UNIVERSITAT DE BARCELONA
SCHOOL OF ECONOMICS

Working Papers

Col·lecció d'Economia E22/426

NON-MANIPULABILITY BY CLONES IN BANKRUPTCY PROBLEMS

Pedro Calleja

Francesc Llerena

ISSN 1136-8365

UNIVERSITAT DE BARCELONA
SCHOOL OF ECONOMICS

Working Papers

UB Economics *Working Paper No. 426*

Title: Non-manipulability by clones in bankruptcy problems

Abstract: In the domain of bankruptcy problems, we show that non manipulability via merging and splitting claims by identical agents characterizes the proportional rule provided claims are positive rational numbers. By adding either claim monotonicity or claims continuity we obtain new characterizations to the whole class of bankruptcy problems.

JEL Codes: C71

Keywords: Rationing problems, manipulability, proportionality

Authors:

Pedro Calleja	Francesc Llerena
Universitat de Barcelona and BEAT	Universitat Rovira i Virgili and ECO-SOS
Email: calleja@ub.edu	**Email:** francesc.llerena@urv.cat

Date: July 2022

Non-manipulability by clones in bankruptcy problems

Pedro Calleja* and Francesc Llerena†

July 13, 2022

Abstract

In the domain of bankruptcy problems, we show that non manipulability via merging and splitting claims by identical agents characterizes the proportional rule provided claims are positive rational numbers. By adding either claim monotonicity or claims continuity we obtain new characterizations to the whole class of bankruptcy problems.

1 Introduction

Bankruptcy problems (O'Neill, 1982; Aumann and Maschler, 1985) deal with situations where an amount of a perfectly divisible resource should be distributed among a group of agents presenting conflicting claims, that is, the total amount to divide is not enough to accomplish all demands. These problems are solved by rules proposing an allocation vector taking into consideration the specific characteristics of the agents.

An important topic in economics is the study of rules that are immune to the strategic behavior of agents by misrepresenting their features. In the bankruptcy problem, O'Neill (1982) introduces *non manipulability* (or *strategy-proofness*[1]) as the combination of *non-manipulability via merging and splitting*. A rule is *non-manipulable via merging* if no group of agents can benefit from consolidating claims and it is *non-manipulable via splitting* if no agent can benefit from dividing its claim into claims of a group of agents. A rule is *non-manipulable* if it is unaffected by both types of manipulation. On the full domain of bankruptcy problems, Moreno-Ternero (2006) shows that *non-manipulability* is equivalent to *additivity by claims* as introduced by Curiel et al. (1987), requiring that merging or splitting the agents' claims do not affect the amounts received by each other agent involved in the problem.

The *proportional* rule makes agents' payments proportional to their demands and it is one of the most commonly used rule in real-life situations when a firm goes bankruptcy. Due to its importance in practice, it has been extensively analyzed from an axiomatic viewpoint. Concerning the merging-splitting proofness requirement, O'Neill (1982) axiomatizes the *proportional* rule making use of *non-manipulability*, together with other axioms. Later on, Chun (1988) shows that the O'Neill's result is not tight and finally de Frutos (1999) concludes that only *non-manipulability* is needed to characterize the *proportional* rule. In a more general class of allocations problems, Ju et al. (2007) investigate the relation between non-manipulability and proportionality.

*Departament de Matemàtica Econòmica, Financera i Actuarial, Universitat de Barcelona-BEAT, Av. Diagonal, 690, 08034 Barcelona. calleja@ub.edu

†Departament de Gestió d'Empreses, Universitat Rovira i Virgili-ECO-SOS, Av. de la Universitat, 1, 43204 Reus. francesc.llerena@urv.cat

[1]O'Neill (1982) introduced this axiom for the class of simple bankruptcy problems where no agent has a claim exceeding the estate.

As originally formulated, *non-manipulability* does not impose conditions on the group of agents that merge or split. Ju (2003) introduces restrictions on the coalition formation by means of weaker forms of non-manipulability just permitting mergers or spin-offs by pairs, and characterizes the set of parametric rules (Young, 1987) that are either non-manipulable via (pairwise) merging or splitting. In this note, we focus on rules that are immune to manipulations involving symmetric agents, that is, agents with the same claim. We call this axiom *non manipulability by clones*. It is quite usual in practice that only agents with some common attributes are allowed to merge or split, while these practices involving very different agents are censured. A natural and simple way to formally accommodate this ideas is to restrict the possibility to manipulate to identical agents or clones. Interestingly, we show that this substantially weaker form of non manipulability is enough to characterize the proportional rule for the realistic case in which all claims are zero o positive rational numbers. Finally, we extend this result to the general domain of bankruptcy problems by adding either *claim monotonicity* or *claims continuity*. While claims continuity enforces that small changes in the claims of the agents do not lead to large changes in the awards recommendation, claim monotonicity requires that if only one agent's claim increases, he should not be worse-off.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and definitions. Section 3 contains the characterization results.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$ (the set of natural numbers) represent the set of all potential agents (claimants) and let $\mathcal{N}$ be the collection of all non-empty finite subsets of $\mathbb{N}$. By $\mathbb{Q}^+ = \{a/b \mid a, b \in \mathbb{N}\}$ we denote the set of positive rational numbers. An element $N \in \mathcal{N}$ describes a finite set of agents where $|N| = n$. For a given $N \in \mathcal{N}$, $\emptyset \neq S \subset N$, and a vector $x \in \mathbb{R}^N$, $x_S = (x_i)_{i\in S} \in \mathbb{R}^S$.

A *bankruptcy problem is a* triple (N, E, c) such that $N \in \mathcal{N}, c \in \mathbb{R}^N_+, E \geq 0$, and $\sum_{i\in N} c_i \geq E$. By $\mathcal{B}$ we denote the set of all bankruptcy problems. If $(N, E, c) \in \mathcal{B}$, then each agent in the set of creditors N has a claim c_i to the net worth or estate $E \geq 0$ of a bankrupt firm. A *bankruptcy rule* is a function $\beta : \mathcal{B} \longrightarrow \bigcup_{N\in\mathcal{N}} \mathbb{R}^N_+$ that associates with every $(N, E, c) \in \mathcal{B}$ a unique vector $\beta(N, E, c) \in \mathbb{R}^N_+$ satisfying $\sum\limits_{i\in N} \beta_i(N, E, c) = E$ (*budget balance* (BB)) and $\beta_i(N, E, c) \leq c_i$ for all $i \in N$ (*claim boundedness* (CB)). BB requires that the sum of the payments should be equal to the net worth and CB means that no creditor receives more than her claim.

Instances of well studied bankruptcy rules are the *proportional* rule (PR), the *constrained equal awards* rule (CEA), and the *constrained equal losses* rule (CEL). The PR rule makes awards proportional to the claims. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $PR_i(N, E, c) = \lambda\, c_i$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \lambda\, c_j = E$. The CEA rule rewards equally to all claimants subject to no one receiving more than her claim. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEA_i(N, E, c) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \min\{c_j, \lambda\} = E$. In contrast, the CEL rule equalizes the losses of claimants subject to no one receiving a negative amount. That is, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEL_i(N, E, c) = \max\{c_i - \lambda, 0\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \max\{c_j - \lambda, 0\} = E$. For a detailed analysis of bankruptcy rules we refer to Thomson (2015).

3 Axiomatizations of the proportional rule

In this part, we provide new axiomatic characterizations of the proportional rule. We first introduce *non-manipulability*, using its equivalent formulation in the full domain of bankruptcy problems,[2] and

[2] See Moreno-Ternero (2006), that names the property as strong non-manipulability.

a mild form of this axiom which we call *non-manipulability by clones*. Formally, a bankruptcy rule β satisfies

- *non-manipulability* (NM) if for all (N, E, c), $(N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c'_m = c_m + \sum_{k \in N \setminus N'} c_k$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$, then $\beta_i(N', E, c') = \beta_i(N, E, c)$ for all $i \in N' \setminus \{m\}$.

- *non-manipulability by clones* (NMC) if for all (N, E, c), $(N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c_i = \frac{c'_m}{|N \setminus N'|+1}$ for all $i \in N \setminus N' \cup \{m\}$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$, then $\beta_i(N', E, c') = \beta_i(N, E, c)$ for all $i \in N' \setminus \{m\}$.

NMC restricts NM to identical agents, that is, agents who have the same claim. It imposes that if an agent splits his claim and appears as a group of symmetric agents, or a group of symmetric agents merge their claims and form a single agent, then the amount received by each other agent in the problem does not change and, as a consequence, neither the payoffs of the agents merging or splitting. In more general settings, such as financial networks or multi-issue problems, the idea to restrict manipulations to agents with some common traits also play a role in characterizing the extension of the proportional rule to these contexts (see, for instance, Csóka and Herings, 2021; Acosta-Vega et all., 2022).

A well established axiom is *equal treatment of equals*, requiring that symmetric agents should receive the same amount. Formally, a bankruptcy rule β satisfies

- *equal treatment of equals* (ETE) if for all $(N, E, c) \in \mathcal{B}$ and all $i, j \in N$, if $c_i = c_j$ then $\beta_i(N, E, c) = \beta_j(N, E, c)$.

De Frutos (1999) shows that NM implies ETE. This result is strengthened in the next lemma establishing that ETE is a consequence of the milder axiom of NMC.

Lemma 1. NMC *implies* ETE.

Proof. Let β be a bankruptcy rule satisfying IM, $\varepsilon_0 = (N^0, E, c^0) \in \mathcal{B}$, and $i, j \in N$ such that $c^0_i = c^0_j = \bar{c}$. Suppose, w.l.o.g., that

$$\beta_i(\varepsilon_0) > \beta_j(\varepsilon_0). \tag{1}$$

We will show that this assumption leads to a contradiction. To do it, we distinguish two cases:

Case 1: $|N^0| \geq 3$. The proof of this case is done in six steps.

Step 1: From N^0 player i splits into players i and i' defining the bankruptcy problem $\varepsilon_1 = (N^1, E, c^1)$, being $N^1 = N^0 \cup \{i'\}$, $c^1_k = c^0_k$ for all $k \in N^1 \setminus \{i, i'\}$, and $c^1_i = c^1_{i'} = \bar{c}/2$. By NMC, for all $k \in N^1 \setminus \{i, i'\}$,

$$\beta_k(\varepsilon_0) = \beta_k(\varepsilon_1), \tag{2}$$

which implies, by BB,

$$\beta_i(\varepsilon_0) = \beta_i(\varepsilon_1) + \beta_{i'}(\varepsilon_1). \tag{3}$$

Step 2: From N^1 player j splits into players j and j' defining the bankruptcy problem $\varepsilon_2 = (N^2, E, c^2)$, being $N^2 = N^1 \cup \{j'\}$, $c^2_k = c^1_k$ for all $k \in N^2 \setminus \{j, j'\}$, and $c^2_j = c^2_{j'} = \bar{c}/2$. Note that $c^2_j = c^2_i = c^2_{j'} = c^2_{i'} = \bar{c}/2$. By NMC, for all $k \in N^2 \setminus \{j, j'\}$,

$$\beta_k(\varepsilon_1) = \beta_k(\varepsilon_2), \tag{4}$$

which implies, by BB,

$$\beta_j(\varepsilon_1) = \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2). \tag{5}$$

Hence,

$$\beta_i(\varepsilon_2)+\beta_{i'}(\varepsilon_2) \underset{(4)}{=} \beta_i(\varepsilon_1)+\beta_{i'}(\varepsilon_1) \underset{(3)}{=} \beta_i(\varepsilon_0) \underset{(1)}{>} \beta_j(\varepsilon_0) \underset{(2)}{=} \beta_j(\varepsilon_1) \underset{(5)}{=} \beta_j(\varepsilon_2)+\beta_{j'}(\varepsilon_2). \tag{6}$$

That is,

$$\beta_i(\varepsilon_2)+\beta_{i'}(\varepsilon_2) > \beta_j(\varepsilon_2)+\beta_{j'}(\varepsilon_2). \tag{7}$$

Assume, w.l.o.g., that

$$\beta_i(\varepsilon_2) > \beta_j(\varepsilon_2). \tag{8}$$

Step 3: From N^2 players i, i' and j' merge under the name of i' defining the bankruptcy problem $\varepsilon_3=(N^3,E,c^3)$ being $N^3=N^2\backslash\{i,j'\}$, $c^3_k=c^2_k$ for all $k\in N^3\backslash\{i'\}$, and $c^3_{i'}=c^2_i+c^2_{i'}+c^2_{j'}=3\,\bar{c}/2$. Note that $c^3_j=c^2_j=\bar{c}/2$. By NMC, for all $k\in N^3\setminus\{i'\}$,

$$\beta_k(\varepsilon_3)=\beta_k(\varepsilon_2), \tag{9}$$

which implies, by BB,

$$\beta_{i'}(\varepsilon_3)=\beta_i(\varepsilon_2)+\beta_{i'}(\varepsilon_2)+\beta_{j'}(\varepsilon_2)=E-\beta_j(\varepsilon_2)-\sum_{k\in N^2\backslash\{i,j,i',j'\}}\beta_k(\varepsilon_2). \tag{10}$$

Step 4: From N^2 players i', j and j' merge under the name of i' defining the bankruptcy problem $\varepsilon_4=(N^4,E,c^4)$ being $N^4=N^2\backslash\{j,j'\}$, $c^4_k=c^2_k$ for all $k\in N^4\backslash\{i'\}$, and $c^4_{i'}=c^2_{i'}+c^2_j+c^2_{j'}=3\,\bar{c}/2$. By NMC, for all $k\in N^4\setminus\{i'\}$,

$$\beta_k(\varepsilon_4)=\beta_k(\varepsilon_2), \tag{11}$$

which implies, by BB,

$$\beta_{i'}(\varepsilon_4)=\beta_{i'}(\varepsilon_2)+\beta_j(\varepsilon_2)+\beta_{j'}(\varepsilon_2)=E-\beta_i(\varepsilon_2)-\sum_{k\in N^2\backslash\{i,j,i',j'\}}\beta_k(\varepsilon_2). \tag{12}$$

Hence,

$$\beta_{i'}(\varepsilon_4) \underset{(8)}{<} E-\beta_j(\varepsilon_2)-\sum_{k\in N^2\backslash\{i,j,i',j'\}}\beta_k(\varepsilon_2) \underset{(10)}{=} \beta_{i'}(\varepsilon_3). \tag{13}$$

Moreover, by BB,

$$E=\beta_i(\varepsilon_4)+\beta_{i'}(\varepsilon_4)+\sum_{k\in N^4\backslash\{i,i'\}}\beta_k(\varepsilon_4) \underset{(11)}{=} \beta_i(\varepsilon_4)+\beta_{i'}(\varepsilon_4)+\sum_{k\in N^4\backslash\{i,i'\}}\beta_k(\varepsilon_2), \tag{14}$$

and

$$\begin{array}{ccc} E & = & \beta_{i'}(\varepsilon_3)+\beta_j(\varepsilon_3)+\sum_{k\in N^3\backslash\{i',j\}}\beta_k(\varepsilon_3) \\ & \underset{(9)}{=} & \beta_{i'}(\varepsilon_3)+\beta_j(\varepsilon_3)+\sum_{k\in N^3\backslash\{i',j\}}\beta_k(\varepsilon_2) \\ & \underset{N^3\backslash\{i',j\}=N^4\backslash\{i,i'\}}{=} & \beta_{i'}(\varepsilon_3)+\beta_j(\varepsilon_3)+\sum_{k\in N^4\backslash\{i,i'\}}\beta_k(\varepsilon_2). \end{array} \tag{15}$$

From (14) and (15),

$$\beta_i(\varepsilon_4)+\beta_{i'}(\varepsilon_4)=\beta_{i'}(\varepsilon_3)+\beta_j(\varepsilon_3)$$

and from (13) we can conclude that

$$\beta_i(\varepsilon_4) > \beta_j(\varepsilon_3). \tag{16}$$

Step 5: From N^4 player i splits into players i and j defining the bankruptcy problem $\varepsilon_5=(N^5,E,c^5)$, being $N^5=N^4\cup\{j\}$, $c^5_k=c^4_k$ for all $k\in N^5\backslash\{i,j\}$, and $c^5_i=c^5_j=c^4_i/2=c^2_i/2=c^1_i/2=\bar{c}/4$. Note that, $c^5_{i'}=c^4_{i'}=3\bar{c}/2$. By NMC, for all $k\in N^5\setminus\{i,j\}$, $\beta_k(\varepsilon_5)=\beta_k(\varepsilon_4)$ which implies, by BB,

$$\beta_i(\varepsilon_4)=\beta_i(\varepsilon_5)+\beta_j(\varepsilon_5). \tag{17}$$

Step 6: From N^3 player j splits into players i and j defining the bankruptcy problem $\varepsilon_6 = (N^6, E, c^6)$, being $N^6 = N^3 \cup \{i\}$, $c^6_k = c^3_k$ for all $k \in N^6 \setminus \{i, j\}$, and $c^6_i = c^6_j = c^3_j/2 = c^2_j/2 = \bar{c}/4$. Note that, $c^6_{i'} = c^3_{i'} = 3\bar{c}/2$. Since $\varepsilon_6 = \varepsilon_5$, we have that

$$\begin{aligned} \beta_j(\varepsilon_3) \quad & \underset{(16)}{<} \quad \beta_i(\varepsilon_4) \\ & \underset{(17)}{=} \quad \beta_i(\varepsilon_5) + \beta_j(\varepsilon_5) \\ & \underset{\varepsilon_5=\varepsilon_6}{=} \quad \beta_i(\varepsilon_6) + \beta_j(\varepsilon_6), \end{aligned} \tag{18}$$

in contradiction with NMC. Thus, we conclude that

$$\beta_i(\varepsilon_0) = \beta_j(\varepsilon_0).$$

Hence, β satisfies ETE for $|N^0| \geq 3$.

Case 2: $|N^0| = 2$. The proof of this case is done in two steps.

Step 1: From N^0 player i splits into players i and i' defining the bankruptcy problem $\varepsilon_1 = (N^1, E, c^1)$, being $N^1 = \{i, i', j\}$, $c^1_i = c^1_{i'} = \bar{c}/2$, and $c^1_j = c^0_j = \bar{c}$. Since $|N^1| = 3$, by Case 1 we know hat β satisfy ETE. Hence, $\beta_i(\varepsilon_1) = \beta_{i'}(\varepsilon_1)$ and, by NMC,

$$\beta_j(\varepsilon_0) = \beta_j(\varepsilon_1). \tag{19}$$

By BB,

$$\beta_i(\varepsilon_0) = \beta_i(\varepsilon_1) + \beta_{i'}(\varepsilon_1) \underset{\text{ETE}}{=} 2\,\beta_i(\varepsilon_1). \tag{20}$$

Step 2: From N^1 player j splits into players j and j' defining the bankruptcy problem $\varepsilon_2 = (N^2, E, c^2)$, being $N^2 = \{i, i', j, j'\}$, $c^2_j = c^2_{j'} = c^1_j/2 = \bar{c}/2$, $c^2_i = c^1_i = \bar{c}/2$, and $c^2_{i'} = c^1_{i'} = \bar{c}/2$. Since $|N^1| \geq 3$, by Case 1 we know hat β satisfy ETE. Hence, $\beta_i(\varepsilon_2) = \beta_j(\varepsilon_2) = \beta_{i'}(\varepsilon_2) = \beta_{j'}(\varepsilon_2)$. By NMC,

$$\beta_i(\varepsilon_2) = \beta_i(\varepsilon_1),\ \beta_{i'}(\varepsilon_2) = \beta_{i'}(\varepsilon_1), \tag{21}$$

and, by BB,

$$\beta_j(\varepsilon_1) = \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2) \underset{\text{ETE}}{=} 2\,\beta_j(\varepsilon_2). \tag{22}$$

Finally, $\beta_j(\varepsilon_0) \underset{(19)}{=} \beta_j(\varepsilon_1) \underset{(22)}{=} 2\,\beta_j(\varepsilon_2)$ and $\beta_i(\varepsilon_0) \underset{(20)}{=} 2\,\beta_i(\varepsilon_1) \underset{(21)}{=} 2\,\beta_i(\varepsilon_2) \underset{\text{ETE}}{=} 2\,\beta_j(\varepsilon_2)$, which concludes the proof.

□

Under NMC, the following lemma states that if two agents have rational claims, then the ratio between what they receive and what they claim remains constant.

Lemma 2. *Let β be a bankruptcy rule satisfying* NMC. *If $(N, E, c) \in \mathcal{B}$ and $i, j \in N$ are such that $c_i, c_j \in \mathbb{Q}^+$, then*

$$\frac{\beta_i(N, E, c)}{c_i} = \frac{\beta_j(N, E, c)}{c_j}. \tag{23}$$

Proof. Let $(N, E, c) \in \mathcal{B}$ and $i, j \in$ such that $c_i, c_j \in \mathbb{Q}^+$, that is, $c_i = p_i/q_i$ and $c_j = p_j/q_j$ for some $p_i, q_i, p_j, q_j \in \mathbb{N}$. Then, $b\,c_i = a\,c_j$ being $a = p_i q_j$ and $b = q_i p_j$. Note that (23) is equivalent to $b\,\beta_i(N, E, c) = a\,\beta_j(N, E, c)$. We distinguish two cases:

Case 1: $b = 1$. That is, $c_i = a\,c_j$. Let $\varepsilon_1 = (N^1, E, c^1) \in \mathcal{B}$ where agent $i \in N$ splits into a identical agents $i, k_1, \ldots, k_{a-1}$, being $N^1 = N \cup \{k_1, \ldots, k_{a-1}\}$, $c_i^1 = c_{k_1}^1 = \ldots = c_{k_{a-1}}^1 = c_i/a$, and $c_l^1 = c_l$ for all $l \in N \setminus \{i\}$. In particular, $c_j^1 = c_j = c_i/a = c_i^1$. By NMC, which implies ETE (Lemma 1), we obtain

$$\beta_i(N^1, E, c^1) \underset{\text{ETE}}{=} \beta_j(N^1, E, c^1) \underset{\text{NMC}}{=} \beta_j(N, E, c)$$

and

$$\begin{aligned}\beta_i(N, E, c) &\underset{\text{NMC}}{=} \beta_i(N^1, E, c^1) + \beta_{k_1}(N^1, E, c^1) + \ldots + \beta_{k_{a-1}}(N^1, E, c^1)\\ &\underset{\text{ETE}}{=} a\,\beta_i(N^1, E, c^1)\\ &\underset{\text{ETE}}{=} a\,\beta_j(N^1, E, c^1)\\ &\underset{\text{NMC}}{=} a\,\beta_j(N, E, c).\end{aligned}$$

Case 2: $b > 1$. Assume, w.l.o.g., that $a < b$. Define $\varepsilon_1 = (N^1, E, c^1)$ where agent $j \in N$ splits into b identical agents $j, k_1, \ldots, k_{b-1}$, being $N^1 = N \cup \{k_1, \ldots, k_{b-1}\}$, $c_j^1 = c_{k_1}^1 = \ldots = c_{k_{b-1}}^1 = c_j/b$, and $c_l^1 = c_l$ for all $l \in N \setminus \{j\}$. In particular, $c_i^1 = c_i = \frac{a}{b} c_j = a\,c_j^1$. Hence, by Case 1,

$$\beta_i(N^1, E, c^1) = a\,\beta_j(N^1, E, c^1). \tag{24}$$

By NMC and ETE,

$$\beta_l(N, E, c) \underset{\text{NMC}}{=} a\,\beta_l(N^1, E, c^1) \text{ for all } l \in N \setminus \{j\}, \tag{25}$$

and

$$\beta_j(N, E, c) \underset{\text{NMC}}{=} \beta_j(N^1, E, c^1) + \beta_{k_1}(N^1, E, c^1) + \ldots + \beta_{k_{b-1}}(N^1, E, c^1) \underset{\text{ETE}}{=} b\,\beta_j(N^1, E, c^1). \tag{26}$$

Combining (24), (25), and (26) we conclude that $b\,\beta_i(N, E, c) = a\,\beta_j(N, E, c)$.

□

Now, we have all the tools to characterize the proportional rule in case all claims are zero or positive rational numbers.

Theorem 1. *Let β be a bankruptcy rule satisfying* NMC. *If $(N, E, c) \in \mathcal{B}$ is such that, for all $i \in N$, c_i is either zero or a positive rational number, then $\beta(N, E, c) = PR(N, E, c)$.*

Proof. Let $(N, E, c) \in \mathcal{B}$ such that all claims are either zero or positive rational numbers. If $|\{i \in N \mid c_i = 0\}| \in \{n-1, n\}$ then, by CB and BB, $\beta(N, E, c) = PR(N, E, c)$. Otherwise, let $i, j \in N$ such that $c_i, c_j \in \mathbb{Q}^+$. In this case, by Lemma 2 we have that $\beta_i(N, E, c)/c_i = \beta_j(N, E, c)/c_j$. Consequently, there exists a constant $r \geq 0$ such that, for all $k \in N$, $\beta_k(N, E, c)/c_k = r$. Finally, by BB $r = E/\sum_{k \in N} c_k$, and thus $\beta_i(N, E, c) = \frac{c_i}{\sum_{k \in N} c_k} E = PR_i(N, E, c)$ for all $i \in N$. □

A way to extend the above characterization including non rational claims is adding one of the following two widely accepted axioms. A bankruptcy rule β satisfies

- *claim monotonicity* (CM) if for all $(N, E, c), (N, E, c') \in \mathcal{B}$ such that $c_i' > c_i$ for some $i \in N$ and $c_j' = c_j$ for all $j \in N \setminus \{i\}$, then $\beta_i(N, E, c') \geq \beta_i(N, E, c)$;
- *claims continuity* (CC) if for all sequences of bankruptcy problems $\{(N, E, c^n)\}_{n \in \mathbb{N}}$ converging to (N, E, c), the sequence $\{\beta(N, E, c^n)\}_{n \in \mathbb{N}}$ converges to $\beta(N, E, c)$.

CM says that if an agent's claim increases, while the claims of the other agents remain equal, his award should not decreases. CC imposes that small variations in the claims imply small variations in the resulting allocation vector. CM and CC are not related to each other, for a discussion see Thomson (2019).

It is well known that the proportional rule satisfies both axioms. So, it remains to show that CM or CC in combination with NMC characterize it.

Theorem 2. *A bankruptcy rule satisfies* NMC *and* CM *if and only if it is the proportional rule.*

Proof. To show uniqueness, let β be a bankruptcy rule satisfying NMC and CM, and $(N, E, c) \in \mathcal{B}$. If c_i equals either zero or a rational number for all $i \in N$, by Theorem 1 $\beta(N, E, c) = PR(N, E, c)$.

Otherwise, we use and induction argument on the number of agents with a non-rational and non-zero claim. Let us denote this set by $N^C_{\neg\mathbb{Q}^+}$. For $|N^C_{\neg\mathbb{Q}^+}| = 1$, let $i^* \in N^C_{\neg\mathbb{Q}^+}$ be the unique agent with $c_{i^*} \notin \mathbb{Q}^+$. Let $\{l^k_{i^*}\}_{k\in\mathbb{N}}$ and $\{r^k_{i^*}\}_{k\in\mathbb{N}}$ be two sequences of rational numbers converging to c_{i^*} such that $l^k_{i^*} \leq l^{k+1}_{i^*} \leq r^{k+1}_{i^*} \leq r^k_{i^*}$ and $l^k_{i^*} < c_{i^*} < r^k_{i^*}$ for all $k \in \mathbb{N}$. Let $\{(N, E, \underline{c}^k)\}_{k\in\mathbb{N}}$ and $\{(N, E, \bar{c}^k)\}_{k\in\mathbb{N}}$ be two associated sequences of bankruptcy problems where, for all $k \in \mathbb{N}$, $\underline{c}^k_i = \bar{c}^k_i = c_i$ for all $i \in N \setminus \{i^*\}$, $\underline{c}^k_{i^*} = l^k_{i^*}$, and $\bar{c}^k_{i^*} = r^k_{i^*}$. By Theorem 1 and CM, for all $k \in \mathbb{N}$,

$$PR_{i^*}(N, E, \underline{c}^k) = \beta_{i^*}(N, E, \underline{c}^k) \leq \beta_{i^*}(N, E, c) \leq \beta_{i^*}(N, E, \bar{c}^k) = PR_{i^*}(N, E, \bar{c}^k).$$

By CC of the PR rule,

$$\lim_{k\to\infty} PR_{i^*}(N, E, \underline{c}^k) = PR_{i^*}(N, E, c) \leq \beta_{i^*}(N, E, c) \leq PR_{i^*}(N, E, c) = \lim_{k\to\infty} PR_{i^*}(N, E, \bar{c}^k),$$

which leads to $\beta_{i^*}(N, E, c) = PR_{i^*}(N, E, c)$.

It remains to see that $\beta_j(N, E, c) = PR_j(N, E, c)$ for all $j \in N \setminus N^C_{\neg\mathbb{Q}^+}$.

If $|\{j \in N \setminus N^C_{\neg\mathbb{Q}^+} \mid c_j \neq 0\}| \leq 1$, by CB and BB, $\beta(N, E, c) = PR(N, E, c)$. Otherwise, there are at least two players in $N \setminus N^C_{\neg\mathbb{Q}^+}$ with a positive rational claim. Thus, by Lemma 2, there exists a constant $r > 0$ such that, for all $j \in N \setminus N^C_{\neg\mathbb{Q}^+}$, $\beta_j(N, E, c) = r\, c_j$, which also holds for players with a zero claim. By BB,

$$E = \sum_{j\in N\setminus N^C_{\neg\mathbb{Q}^+}} \beta_j(N, E, c) + PR_{i^*}(N, E, c) = r \sum_{j\in N\setminus N^C_{\neg\mathbb{Q}^+}} c_j + \frac{c_{i^*}}{\sum_{j\in N} c_j} E,$$

which implies $r = E/\sum_{j\in N} c_j$, and hence $\beta(N, E, c) = PR(N, E, c)$.

Induction hypothesis: if $|N^C_{\neg\mathbb{Q}^+}| < k$, for all $1 \leq k \leq n-1$ then $\beta(N, E, c) = PR(N, E, c)$.

If $|N^C_{\neg\mathbb{Q}^+}| = k + 1$, select an arbitrary agent $i^* \in N^C_{\neg\mathbb{Q}^+}$ and, following the same arguments as in case $|N^C_{\neg\mathbb{Q}^+}| = 1$, construct two sequences of bankruptcy problems converging to (N, E, c). Then, by induction hypothesis, following identical reasoning we may conclude that $\beta_{i^*}(N, E, c) = PR_{i^*}(N, E, c)$. To see that $\beta_j(N, E, c) = PR_j(N, E, c)$ for all $j \in N \setminus N^C_{\neg\mathbb{Q}^+}$, apply the same arguments as for the case $|N^C_{\neg\mathbb{Q}^+}| = 1$.

□

To conclude, replacing CM by CC we obtain a new characterization.

Theorem 3. *A bankruptcy rule satisfies* NMC *and* CC *if and only if it is the proportional rule.*

Proof. Let β be a bankruptcy rule satisfying NMC and CC, and $(N, E, c) \in \mathcal{B}$. If, for all $i \in N$, $c_i \in \mathbb{Q}^+$ then, by Theorem 1, $\beta(N, E, c) = PR(N, E, c)$. Otherwise, let $\{(N, E, c^n)\}_{n\in\mathbb{N}}$ be a sequence of bankruptcy problems with $c^n_i \in \mathbb{Q}^+$ for all $i \in N$ and all $n \in \mathbb{N}$ converging to (N, E, c). Then, any subsequence of $\{(N, E, c^n)\}_{n\in\mathbb{N}}$ converges to (N, E, c). Hence, w.l.o.g., by CC we have that $PR(N, E, c) = \lim_{n\to\infty} PR(N, E, c^n) = \lim_{n\to\infty} \beta(N, E, c^n) = \beta(N, E, c)$. □

In the following remark we show the axioms in Theorems 3 and 2 are logically independent.

Remark 1. *The* CEA *rule satisfies* CM*, and thus* CC*, but not* NMC*. Now we define a bankruptcy rule* β^* *meeting* NMC *but not* CC*, and thus neither* CM*. Let* $(N, E, c) \in \mathcal{B}$*. If* $E = 0$*,* $\beta_i^*(N, E, c) = 0$ *for all* $i \in N$*. Otherwise, if* $c_i \in \mathbb{Q}^+$ *for all* $i \in N$*,* $\beta^*(N, E, c) = PR(N, E, c)$*. If* $N^* = \{i \in N \mid c_i \in \mathbb{R} \setminus \mathbb{Q}^+\} \neq \emptyset$ *we distinguish two cases. Case (a):* $\sum_{k \in N \setminus N^*} c_k \geq E$*, then* $\beta_i^*(N, E, c) = \frac{c_i}{\sum_{k \in N \setminus N^*} c_k} E$ *for all* $i \in N \setminus N^*$*, and* $\beta_i^*(N, E, c) = 0$ *for all* $i \in N^*$*. Case (b):* $\sum_{k \in N \setminus N^*} c_k < E$*, then* $\beta^*(N, E, c) = PR(N, E, c)$*.*

References

[1] Acosta-Vega, R.K., Algaba, E. and Sánchez-Soriano, J. (2022) On proportionality in multi-issue problems with crossed claims. Working Paper.

[2] Aumann, R. and Maschler M. (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory, 36: 195-213.

[3] Chun, Y. (1988) The proportional solution for rights problems. Mathematical Social Sciences, 15: 231-246.

[4] Csóka, P., Herings, PJ-J. (2021) An Axiomatization of the proportional rule in financial networks. Management Science, 67(5): 2799-2812.

[5] Curiel, I.J., Maschler, M. and Tijs, S.H. (1987) Bankruptcy games. Zeitschrift für Operations Research 31: A 143-A 159.

[6] de Frutos, M.A. (1999) Coalitional manipulations in a bankruptcy problem. Review of Economic Design, 4: 255-272.

[7] Ju, B.-G. (2003) Manipulation via merging and splitting in claims problems. Review of Economic Design, 8: 205-215.

[8] Ju, B.-G., Miyagawa, E. and Sakai, T. (2007) Non-manipulable division rules in claims problems and generalizations. Journal of Economic Theory, 132: 1-26.

[9] Moreno-Ternero, J.D. (2006) Proportionality and non-manipulability in bankruptcy problems. International Game Theory Review, 1: 127-139.

[10] O'Neill, B. (1982) A problem of rights arbitration from the Talmud, Mathematical Social Sciences, 2: 345-371.

[11] Thomson, W. (2015) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update. Mathematical Social Sciences, 74: 41-59.

[12] Young, H.P. (1987) On dividing an amount according to individual claims or liabilities. Mathematics of Operations Research 12(3): 398-414.