of a numerical algorithm for solving the relevant integral equation. It is therefore natural to inquire as to whether the problem cannot be modified to lead to a solution not based on either simulation or a perturbation expansion.

In the present paper we consider the same problem, replacing the pure sinusoid in Eq. (1) by the telegraph signal

$$v(t)=\begin{cases}+v_0, & t\in[2n\Delta\tau,(2n+1)\Delta\tau]\\ -v_0, & t\in[(2n+1)\Delta\tau,(2n+2)\Delta\tau] ,\end{cases} \tag{2}$$

v_0 being a constant. This model is developed in order to simplify the mathematical development and to derive results without resorting to an assumption of a small amplitude for the forcing term. The basic idea behind our calculation is that a diffusion equation with a constant bias can be solved exactly. A solution valid over the entire time axis is then obtained by matching solutions at the times at which the telegraph signal changes sign. Notice that the problem discussed here is not quite the same as that posed in Eq. (1) since a Fourier expansion of the telegraph signal has an infinite number of frequencies rather than the single frequency inherent in that formulation. Nevertheless, we can define an effective frequency for such a signal as $\omega=(2\Delta\tau)^{-1}$ and show that the same CSR exists for this form of the bias field. Thus, we seek the solution to a dynamical system whose evolution is governed by the equation

$$\dot{X}=\xi(t)+v(t) , \tag{3}$$

where $\xi(t)$ is Gaussian white noise and $v(t)$ is the telegraph signal defined in Eq. (2). We assume traps set at $x=0$ and $x=L$, and show that the first two cumulants of the survival probability exhibit resonant behavior at the same frequency.

II. DESCRIPTION OF THE BASIC FUNCTIONS

The endpoint for our analysis consists of expressions for the MFPT and the variance of the first-passage time diffusing particle for which $X(0)=x_0$. The advantage of using a telegraph signal to model the bias term is that an exact solution to the diffusion equation can be derived. Hence we derive the following results by decomposing the time into periods during which the bias is constant, find a solution in each interval and match solutions at the change points.

Since the resulting dynamical system is a regenerative one in the sense of Smith [8], it is necessary to study the evolution of the system at the change points. For this purpose we define a pair of sets of state densities at the change points, $\{P_{2n+1}(x)\}$ and $\{P_{2n}(x)\}$, $n=0,1,2,\ldots$, where, for example,

$$P_{2n+1}(x)dx \equiv \Pr\{x < X[(2n+1)\Delta\tau] \leq x+dx\} \ , \quad (4)$$

with the set $\{P_{2n}(x)\}$ defined similarly. We need also to define two propagators, denoted by $p_+(x,t|y)$ and $p_-(x,t|y)$, in which

$$p_+(x,t|y)dx \equiv \Pr\{x < X(t) \leq x+dx | X(0)=y \ ; \ v(\tau)=+v_0, \ \tau\in(0,t)\} \ ; \quad (5)$$

a similar definition holds for $p_-(x,t|y)$, with the sign of $v(t)$ being negative. Since the bias field is piecewise constant it follows that $\{P_{2n+1}(x)\}$ and $\{P_{2n}(x)\}$ satisfy the recurrence relations

$$\begin{aligned} P_{2n+1}(x) &= \int_0^L P_{2n}(y) p_+(x,t|y)dy \ , \\ P_{2n}(x) &= \int_0^L P_{2n-1}(y) p_-(x,t|y)dy, \quad n \geq 1 \ . \end{aligned} \quad (6)$$

The initial condition implies the additional relation $P_0(x)=\delta(x-x_0)$.

In order to calculate the moments of the time to trapping we need an expression for the survival probability for a particle that is initially at x_0. This will be denoted by $S(t|x_0)$. Let $\langle T^m(x_0)\rangle$ be the mth moment of the time to trapping. This can be related to $S(t|x_0)$ by

$$\langle T^m(x_0)\rangle = m\int_0^\infty t^{m-1} S(t|x_0)dt \ . \quad (7)$$

The function $S(t|x_0)$ will, in turn, be decomposed into the contributions from intervals in which $v(t|x_0)$ has a constant sign as

$$S(t|x_0) = \sum_{n=0}^\infty [S_{2n}(t|x_0) + S_{2n+1}(t|x_0)] \ , \quad (8)$$

where, for example, $S_{2n}(t|x_0)$ is the survival probability during the $2n$th interval. The term $S_{2n}(t|x_0)$ can be expressed in terms of the probability densities defined in the preceding paragraph as

$$\begin{aligned} S_0(t|x_0) &= \int_0^L p_-(x,t|x_0)dx \ , \\ S_{2n}(t|x_0) &= \int_0^L P_{2n}(y)dy \\ &\quad \times \int_0^L p_-(x,t-2n\Delta\tau|y)dx, \quad n \geq 1 \ , \end{aligned} \quad (9)$$

and $S_{2n+1}(t|x_0)$ can similarly be related to the functions $P_{2n+1}(x)$ and $p_+(x,t|y)$. The combination of the last three equations implies that the mean first-passage time is equal to

$$\begin{aligned} \langle T(x_0)\rangle &= \int_0^{\Delta\tau} dt \int_0^L p_-(x,t|x_0)dx \\ &\quad + \sum_{n=1}^\infty \int_0^L dx \int_0^L P_{2n}(y)dy \int_0^{\Delta\tau} p_+(x,t|y)dt \\ &\quad + \sum_{n=0}^\infty \int_0^L dx \int_0^L P_{2n+1}(y)dy \\ &\qquad \times \int_0^{\Delta\tau} p_-(x,t|y)dt \ . \end{aligned} \quad (10)$$

Because of the infinite sums that appear here and in representations of other quantities, it proves convenient to introduce generating functions for $P_j(x)$. Accordingly, we define the generating functions

$$\begin{aligned} U_=(x;s|x_0) &= \sum_{n=0}^\infty P_{2n+1}(x)s^{2n+1} \ , \\ U_-(x;s|x_0) &= \sum_{n=0}^\infty P_{2n}(x)s^{2n} \ , \end{aligned} \quad (11)$$

which, because of Eq. (6), are related by

$$\begin{aligned} U_+(x;s|x_0) &= s\int_0^L U_-(y,s|x_0)p_+(x,\Delta\tau|y)dy \ , \\ U_-(x;s|x_0) &= \delta(x-x_0) \\ &\quad + s\int_0^L U_+(y;s|x_0)p_-(x,\Delta\tau|y)dy \ . \end{aligned} \quad (12)$$

These can be combined into a single integral equation for $U_+(x;s|x_0)$:

$$\begin{aligned} U_+(x;s|x_0) &= sp_+(x,\Delta\tau|x_0) \\ &\quad + s^2\int_0^L U_+(z;s|x_0)K(x,z)dz \ , \end{aligned} \quad (13)$$

in which the kernel $K(x,z)$ is

$$K(x,z) = \int_0^L p_+(x,\Delta\tau|y)p_-(y,\Delta\tau|z)dy \ . \quad (14)$$

The generating functions defined in Eq. (11) allow $\langle T(x_0)\rangle$ to be expressed as

$$\begin{aligned} \langle T(x_0)\rangle &= \int_0^L U_-(x;1|x_0)q_+(x)dx \\ &\quad + \int_0^L U_+(x;1|x_0)q_-(x)dx \ , \end{aligned} \quad (15)$$

where the functions $q_\pm(x)$ are defined by

$$q_\pm(x) \equiv \int_0^L dy \int_0^{\Delta\tau} p_\pm(y,t|x)dx \ . \quad (16)$$

Relations similar to Eq. (15) can also be written for moments of order higher than the first. Since it is easy to solve the diffusion equation in a constant field, it is possible to write a straightforward algorithm to generate the functions that exhibit CSR.

III. NUMERICAL RESULTS

A. First moment

The results to follow will always be expressed in terms of dimensionless units, which is equivalent to setting the coefficients of the biased diffusion equation equal to 1. A change of variables that ensures this end consists of defining a dimensionless spatial variable ξ and a dimensionless time θ by

$$\xi=\frac{v}{D}x,\quad \theta=\frac{v^2}{D}t\ , \tag{17}$$

in which case the equations satisfied by the two functions $p_\pm(\xi,\theta|\xi_0)$ become

$$\frac{\partial p_\pm}{\partial\theta}=\frac{\partial^2 p_\pm}{\partial\xi^2}\pm\frac{\partial p_\pm}{\partial\xi}\ . \tag{18}$$

The density p_+ satisfies the initial conditions $p_+(\xi,0|\xi_0)=\delta(\xi-\xi_0)$ and the two densities must be found subject to boundary conditions that reflect the fact that $\xi=$ and $\xi=\Lambda(\equiv v_0L/D)$ are trapping points, which is to say that $p_\pm(0,\theta|\xi_0)=p_\pm(\Lambda,\theta|\xi_0)=0$. To conveniently write the required solutions, define the variables $\beta_n=n\pi/\Lambda$. Because Eq. (18) has constant coefficients the solution to it in any segment with a constant bias is found quite straightforwardly as

$$p_\pm(\xi,\theta|\xi_0)=\frac{2}{\Lambda}\exp[\pm\tfrac{1}{2}(\xi-\xi_0)]\times\sum_{n=0}^{\infty}\exp[-(\beta_n^2+\tfrac{1}{4})\theta]\times\sin(\beta_n\xi_0)\sin(\beta_n\xi)\ . \tag{19}$$

A tedious but straightforward calculation outlined in the Appendix and based on this solution allows us to express the mean first-passage time in the form of a Fourier series:

$$\langle T(\omega|\xi_0)\rangle=e^{-\xi_0/2}\sum_{k=1}^{\infty}T_k(\omega)\sin(\beta_k\xi_0)\ , \tag{20}$$

where the frequency ω has been defined earlier. The functions $T_k(\omega)$ are, in turn, expressed as a double sum whose exact form is given in the Appendix. A similar expansion can be derived for the second moment $\langle T^2(\omega|\xi_0)\rangle$, which, together with Eq. (20), can be used to find the variance of the first-passage time. It is trivial to calculate similar results from Eq. (20) when ξ_0 is uniformly distributed over the interval $(0,\Lambda)$.

Figure 1 indicates the degree of resonant behavior exhibited in $\langle T(\omega|\xi_0)\rangle$. This is plotted as a function of ω for a peak initially at $\xi_0=\Lambda/2$. If the resonant frequency is denoted by $\omega_{\text{res}}(\Lambda)$ then this function is found to have the scaling form

$$\omega_{\text{res}}(\Lambda)\sim v_0/\Lambda \tag{21}$$

for large Λ, where v_0 is a constant which agrees with results of the calculation in [2]. The reason for this behavior is, as in the cited reference, due to coherent motion orchestrated by the periodic bias. At early times, since $v(0)>0$, trapping is enhanced at $\xi=\Lambda$. When the signal reverses sign trapping is enhanced at $\xi=0$, and so forth. When ω becomes very large this enhanced trapping effect operates over short time intervals, which tends to enhance the effect of diffusion. High frequency bias

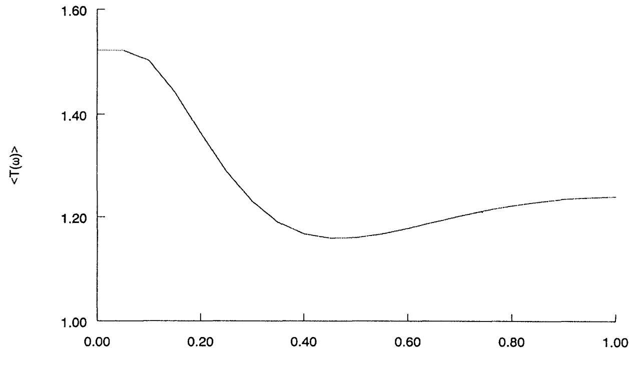

FIG. 1. The MFPT $\langle T(\omega|\xi_0)\rangle$ plotted as a function of ω for $\Lambda=4$ and $\xi_0=2$.

therefore causes the system to behave as if there were no periodic forcing term.

Another result first demonstrated in [2] is the dependence of $\langle T(\Lambda|\omega)\rangle$ on length, the notation signifying that we have now emphasized the behavior of the MFPT as a function of the segment length Λ rather than of ω. Our numerical calculations indicate that, when $\omega\neq\omega_{\text{res}}$,

$$\langle T(\Lambda|\omega)\rangle\approx\alpha\Lambda^2 \tag{22}$$

as indicated in Fig. 2(a), which switches over to

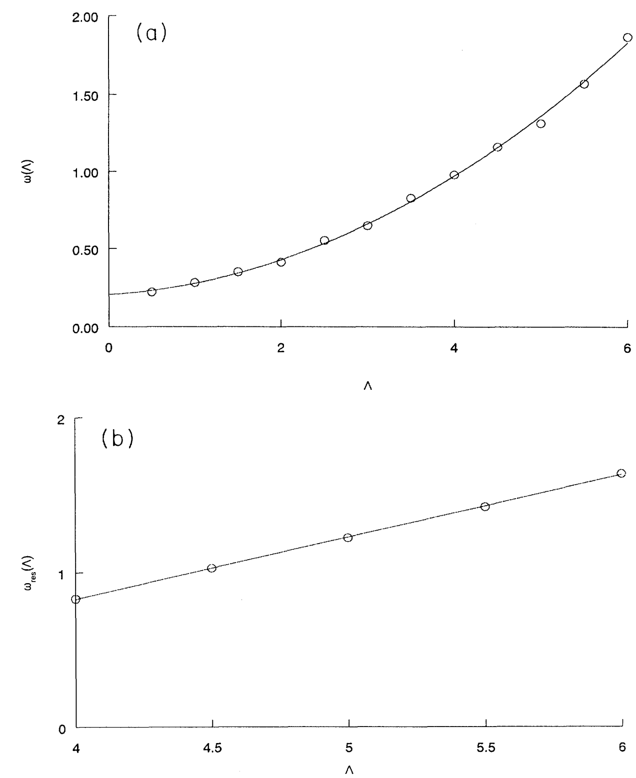

FIG. 2. (a) The mean-first-passage time plotted as a function of the interval length for $\omega\neq\omega_{\text{res}}$. The curve shown is fitted to a quadratic function of the length and indicates that the system is primarily diffusive. (b) The same plot for $\omega=\omega_{\text{res}}$. The fit to a straight line is consistent with coherent behavior.

$$\langle T(\Lambda|\omega_{\rm res})\rangle \approx \beta\Lambda \quad (23)$$

at the resonant frequency, where α and β are constants. This behavior is illustrated by the curve shown in Fig. 2(b). Notice that Eq. (22) is the dependence expected for unbiased diffusion, while the form of Eq. (23) coincides with the result expected for biased diffusion.

B. Variance of the first-passage time

In this section we show that not only does the MFPT exhibit resonant behavior as the frequency of the driving signal is changed, but the associated variance,

$$\sigma^2(\omega|\xi_0) \equiv \langle T^2(\omega|\xi_0)\rangle - \langle T(\omega|\xi_0)\rangle^2 , \quad (24)$$

also exhibits a minimum which appears also to be located at $\omega = \omega_{\rm res}$. An argument that closely parallels that leading to Eq. (15) leads to an integral representation for $\langle T^2\rangle$ that can be expressed as

$$\langle T^2\rangle = 2\int_0^L \{[U_+(\xi;1|\xi_0)r_-(\xi) + U_-(\xi;1|\xi_0)r_+(\xi)] + \Delta\tau[V_+(\xi;1|\xi_0)q_-(\xi) + V_-(\xi;1|\xi_0)q_+(\xi)]\}d\xi , \quad (25)$$

in which the functions $r_\pm(\xi)$ and $V_\pm(\xi;1|\xi_0)$ are defined by

$$r_\pm(\xi) \equiv \int_0^L d\xi_0 \int_0^{\Delta t} t p_\pm(\xi_0,t|\xi)dt \quad (26)$$

and

$$V_\pm(\xi;1|\xi_0) = \left.\frac{\partial U_\pm(\xi;s|\xi_0)}{\partial s}\right|_{s=1} . \quad (27)$$

Finally, we can write an expression for $\langle T^2(\omega|\xi_0)\rangle$ which has the same form as a Fourier series as Eq. (20), except that the function $T_k(\omega)$ is to be replaced by another function somewhat more complicated in form.

Numerical evaluations of the resulting formulas suggest that both the second moment and the variance exhibit resonant behavior and the resonant frequency in both cases is the same as that for the first moment. A plot of $\sigma^2(\omega|\xi_0)$ as a function of ω is shown in Fig. 3(a). The same qualitative resonant behavior is observed for an initial condition uniformly distributed over the initial interval,

$$\sigma^2(\omega) = \frac{1}{L}\int_0^L \sigma^2(\omega|\xi_0)d\xi_0 , \quad (28)$$

as is exemplified by the plot in Fig. 3(b).

IV. CONCLUSIONS

We have studied the behavior of the first two moments of the first-passage time for a particle simultaneously driven by additive white noise and a periodic square wave. The use of a dichotomous signal rather than a pure sinusoid as in [2] simplifies many of the calculations in this system and shows that the observed resonant behavior is not a unique feature of the purely sinusoidal forcing term. Generally, the phenomenon of stochastic resonance requires that the system dynamics be nonlinear. In the present case resonant behavior is due to coherence of motion induced by the periodic forcing term. We have also shown that the same type of coherent resonance occurs in the second moment and the variance, the resonant frequency apparently coinciding with the one found in the case of the first moment. A natural conjecture is that this will also be true for moments higher than the second, but we have not investigated this point. We have also looked for CSR when the periodic telegraph signal is replaced by a random telegraph signal, but have so far been unable to demonstrate the existence of such an effect. One might, for example, expect to see such behavior, at least over some interval of time, in a model in which the times between successive change points is random. Such randomness would be described by a sequence of probability densities $\{\psi_n(t)\}$, where, for example, $\psi_1(t) = ke^{-kt}$ and $\lim_{n\to\infty}\psi_n(t) = \delta(t-\Delta t)$. Finally, we point out that the present work is related to results in [4]. However, while the results in that paper are qualitatively correct, the authors there have used the method of images inappropriately [8], leading to some incorrect quantitative results.

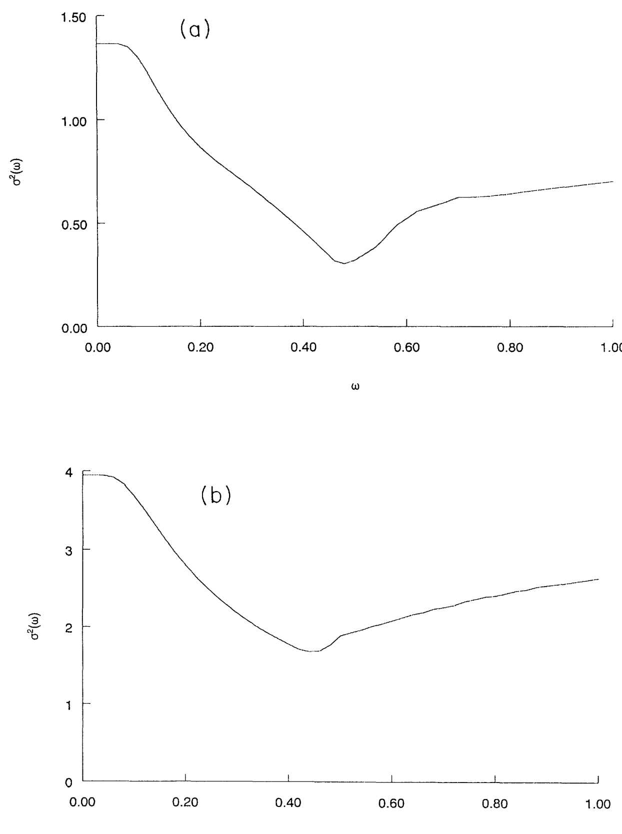

FIG. 3. (a) A plot of the variance of the first-passage time, $\sigma^2(\omega|\xi_0)$, plotted as a function of ω for the parameters $\Lambda = 4$ and $\xi_0 = 2$. To graphical accuracy the parameter $\omega_{\rm res}$ is the same as is found for the mean-first-passage time as indicated in Fig. 1. (b) The variance uniformly weighted with respect to ξ_0 over the interval for $\Lambda = 4$.

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APPENDIX

An explicit form of the solution for $p_{\pm}(\xi,\theta|\xi_0)$ is given in Eq. (19) from which it is possible to show that the kernel $K(u,v)$ in Eq. (14) has the form

$$K(u,v)=\frac{4}{L^2}e^{(1/2)(u+v)}\sum_{i,j=1}^{\infty}e^{-(\beta_i^2+\beta_j^2+1/2)\Delta\tau}\times I_{ij}\sin(\beta_n u)\sin(\beta_n v)\ , \tag{A1}$$

in which the constants I_{ij} are

$$I_{ij}\equiv\int_0^L e^{-\xi}\sin(\beta_i\xi)\sin(\beta_j\xi)d\xi =\tfrac{1}{2}[1-(-1)^{i+j}e^{-L}]\left[\frac{1}{1+(\beta_i-\beta_j)^2}+\frac{1}{1+(\beta_i+\beta_j)^2}\right]. \tag{A2}$$

Because β_m^2 is proportional to m^2 the series in Eq. (29) will generally be convergent, requiring only a few terms for graphical accuracy.

Notice that the expression for $\langle T(\xi_0)\rangle$ in Eq. (15) involves only the functions $U_{\pm}(\xi;s|\xi_0)$ evaluated at $s=1$. It therefore suffices to consider the integral equations for $U_{\pm}(\xi;1|\xi_0)$ to calculate this function. The integral equation satisfied by $U_+(\xi;1|\xi_0)$ is

$$U_+(\xi;1|\xi_0)=p_+(\xi,\Delta\tau|\xi_0)+\int_0^L K(\xi,z)U_+(z;1|\xi_0)dz\ . \tag{A3}$$

Once $U_+(\xi;1|\xi_0)$ is known $U_-(\xi;1|\xi_0)$ can be found from it by the relation

$$U_-(\xi;1|\xi_0)=\delta(\xi-\xi_0)+\int_0^L U_+(z;1|\xi_0)p_-(\xi,\Delta\tau|z)dz\ . \tag{A4}$$

Since Eq. (A3) is a Fredholm equation one can write its solution in terms of a resolvent function $R(\xi,z)$ [9] as

$$U_+(\xi;1|\xi_0)=p_+(\xi,\Delta\tau|\xi_0)+\int_0^L R(\xi,z)p_+(z,\Delta\tau|\xi_0)dz\ , \tag{A5}$$

where the resolvent can be written as a double sum

$$R(\xi,z)=e^{(1/2)(\xi+z)}\sum_{i,j}e^{-(\beta_i^2+\beta_j^2)\Delta\tau}R_{ij}\sin(\beta_i\xi)\sin(\beta_j z)\ . \tag{A6}$$

The R_{ij}, in turn are written in terms of a sequence of functions $I_{ij}^{(n)}$ which are calculated from a recurrence relation of the form

$$I_{ij}^{(n)}=\sum_{r,s}e^{-(\beta_i^2+\beta_j^2)\Delta\tau}J_{rs}I_{ir}I_{js}^{(n-1)}\ , \tag{A7}$$

with $I_{ij}^{(1)}=I_{ij}$, this being defined in Eq. (A2), and

$$J_{ij}=\int_0^L e^{\xi}\sin(\beta_i\xi)\sin(\beta_j\xi)d\xi =\tfrac{1}{2}[(-1)^{i+j}e^{L}-1]\left[\frac{1}{1+(\beta_i-\beta_j)^2}-\frac{1}{1+(\beta_i+\beta_j)^2}\right]. \tag{A8}$$

These definitions allow us to write R_{ij} as a single sum,

$$R_{ij}=\sum_{n=1}^{\infty}\left[\frac{2}{L}\right]^{2n}I_{ij}^{(n)}e^{-n\Delta\tau/2}\ . \tag{A9}$$

Again, because of the exponential terms in Eqs. (A7) and (A9), a numerical evaluation of the indicated series offers no difficulties in practice.

The function $U_+(\xi;1|\xi_0)$ is then expressed in terms of two functions which we denote by ρ_{il} and $u_i^{(+)}$. The first of these can be expressed in terms of R_{ij} through the relation

$$\rho_{il}=\sum_j e^{-\beta_j^2\Delta\tau}J_{lj}R_{ij} \tag{A10}$$

and the second is defined in terms of the ρ_{il} by

$$u_i^{(+)}=\frac{2}{L}e^{-\xi_0/2}\sum_l[\delta_{il}+e^{-\beta_i^2\Delta\tau}\rho_{il}]\sin(\beta_i\xi_0)\ . \tag{A11}$$

Finally, the function $U_+(\xi;1|\xi_0)$ is related to the resolvent $R(\xi,z)$ by

$$U_+(\xi;1|\xi_0)=e^{-\xi/2}\sum_i e^{-(\beta_i^2+1/2)\Delta\tau}u_i^{(+)}\sin(\beta_i\xi)\ . \tag{A12}$$

Likewise, it is readily shown that $U_-(\xi;1|\xi_0)$ can be written in terms of a function $u_i^{(-)}$, which is defined by

$$u_i^{(-)}=\frac{2}{\Lambda}\sum_i e^{-\beta_j^2\Delta\tau}J_{lj}u_i^{(+)}\ . \tag{A13}$$

The relation is

$$U_-(\xi;1|\xi_0)=\delta(\xi-\xi_0)+e^{-\xi/2}\times\sum_i e^{-(\beta_i^2+1/2)\Delta\tau}u_i^{(-)}\sin(\beta_i\xi)\ . \tag{A14}$$

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