Amplification and displacement of chaotic attractors by means of unidirectional chaotic driving

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Chaotic systems, when used to drive copies of themselves (or parts of themselves) may induce interesting behaviors in the driven system. In case the later exhibits invariance under amplification or translation, they may show amplification (reduction), or displacement of the attractor. It is shown how the behavior to be obtained is implied by the symmetries involved. Two explicit examples are studied to show how these phenomena manifest themselves under perfect and imperfect coupling. [S1063-651X(98)13206-3]

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Pecora and Carroll [1] have reported a driving method (PCM for short) that allows synchronization between two identical chaotic systems. This has been successfully implemented in experiments; in particular, it has been observed in electric circuits, [2,3] and used in telecommunication devices [4]. Further developments have focused towards generalized forms of synchronization [5,6], understood as situations where the two systems evolve in perfect correlation despite that their distance in phase space does not go to zero, as in the Pecora and Carroll synchronization. Recently, there have appeared two interesting behaviors of a response under chaotic driving [7]: the amplification (reduction) of a chaotic attractor, and the sustained evolution of the driven system, repeating the drive attractor, in a region of phase space far from where the stable attractor lies. Because the driving scheme proposed is the same as that used by Pecora and Carroll for synchronization, there is no doubt about the possibility of preparing an experimental setting for their observation. Moreover, such driving situations might occur in nature (e.g., the case of neuronal systems has been the object of speculation [1,8]). The possibilities of getting amplified (shrunken) copies of a given chaotic signal, or copies of a system steadily evolving within variable ranges different from those imposed by its constrains, enriches the range of behaviors expected from driven chaotic systems, and then provides new tools for prediction, explanation, or application in science and technology. The aims of this report are to show how the behavior to be obtained is implied by the system symmetries, to study these systems to gain insight into how these phenomena manifest themselves, and to show that they are robust enough to be observed in experiments.

Consider a dissipative nonlinear dynamical system

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(\mathbf{x}), \tag{1}$$

with $\mathbf{x}\in\mathbb{R}^n$, whose evolution is chaotic and occurs in a strange attractor, $\mathcal{A}$ [9]. Divide the set of $\mathbf{x}$ coordinates in two subsets $\mathbf{x}_1\in\mathbb{R}^{n_1}$ and $\mathbf{x}_2\in\mathbb{R}^{n_2}$, where $n_1+n_2=n$, such that the latter is invariant under a set of transformations of coordinates, $T_{\mathbf{P}}$, that act only on the variables, $\mathbf{x}_2$. Then, the dynamical equations are rewritten as

$$\frac{d\mathbf{x}_1}{dt}=\mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2), \tag{2a}$$

$$\frac{d\mathbf{x}_2}{dt}=\mathbf{f}_2(\mathbf{x}_1,\mathbf{x}_2), \tag{2b}$$

In particular, given this decomposition, the $\mathbf{x}_2$ subsystem will be attracted to a set of points $\mathcal{A}_2\subset\mathbb{R}^{n_2}$.

The set of transformations of coordinates $T_{\mathbf{P}}:\mathbf{x}_2\rightarrow\mathbf{x}_2^*$, from $\mathbb{R}^{n_2}\rightarrow\mathbb{R}^{n_2}$, all have the same functional form, each particular transformation is specified by the values of a set of parameters $\mathbf{P}=(P_1,P_2,\ldots,P_q)$, which change continuously in a given interval, $P_i\in[a_i,b_i]\subset\mathbb{R}$, such that $0\in[a_i,b_i]$, to produce the different transformations of the set. It is assumed that these transformations are continuous in the sense that $|\mathbf{P}-\mathbf{P}'|\rightarrow 0$ implies $|T_{\mathbf{P}}(\mathbf{x}_2)-T_{\mathbf{P}'}(\mathbf{x}_2)|\rightarrow 0$ for all points $\mathbf{x}_2\in\mathcal{A}_2$. Moreover, $T_{\mathbf{P}}\rightarrow I$, when $\mathbf{P}\rightarrow 0$, where $\mathbf{I}$ is the identity. Under any one of this transformations Eq. (2b) remains unchanged; i.e., Eq. (2b) holds for $\mathbf{x}_2^*$ as well as for $\mathbf{x}_2$. Two particular transformations of that type will be studied: an amplitude transformation, defined by $T_A(\mathbf{x}_2)\equiv A\cdot\mathbf{x}_2$, with $A\in\mathbb{R}$, and a displacement transformation, defined by $T_{\mathbf{D}}(\mathbf{x}_2)\equiv\mathbf{x}_2+\mathbf{D}$, with $\mathbf{D}\in\mathbb{R}^{n_2}$.

In the PCM [1,8], for that type of system, the drive is described by Eqs. (2), and a copy of the symmetric subsystem $\mathbf{x}_2$, denoted by $\mathbf{x}_2'$, called the response, is prepared so that its dynamics is governed by the equation

$$\frac{d\mathbf{x}_2'}{dt}=\mathbf{f}_2(\mathbf{x}_1,\mathbf{x}_2'), \tag{3}$$

which is driven by the variables $\mathbf{x}_1$, that constitute the drive signal. The trajectory $\mathbf{x}_2'(t)=\mathbf{x}_2(t)$ is a solution of Eq. (3) if $\mathbf{x}_2(t)$ is a solution of Eqs. (2) in $\mathcal{A}_2$. However, for this synchronized state to be asymptotically stable, all Lyapunov exponents of the response, called conditional Lyapunov exponents, have to be negative. Synchronization can be tested by computing the largest conditional Lyapunov exponent Λ, which can be obtained from the time evolution of the drive as

$$\Lambda=\lim_{t\rightarrow\infty}\frac{1}{t}\ln\frac{|\delta\mathbf{x}_2(t)|}{|\delta\mathbf{x}_2(0)|}, \tag{4}$$

where $\delta\mathbf{x}_2\equiv|\mathbf{x}_2'-\mathbf{x}_2|$ is an infinitesimal deviation of $\mathbf{x}_2'$ from $\mathbf{x}_2$. If $\Lambda<0$, it is guaranteed that there will be asymptotically

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stable synchronization from an appropriate subset of initial conditions of drive and response [8,10].

For the type of symmetric systems studied in this paper, when the coordinate transformation $T_{\mathbf{P}}$ is applied to the $\mathbf{x}_2$ subsystem attractor, $\mathcal{A}_2$, one obtains a copy of it, $T_{\mathbf{P}}(\mathcal{A}_2)$, which somehow mimics $\mathcal{A}_2$. For the amplitude transformation $T_{\mathbf{P}}(\mathcal{A}_2)$ is an enlarged or shrunken copy of $\mathcal{A}_2$, while for the displacement transformation it is a fair copy of $\mathcal{A}_2$ displaced from the region where the stable attractor evolves. We know that when initial conditions of drive and response are such that $\mathbf{x}_2'(0)=\mathbf{x}_2(0)$ the evolution of $\mathbf{x}_2$ and $\mathbf{x}_2'$ is such that $\mathbf{x}_2'(t)=\mathbf{x}_2(t)$ for $t>0$. Therefore, because Eq. (2b) holds for $\mathbf{x}_2^*$, it must happen that if the response initial condition is $T_{\mathbf{P}}[\mathbf{x}_2(0)]$, it will evolve in $T_{\mathbf{P}}(\mathcal{A}_2)$ following a trajectory in perfect synchrony with $\mathbf{x}_2(t)$ that follows some type of copy of $\mathcal{A}_2$, amplified (shrunken) or displaced.

For these systems, the largest conditional Lyapunov exponent has to be zero. Because $T_{\mathbf{P}}$ is continuous, a single small perturbation applied to the response, when it is following a trajectory in $T_{\mathbf{P}}(\mathcal{A}_2)$, will send it to another trajectory, in $T_{\mathbf{P}+\delta\mathbf{P}}(\mathcal{A}_2)$, similar and close to the former where it will stay. Because $T_{\mathbf{P}}\to I$, when $\mathbf{P}\to 0$, one can have the points of $T_{\mathbf{P}}(\mathcal{A}_2)$ as close as the point of $\mathcal{A}_2$ as desired. Therefore, if the initial condition for the response is $T_{\mathbf{0}}[\mathbf{x}_2'(0)]$, i.e., $\mathbf{x}_2'(0)=\mathbf{x}_2(0)$, it will follow a trajectory in $T_{\mathbf{0}}(\mathcal{A}_2)=\mathcal{A}_2$ that exactly reproduces $\mathbf{x}_2(t)$. Then, a small perturbation applied to the response evolving in $\mathcal{A}_2$ will send it to a trajectory in $T_{\delta\mathbf{P}}(\mathcal{A}_2)$, with $\delta\mathbf{P}$ small, so that it is a close reproduction of the unperturbed trajectory in $\mathcal{A}_2$. Therefore, there is neither divergence, $\Lambda>0$, nor convergence, $\Lambda<0$, of close trajectories, and then, $\Lambda=0$.

In particular, for the case of the amplitude transformation, every initial condition of the response will evolve in a set, $T_A(\mathcal{A}_2)$, whose points are related to those of the original system attractor by means of $\mathbf{x}_2'=A\mathbf{x}_2$, with the value of A determined by the values of the initial condition. Therefore, $\delta\mathbf{x}_2(t)=|\mathbf{x}_2'(t)-\mathbf{x}_2(t)|=|A-1||\mathbf{x}_2(t)|$, and with $|\delta\mathbf{x}_2(0)|\simeq|A-1||\mathbf{x}_2(0)|$, with both $\mathbf{x}_2(t)$ and $\mathbf{x}_2(0)$ points in $\mathcal{A}_2$, it must be

$$\Lambda=\lim_{t\to\infty}\frac{1}{t}\ln\frac{|A-1||\mathbf{x}_2(t)|}{|\delta\mathbf{x}_2(0)|}=0 \tag{5}$$

because the argument in the logarithm is a fluctuating bounded quantity. For the case of the displacement transformation, every initial condition of the response will evolve in a set of points, $T_{\mathbf{D}}(\mathcal{A}_2)$, which is related to the original system attractor by means of $\mathbf{x}_2'=\mathbf{x}_2+\mathbf{D}$, with the value of $\mathbf{D}$ determined by the values of the initial condition. Therefore, $\delta\mathbf{x}_2(t)=|\mathbf{x}_2'(t)-\mathbf{x}_2(t)|=|\mathbf{D}|$, and with $|\delta\mathbf{x}_2(0)|\simeq|\mathbf{D}|$, it must be

$$\Lambda=\lim_{t\to\infty}\frac{1}{t}\ln\frac{|\mathbf{D}|}{|\delta\mathbf{x}_2(0)|}=0 \tag{6}$$

because the logarithm argument is a bounded constant.

The existence of this non-negative conditional Lyapunov exponent means that, in the present case, we will not have asymptotically stable behaviors; i.e., $|\mathbf{x}_2'(t)-T_{\mathbf{P}}[\mathbf{x}_2(t)]|\to 0$ for $|\mathbf{x}_2'(0)-T_{\mathbf{P}}[\mathbf{x}_2(0)]|\leqslant\delta$, as would occur if $\Lambda<0$. Instead we have what it is known as uniform stability [9], which is defined by the following condition: if $|\mathbf{x}_2'(0)-T_{\mathbf{P}}[\mathbf{x}_2(0)]|\leqslant\delta$ for some small positive real number δ, there must exist another small positive real number ε, such that $|\mathbf{x}_2'(t)-T_{\mathbf{P}}[\mathbf{x}_2(t)]|\leqslant\varepsilon$ for $t\geqslant 0$. It is to be noted that uniform stability, although weaker then asymptotic stability, is stronger than another type of stability, frequently found in nature and technology, orbital stability. This is defined [9] as the case in which if $|\mathbf{x}_2'(0)-T_{\mathbf{P}}[\mathbf{x}_2(0)]|\leqslant\delta$, there must exist an ε and some function $t^*(t)$ such that $|\mathbf{x}_2'(t^*)-T_{\mathbf{P}}[\mathbf{x}_2(t)]|\leqslant\varepsilon$ for $t\geqslant 0$; so, in this last case, there is no isochronous correspondence between the two time evolutions that one finds in the cases of asymptotic and uniform stability.

The amplitude transformation has been studied in the Lorenz model (LM for short) for convection in fluids [11] given by $\dot{x}=\sigma(y-x)$, $\dot{y}=(r-z)x-y$, and $\dot{z}=xy-bz$. The values of the parameters are $\sigma=16$, $r=45.92$, and $b=4$, as in [8]. The equations for x and y are invariant under an amplitude transformation $T_A(x,y)\equiv(Ax,Ay)$. Therefore, one can observe the amplification of the signal in the x-y plane. Then, the appropriate drive signal is z, and the response subsystem is described by

$$\dot{x}'=\sigma(y'-x'), \tag{7a}$$

$$\dot{y}'=(r-z)x'-y'. \tag{7b}$$

The displacement transformation is studied in the double scroll (DS for short), which is a model of an electric circuit [12,3]: $\dot{x}=\alpha[y-x-f(x)]$, $\dot{y}=x-y+z$, and $\dot{z}=-\beta y$, where $f(x)=bx+\frac{1}{2}(a-b)[|x+1|+|x-1|]$. The parameter values studied are $\alpha=10$, $\beta=14.87$, $a=-1.27$, and $b=-0.68$ as in [3]. The equations for x and z are invariant under $T_D(z)\equiv z+D$; therefore, one may obtain a displacement of the attractor in the z direction using as the response a copy of the (x,z) subsystem. Then, the drive variable will be y and the response equations will be given by

$$\dot{x}'=\alpha[y-x'-f(x')], \tag{8a}$$

$$\dot{z}'=-\beta y. \tag{8b}$$

The numerical results presented have been obtained by means of numerical integrations of the above equations using a fourth-order Runge-Kutta algorithm with a time step of 0.003 for LM, and 0.02 for DS.

The conditional Lyapunov exponents have been obtained from the integration of the variational equation

$$\frac{d(\delta\mathbf{x}_2)}{dt}=\mathbf{D}_{\mathbf{w}}(\mathbf{h}(\mathbf{x}_1,\mathbf{x}_2'))\delta\mathbf{x}_2, \tag{9}$$

where $\mathbf{D}_{\mathbf{w}}(\mathbf{h}(\mathbf{x}_1,\mathbf{x}_2))$ is the Jacobian of the response at $\mathbf{x}_2'=\mathbf{x}_2$, where the time evolution of $\mathbf{x}_1$ and $\mathbf{x}_2$ is given by Eqs. (2). In Fig. 1(a), for LM, and Fig. 1(b), for DS, there appear some representative results for the parameter dependence of the conditional Lyapunov exponents for the response, and the Lyapunov exponents for the original three-dimensional system. These figures show that the largest conditional Lyapunov exponent, being determined by the symmetries of

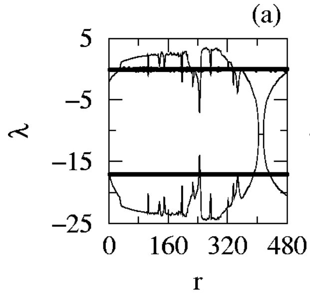

FIG. 1. Lyapunov exponents (thin lines) and conditional Lyapunov exponents (thick lines) as functions of the parameters of the system: (a) r dependence for LM, and (b) β dependence for DS.

the system, is null and does not change when the system parameters change, despite that this changes the behavior of the system.

To observe the amplification of the attractor, in LM, and the displacement, in DS, the appropriate equations of motion have been integrated in each case. The corresponding phenomena have been monitored by means of parametric plots of the variables of the response versus the variables of the drive, which appear as straight lines when both signals evolve in perfect synchrony. The amplification A is measured by the slope of the straight line, and the displacement D by its ordinate in the origin. A case for LM with initial conditions such that the $A \simeq 5$ is displayed in Fig. 2(a), which shows how the y' signal is an amplified copy, by a factor 5, of the y signal (a plot for the x signal would have to look almost the same). The results in Fig. 2(b), for DS with initial conditions appropriate to obtain $D \simeq 10$, show how $x'(t) = x(t)$, while $z'(t) \simeq z(t) + 10$. Similar calculations performed for the same systems using different initial conditions resulted in the same types of behaviors, but with different numerical values for A or D.

The magnitude of the amplification, or displacement, is determined by the initial conditions of the systems. Because of the uniform stability that characterizes this case, we have to expect a linear relation between A, or D, and the initial distance between the systems. To observe this dependence I have chosen a point in the stable attractor as a fixed initial condition for the drive, and studied the degree of amplification, or displacement, for initial conditions of the response evenly distributed on a rectangular grid in the x'-y' plane, for LM, or the x'-z' plane, for DS. For a given drive initial condition, I obtained that, for LM, the functions $A = A(x_0', y_0')$ are given by two symmetric planes that intersect the $A = 0$ plane along a straight line that crosses the origin of coordinates. The inclination and orientation of these planes, given respectively by the angle θ between the planes $A = A(x_0', y_0')$ and $A = 0$, and by the angle ϕ between the straight line made by the intersection between those planes and the x_0' axis, change with the values of the drive initial condition. For DS, $D = D(x_0', z_0')$ is given by a plane whose intersection with the $D = 0$ plane is a straight line parallel to the x_0' axis. The inclination and orientation of this plane does not change with the initial condition of the drive, while the distance between its intersection with the $D = 0$ plane and the x_0' axis, δ, changes with the drive initial condition. To explore those dependencies on drive initial conditions I have determined the functions $A = A(x_0', y_0')$ and $D = D(x_0', z_0')$ for sets of drive initial conditions made of points taken consecutively along a trajectory in the stable attractor. Figure 3(a), for $A(x_0', y_0')$, and Fig. 3(b), for $D(x_0', z_0')$, show that θ and ϕ, as well as δ, change continuously and smoothly as the drive evolves in its attractor.

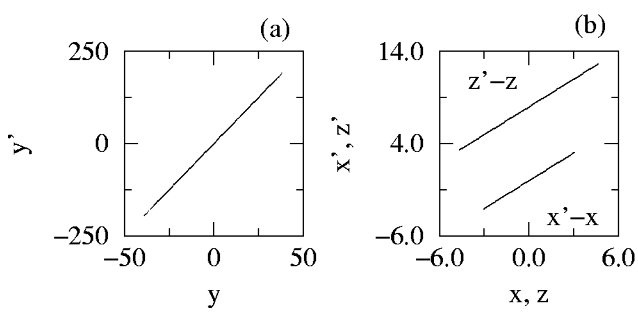

FIG. 2. Amplification and displacement monitored by means of parametric plots: (a) y signal for LM with initial conditions chosen to get $A \simeq 5$, and (b) x and z signals for DS with initial conditions chosen to obtain $D \simeq +10$.

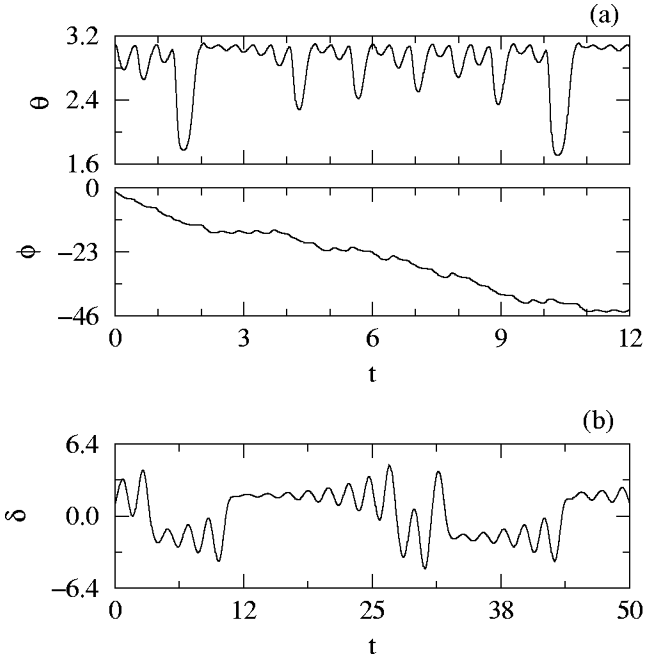

FIG. 3. (a) Time dependence of the angles, θ and ϕ, for LM. (b) The same for the distance δ for DS.

To study the behavior of the response trajectories in the presence of external noise, a series of time evolutions have been performed adding a Gaussian white noise δ_t to the drive signal. The control parameter in this study is the dispersion in the distribution of δ_t, σ, given in units of the amplitude of the signal considered. Calculations made for a fixed observational time window of 10^4 time steps (about 100 orbits) showed that, once the time window is fixed, there is a level of noise below which one obtains a response trajectory that accurately reproduces the drive attractor amplified (shrunken), for LM, or displaced, for DS. Once that level of noise is overcome these trajectories exhibit a diffusivelike behavior: for LM one obtains amplified copies of the drive attractor with an amplitude that fluctuates, and for DS one obtains displaced reproductions of the drive that shift up and down along the z axis. Moreover, I have studied the average size of the window, W, in which the response reproduction of the attractor is acceptable, as a function of the amplitude of the noise. The quantity W was defined as the average length of the time interval in which a fit to a straight line of $x'=x'(x)$ and $y'=y'(y)$ (for LM), or $x'=x'(x)$ and $z'=z'(z)$ (for DS) starts to fail for each level of noise. Such a breakdown of the fit was defined as the case when the correlation coefficient of the fit drops below 0.9999. This choice, although somewhat arbitrary, is based on results obtained for the dependence of this quantity with the level of noise for a fixed time window (10^4 time steps) as those displayed in Fig. 4(a). The results obtained for $W(\sigma_z)$ and $W(\sigma_y)$, displayed in Fig. 4(b), show that these functions are potential laws, and increase as the noise level decreases.

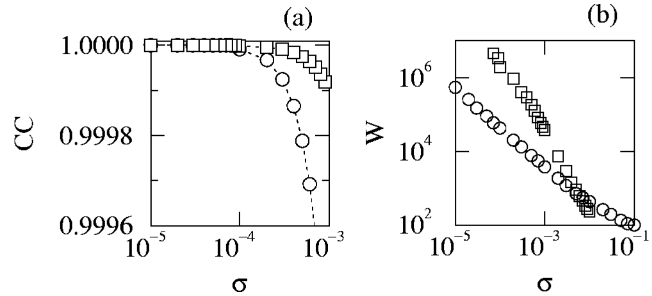

FIG. 4. (a) Breakdown of the synchronizationlike behavior measured by the correlation coefficient (CC) for LM (circles) and DS (squares). (b) Window size where the response is a fair copy of the drive attractor as a function of the amplitude of the noise for LM (circles) and DS (squares).

In conclusion, when chaotic systems that exhibit invariance properties under a special type of continuous transformation are subject to chaotic driving under a PCM, they can undergo interesting synchronizationlike phenomena. These include partial amplification of the attractor, and displacement of it to a region of phase space where the original system is unstable. The particular orbit is determined by the initial conditions. It is a consequence of these symmetries that the largest conditional Lyapunov has to be null; then, the stability is not asymptotic, but uniform. The numerical study of these phenomena has shown that under perfect coupling the largest conditional Lyapunov exponent is indeed null, it is possible to observe amplified or displaced trajectories in the computer simulations, and the degree of amplification or displacement obtained is smoothly dependent on the initial conditions. Computer simulations under situations of external noise have shown that the phenomena studied here can allow experimental observation.

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