Solution to the telegrapher's equation in the presence of reflecting and partly reflecting boundaries

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We show that the reflecting boundary condition for a one-dimensional telegrapher's equation is the same as that for the diffusion equation, in contrast to what is found for the absorbing boundary condition. The radiation boundary condition is found to have a quite complicated form. We also obtain exact solutions of the telegrapher's equation in the presence of these boundaries.

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The telegrapher's equation is the simplest example of a diffusionlike process which deals not only with the position of a particle but also with a crude form of momentum [1,2]. It has been applied to a number of problems in solid-state physics and thermodynamics [3,4] and has been suggested as a model for the diffusion of light in turbid media because it can describe effects of forward scattering [5,6]. The equation can be derived, at least in the case of one dimension, as a continuum limit of the persistent random walk [7]. It is also known that telegrapher's processes are equivalent to free processes driven by dichotomous Markov noise [8].

The telegrapher's equation has most often been solved for particles that diffuse in an unbounded space. More recently there has been some interest in the effects of boundaries on diffusion in which the probability density for the position of particles is found by solving a telegrapher's equation. This interest has at least two motivating considerations, the first being the purely mathematical one of characterizing the maximum value of such random variables as the maximum displacement of a particle at time t [9,10]. A second motivation for the study of the telegrapher's equation in the presence of boundaries is because of potential applications of such equations in the area of the medical use of laser light for diagnostic purposes [6].

We have recently derived the appropriate boundary conditions as well as the solution of the one-dimensional telegrapher's equation in the presence of one or two trapping points [11]. The boundary conditions differ in certain important aspects from those found for diffusion equation in the presence of trapping boundaries, and in fact the problem of finding the correct boundary conditions in dimensions greater than one has not, to our knowledge, been solved in generality.

In this paper we discuss the problem of a one-dimensional telegrapher's equation in the presence of either reflecting or partly reflecting boundary conditions. The reflecting boundary condition (which has been previously found in the context of dichotomous noise [12]) is shown to coincide with that for the standard diffusion process in the presence of a reflecting point. On the other hand, the radiation boundary condition, suitable for a partly reflecting boundary, is shown to have a similar structure to that of the absorbing boundary for telegrapher's processes. We obtain in both cases the solution of the telegrapher's equation.

In order to formulate the initial and boundary conditions to be imposed on the telegrapher's equation it is convenient to decompose the probability density for the position of the particle at time t into a sum of two terms. Let $X(t)$ be the position of the diffusing particle and let $a(x,t|x_0)$ be the probability density for the position of the particle at time t subject to $X(0)=x_0$ together with the condition that the particle is moving in the positive x direction at time t, and let $b(x,t|x_0)$ be the analogous function for decreasing $X(t)$. Let c be a velocity and T a time. A standard argument can be evoked to show that the functions a and b satisfy

$$\frac{\partial a}{\partial t} = -c\frac{\partial a}{\partial x} + \frac{1}{2T}(b-a), \tag{1a}$$

$$\frac{\partial b}{\partial t} = c\frac{\partial b}{\partial x} + \frac{1}{2T}(a-b). \tag{1b}$$

The probability density for the position of the particle independent of the direction in which it is moving at that time will be denoted by $p(x,t|x_0)$ and is given in terms of a and b by

$$p(x,t|x_0) = a(x,t|x_0) + b(x,t|x_0).$$

A simple consequence of Eqs. (1a) and (1b) is that the functions a, b, and p all satisfy the telegrapher's equation

$$\frac{\partial^2 p}{\partial t^2} + \frac{1}{T}\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial x^2}. \tag{2}$$

We now derive a radiation boundary condition for this equation, which must be used for problems involving partly reflecting boundaries. For this purpose it is expedient to consider first a persistent random walk on a

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lattice, whose continuum limit is the telegrapher's equation. Let $a_n(j)$ be the probability that a lattice random walk is at site j at step n, its position at the immediately preceding step having been $j-1$, and let $b_n(j)$ be the analogous probability with the immediately preceding position having been at $j+1$. Let μ be the probability that the direction of a step is the same as that of the preceding step. Then, in the absence of a boundary the discrete evolution equation is

$$a_{n+1}(j) = \mu a_n(j-1) + (1-\mu)b_n(j-1), \tag{3a}$$

$$b_{n+1}(j) = (1-\mu)a_n(j+1) + \mu b_n(j+1). \tag{3b}$$

The telegrapher's equation is obtained from this set by scaling the time by $t = n\Delta t$, space by $x = j\Delta x$, and probability μ by $\mu = 1 - \Delta t/(2T)$. Finally, let both Δt and Δx tend towards 0 subject to $\Delta x/\Delta t = c$.

Consider, in the discrete formulation, the effect of a point at $j = 0$ that traps a particle reaching it with probability equal to β, or reflects it with probability $1 - \beta$ where this reflection is immediate. Other situations where the "immediate reinjection" does not take place have been considered elsewhere [12]. The equation for $a_n(1)$ contains the effect of the partial reflection and can be written

$$a_{n+1}(1) = (1-\beta)b_n(0), \tag{4}$$

which, in the continuum limit, implies the boundary condition

$$a(0,t|x_0) = (1-\beta)b(0,t|x_0). \tag{5}$$

A trivial observation is that the known boundary condition for pure trapping or reflection corresponds, respectively, to setting β equal 1 or 0 in this relation.

Let us now find the radiation boundary condition for the total probability density function $p(x,t|x_0)$. We proceed as follows. From Eqs. (1a) and (1b) we get

$$\frac{\partial}{\partial t}(a-b) = -c\frac{\partial p}{\partial x} - \frac{1}{T}(a-b), \tag{6}$$

but at the boundary

$$p(0,t|x_0) \equiv a(0,t|x_0) + b(0,t|x_0) = (2-\beta)b(0,t|x_0)$$

and

$$a(0,t|x_0) - b(0,t|x_0) = -\frac{\beta}{2-\beta}p(0,t|x_0). \tag{7}$$

The substitution of Eq. (7) into Eq. (6) yields the radiation boundary condition:

$$c\left.\frac{\partial p}{\partial x}\right|_{x=0} = \frac{\beta}{2-\beta}\left.\left(\frac{\partial p}{\partial t} + \frac{1}{T}p\right)\right|_{x=0}. \tag{8}$$

In the case of an absorbing boundary $\beta = 1$ and from (8) we obtain the boundary condition for pure trapping:

$$c\left.\frac{\partial p}{\partial x}\right|_{x=0} = \left.\left(\frac{\partial p}{\partial t} + \frac{1}{T}p\right)\right|_{x=0}. \tag{9}$$

We note that this condition is equivalent to [cf. Eq. (5)]

$$a(0,t|x_0) = 0.$$

This is the condition for pure trapping, which was obtained in our previous work [11].

For the case of reflection $\beta = 0$ and we get

$$\left.\frac{\partial p}{\partial x}\right|_{x=0} = 0, \tag{10}$$

which is the same as that found in the case of ordinary diffusion. At this point we observe that in the case of a single reflecting point the boundary condition (10) can be derived by integrating the telegrapher's equation (2) and noting that with perfect reflection probability must be conserved. This global derivation does not apply to the case of two reflecting boundaries where, in addition to conservation of probability, another condition is needed.

A derivation of the initial conditions requires a return to the form of the equation given in Eqs. (1a) and (1b) in terms of separate components to take into account any asymmetry inherent in the formulation of the problem. If we suppose that the diffusing particle is initially at x_0, but that the probabilities of initial directions of motion are not necessarily equal, then we may write for the initial conditions

$$a(x,0|x_0) = \alpha\delta(x-x_0),$$
$$b(x,0|x_0) = (1-\alpha)\delta(x-x_0), \tag{11}$$

in which $0 \leq \alpha \leq 1$. Thus the probability density $p(x,t|x_0)$ must satisfy the obvious initial condition

$$p(x,0|x_0) = \delta(x-x_0). \tag{12}$$

A second initial condition is found by adding Eqs. (1a) and (1b). This yields the relation

$$\frac{\partial p}{\partial t} = -c\frac{\partial}{\partial x}(a-b),$$

which implies that the form of the second initial condition is

$$\left.\frac{\partial p}{\partial t}\right|_{t=0} = -(2\alpha-1)c\delta'(x-x_0). \tag{13}$$

The case of a single reflecting point at $x = 0$ is easily solved because the simple form of the boundary condition in Eq. (10) enables us to use the method of images to express the solution. Let $\Phi(x,t|x_0;\alpha)$ denote the solution to the telegrapher's equation in an unbounded space subject to the two initial conditions given earlier. The solution for $p(x,t|x_0)$ is then expressible as

$$p(x,t|x_0) = \Phi(x,t|x_0;\alpha) + \Phi(x,t|-x_0;1-\alpha). \tag{14}$$

The most expedient way to find $\Phi(x,t|x_0;\alpha)$ is to invert the joint Fourier-Laplace transform of Eq. (2). Let ω be the Fourier parameter and s be the Laplace parameter. The joint transform $\hat{p}(\omega,s|x_0)$ is then found as the sum of the transforms relevant for the free-space propagators

$$\hat{\Phi}(\omega, s|x_0;\alpha) = \frac{s + \frac{1}{T} + i(2\alpha - 1)c\omega}{s^2 + \frac{s}{T} + \omega^2 c^2} e^{-i\omega x_0}. \tag{15}$$

It is possible to invert the two transformations in closed form, which then gives

$$\Phi(x,t|x_0;\alpha) = e^{-t/(2T)} \Bigg\{ \alpha\delta(x - x_0 - ct) + (1-\alpha)\delta(x - x_0 + ct) + \frac{1}{4cT}\Theta\left(ct - |x - x_0|\right)\left[I_0(\lambda) + \frac{ct + (2\alpha - 1)(x - x_0)}{2cT}\,\frac{I_1(\lambda)}{\lambda}\right]\Bigg\}, \tag{16}$$

where Θ is the Heaviside step function, $I_0(\lambda)$ and $I_1(\lambda)$ are modified Bessel functions, and λ is the following function of t and x:

$$\lambda = \frac{\sqrt{c^2t^2 - (x - x_0)^2}}{2cT}. \tag{17}$$

In Fig. 1 we show some curves of the function $p(x,t|x_0)$ given by (14) as a function of x for two different times along with data obtained by means of the "exact enumeration method" [13]. At the earlier time ($t = 0.5T$) the diffusing particle has not reached the reflecting point and the profile is what one would calculate as the solution to the telegrapher's equation in free space. Since the telegrapher's equation has a wavelike term a discontinuous profile results from the reflection as is apparent from the figure.

Because of the relatively simple form of the reflecting boundary condition one can easily calculate an expression for $p(x,t|x_0)$ when there are two reflecting boundaries, one at $x = 0$ and the second at $x = L$. The solution can be found by a separation of variables and expressed in terms of Fourier cosine series. A straightforward calculation based on this approach suffices to show that the solution for $p(x,t|x_0)$ can be written in terms of the parameters

$$\theta_n = \left(\frac{1}{4T^2} - \frac{\pi^2c^2}{L^2}n^2\right)^{1/2} \tag{18}$$

as

$$p(x,t|x_0) = \frac{1}{L} + \frac{2}{L}e^{-t/(2T)}\sum_{n=1}^{\infty}\Bigg\{\left[\cosh(\theta_n t) + \frac{1}{2T\theta_n}\sinh(\theta_n t)\right]\cos\left(\frac{\pi n x_0}{L}\right) - \frac{\pi n(2\alpha - 1)c}{L\theta_n}\sinh(\theta_n t)\sin\left(\frac{\pi n x_0}{L}\right)\Bigg\}\cos\left(\frac{\pi n x}{L}\right). \tag{19}$$

As is well known, the Fourier series solution of a partial differential equation is better suited for obtaining its long-time behavior. Thus, in Fig. 2 we show the solution (19) when $t \gg T$. Note from the figure that at $t = 9T$ the density $p(x,t|x_0)$ is practically equal to the equlibrium value $1/L$, except for the δ-function terms, which still survive but with an intensity decreased by the factor $e^{-t/(2T)}$.

In the case that one is interested in the short-time behavior of $p(x,t|x_0)$ it is more convenient to use an alternative solution which can be also found by the method of images. A straightforward calculation yields the expression

$$p(x,t|x_0) = \sum_{n=0}^{\infty}\Big\{\Phi(x,t|x_0 - 2nL;\alpha) + \Phi(x,t|-x_0 - 2nL; 1-\alpha)\Big\} + \sum_{n=1}^{\infty}\Big\{\Phi(x,t|x_0 + 2nL;\alpha) + \Phi(x,t|-x_0 + 2nL; 1-\alpha)\Big\}, \tag{20}$$

where $\Phi(x,t|x_0;\alpha)$ is given by Eq. (16) (see Fig. 3).

We will now solve the case of a single partly absorbing point at $x = 0$ assuming, without loss of generality, that $x \geq 0$, $x_0 \geq 0$. A convenient way of finding the solution in this case is by means of the Laplace transformed version of the telegrapher's equation (2) together with the boundary condition (8), which reduces the problem to solving an ordinary differential equation. If we assume symmetric initial conditions ($\alpha = 1/2$), this solution can be written in the form

$$p(x,t|x_0) = e^{-t/(2T)}P(x,t|x_0), \tag{21}$$

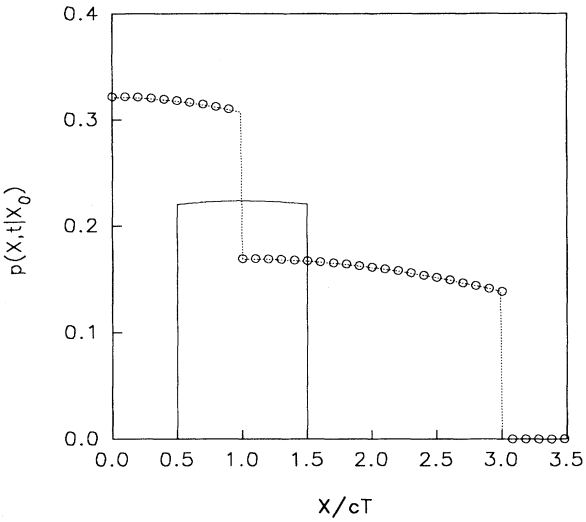

FIG. 1. Two graphs of solutions to telegrapher's equation with a single reflecting point at $x = 0$. The initial position is at $x_0 = cT$, and the solid line corresponds to the solution for a time before the particle can reach 0 ($ct < x_0$) while the dashed curve is a representative solution after that time. Because of the wavelike character of the equation there is a step discontinuity as a result of a reflection from the origin. Circles represent data from the exact enumeration method. Note that we have omitted the δ-function contribution to the curves and that this omission is responsible for the apparent lack of conservation of probability shown in the figure.

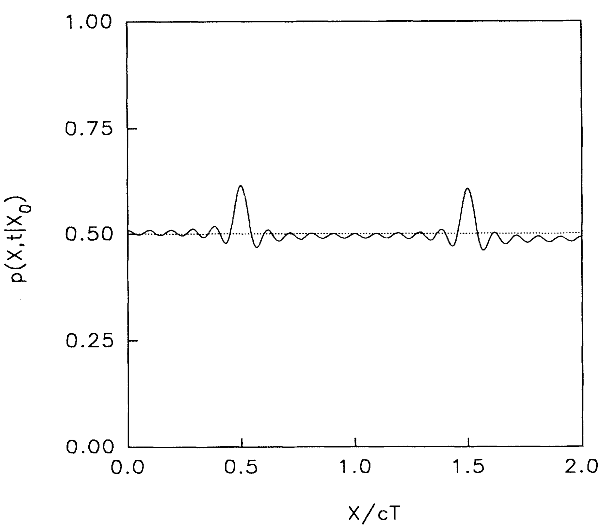

FIG. 2. The solid line is the solution to telegrapher's equation with two reflecting points at $x = 0$ and $x = 2cT$ when $t = 9T$ as calculated from the Fourier series (19) with $n = 40$. The initial position is at $x_0 = 0.5cT$. The δ-function terms are represented by the two spikes appearing at $x = 0.5cT$ and $x = 1.5cT$. The dotted line corresponds to the stationary density.

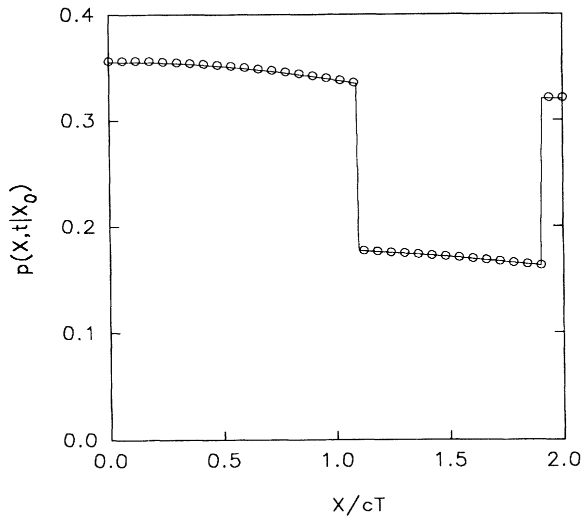

FIG. 3. Solution to telegrapher's equation with two reflecting points at $x = 0$ and $x = 2cT$ when $t = 1.6T$ as calculated from Eq. (20) without δ-function terms. The initial position is at $x_0 = 0.5cT$. Circles represent data from the exact enumeration method.

where the Laplace transform of $P(x, t|x_0)$ reads

$$\hat{P}(x, s|x_0) = \frac{s + 1/(2T)}{2\rho(s)}\left[e^{-\rho(s)|x-x_0|} - \Omega(s)e^{-\rho(s)(x+x_0)}\right], \tag{22}$$

where

$$\rho(s) = \sqrt{s^2 - 1/(4T^2)} \tag{23}$$

and

$$\Omega(s) = \frac{\beta\,[s + 1/(2T)] - (2-\beta)\rho(s)}{\beta\,[s + 1/(2T)] + (2-\beta)\rho(s)}. \tag{24}$$

When $\beta = 1$ one obtains, after inverting the Laplace transform, the solution for pure trapping [11]:

$$p_a(x, t|x_0) = e^{-t/(2T)}\left[f_0(t, |x - x_0|) - f_1(t, x + x_0)\right], \tag{25}$$

where

$$f_0(t, x) = \frac{1}{2}\delta(ct - x) + \frac{\Theta(ct - x)}{4cT}\left[I_0(u) + \frac{t}{2uT}I_1(u)\right] \tag{26}$$

and

$$f_1(t, x) = \frac{\Theta(ct - x)}{8cT}\left[I_0(u) + 2\left(\frac{ct - x}{ct + x}\right)^{1/2} I_1(u) + \left(\frac{ct - x}{ct + x}\right) I_2(u)\right], \tag{27}$$

where

$$u = \frac{\sqrt{c^2t^2 - x^2}}{2cT}. \tag{28}$$

In the case of pure reflection we have $\beta = 0$ and Eq. (21) reduces to Eq. (14) with $\alpha = 1/2$, that is,

$$p_r(x, t|x_0) = e^{-t/(2T)} \left[f_0(t, |x - x_0|) + f_0(t, x + x_0) \right]. \tag{29}$$

When $\beta \neq 1$ we show in the Appendix that the inverse transform of Eq. (22) reads

$$p(x, t|x_0) = \beta p_a(x, t|x_0) + (1 - \beta) p_r(x, t|x_0) - \frac{1}{8}\beta(1 - \beta)e^{-t/(2T)} \sum_{n=0}^{\infty} c_n g_n(t, x + x_0), \tag{30}$$

where $p_a(x, t|x_0)$ and $p_r(x, t|x_0)$ are given by Eqs. (25) and (29), and

$$c_0 = 1,$$
$$c_1 = (4 - \beta),$$
$$c_2 = (2 - \beta)(3 - \beta) + 1,$$
$$c_n = (2 - \beta)^3(1 - \beta)^{n-3}, \qquad n \geq 3,$$

$$g_n(t, y) = \Theta(ct - y) \left(\frac{ct - y}{ct + y} \right)^{n/2} I_n \left(\frac{\sqrt{c^2t^2 - y^2}}{2cT} \right), \tag{31}$$

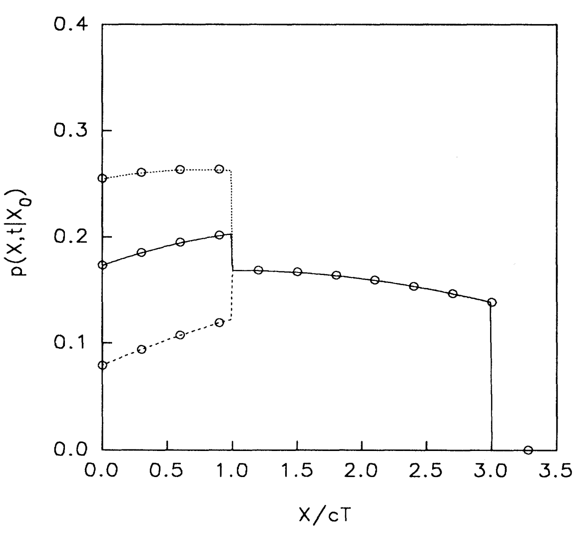

FIG. 4. Three graphs of solutions to telegrapher's equation with a single partly reflecting point at $x = 0$ and when $t = 2T$. The initial position is at $x_0 = cT$. Dotted line corresponds to $\beta = 0.2$, solid line to $\beta = 0.5$, and dashed line to $\beta = 1$. Note that for $x > cT$ all graphs merge because of the finite speed of propagation c. Circles represent data from the exact enumeration method.

where $I_n(z)$ are modified Bessel functions.

In Fig. 4 we plot Eq. (30) for three values of β. Note that when $\beta \neq 1$ the δ-function peaks traveling with the wave front are also partly reflected, although they are not shown in the figure.

As a final remark we mention that a derivation of boundary conditions for the telegrapher's equation in a greater number of dimensions gets us into difficulties generally associated with boundaries related to transport equations (cf., for example, the discussion in [14]).

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APPENDIX: DERIVATION OF EQ. (30)

The starting point of our derivation will be Eq. (24). If we set

$$\beta = 1 - \eta \qquad (0 \leq \eta \leq 1), \tag{A1}$$

and observe that

$$\frac{[s + 1/(2T)] - \rho(s)}{[s + 1/(2T)] + \rho(s)} = 2T\left[s - \rho(s)\right], \tag{A2}$$

then Eq. (24) can be written as

$$\Omega(s) = \frac{2T\left[s - \rho(s)\right] - \eta}{1 - 2\eta T\left[s - \rho(s)\right]}. \tag{A3}$$

One can easily see from Eq. (A2) that

$$2T|s - \rho(s)| \leq 1,$$

which allows us to expand the denominator of Eq. (A3) in powers of η, with the result

$$\Omega(s) = \sum_{n=0}^{\infty} \left\{ \eta^n \left(2T\left[s - \rho(s)\right]\right)^{n+1} - \eta^{n+1} \left(2T\left[s - \rho(s)\right]\right)^n \right\}. \tag{A4}$$

The substitution of this equation into Eq. (22) yields

$$p(x, t|x_0) = f_0(t, |x - x_0|) + \eta f_0(t, x + x_0) - \sum_{n=1}^{\infty} (1 - \eta^2)\eta^{n-1} f_n(t, x + x_0), \tag{A5}$$

where the functions $f_n(t, x)$ are defined as follows [11,15]:

$$f_n(t,x) \equiv \mathcal{L}^{-1}\left\{\left[\frac{s+1/(2T)}{2\rho(s)}\right](2T[s-\rho(s)])^n \times \exp[-\rho(s)x]\right\}, \tag{A6}$$

and can be expressed in terms of functions $g_n(t,x)$ as [cf. Eq. (31)]

$$f_n(t,x) = \frac{1}{8}[g_{n-1}(t,x) + 2g_n(t,x) + g_{n+1}(t,x)], \quad n \geq 1, \tag{A7}$$

and $f_0(t,x)$ as in Eq. (26).

The substitution of Eq. (A7) into Eq. (A5) finally yields Eq. (30).

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